Lecture 11: FROM NON-RELATIVISTIC TO RELATIVISTIC FERMIONS

The assumption of a finite-dimensional vector space of one-particle states is quite restrictive for fermions since the total number of fermions is then bounded. More interesting applications require replacing, for one-particle states, finite dimensional vector spaces by Hilbert spaces.

We thus generalize the formalism of the preceding lecture, replacing one-particle finite-dimensional complex vector spaces by Hilbert spaces. We first determine the equation of state for independent fermions and then extend the analysis to interacting fermions. We show how a generalization of the path integral of section 10.14 allows expressing the partition function of fermion systems as an integral over Grassmann fields.

In a first part we discuss non-relativistic quantum statistical physics of fermions in the grand canonical formulation and then extend the construction to relativistic quantum field theory.

11.1 Fermion states: Hilbert space

11.1.1 Independent fermions

For independent fermions, the equation of state can still be expressed in terms of the one-particle quantum Hamiltonian $H^{(1)}$ in the form (10.63),

$$\langle \mathbf{N} \rangle = \operatorname{tr} \frac{1}{e^{\beta(H^{(1)}-\mu)}+1} \,.$$

As an illustration, we consider a gas of free fermions of mass m in a cubic box of linear size L and, thus, of volume L^d in dimension d. The one-particle quantum Hamiltonian is

$$H^{(1)} = \hat{\mathbf{p}}^2/2m \,.$$

In a box, momenta are quantized, the precise form depending on the boundary conditions.

Assuming periodic boundary conditions for convenience, but this plays no role in the large volume limit, one finds

$$\mathbf{p} = 2\pi\hbar\,\mathbf{n}/L\,, \quad \mathbf{n} \in \mathbb{Z}^d,$$

the corresponding energy being $E = \mathbf{p}^2/2m$.

The derivation of the equation of state in d space dimensions then follows from the arguments presented in section 5.1.3. In the infinite volume limit $L \to \infty$, one finds for the density

$$\rho(\beta,\mu) \underset{L\to\infty}{=} \frac{\langle \mathbf{N} \rangle}{L^d} = \frac{1}{(2\pi\hbar)^d} \int \frac{\mathrm{d}^d p}{\mathrm{e}^{\beta(\mathbf{p}^2/2m-\mu)}+1}\,. \tag{11.1}$$

In isotropic space the Fermi surface thus is the sphere $\mathbf{p}^2/2m = \mu$ for $\mu > 0$.

11.1.2 Fock space

We now generalize the construction of sections 10.11-10.12 to Hilbert spaces.

We consider systems of identical fermions that are described by wave functions $\psi_n(x_1, \dots, x_n)$, $x_i \in \mathbb{R}^d$, antisymmetric functions of their arguments (because fermions have no other quantum numbers). Therefore, the arguments of a generating functional must be functions $\varphi(x)$ that are generators of an infinite dimensional Grassmann algebra and satisfy

$$\varphi(x)\varphi(x') + \varphi(x')\varphi(x) = 0\,.$$

We then define the functional (the product of φ's is ordered)

$$\Psi(\varphi) = \sum_{n=0} \frac{1}{n!} \left(\int \prod_i \mathrm{d}^d x_i \, \varphi(x_i) \right) \psi_n(x_1, \dots, x_n).$$

The scalar product of two generating functionals is defined in terms of a Grassmann field integral, which generalizes expression (10.34):

$$(\Psi_1, \Psi_2) = \int [\mathrm{d}\varphi(x)\mathrm{d}\bar{\varphi}(x)] \overline{\Psi_1(\varphi)} \Psi_2(\varphi) \exp\left[\int \mathrm{d}^d x\, \bar{\varphi}(x)\varphi(x)\right],$$

normalized by $(1, 1) = 1$. The explicit calculation of the scalar product requires the corresponding Gaussian two-point function, which can be written as a scalar product. One finds

$$\big(\varphi(x), \varphi(y)\big) = \delta^{(d)}(x - y).$$

In particular, the Fock space is the space of functionals with finite norm, that is, such that

$$\|\Psi\|^2 = (\Psi, \Psi) = \sum_{n=0} \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, |\psi_n(x_1, \ldots, x_n)|^2 < \infty\,.$$

When $\|\Psi\| = 1$, the nth term in the sum is the probability that the fermion system is in an n-particle state.

