QUANTUM FIELD THEORY: AN INTRODUCTION

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Contents

The construction of Quantum Field Theories is one of main intellectual achievement of the 20th century physics. In particular, Quantum Field Theories have played an essential role in in two apparently unrelated domains of physics: the theory of fundamental interactions at the microscopic scale, which has taken the form of the so-called the Standard Model, and the theory of continuous macroscopic phase transitions.

Relativistic quantum field theory, although it incorporates both the theory of Relativity and Quantum Mechanics, is not a straightforward extension of non-relativistic quantum mechanics in the sense that the dynamical variables are not related to individual particles but to fields and fields have an infinite number of degrees of freedom.

This is the origin of unusual properties, like the appearance of infinities in a straightforward interpretation of quantum field theory.

The problem of the removal of infinities has then led to the method of renormalization, a method in which the initial parameters of the Lagrangian are replaced by physical observables. From the freedom of defining the parameters of the renormalized theory at different momentum scales emerged another important concept: the renormalization group (RG).

Much insight has been gained later from the study of continuous phase transitions, where the physics is initially defined in terms of a microscopic model that has no short distance singularities.

However, the large distance physics near the transition can be derived from an effective quantum field theory in imaginary (Euclidean) time equipped with a short-distance cut-off, reflection of the defining microscopic scale of the initial theory.

Both the large momentum divergences of the quantum field theory and the singularities of phase transitions at the critical temperature are a reflection of the non-decoupling of large distance physics from microscopic scales. A more general RG, based on a recursive averaging over short distance degrees of freedom, has provided the necessary tool to understand why the large distance physics is nevertheless to a large extent insensitive to the specific form of the short distance structure or regularization.

The RG of quantum field theory can now be understood as the asymptotic form of the general RG in some neighbourhood of the Gaussian fixed point or free field theory in the particle terminology.

The first set of lectures is mainly devoted to the scalar boson and Dirac fermion relativistic fields. Then, we will discuss gauge theories and topics relevant to the Standard Model.

As the programme tries to indicate, we will adopt a constructive, and not only descriptive, approach to quantum field theory.

We will emphasize its characteristic features, beyond a specific realization.

For this purpose, some tools are needed. One important tool is functional integration, that is, integration over paths or fields together with related functional methods, motivated by Feynman path integral representation of quantum mechanics.

Finally, most of the time, the convention of imaginary or Euclidean time will be used, that is, we will deal with the operator $\mathrm{e}^{-\beta H}$ of statistical physics rather that the evolution operator $\mathrm{e}^{-itH/\hbar}$. The reasons will be explained later. The quantum evolution is then obtained by analytic continuation $\beta \mapsto it/\hbar$.

For an introductory note on path integrals in physics see, for example,

J. Zinn-Justin, *Path integral,* Scholarpedia, 4(2): 8674 (2009) (www.scholarpedia.org).

For details and more references see, for example,

J. Zinn-Justin, *Path integrals in Quantum Mechanics*, French version EDP Sciences et CNRS Editions (Les Ulis 2003), English version Oxford Univ. Press (Oxford 2005); Russian translation (Fizmatlit 2007).

In www.scholarpedia.org, see

Jean Zinn-Justin (2010), *Critical Phenomena: field theoretical approach*, Scholarpedia, 5(5):8346.

More advanced material can be found in

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989 (Oxford 4th ed. 2002).

Lecture 1: RANDOM WALK, CONTINUUM LIMIT AND RENORMALIZATION GROUP

As an introduction, we first illustrate the notion of universality and continuum limit with the elementary example of random walk.

The asymptotic properties at large time and large distance of the random walk can be easily determined by methods reminiscent of the proof of the central limit theorem of probabilities.

The universality of the large scale behaviour and, correspondingly, the existence of a macroscopic continuum limit, emerges as a collective property of systems involving a large number of microscopic random variables, when the probability of large steps decreases fast enough (the property of locality).

We revisit the problem with RG inspired methods, introducing in this way the RG terminology. The main result that the large scale behaviour is universal and defines a continuum limit, the Brownian motion, that can be described by a Gaussian path integral, will be recovered.

1.1 Translation invariant local random walk

We consider a stochastic process, a random walk, in discrete times, first on the real axis and then, briefly, on the lattice of points with integer coordinates.

The random walk is specified by a probability distribution $P_0(q)$ (q being a position) at initial time $n = 0$ and a density of transition probability $\rho(q, q') \geq 0$ from the point q' to the point q, which we assume independent of the (integer) time n.

These conditions define a Markov chain, a Markovian process, in the sense that the displacement at time n depends only on the position at time n, but not on the positions at prior times, homogeneous or stationary, that is, invariant under time translation, up to the boundary condition.

1.1.1 Walk in continuum space

The probability distribution $P_n(q)$ for a walker to be at point q at time n satisfies the evolution equation

$$P_{n+1}(q) = \int \mathrm{d}q' \, \rho(q, q') P_n(q') \,.$$

Probability conservation implies

$$\int \mathrm{d}q \, \rho(q, q') = 1 \,. \tag{1.1}$$

To slightly simplify the analysis, we take as initial distribution

$$P_0(q) = \delta(q).$$

where δ is Dirac's distribution.

Translation symmetry. We have already assumed ρ independent of n and, thus, the random walk transition probability is invariant under time translation. In addition, we now assume that the transition probability is also invariant under space translation and, thus,

$$\rho(q, q') \equiv \rho(q - q').$$

As a consequence, the evolution equation takes the form of the convolution equation,

$$P_{n+1}(q) = \int \mathrm{d}q'\, \rho(q - q')P_n(q'),$$

which also appears in the discussion of the central limit theorem of probabilities.

Locality. We consider only transition functions piecewise differentiable and with bounded variation, and satisfying a property of locality in the form of an exponential decay: qualitatively, large displacements have a very small probability. More precisely, we assume that the transition probabilities $\rho(q)$ satisfy a bound of the exponential type,

$$\rho(q) \leq M \, \mathrm{e}^{-A|q|}, \quad M, A > 0 \,.$$

1.1.2 Fourier representation

The evolution equation is an equation that simplifies after Fourier transformation. We thus introduce

$$\tilde{P}_n(k) = \int \mathrm{d}q \; \mathrm{e}^{-ikq} \, P_n(q) \,,$$

which is a generating function of the moments of the distribution $P_n(q)$.

The reality of $P_n(q)$ and the normalization of the total probability imply

$$\tilde{P}_n^*(k) = \tilde{P}_n(-k), \quad \tilde{P}_n(k=0) = 1\,.$$

Similarly, we introduce

$$\tilde{\rho}(k) = \int \mathrm{d}q \; \mathrm{e}^{-ikq} \, \rho(q),$$

which is a generating function of the moments of the distribution $\rho(q)$. Finally, the exponential decay condition implies that the function $\tilde{\rho}(k)$ is a function analytic in the strip $|\operatorname{Im} k| < A$.

The evolution equation then becomes

$$\tilde{P}_{n+1}(k) = \tilde{\rho}(k)\tilde{P}_n(k)$$

and since with our choice of initial conditions $\tilde{P}_0(k) = 1$,

$$\tilde{P}_n(k) = \tilde{\rho}^n(k).$$

1.1.3 Generating function of cumulants

We also introduce the function

$$w(k) = \ln \tilde{\rho}(k) \Rightarrow \ w^*(k) = w(-k), \quad w(0) = 0 \,, \qquad (1.2)$$

a generating function of the cumulants of $\rho(q)$. The analyticity of $\tilde{\rho}(k)$ and the condition $\tilde{\rho}(0) = 1$ imply that $w(k)$ is analytic at $k = 0$. It has an expansion at $k = 0$ of the form

$$w(k) = -iw_1 k - \frac{1}{2} w_2 k^2 + \sum_{r=3} \frac{(-i)^r}{r!} w_r k^r \,, \ w_2 > 0 \,,$$

where w_r is the rth cumulant. Then,

$$\tilde{P}_n(k) = \mathrm{e}^{nw(k)} \,.$$

1.1.4 Random walk: Asymptotic behaviour from a direct calculation

With the hypotheses satisfied by P_0 and ρ, the determination of the asymptotic behaviour for $n \to \infty$ follows from arguments identical to those leading to the central limit theorem of probabilities. One finds the asymptotic behaviour

$$P_n(q) \sim \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq - n(iw_1 k + w_2 k^2/2)} = \frac{1}{\sqrt{2\pi n w_2}} \, \mathrm{e}^{-(q - nw_1)^2/2nw_2} \, .$$

At q fixed, the probability converges exponentially to zero for all $w_1 \neq 0$. By contrast, the random variable $s = q/n$ has the asymptotic distribution

$$R_n(s) = nP_n(ns) \underset{n \to \infty}{\sim} \sqrt{\frac{n}{2\pi w_2}} \, \mathrm{e}^{-n(s - w_1)^2/2w_2} \, . \tag{1.3}$$

The average value s is thus a random variable that converges with probability 1 toward the expectation value $\langle s \rangle = w_1$ (the mean velocity).

Finally, the random variable that characterizes the deviation with respect to the mean trajectory,

$$X = (s - w_1)\sqrt{n} = \frac{q}{\sqrt{n}} - w_1\sqrt{n}, \qquad (1.4)$$

and thus $\langle X \rangle = 0$, has, as limiting distribution, the universal Gaussian distribution

$$L_n(X) = \sqrt{n}P_n(nw_1 + X\sqrt{n}) \sim \frac{1}{\sqrt{2\pi w_2}}\, e^{-X^2/2w_2}\,.$$

The neglected terms are of two types, multiplicative corrections of order $1/\sqrt{n}$ and additive corrections decreasing exponentially with n.

The result implies that the mean deviation from the mean trajectory increases as the square root of time, a characteristic property of the Brownian motion.

1.1.5 Continuum time limit

The asymptotic Gaussian distribution of the deviation $\bar{q} = q - nw_1$ from the mean trajectory is

$$P_n(\bar{q}) \sim \frac{1}{\sqrt{2\pi n w_2}}\, \mathrm{e}^{-\bar{q}^2/2n w_2} \,.$$

By changing the time scale and by a continuous interpolation, one can define a diffusion process or Brownian motion in continuous time.

Let t and ε be two real positive numbers and n the integer part of t/ε:

$$n = [t/\varepsilon] \,. \tag{1.5}$$

One then takes the limit $\varepsilon \to 0$ at t fixed and thus $n \to \infty$.

If the time t is measured with a finite precision Δt, as soon as $\Delta t \gg \varepsilon$, time can be considered as a continuous variable for what concerns all expectation values continuous functions of time.

One then performs the change of distance scale

$$\bar{q} = x/\sqrt{\varepsilon}.$$

Since the Gaussian function is continuous, the limiting distribution takes the form

$$P_n(q)/\sqrt{\varepsilon} \underset{\varepsilon \to 0}{\sim} \Pi(t,x) = \frac{1}{\sqrt{2\pi t w_2}} \, \mathrm{e}^{-x^2/2tw_2} \, . \qquad (1.6)$$

(The change of variables $q \mapsto x$ implies a change of normalization of the distribution.) This distribution is a solution of the diffusion or heat equation:

$$\frac{\partial}{\partial t}\Pi(t,x) = \tfrac{1}{2}w_2 \frac{\partial^2}{(\partial x)^2}\Pi(t,x).$$

In the limit $n \to \infty$ and in suitable macroscopic variables, one thus obtains a universal diffusion process that can entirely be described in continuum time and space.

The limiting distribution $\Pi(t, x)$ implies a scaling property characteristic of the Brownian motion. The moments of the distribution satisfy

$$\langle x^{2m} \rangle = \int \mathrm{d}x \, x^{2m} \Pi(t, x) \propto t^m. \qquad (1.7)$$

The variable $x/\sqrt{t}$ has time-independent moments. As the change $\bar{q} = x/\sqrt{\varepsilon}$ also indicates, one can thus assign to the position x a dimension $1/2$ in time unit (this also corresponds to assign a Hausdorff dimension two to a Brownian trajectory in higher dimensions).

Remark. The dimensional relation between the dynamical variable x and the coordinate t will be called power counting in quantum field theory.

13

1.1.6 Corrections to continuum limit

One can also study how perturbations to the limiting Gaussian distribution decrease with ε.

We can express the distribution of q in terms of $w(k) = \ln \tilde{\rho}(k)$. Correspondingly, we introduce $\bar{w}(k)$ the equivalent quantity for $\bar{q} = q - nw_1$. We then obtain the relation

$$\int \mathrm{d}k \; \mathrm{e}^{ik\bar{q}+iknw_1} \, \mathrm{e}^{nw(k)} = \int \mathrm{d}k \; \mathrm{e}^{ik\bar{q}+n\bar{w}(k)}$$

with

$$\bar{w}(k) = w(k) + ikw_1 \, .$$

With our assumptions, the expansion of the regular function $\bar{w}(k)$ in powers of k reads

$$\bar{w}(k) = -\frac{1}{2}w_2 k^2 + \sum_{r=3} \frac{(-i)^r}{r!} w_r k^r \, .$$

After the introduction of macroscopic variables, which for the Fourier variables corresponds to $k = \kappa\sqrt{\varepsilon}$, one finds

$$n\bar{w}(k) = t\omega(\kappa) \text{ with } \omega(\kappa) = -\frac{w_2}{2!}\kappa^2 + \sum_{r=3} \varepsilon^{r/2-1}\frac{(-i)^r}{r!}w_r\kappa^r \,.$$

One observes that, when $\varepsilon = t/n$ goes to zero, each additional power of κ goes with an additional power of $\sqrt{\varepsilon}$.

In the continuum limit, the distribution becomes

$$\Pi(t,x) = \frac{1}{2\pi}\int \mathrm{d}\kappa\, \mathrm{e}^{-i\kappa x}\, \mathrm{e}^{tw(\kappa)} \,.$$

Differentiating with respect to the time t, one obtains

$$\frac{\partial}{\partial t}\Pi(t,x) = \frac{1}{2\pi}\int \mathrm{d}\kappa\, w(\kappa)\, \mathrm{e}^{-i\kappa x}\, \mathrm{e}^{tw(\kappa)}$$

and in $w(\kappa)$, κ can then be replaced by the differential operator $i\partial/\partial x$.

One thus finds that $\Pi(t, x)$ satisfies the linear 'partial differential equation'

$$\frac{\partial}{\partial t} \Pi(t, x) = \left[\frac{w_2}{2!} \left(\frac{\partial}{\partial x} \right)^2 + \sum_{r=3} \varepsilon^{r/2-1} \frac{1}{r!} w_r \left(\frac{\partial}{\partial x} \right)^r \right] \Pi(t, x).$$

In the expansion, each additional derivative implies an additional factor $\sqrt{\varepsilon}$ and, thus, the contributions that contain more derivatives decrease faster to zero.

1.2 Universality: the renormalization group strategy

We now explain the universality property, that is, the existence of a limiting Gaussian distribution that is independent of the initial distribution, and its scaling properties by a quite different method, that is, without calculating the asymptotic distribution explicitly.

For simplicity, we assume that initially the number of time steps is of the form $n = 2^m$.

The idea then is to recursively combine the time steps two by two, decreasing the number of steps by a factor two at each iteration. This provides a very simple application of RG ideas to the derivation of universality properties.

At each iteration one thus replaces $\rho(q - q')$ by

$$[\mathcal{T}\rho](q - q') \equiv \int \mathrm{d}q'' \rho(q - q'')\rho(q'' - q').$$

The transformation of the distribution $\rho(q)$ is non-linear but applied to the function $w(k)$, it becomes the linear transformation

$$[\mathcal{T}w](k) \equiv 2w(k).$$

This transformation has an important property: it is independent of m or n. In the terminology of dynamical systems, its repeated application generates a stationary, or invariant under time translation, Markovian dynamics.

We now study the properties of the iterated transformation $\mathcal{T}^m$ for $m \to \infty$. A limiting distribution necessarily is a fixed point of the transformation. Thus, it corresponds to a function $w_*(k)$ (where the notation $_*$ is not related to complex conjugation) that satisfies

$$[\mathcal{T}w_*](k) \equiv 2w_*(k) = w_*(k).$$

Expanding in powers of k, one verifies that such a transformation has, with our assumptions, only the trivial fixed point $w_*(k) \equiv 0$.

But a larger class of fixed points becomes available if the transformation is combined with a renormalization of the distance scale, $q \mapsto \lambda q$, with $\lambda > 0$. We thus consider the transformation

$$[\mathcal{T}_\lambda w](k) \equiv 2w(k/\lambda)\,.$$

The transformation $\mathcal{T}_\lambda$ provides a simple example of a RG transformation, a concept that we describe in more detail in the framework of phase transitions.

1.2.1 Generic situation

The fixed point equation then becomes

$$[\mathcal{T}_\lambda w_*](k) \equiv 2w_*(k/\lambda) = w_*(k).$$

For the class of fast decreasing distributions, the functions $w(k)$ are regular at $k = 0$.