11.1.3 Operators and kernels

The identity kernel that generalizes expression (10.37), is

$$\mathcal{I}(\varphi,\bar\varphi)=\exp\left[-\int \mathrm{d}^d x\,\bar\varphi(x)\varphi(x)\right].$$

We now consider the interacting Hamiltonian $\mathbf{H}$ of section 5.2 with the form (5.8) in the n-particle space.

We set $\mathbf{H}=\mathbf{T}+\mathbf{V}$, where $\mathbf{T}$ is the kinetic term, which in the sub-space of n-particle wave functions is represented by

$$T_n=-\frac{\hbar^2}{2m}\sum_{i=1}^{n}\nabla^2_{x_i}$$

and $\mathbf{V}$ is a pair interaction represented by

$$V_n=\sum_{i<j\le n}V(x_i-x_j)\quad\text{with}\quad V(x)=V(-x).$$

The formal expression of the kinetic term is the same as in the boson case. The potential term remains also the same, but with a specific order of fields in products.

With the conventions of section 10.12, the particle number operator and the Hamiltonian have the kernel representation

$$\langle\varphi|\mathbf{N}|\bar\varphi\rangle = \varphi(x)\bar\varphi(x)\mathcal{I}(\varphi,\bar\varphi),\quad \langle\varphi|\mathbf{H}|\bar\varphi\rangle = H(\varphi,\bar\varphi)\mathcal{I}(\varphi,\bar\varphi)$$

with

$$H(\varphi,\bar\varphi) = -\frac{\hbar^2}{2m}\int \mathrm{d}^d x\,\varphi(x)\nabla_x^2\bar\varphi(x) + \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\varphi(x)\varphi(y)V(x-y)\bar\varphi(y)\bar\varphi(x).$$

11.2 Partition function: the field integral

The derivation of a field integral representation then follows closely the arguments already presented in the boson case in section 5.2, except that it is necessary to carefully keep track of the ordering of factors in field products and of the signs. As a generalization of expression (10.61), the field integral representation of the partition function of the Fermi gas follows:

$$\mathcal{Z}(\beta,\mu)=\int[\mathrm{d}\varphi(t,x)\mathrm{d}\bar{\varphi}(t,x)]\exp[-\mathcal{S}(\bar{\varphi},\varphi)] \tag{11.2}$$

with the anti-periodic boundary conditions

$$\varphi(\beta/2,x)=-\varphi(-\beta/2,x),\quad \bar{\varphi}(\beta/2,x)=-\bar{\varphi}(-\beta/2,x),$$

and the euclidean action

$$\begin{aligned}\mathcal{S}(\bar{\varphi},\varphi)=&\int\mathrm{d}t\,\mathrm{d}^d x\,\bar{\varphi}(t,x)\left(\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla_x^2+\mu\right)\varphi(t,x)\\&+\tfrac{1}{2}\int\mathrm{d}t\,\mathrm{d}^d x\,\mathrm{d}^d y\,\bar{\varphi}(t,x)\varphi(t,x)V(x-y)\bar{\varphi}(t,y)\varphi(t,y)\,.\end{aligned} \tag{11.3}$$

The equation of state. Quite generally, the equation of state is obtained by differentiating the partition function (11.2). Assuming a periodic box of linear size L, one finds the density

$$\rho(\beta,\mu)=\frac{1}{\beta L^d}\frac{\partial \ln \mathcal{Z}}{\partial \mu}=\frac{1}{\beta L^d \hbar}\int \mathrm{d}t\,\mathrm{d}^d x\,\langle \varphi(t,x)\bar{\varphi}(t,x)\rangle=\langle \varphi(0,0)\bar{\varphi}(0,0)\rangle\,, \tag{11.4}$$

where translation invariance in space and time has been used.