Thus, $w_*(k)$ has an expansion in powers of k of the form $(w(0) = 0)$

$$w_*(k) = -iw_1 k - \tfrac{1}{2}w_2 k^2 + \sum_{\ell=3} \frac{(-i)^\ell}{\ell!} w_\ell k^\ell \,, \ \ w_2 > 0 \,.$$

In the generic situation $w_1 \neq 0$. Expanding the equation, at order k one finds

$$2w_1/\lambda = w_1 \ \Rightarrow \ \lambda = 2 \,.$$

Then, identifying the terms of higher degree, one obtains

$$2^{1-\ell} w_\ell = w_\ell \ \Rightarrow \ w_\ell = 0 \text{ for } \ell > 1 \,.$$

Therefore,

$$w_*(k) = -iw_1 k \,.$$

The fixed points form a one-parameter family, but the parameter w_1 can be absorbed into a normalization of the random variable q.

Since

$$\rho_*(q) = \frac{1}{2\pi} \int dk \; e^{ikq - iw_1 k} = \delta(q - w_1),$$

fixed points correspond to the certain distribution $q = \langle q \rangle = w_1$. Since space and time are rescaled by the same factor 2, the fixed point corresponds to $q(t) = w_1 t$, the equation of the mean path.

Convergence and fixed point stability. For a non-linear transformation, a global stability analysis is often impossible. One can only linearize the transformation near the fixed point and perform a local study.

Here, this is not necessary since the transformation for $w(k)$ is linear. Setting

$$w(k) = w_*(k) + \delta w(k),$$

then,

$$[\mathcal{T}_2 \delta w](k) = 2\delta w(k/2) \,.$$

The function δw is regular and, thus, can be expanded in a Taylor series of the form

$$\delta w(k) = \sum_{\ell=1} \frac{(-i)^\ell}{\ell!} \delta w_\ell k^\ell \,.$$

Then,

$$[\mathcal{T}_2 \delta w](k) = 2\delta w(k/2) = 2 \sum_{\ell=1} \frac{(-ik)^\ell}{\ell!} 2^{-\ell} \delta w_\ell \,.$$

The expression shows that the functions k^ℓ with $\ell > 0$, are the eigenvectors of the transformation $\mathcal{T}_2$ and the corresponding eigenvalues are

$$\tau_\ell = 2^{1-\ell}.$$

Since at each iteration the number of variables is divided by two, one can relate the eigenvalues to the behaviour as a function of the initial number n of variables. One defines an associated exponent

$$\alpha_\ell = \ln \tau_\ell / \ln 2 = 1 - \ell \,.$$

After m iterations, the component δw_ℓ is multiplied by n^{α_ℓ} since

$$\mathcal{T}_2^m k^\ell = 2^{m(1-\ell)} k^\ell = n^{\alpha_\ell} k^\ell.$$

The behaviour, for $n \to \infty$, of a component of δw on the eigenvectors thus depends on the sign of α_ℓ.

We now adopt the RG terminology to discuss eigenvalues and eigenvectors.

We examine the various values of ℓ:

(i) $\ell = 1 \Rightarrow \tau_1 = 1$, $\alpha_1 = 0$. If one adds a term δw proportional to the eigenvector k to $w_*(k)$, $\delta w(k) = -i\delta w_1 k$, then

$$w_1 \mapsto w_1 + \delta w_1,$$

which correspond to a new fixed point. This change has also the interpretation of a linear transformation on k or on the random variable q.

An eigen-perturbation corresponding to the eigenvalue 1 and, thus to an exponent $\alpha_1 = 0$, is called marginal.

Quite generally, the existence of a one-parameter family of fixed points implies the existence of an eigenvalue $\tau = 1$ and, thus, an exponent $\alpha = 0$. Indeed, let us assume the existence of one-parameter family of fixed points $w_*(s)$,

$$\mathcal{T} w_*(s) = w_*(s)\,,$$

where $w_*(s)$ is a differentiable function of the parameter s. Then,

$$\mathcal{T} \frac{\partial w_*}{\partial s} = \frac{\partial w_*}{\partial s}.$$

(ii) $\ell > 1 \Rightarrow \tau_\ell = 2^{1-\ell} < 1$, $\alpha_\ell < 0$. The components of δw on such eigenvectors converge to zero for n or $m \to \infty$.

In the RG terminology, the eigen-pertubations that correspond to eigenvalues smaller in modulus than 1 and, thus, to negative exponents (more generally with a negative real part), are called irrelevant.

Universality, in the RG formulation, is a consequence of the property that all eigenvectors, but a finite number, are irrelevant.

Dimension of a random variable. To the random variable that has a limiting distribution, one can attach a dimension d_q defined by

$$d_q = \ln \lambda / \ln 2 \,. \tag{1.8}$$

This corresponds to dividing the sum by n^{d_q}. One here finds $d_q = 1$, which is consistent with $q(t) \propto t$.

1.2.2 Centred distribution

For a centred distribution, $w_1 = 0$, one has to expand to order k^2. One finds the equation

$$w_2 = 2w_2/\lambda^2 \,.$$

Since the variance w_2 is strictly positive, except for a certain distribution, a case that we now exclude, the equation implies $\lambda = \sqrt{2}$.

Again, the coefficients w_ℓ vanish for $\ell > 2$ and the fixed points have the form

$$w_*(k) = -\tfrac{1}{2}w_2 k^2.$$

Therefore, one finds the Gaussian distribution

$$\rho_*(q) = \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq - w_2 k^2/2} = \frac{1}{\sqrt{2\pi w_2}} \, \mathrm{e}^{-q^2/2w_2} \; .$$

Since, in the transformation $\mathcal{T}$, the number n of time steps is divided by two, this value of the renormalization factor λ corresponds to dividing space by a factor $\sqrt{2}$. This is consistent with the scaling dimension of the space variable in time unit $x \propto \sqrt{t}$ of the Brownian motion:

$$d_q = \ln \lambda / \ln 2 = \tfrac{1}{2} \; .$$

The two essential asymptotic properties of the random walk, convergence toward a Gaussian distribution and scaling property are thus reproduced by the RG type analysis.

Fixed point stability. One can now study the stability of the fixed point corresponding to the transformation $\mathcal{T}_{\sqrt{2}}$. One sets

$$w(k) = w_*(k) + \delta w(k),$$

and looks for the eigenvectors and eigenvalues of the transformation

$$[\mathcal{T}_{\sqrt{2}}\, \delta w](k) \equiv 2\delta w(k/\sqrt{2}) = \tau \delta w(k).$$

Clearly, the eigenvectors have still the form

$$\delta w(k) = k^{\ell} \;\Rightarrow\; \tau_{\ell} = 2^{1-\ell/2}.$$

The correspondent exponent is

$$\alpha_{\ell} = \ln \tau_{\ell} / \ln 2 = 1 - \ell/2\,.$$

The values can be classified as:

(i) $\ell = 1 \Rightarrow \tau_1 = \sqrt{2}$, $\alpha_1 = \frac{1}{2}$. This corresponds to an unstable direction; a component on such a eigenvector diverges for $m \to \infty$.

In the RG terminology, a perturbation corresponding to a positive exponent α, and which thus moves away from the fixed point, is called relevant.

Here, a perturbation linear in k violates the condition $w_1 = 0$. One is then brought back to the study of fixed points with $w_1 \neq 0$.

(ii) $\ell = 2 \Rightarrow \tau_2 = 1$, $\alpha_2 = 0$. A vanishing eigenvalue characterizes a marginal perturbation. Here, the perturbation only modifies the value of w_2 and, again, has an interpretation as a linear transformation on the random variable.

(iii) $\ell > 2 \Rightarrow \tau_\ell = 2^{1-\ell/2} < 1$, $\alpha_\ell = 1 - \ell/2 < 0$. Finally, all perturbations $\ell > 2$ correspond to stable directions in the sense that their amplitudes converge to zero for $m \to \infty$ and are irrelevant.

Redundant perturbations. In the examples examined here, the marginal perturbations correspond to simple changes in the normalization of the random variables. In many problems, this normalization plays no role. One can then consider that fixed points corresponding to different normalizations should not be distinguished. From this viewpoint, in both cases one has found really only one fixed point. The perturbation corresponding to the vanishing eigenvalue is then no longer called marginal but redundant, in the sense that it changes only an arbitrary normalization.

Other fixed points. Other values of $\lambda = 2^{1/\mu}$, correspond formally to new fixed points of the form $|k|^\mu$, $0 < \mu < 2$ ($\mu > 2$ is excluded because the coefficient of k^2 is strictly positive). However, these fixed points are no longer regular functions of k. They correspond to distributions that have no second moment $\langle q^2 \rangle$ and thus no variance: they decay only algebraically for large values of q. In the RG terminology, they correspond to different universality classes, distributions with other decay properties.

Random walk on the lattice of points with integer coordinates. The analysis can also be generalized to a random walk on the points of integer coordinate. The main difference is that $w(k)$ is a periodic function of period 2π. However, at each iteration the period is multiplied by a factor $\lambda > 1$. Thus, asymptotically, the period diverges and, at least for continuous observables, the discrete character of the initial lattice disappears.

In the d-dimensional lattice $\mathbb{Z}^d$, if the random walk has hypercubic symmetry, the leading term in the expansion of $w(\mathbf{k})$ for $\mathbf{k}$ small is again $\frac{1}{2}w_2\mathbf{k}^2$ because it is the only quadratic hypercubic invariant. Therefore, asymptotically the random walk is Brownian with rotation symmetry. The lattice structure is only apparent in the first irrelevant perturbation because there exists two independent cubic invariant monomials of degree four:

$$\sum_{\mu=1}^{d} k_\mu^4, \quad \left(\mathbf{k}^2\right)^2 .$$

1.3 Brownian motion and path integral

To derive the universal properties of the asymptotic distribution, which have been shown to be independent of the initial transition probability, one can directly start from Gaussian transition probabilities of the form

$$\rho(q) = \frac{1}{(2\pi w_2)^{1/2}} \, \mathrm{e}^{-q^2/2w_2} . \tag{1.9}$$

An iteration of the general evolution equation leads to

$$P_n(q) = \int \mathrm{d}q' \mathrm{d}q_1 \, \mathrm{d}q_2 \, \dots \mathrm{d}q_{n-1} \, \rho(q, q_{n-1}) \dots \rho(q_2, q_1)\rho(q_1, q')P_0(q').$$

For a certain initial position $q' = q_0 = 0$ and the transition probability (1.9), it becomes $(q_n = q)$

$$P_n(q) = \frac{1}{(2\pi w_2)^{n/2}} \int \mathrm{d}q_1 \, \mathrm{d}q_2 \, \dots \mathrm{d}q_{n-1} \, \mathrm{e}^{-\mathcal{S}(q_0,q_2,\dots,q_n)} \tag{1.10}$$

$$\mathcal{S}(q_0, q_2, \dots, q_n) = \sum_{\ell=1}^{n} \frac{(q_\ell - q_{\ell-1})^2}{2w_2} .$$

We now introduce macroscopic time variables,

$$\tau_\ell = \ell\varepsilon \,, \quad \tau_n = n\varepsilon = t \,,$$

and a continuous, piecewise linear path $x(\tau)$ (*cf.* Fig. 1.1)

$$x(\tau) = \sqrt{\varepsilon} \left[q_{\ell-1} + \frac{\tau - \tau_{\ell-1}}{\tau_\ell - \tau_{\ell-1}} \left(q_\ell - q_{\ell-1} \right) \right] \quad \text{for } \tau_{\ell-1} \leq \tau \leq \tau_\ell \,.$$

One verifies that $\mathcal{S}$ can be written as (with the notation $\dot{x}(\tau) \equiv \mathrm{d}x/\mathrm{d}\tau$)

$$\mathcal{S}\big(x(\tau)\big) = \frac{1}{2w_2} \int_0^t \big(\dot{x}(\tau)\big)^2 \mathrm{d}\tau$$

with the boundary conditions

$$x(0) = 0 \,, \quad x(t) = \sqrt{\varepsilon}q = x \,.$$
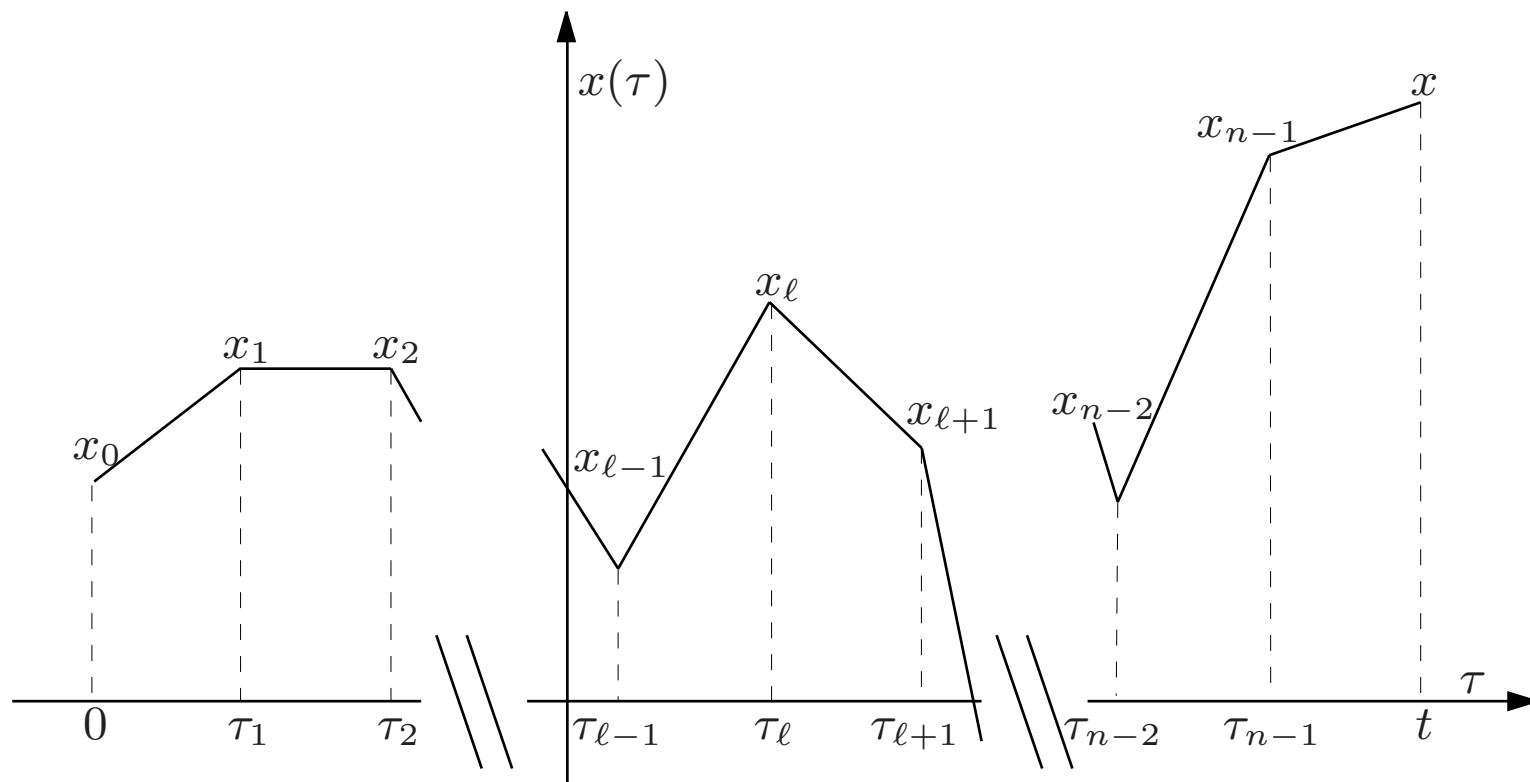
Fig. 1.1 – A piecewise linear path contributing to the time-discretized path integral (1.10) with $x_\ell \equiv x(\tau_\ell)$. The continuum limit is reached by taking the limit $n \to \infty$ at $n\varepsilon$ fixed.

Moreover,

$$P_n(q) = \frac{1}{(2\pi w_2)^{1/2}} \int \left(\prod_{\ell=1}^{n-1} \frac{\mathrm{d}x(\tau_\ell)}{(2\pi w_2 \varepsilon)^{1/2}} \right) \mathrm{e}^{-\mathcal{S}(x)} \,.$$

In the continuum limit $\varepsilon \to 0$, $n \to \infty$ with t fixed, the expression yields a representation of the distribution of the continuum limit,

$$\Pi(t, x) \sim \varepsilon^{-1/2} P_n(q),$$

in the form of a path integral, which we denote symbolically

$$\Pi(t, x) = \int [\mathrm{d}x(\tau)] \, \mathrm{e}^{-\mathcal{S}(x(\tau))} \,, \quad \mathcal{S}(x(\tau)) = \frac{1}{2w_2} \int_0^t (\dot{x}(\tau))^2 \mathrm{d}\tau \,,$$

where $\int [\mathrm{d}x(\tau)]$ means sum over all continuous paths that start from the origin at time $\tau = 0$ and reach x at time t.

The trajectories that contribute to the path integral correspond to a Brownian motion, a random walk in continuum time and space. The representation of the Brownian motion by path integrals, initially introduced by Wiener, is also called Wiener integral.

Wiener and Feynman path integral. The Wiener path integral can be compared with Feynman path integral for a free non-relativistic particle in one dimension, where $\mathcal{S}(x)$ is replaced by

$$\mathcal{A}(q) = \frac{i}{\hbar} \int \mathrm{d}\tau \, \tfrac{1}{2} m \big(\dot{x}(\tau) \big)^2 \mathrm{d}\tau \,,$$

where $\mathcal{A}$ is the classical action.