We now verify that, in free field theory, the equation of state reduces to the equation (11.1) of free fermions, and then comment briefly about the effect of interactions.

11.2.1 The free field theory

The action of the free theory reduces to

$$\mathcal{S}(\bar{\varphi},\varphi)=\int \mathrm{d}t\,\mathrm{d}^d x\,\bar{\varphi}(t,x)\left(\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla_x^2+\mu\right)\varphi(t,x).$$

In a free (Gaussian) theory, all quantities can be expressed in terms of the two-point function. The two-point function

$$\langle\bar{\varphi}(t,x)\varphi(t',x')\rangle\equiv\Delta(t-t',x-x')$$

satisfies the equation

$$\left(\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\nabla_x^2-\mu\right)\Delta(t,x)=\delta(t)\delta^{(d)}(x),$$

with anti-periodic boundary conditions.

Setting

$$\Delta(t,x)=\frac{1}{(2\pi\hbar)^d}\int \mathrm{d}^d p\ \mathrm{e}^{ipx/\hbar}\,\tilde{\Delta}(t,p),$$

one finds

$$\left(\frac{\partial}{\partial t}+\frac{p^2}{2m}-\mu\right)\tilde{\Delta}(t,p)=\delta(t).$$

Using the result (10.55), one obtains

$$\tilde{\Delta}(t,p)=\mathrm{e}^{-\kappa(p)t}\left[\theta(t)-\frac{1}{\mathrm{e}^{\kappa(p)\beta}+1}\right],$$

where $\kappa(p)=p^2/2m-\mu$ and $\theta(t)$ is here the step function.

The equation of state. Introducing this result in equation (11.4), one obtains

$$\begin{aligned}\rho(\beta,\mu)&=-\frac{1}{(2\pi\hbar)^d}\int \mathrm{d}^d p\,\tilde{\Delta}(0,p)\\ &=\frac{1}{(2\pi\hbar)^d}\int \mathrm{d}^d p\left[-\theta(0)+\frac{1}{\mathrm{e}^{\kappa(p)\beta}+1}\right].\end{aligned}$$

This expression coincides with the result (11.1) obtained directly when $\theta(0)$ is chosen to vanish,

$$\rho(\beta,\mu)=\frac{1}{(2\pi\hbar)^d}\int\frac{\mathrm{d}^d p}{\mathrm{e}^{\beta(\mathbf{p}^2/2m-\mu)}+1}$$

and, otherwise, differs by an infinite constant that can be removed by adding a constant linear in μ to the action.

11.2.2 Interactions: the δ-function potential

An interesting interacting example is provided by the two-body pseudo-potential

$$V(x)=g\,\delta^{(d)}(x),$$

where $\delta^{(d)}$ is Dirac'sd-dimensional function.

The action then becomes local, in the sense that it becomes the integral of a Lagrangian density depending only on the field and its derivatives.

In the case of fermions without internal degrees of freedom, the two-body interaction then vanishes since it involves the squares of Grassmann variables, and the fermions are free. A more interesting example is provided by systems where fermions have an internal degree of freedom with two possible values (like the spin for an electron). The action then depends on two pairs of fields $\varphi_\alpha(t,x)$, $\alpha = 1,2$ and the interaction no longer vanishes:

$$\mathcal{S}(\bar\varphi,\varphi) = \int \mathrm{d}t\,\mathrm{d}^d x \left[\sum_\alpha \bar\varphi_\alpha(t,x)\left(\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla_x^2 + \mu\right)\varphi_\alpha(t,x)\right.$$
$$\left. + g\bar\varphi_1(t,x)\varphi_1(t,x)\bar\varphi_2(t,x)\varphi_2(t,x)\right]. \tag{11.5}$$