From the viewpoint of the path Feynman integral, the Wiener path integral corresponds to motion in imaginary time.

Remarks

(i) Note that, after the rescaling from microscopic to macroscopic variables, $x = q\sqrt{\varepsilon}$, the transition probability (1.9) becomes

$$\rho(x) = \frac{1}{(2\pi w_2 \varepsilon)^{1/2}}\, \mathrm{e}^{-x^2/2w_2\varepsilon}\ .$$

It vanishes for $|x| > 0$ when the time step ε goes to zero, a characteristic property of locality of short-time evolution.

The parameter $1/\varepsilon$ plays the role of the cut-off in quantum field theory.

(ii) We have determined the path integral at leading order in the large time, large space limit. The leading corrections are of two types (from time translation invariance)

$$\varepsilon \int d\tau \, (\dot{x}(\tau))^4 \, , \ \varepsilon^2 \int d\tau \, (\ddot{x}(\tau))^2.$$

The powers of ε can be determined by power counting, which a form of dimensional analysis, using the property that the action is dimensionless:

For example, the first term above, in time units has a dimension $4 \times \frac{1}{2} - 4 + 1 = -1$ since x has time dimension $\frac{1}{2}$. A factor ε is required to make it dimensionless. The second term has dimension $2 \times \frac{1}{2} - 4 + 1 = -2$. A factor ε^2 is needed.

In the QFT terminology, $\mathcal{S}(x)$ is an effective (Euclidean) action.

Exercises

Exercise 1.1

Study the local stability of the Gaussian fixed point

$$\rho_{\mathrm{G}}(q) = \mathrm{e}^{-q^2/2}/\sqrt{2\pi}\,,$$

by starting directly from the equation

$$[\mathcal{T}_\lambda \rho](q) = \lambda \int \mathrm{d}q'\, \rho(q')\rho(\lambda q - q').\tag{1.11}$$

Determine the value of the renormalization factor λ for which the Gaussian probability distribution ρ_{G} is a fixed point of $\mathcal{T}_\lambda$.

Setting $\rho = \rho_{\mathrm{G}} + \delta\rho$, expand equation (1.11) to first order in $\delta\rho$. Show that the eigenvectors of the linear operator acting on $\delta\rho$ have the form

$$\delta\rho_p(q) = (\mathrm{d}/\mathrm{d}q)^p\, \rho_{\mathrm{G}}(q), \quad p > 0\,.$$

Infer the corresponding eigenvalues.

Exercise 1.2

Random walk on a circle. To exhibit the somewhat different asymptotic properties of a random walk on compact manifolds, it is proposed to study random walk on a circle. One still assumes translation invariance. The random walk is then specified by a transition function $\rho(q - q')$, where q and q' are two angles corresponding to positions on the circle. Moreover, the function $\rho(q)$ is assumed to be periodic and continuous. Determine the asymptotic distribution of the walker position. At initial time $n = 0$, the walker is at the point $q = 0$.

Exercise 1.3

Another universality class

One considers now the transition probability $\rho(q - q')$ with

$$\rho(q) = \frac{2}{3\pi} \frac{2 + q^2}{(1 + q^2)^2}.$$

The initial distribution is again

$$P_0(q) = \delta(q).$$

Evaluate the asymptotic distribution $P_n(q)$ for $n \to \infty$.

Lecture 2: THE ESSENTIAL ROLE OF FUNCTIONAL INTEGRALS IN THEORETICAL PHYSICS

Twentieth century has seen the emergence of the physics of fluctuating systems, statistical and quantum. This explains, to some extent, the important role played by functional integrals in modern physics.

Path integrals: The origins

The first path integral seems to have been defined by Wiener (1923), as a tool to describe the statistical properties of the Brownian motion, inspired by the well-known work of Einstein.

Wentzel (1924), in a less-known work, introduces, in the framework of quantum optics, the notions of sums over paths weighted by a phase factor, of destructive interference between paths that do not satisfy classical equations of motion, and the interpretation of the sum as a transition probability amplitude.

Dirac (1933) has written a first expression of the quantum evolution operator that resembles a path integral, but he did not go beyond the approximation of discrete time intervals.

Of course, the modern history of path integrals begins with the articles of Feynman (1948) who formulates quantum evolution in terms of sums over a set of trajectories weighted by $\mathrm{e}^{i\mathcal{A}/\hbar}$, where $\mathcal{A}$ is the value of the corresponding classical action (time-integral of the Lagrangian).

Later, in the framework of quantum field theory (QFT), the Russian school was specially active (Berezin, Faddeev, Lipatov, Polyakov, Popov, Slavnov, Vasiliev...). From the beginning of the seventies on, the use of field integrals slowly spread to the whole theoretical community.

We describe now a few striking examples of physics problems where the use of path or field integrals has proven essential from the conceptual or the technical point of view.

2.1 Classical equations: The mysterious variational principle

2.1.1 Euler–Lagrange equations

Following the intuition of Maupertuis and the formulation of the variational principle by Euler and Lagrange, Lagrange showed that the equations of motion of Newtonian mechanics can be derived from a variational principle.

In modern form, Hamilton's principle states that classical equations of motion can be derived from a purely mathematical quantity, the action, integral of a Lagrangian,

$$\mathcal{A}(q) = \int_{t'}^{t''} \mathrm{d}t\, \mathcal{L}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t); t\big),$$

by expressing the stationarity of the action with respect to variations of the trajectory $\mathbf{q}(t)$. The equations then take the form of Euler–Lagrange equations:

$$\delta\mathcal{A} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q_i}.$$

The simplest example is

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2}m\dot{q}^2 - V(q) \;\Rightarrow\; m\ddot{q} = -V'(q).$$

Technically, this formalism happens to be very efficient, for example, for systems with constraints, or to establish conservation laws generated by continuous symmetries. However, in this classical framework, the action and the Lagrangian remain purely mathematical quantities.

2.1.2 The particle in a static magnetic field

Later it was discovered that, quite remarkably, the equation of motion of a particle in a static magnetic field $\mathbf{B}$, which takes the form

$$m\ddot{\mathbf{q}} = e\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{where} \quad \nabla \cdot \mathbf{B}(\mathbf{q}) = 0\,,$$

can also be derived from an action principle, provided an additional mathematical quantity is introduced, the vector potential:

$$\mathbf{B}(\mathbf{q}) = -\nabla \times \mathbf{A}(\mathbf{q}).$$

The Lagrangian can be written as

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \tfrac{1}{2}m\dot{\mathbf{q}}^2 - e\mathbf{A}(\mathbf{q}) \cdot \dot{\mathbf{q}}\,.$$

Classically, the vector potential is not considered as a physical quantity since it is defined only up to a shift by a gradient: $\mathbf{A}(\mathbf{q}) \mapsto \mathbf{A}(\mathbf{q}) + \nabla\Omega(\mathbf{q})$.

2.1.3 Electromagnetism and Maxwell's equations

Maxwell's equations (in the vacuum) can be written as

$$\nabla \cdot \mathbf{E} = \rho\,, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}\,,$$

$$\nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,,$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, resp., ρ the charge and $\mathbf{J}$ the current densities, resp..

In quadri-covariant notation where $(i, j = 1, 2, 3)$

$$t \equiv x_0 \,, \ \ F_{i0} = E_i \,, \ \ F_{ij} = -\sum_k \epsilon_{ijk} B_k \,, \ \ J_0 = \rho \,,$$

they take the form

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = J^\nu \ \Rightarrow \ \sum_{\nu=0}^{3} \partial_\nu J^\nu = 0 \,.$$

These equations imply that the tensor $F_{\mu\nu}$ can be expressed in terms of a vector potential, or gauge field, $A_\mu(x)$ under the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

The gauge is defined only up to an Abelian gauge transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \Omega(x).$$

Then again, remarkably enough, with the introduction of this new mathematical quantity, the gauge field, Maxwell's equations can be derived from an action principle with the Lagrangian density

$$\mathcal{L}(\mathbf{A}, \dot{\mathbf{A}}) = -\tfrac{1}{4} \sum_{\mu,\nu} F^{\mu\nu} F_{\mu\nu} - \sum_{\mu} J^{\mu} A_{\mu} \ \text{with} \ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \,,$$

and gauge invariant action,

$$\mathcal{A} = \int \mathrm{d}^4 x \, \mathcal{L}(\mathbf{A}, \dot{\mathbf{A}}).$$

2.1.4 General Relativity

In Einstein's relativistic theory of gravitation (or General Relativity), the equations of motion can also be derived from an action principle.

For example, in the absence of matter, in terms of metric tensor $g_{\mu\nu}(x)$ they read

$$R_{\mu\nu}\big(\mathbf{g}(x)\big) - \tfrac{1}{2}R\big(\mathbf{g}(x)\big)g_{\mu\nu} = 0\,,$$

where R is the scalar curvature and $R_{\mu\nu}$ the Ricci tensor.

These equations can be derived by varying Einstein–Hilbert's action,

$$\mathcal{A}(\mathbf{g}) = \int \mathrm{d}^4x \,(-g(x))^{1/2}R\big(\mathbf{g}(x)\big),$$

where $g(x)$ is the determinant of the metric tensor. The variational property still holds in presence of a cosmological constant and matter when the metric is replaced by the spin connection.

The question then arises: why can all fundamental classical equations be derived from a variational principle, by expressing the stationarity of a local action?

At first sight, quantum mechanics in its Hamiltonian formulation, gives no direct answer to the question. It should be considered as a major success of quantum mechanics in the path integral formulation, quantum field theory in the field integral formulation, that it provides a simple explanation to this property.

2.1.5 Variational principle and path integral

According to Feynman, quantum evolution is given by a path integral of the form

$$\langle q'' |\, U(t'', t')\, | q' \rangle = \mathcal{N} \int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}q(t)]\, \mathrm{e}^{i\mathcal{A}(q)/\hbar},$$

where $U(t'', t')$ is the evolution operator, $\mathcal{A} = \int \mathrm{d}t\, \mathcal{L}$ is the classical action, time-integral of the classical Lagrangian, and one sums over all possible trajectories $q(t)$ satisfying the boundary conditions at times t' and t''.

In the classical limit, for $\hbar \to 0$, the path integral is dominated by paths that leave the action stationary (the stationary phase method): these are precisely the classical paths.

The property generalizes to the relativistic quantum field theory.

2.1.6 Additional remarks

In the framework of path or field integrals, Lagrangians, Hamiltonians, vector potentials, all initially *ad hoc* mathematical quantities, acquire now a physical status.

Also, it has been argued that the variational principle of classical mechanics seems to violate causality, since it requires the knowledge of the initial and final position. The argument does not apply to the path integral, which is a sum over all possible trajectories and classical paths are selected because they leave the action stationary.

The formulation by path integrals thus leads to a more intuitive picture of quantum mechanics, gives a direct meaning to the notion of quantum fluctuations and allows understanding the origin of the variational principle in classical mechanics.

2.1.7 Quantum gravity

As another potential non-trivial consequence, one can conclude that since the classical equations of General Relativity follow from a variational principle, the field integral over metrics (or, more generally, spin connection) of $\mathrm{e}^{i\mathcal{A}_{\mathrm{EH}}/\hbar}$, involving Einstein–Hilbert's action $\mathcal{A}_{\mathrm{EH}}$, properly regularized at short distance (of course, a non-trivial issue), should be directly relevant to quantum gravity.

Einstein–Hilbert's action is most likely one of the first terms of the expansion of an effective quantum action.

2.2 Relativistic quantum field theory: unitarity and covariance

The standard Hamiltonian formulation of relativistic quantum theory is explicitly unitary but not explicitly covariant. However, as first noticed by Dirac, the short time evolution can be expressed in terms of the Lagrangian and is thus explicitly covariant.

In the Hamiltonian formulation, for a Hamiltonian $H(p, q)$ where p and q are phase space variables (position and conjugate momentum), the matrix elements of the quantum evolution operator $U(t'', t')$ between times t' and t'' are given by an integral over phase space trajectories,

$$\langle q'' \left| U(t'', t') \right| q' \rangle = \int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}p(t)\mathrm{d}q(t)] \exp\left(\frac{i}{\hbar} \mathcal{A}(p, q) \right),$$

where $\mathcal{A}(p, q)$ is the classical action in the Hamiltonian formalism:

$$\mathcal{A}(p, q) = \int_{t'}^{t''} \left[p(t)\dot{q}(t) - H\big(p(t), q(t); t\big) \right] \mathrm{d}t.$$

When the classical Hamiltonian H is a quadratic form in p, like $p^2/2m + V(q)$, the integral over p is Gaussian and can be performed explicitly:

$$\int [\mathrm{d}p(t)] \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} \mathrm{d}t\left(p(t)\dot{q}(t) - p^2(t)/2m\right)\right] \propto \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} \mathrm{d}t\,\tfrac{1}{2}m\dot{q}^2(t)\right].$$

The integration amounts to replacing $p(t)$ by the solution $m\dot{q}(t)$ of the classical equation and thus generates the Lagrangian:

$$\langle q''\,|U(t'',t')|\,q'\rangle = \int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}q(t)] \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} \mathrm{d}t\,\mathcal{L}(q,\dot{q})\right)$$

with

$$\mathcal{L}(q,\dot{q}) = \tfrac{1}{2}m\dot{q}^2 - V(q).$$

In a relativistic quantum theory, the Lagrangian formulation is explicitly relativistic covariant, in contrast with the Hamiltonian formulation, which is explicitly unitary.

2.2.1 Path integrals as algebraic tools: application to perturbation theory

Functional integrals allow a smooth transition between ordinary integrals, path integrals and field integrals, unlike any other formulation of non-relativistic quantum mechanics or relativistic quantum field theory.

As a consequence, functional integrals yield a simple derivation of Wick's theorem and provide a convenient tool to generate perturbative expansions. They can be used to sum partially the perturbative expansion in the form of systematic loop expansion.

They allow easy proofs of Ward–Takahashi identities, relations between correlation functions induced by symmetries of the action, to all orders of the perturbative expansion. In particular, relations between renormalization constants can be established, which are not obvious when the symmetry is broken by soft terms (or spontaneously) in the action.

2.3 The non-linear σ-model

The non-linear σ-model is a model with global $O(N)$ symmetry and an N-component scalar field $\boldsymbol{\phi}(x)$ that lives on the sphere S_{N-1}:

$$\boldsymbol{\phi}^2(x) = 1 \,. \tag{2.1}$$

In terms of $\boldsymbol{\phi}$, the action takes the form of a free action,

$$\mathcal{S}(\phi) = \frac{1}{2g} \int \mathrm{d}^d x \, [\partial_\mu \boldsymbol{\phi}(x)]^2 \,,$$

but the constraint (2.1) generates interactions. Within the perturbative expansion, the $O(N)$ symmetry is realized in the phase of spontaneous symmetry breaking and the dynamical fields correspond to Goldstone modes. In two $(1+1)$ dimensions, only the perturbative expansion of $O(N)$ invariant quantities is IR finite. The one-loop calculation of the RG β-function shows that the model is asymptotically free.

Moreover, perturbation theory is misleading since the $O(N)$ symmetry is not broken, the N-component field is massive and the dependence of the coupling constant is given by RG arguments. Several of these properties are reminiscent of Quantum ChromoDynamics in four dimensions.

First (one-loop) calculations seemed to indicate that the $O(N)$ symmetry was explicitly broken by perturbative corrections, the dynamical fields becoming massive. Within the canonical formulation, a more careful and involved calculation showed that the breaking terms actually cancelled.

However, the field integral representation gave both the correct quantized form to all orders (Meetz and Honerkamp),

$$\mathcal{Z} = \int e^{-\mathcal{S}(\boldsymbol{\phi})} \prod_x \delta(\boldsymbol{\phi}^2(x) - 1)[\mathrm{d}\phi],$$

and a geometric interpretation of the initial error: in the field integral framework, it amounted to replacing the $O(N)$ invariant measure by a flat Euclidean measure, an error that obviously breaks the $O(N)$ symmetry.

2.4 Quantum field theory and lattice regularization

The field integral formulation of quantum field theory emphasizes its relation with classical statistical physics.

To regularize field theories and to define them beyond perturbation theory, the field integral representation suggests introducing a lattice regularization. In the example of the field theory with imaginary time action,

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x \left[\tfrac{1}{2} \left(\nabla_x \phi(x) \right)^2 + \tfrac{1}{2} r \phi^2(x) + \frac{g}{4!} \phi^4(x) \right],$$

one obtains a lattice spin model of the form (a is the lattice spacing)

$$\mathcal{Z} = \int \exp \left(a^2 \sum_{i,j \text{ n.n.}} \phi_i \phi_j \right) \prod_{i \in \mathbb{Z}^4} \rho(\phi_i) \mathrm{d}\phi_i$$

(i, j are lattice sites and n.n. stands for nearest neighbour) with

$$\rho(\phi) = \exp \left[-a^2 (\tfrac{1}{2} r a^2 + 1) \phi^2 - \frac{a^4}{4!} g \phi^4 \right].$$

This lattice model can then be handled by all methods of statistical physics, including MC type computer simulations.