The action and the corresponding field integral are then invariant under $U(2)$ unitary transformations

$$\varphi_\alpha \mapsto \sum_\beta U_{\alpha\beta}\varphi_\beta\,, \quad \bar\varphi_\alpha \mapsto \sum_\beta U^*_{\alpha\beta}\bar\varphi_\beta \text{ with } UU^\dagger = \mathbf{1}\,.$$

Indeed, the kinetic term is a complex scalar product, and for the interaction term one obtains $|\det U|^2 = 1$. This symmetry is a combination of the $U(1)$ particle number conservation and the $SU(2)$ spin group symmetry.

The one-dimensional quantum system is completely integrable, in the sense that all eigenstates of the Hamiltonian are linear combinations of a finite number of plane waves (Bethe ansatz).

Finally, note that this system has a relativistic generalization, the Thirring model, which is also integrable in one space dimension.

Mean-field approximation. Interesting physics is associated to an attractive interaction, that is $g < 0$. However, unlike for bosons, the steepest descent method yields no direct insight into this problem. A possible strategy relies on introducing an auxiliary boson field $\chi(t, x)$ and rewriting the quartic fermion interaction as an integral over χ with an action quadratic in the fermions. The fermion integral becomes Gaussian and can be performed. The remaining χ integral can be evaluated by the steepest descent method.

11.3 Fermi gas: evolution operator

The Fermi gas. The generalization of the formalism of section 10.14.4 to fields allows to describe the time evolution of the Fermi gas in the formalism of second quantization. The evolution operator for the non-relativistic Fermi gas in the presence of a chemical potential μ coupled to the particle number $\mathbf{N}$, is given by a field integral, continuation to real time of the expression (11.3). Here we find

$$\begin{aligned}\langle \varphi'' |\, \mathbf{U}(t'', t')\, |\bar{\varphi}'\rangle &= \langle \varphi'' |\, \mathrm{e}^{-i(t''-t)(\mathbf{H}-\mu\mathbf{N})/\hbar}\, |\bar{\varphi}'\rangle \\ &= \int [\mathrm{d}\bar{\varphi}(t,x)\mathrm{d}\varphi(t,x)] \exp[i\mathcal{A}(\varphi,\bar{\varphi})/\hbar],\end{aligned}$$

where the fields $\{\varphi(t,x), \bar{\varphi}(t,x)\}$ are generators of a Grassmann algebra, and satisfy the boundary conditions

$$\bar{\varphi}(t,x') \equiv \bar{\varphi}'(x), \quad \varphi(t,x'') \equiv \varphi''(x).$$

In the example of an external potential V_1 (and then $\mu = 0$) and a pair potential V_2, the action $\mathcal{A}(\varphi, \bar{\varphi})$ is

$$\begin{aligned}\mathcal{A}(\varphi,\bar{\varphi}) = & -i\hbar\bar{\varphi}(t,x')\varphi(t,x') \\ & + \int \mathrm{d}t\,\mathrm{d}^d x\,\bar{\varphi}(t,x)\left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla_x^2 - V_1(x)\right)\varphi(t,x) \\ & - \tfrac{1}{2}\int \mathrm{d}t\,\mathrm{d}^d x\,\mathrm{d}^d y\,\bar{\varphi}(t,x)\varphi(t,x)V_2(x,y)\bar{\varphi}(t,y)\varphi(t,y).\end{aligned} \tag{11.6}$$