In particular, the existence of a continuum limit requires a continuous phase transition, thus a fine-tuning of the coefficient of ϕ^2 in $\rho(\phi)$ to some negative value.

In the case of the non-linear σ model,

$$\rho(\boldsymbol{\phi}) = \delta(\boldsymbol{\phi}^2 - 1).$$

The similarity between the two regularizations suggests a relation between the non-linear σ-model and the $O(N)$-symmetric $(\boldsymbol{\phi}^2)^2$ statistical field theory, a relation that is confirmed by RG arguments or by a large N expansion.

2.2.1 Critical phenomena and quantum field theory

Conversely, following Wilson, it has been realized that universal properties near a continuous phase transition of a large class of statistical models can be described by an Euclidean quantum or statistical field theory.
For example, the critical properties of the d-dimensional Ising model

$$\mathcal{Z}_{\text{Ising}} = \sum_{\{S_i\} = \pm 1} \exp\left(J \sum_{i,j \text{ n.n.}} S_i S_j \right)$$

are described by the ϕ^4 quantum field theory (in imaginary time)

$$\mathcal{Z}_{\phi^4} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) \right],$$

where one now integrates over all fields $\phi(x)$, $x \in \mathbb{R}^d$ and $\mathcal{S}(\phi)$ is the Euclidean action:

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2} \sum_\mu \left(\partial_\mu \phi(x) \right)^2 + \tfrac{1}{2} r \phi^2(x) + \tfrac{1}{4!} g \phi^4(x) \right].$$

This property generalizes to classical spin models with $O(N)$ symmetry like

$$\mathcal{Z} = \sum_{\{|\mathbf{S}_i|\}=1} \exp\left(J \sum_{i,j \text{ n.n.}} \mathbf{S}_i \cdot \mathbf{S}_j \right).$$

In addition, path integral techniques allow proving directly that the formal $N = 0$ limit of the $O(N)$ symmetric field theory describes the statistical properties of large polymers (or SAW).

The quantum field theory renormalization group has then be used to calculate precisely universal critical properties of classical statistical systems near a continuous phase transition.

2.3 Lattice gauge theories and numerical simulations

The correspondence between classical statistical models and quantum field theory has led to the application of statistical methods to the non-perturbative study of quantum field theories. By replacing the continuum field integral by a lattice regularized form, non-perturbative numerical techniques become available, like Monte-Carlo type simulations.

An outstanding example is QCD. On the lattice, gauge fields are replaced by group elements U_{ij} associated to links joining next neighbour sites ij. In the absence of matter, the partition function $\mathcal{Z}$ can be calculated with Wilson's plaquette action:

$$\mathcal{Z} = \int e^{-\beta_p \mathcal{S}_{\mathrm{W}}(\mathbf{U})} \prod_{\mathrm{links}\{ij\}} d\mathbf{U}_{ij}\,, \quad \mathcal{S}_{\mathrm{W}}(\mathbf{U}) = - \sum_{\substack{\mathrm{plaquettes} \\ (ijkl)}} \mathrm{tr}\, \mathbf{U}_{ij}\mathbf{U}_{jk}\mathbf{U}_{kl}\mathbf{U}_{li}.$$

After addition of matter fields, the lattice formulation yields a non-perturbative definition of QCD, which can be investigated by MC simulations.

2.4 Instantons, barrier penetration and vacuum instability

In non-relativistic quantum mechanics, barrier penetration effects can evaluated in the semi-classical limit by WKB methods.

Alternatively, in the same way as for simple integrals, barrier penetration effects can be determined by applying the steepest descent method to the path integral in the imaginary time framework. Saddle points then are finite action solutions of Euclidean (imaginary time) equations of motion (instantons).

However, the instanton method generalizes simply to quantum field theory, unlike methods based on the Schrödinger equation.

Important physics phenomena, like vacuum instability, the periodic structure of the QCD vacuum and the strong CP problem, the solution of the $U(1)$ problem are related to instantons.

2.5 Instantons, large order behaviour and the problem of Borel summability

In quantum field theory, the behaviour of the perturbative expansion at large orders is related to vacuum instability. Field integral techniques (instantons) (as initiated by Lipatov) have led to the determination of the large order behaviour of the perturbative expansion in many quantum field theories.

An important outcome is that perturbation series are always divergent. Moreover, the knowledge of the large order behaviour has then direct implications on the problem of Borel summability.

A direct application has been the summation of the perturbative expansion of the $O(N)$ symmetric $(\phi^2)^2$ QFT to determine critical exponents.

In the table, $N = 0$ corresponds to the statistical properties of polymers, $N = 1$ to the Ising model universality class, which also contains liquid-vapour transition and binary mixtures mixing transition, $N = 2$ to the Helium superfluid transition and $N = 3$ to isotropic ferromagnets.

Reliable *critical exponents from $O(N)$ symmetric $(\phi^2)_3^2$ field theory*
(Le Guillou and Z.-J. (1980) updated by Guida and Z.-J. (1998))

N	0	1	2	3
$\tilde{g}^*$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

2.2.1 Potentials with degenerate minima

In the case of potentials with discrete degenerate classical minima, the large order behaviour (derived from instantons) indicates that the perturbative expansion is not Borel summable and, as a consequence, does not determine unique functions. In simple quantum mechanics with analytic potentials, the problem can be studied systematically and it can be shown that all multi-instanton configurations must be taken into account and a generalized summation procedure introduced.

J. Zinn-Justin, *Multi-instanton contributions in quantum mechanics*, Nucl. Phys. B192, 125-140 (1981).

U.D. Jentschura and J. Zinn-Justin, *Multi-instantons and exact results I, II: conjectures, WKB expansions, and instanton interactions*, Ann. Phys. 313 (2004) 197; *ibidem,* (2004) 269-325.

2.3 Relation between classical and quantum statistical physics

For a scalar quantum field ϕ in d space dimensions at temperature $T = 1/\beta$, the quantum partition function reads

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp \left[- \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x \, \mathcal{S}(\phi) \right],$$

where $\mathcal{S}$ is the Euclidean (imaginary time) action, and the (Bose) fields satisfy the periodic boundary conditions

$$\phi(0, x) = \phi(\beta, x).$$

However, the field integral representation immediately shows that the same partition function has also the interpretation of a classical partition function in $(d+1)$ space dimensions with finite size β and periodic boundary conditions in one space direction.

This observation has important implications for the theory of continuous phase transitions: it relates a class of classical transitions in $(d+1)$ space dimensions to quantum transitions at zero temperature $(\beta = \infty)$ in d dimensions.

More generally, this relation between classical and quantum statistical physics maps finite temperature quantum effects to finite size effects in the classical theory. This is most useful from the renormalization group viewpoint.

2.4 Finite temperature QFT, finite size effects in statistical field theory and dimensional reduction

In particular, in this framework, high temperature is associated to dimensional reduction. Technically, one expands the periodic field in Fourier (Matsubara) modes

$$\phi(t, x) = \sum_{\nu} e^{i2\pi\nu t/\beta} \phi_\nu(x).$$

At high temperature, near a continuous phase transition, when the correlation length is much larger than the thermal wave length $\lambda = \hbar\sqrt{2\pi/mT}$, only the zero-mode is critical. One can then integrate perturbatively over all non-zero modes:

$$e^{-\mathcal{S}_{\text{eff.}}(\phi_0)} = \int \prod_{\nu \neq 0} [d\phi_\nu] \, e^{-\mathcal{S}(\phi)} \text{ with } \mathcal{Z} = \int [d\phi_0] \, e^{-\mathcal{S}_{\text{eff}}(\phi_0)},$$

but must treat the zero-mode ϕ_0 non-perturbatively.

2.4.1 The dilute (thus weakly interacting) Bose gas

As an example, the technique has been applied to the dilute Bose gas. The initial field integral over fields ψ^*, ψ periodic in Euclidean time reads

$$\mathcal{Z} = \int [\mathrm{d}\psi(t,x)\mathrm{d}\psi^*(t,x)] \, \mathrm{e}^{-\mathcal{S}(\psi^*,\psi)/\hbar},$$

Since one is interested only in long wavelength phenomena, the two-body potential can be replaced by a delta-function and parametrized in terms of the s-wave scattering length a (positive because the interaction is assumed repulsive).

For $d = 3$, the effective Euclidean action of the system may then be written as (μ is the chemical potential)

$$\mathcal{S}(\psi^*,\psi) = -\int_0^\beta \mathrm{d}t \int \mathrm{d}^3 x \left[\psi^*(t,x) \left(\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_x^2 + \mu \right) \psi(t,x) \right.$$
$$\left. + \frac{2\pi\hbar^2 a}{m} \left(\psi^*(t,x)\psi(t,x) \right)^2 \right].$$

The reduced partition function, at leading order when only the zero-mode is kept, takes the form of the field integral

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}(x)] \exp\left[-\mathcal{S}(\boldsymbol{\phi})\right]$$

with

$$\mathcal{S}(\boldsymbol{\phi}) = \int \left\{ \frac{1}{2} [\partial_\mu \boldsymbol{\phi}(x)]^2 + \frac{1}{2} r \boldsymbol{\phi}^2(x) + \frac{u}{4!} \left[\boldsymbol{\phi}^2(x)\right]^2 \right\} \mathrm{d}^d x \,,$$

where $\psi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2}$, $\psi^*(x) = (\phi_1(x) - i\phi_2(x))/\sqrt{2}$, $r = -2mT\mu$ and, for $d = 3$, $u = 96\pi^2 a/\lambda^2$.

The Euclidean action reduces to the ordinary $O(2)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory, which also describes the universal properties of the superfluid Helium transition.

2.4.2 Large N non-perturbative techniques

In quantum field theories with $O(N)$ or $U(N)$ symmetries and fields in the vector representation, physical quantities can be calculated in the large N limit, yielding non-perturbative results. At leading order, the results can be obtained by summing Feynman diagrams, but field integral techniques are much simpler and can easily be extended to higher orders in $1/N$. Applications include the study of the $(\phi^2)^2$ theory, the Gross–Neveu model....

For example, in $U(\phi^2)$ scalar field theories the starting point is the insertion into the ϕ-field integral of the identity

$$1 = \int [\mathrm{d}\lambda \mathrm{d}\rho] \exp \left\{ i \int \mathrm{d}^d x \, \lambda(x) \left[\rho(x) - \phi^2(x) \right] \right\},$$

which then allows a Gaussian integration over ϕ.

For a review see, for example,

M. Moshe, J. Zinn-Justin, *Quantum field theory in the large N limit: a review*, Phys. Rept. 385 (2003) 69 [hep-th/0306133].

2.5 Quantization of non-Abelian gauge theories

By contrast with QED, the quantization of non-Abelian gauge theories, even without matter fields, does not follow from simple heuristic arguments.

Due to gauge invariance, not all components of the gauge field $\mathbf{A}_\mu(x)$ are dynamical and simple canonical quantization is impossible. Tricks that worked for QED, like the direct elimination of the auxiliary components in the Coulomb gauge fail.

All concepts required to quantize non-Abelian gauge theories, like the so-called Faddeev–Popov method and ghosts are based on field integrals.

BRST symmetry, a new fermion-like symmetry has emerged from this formalism. BRST symmetry has opened the way for supersymmetry.

2.5.1 The quantization problem

The gauge field $\mathbf{A}_\mu(x)$ transforms under the adjoint representation of a Lie group G (*e.g.*, $SU(N)$). If it is represented as an element of the Lie algebra $\mathfrak{L}(G)$ and $\mathbf{g}(x)$ is a space-time dependent group element, a gauge transformation, an affine transformation, reads

$$\mathbf{A}_\mu(x) \mapsto \mathbf{g}(x)\mathbf{A}_\mu(x)\mathbf{g}^{-1}(x) + \mathbf{g}(x)\partial_\mu\mathbf{g}^{-1}(x).$$

For matter fields, gauge invariance is enforced by replacing derivatives by covariant derivatives: $\mathbf{D}_\mu = \mathbf{1}\,\partial_\mu + \mathbf{A}_\mu$.

For the gauge field action, the associated curvature

$$\mathbf{F}_{\mu\nu}(x) = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu],$$

is a tensor (*i.e.*, transforms linearly) for gauge transformations:

$$\mathbf{F}_{\mu\nu}(x) \mapsto \mathbf{g}(x)\mathbf{F}_{\mu\nu}(x)\mathbf{g}^{-1}(x).$$

The local gauge action

$$\mathcal{A}(\mathbf{A}) = \frac{1}{4g^2} \int d^4x \, \mathrm{tr}\, \mathbf{F}_{\mu\nu}(x)\mathbf{F}^{\mu\nu}(x),$$

then is gauge-invariant.

In the field integral formalism, the straightforward field integral representation of the evolution operator

$$U = \int [d\mathbf{A}_\mu] \exp\left[i\mathcal{A}(\mathbf{A})/\hbar\right],$$

is not defined because the action $\mathcal{A}$, being gauge-invariant, does not depend on one of the integration fields.

It is necessary to integrate only once over each gauge copy, that is to limit the integration to a section of gauge field space cutting each gauge-orbit once.

A typical covariant section is (Landau gauge)

$$\Phi(\mathbf{A}) \equiv \sum_\mu \partial_\mu \mathbf{A}^\mu(x) = 0\,.$$

The difficult problem then is to determine the measure on the gauge section. Quite remarkably, and this is typical for path integral methods, the method can be inferred from the consideration of finite dimensional integrals.

2.5.2 Faddeev–Popov method: the idea

The goal is to factorize the integration over gauge transformations. One starts from a non-gauge invariant equation for the space-dependent group element $\mathbf{g}(x)$, for example,

$$\Phi(\mathbf{A}_\mu^g) \equiv \partial_\mu \mathbf{A}_\mu^g(x) - \boldsymbol{\nu}(x) = 0\,,$$

where $\mathbf{A}_\mu^g$ is the gauge transform by $\mathbf{g}$ of $\mathbf{A}_\mu$ and $\boldsymbol{\nu}(x)$ an arbitrary field.

The variation of the equation with respect to $\mathbf{g}$: $\delta\mathbf{g}(x) = \boldsymbol{\omega}(x)\mathbf{g}(x)$, $\boldsymbol{\omega}(x)$ belonging to the Lie algebra, has the form

$$\delta\Phi(\mathbf{A}_\mu^g) = [\mathbf{M}(\mathbf{A}_\mu^g)\boldsymbol{\omega}](x), \quad \mathbf{M} = \partial_\mu \mathbf{D}_\mu\,.$$

One then introduces spinless fermions $\bar{\mathbf{C}}$ and $\mathbf{C}$, the Faddeev–Popov 'ghosts', and a boson field $\boldsymbol{\lambda}$ all transforming under the adjoint representation.

One uses the identity

$$1 = \int [\mathrm{d}\mathbf{g}\,\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}\,\mathrm{d}\boldsymbol{\lambda}] \exp\left[-\mathcal{S}_{\mathrm{gauge}}(\mathbf{A}_\mu^g, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu})\right]$$

with

$$\mathcal{S}_{\mathrm{gauge}} = \int \mathrm{d}^d x\,\mathrm{tr}\left\{\boldsymbol{\lambda}(x)\left[\Phi(\mathbf{A}_\mu)(x) - \boldsymbol{\nu}(x)\right] + \mathbf{C}(x)\mathbf{M}(\mathbf{A})\bar{\mathbf{C}}(x)\right\}.$$

The identity involves the generalization of field integration to Grassmann fields.

78

Introducing the identity in the formal representation of the partition function, one obtains

$$\mathcal{Z} = \int [\mathrm{d}\mathbf{g}\,\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}\,\mathrm{d}\boldsymbol{\lambda}\,\mathrm{d}\mathbf{A}_\mu]$$
$$\times \exp\left[\frac{1}{4g^2}\int \mathrm{d}^d x\,\mathrm{tr}\,\mathbf{F}^2_{\mu\nu}(x) - \mathcal{S}_{\mathrm{gauge}}(\mathbf{A}^g_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu})\right].$$

After the change variables $\mathbf{A}^g_\mu \mapsto \mathbf{A}_\mu$, the integration over $\mathbf{g}(x)$ factorizes and yields an infinite multiplicative constant.

After a few additional simple manipulations, one obtains the quantized partition function

$$\mathcal{Z} = \int \left[\mathrm{d}\mathbf{A}_\mu \, \mathrm{d}\bar{\mathbf{C}} \, \mathrm{d}\mathbf{C} \, \mathrm{d}\boldsymbol{\lambda} \right] \exp \left[-\mathcal{S}(\mathbf{A}_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}) \right],$$

where $\mathcal{S}$, in the covariant gauge $\Phi = \partial_\mu \mathbf{A}_\mu$, is the local action:

$$\mathcal{S}(\mathbf{A}_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda})$$
$$= \int \mathrm{d}^d x \, \mathrm{tr} \left[-\frac{1}{4e^2} \mathbf{F}_{\mu\nu}^2 + \frac{\xi e^2}{2} \boldsymbol{\lambda}^2(x) + \boldsymbol{\lambda}(x) \partial_\mu \mathbf{A}_\mu(x) + \mathbf{C}(x) \partial_\mu \mathbf{D}_\mu \bar{\mathbf{C}}(x) \right].$$