For example, in the absence of an external potential V_1, the action for N-component fermions $\varphi^\alpha, \bar{\varphi}^\alpha$ and a pseudo-potential $V_2 = G\delta_{\alpha\beta}\delta(x-y)$ becomes local:

$$\begin{aligned}\mathcal{A}(\varphi,\bar{\varphi}) = \int \mathrm{d}t\,\mathrm{d}^d x\Bigg[\sum_\alpha \bar{\varphi}^\alpha(t,x)\left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla_x^2 + \mu\right)\varphi^\alpha(t,x) \\ - \tfrac{1}{2}G\left(\sum_\alpha \bar{\varphi}^\alpha(t,x)\varphi^\alpha(t,x)\right)^2\Bigg] - i\hbar\sum_\alpha \bar{\varphi}^\alpha(t,x')\varphi^\alpha(t,x').\end{aligned}$$

11.4 Relativistic fermions

In previous sections, we have described the basic tools required for constructing theories with Fermi fields: quantum mechanics with Grassmann variables in section 10.13, a field integral representation of the statistical operator $\mathrm{e}^{-\beta H}$ for the non-relativistic Fermi gas in the formalism of second quantization and an expression for the evolution operator in section 10.14.4.

In the case of relativistic fermions, the new feature is the specific role played by the spin structure of fermions. Therefore, unlike what we have done with scalar bosons, we begin with the real time formalism, in the spirit of section 3.11, a direct euclidean presentation being less intuitive.

We analyse the free action for Dirac fermions and explain the relation between fields and particles. We derive an expression for the scattering matrix. We discuss the non-relativistic limit of a model of self-coupled massive Dirac fermions.

We have devoted lecture 6 to perturbation theory and general functional methods. We outline here only the aspects that are specific to fermions. As for the scalar field theory, we first calculate the Gaussian integral, which corresponds to a free field theory. Then adding a source term to the action, we obtain the generating functional of correlation functions. The field integral corresponding to a general action with an interaction expandable in powers of the field, can be expressed in terms of a series of Gaussian expectation values, which can be calculated, for example, with the help of Wick's theorem.

In section 11.4.1, we note the connection between spin and statistics for free fermions. In section 11.11, we verify the property in a simple perturbative calculation.

For completeness, in appendix A11, we describe a few properties of the spin group, the algebra of γ matrices, and the corresponding spinors.

11.4.1 Massive Dirac fermions

We consider massive relativistic fermions in four dimensions. We describe in the real time formulation the structure of fermion states in the free field theory, and construct the S-matrix.

In the formalism of field integrals, fermion fields are generators of a Grassmann algebra. In a relativistic invariant theory, they are also spinors $\bar{\psi}_\alpha(x), \psi_\alpha(x)$, $1 \le \alpha \le 4$, four-component vectors transforming under the action of the spin group.

In real time the spin group is $\mathrm{Spin}(1,3)$, which we do not discuss extensively. It is associated with the relativistic group (Lorentz group) $SO(1,3)$ of space–time pseudo-orthogonal transformations. In section 11.8 and in more detail in appendix A11, we describe instead the euclidean analytic continuation $\mathrm{Spin}(4)$ of the spin group. We use throughout the corresponding four hermitian γ-matrices. With our conventions, they satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = \delta_{\mu\nu} \,, \quad \gamma_1 = \gamma_1^T \,, \ \gamma_3 = \gamma_3^T \,, \ \gamma_2 = -\gamma_2^T \,, \ \gamma_4 = -\gamma_4^T \,. \quad (11.7)$$

The free action. The action for a free massive Dirac field can be written as

$$\mathcal{A}_0(\bar{\psi},\psi)=\int \mathrm{d}t\,\mathrm{d}^3x\sum_{\alpha,\beta}\bar{\psi}_\alpha(t,x)\left[\frac{1}{i}(\gamma_0)_{\alpha\beta}\partial_t+\boldsymbol{\gamma}_{\alpha\beta}\cdot\nabla_x+m\delta_{\alpha\beta}\right]\psi_\beta(t,x)\,, \tag{11.8}$$

where, here, we denote by γ_0 the matrix γ_4 associated with the time variable $t\equiv x_0\equiv -ix_4$, $\boldsymbol{\gamma}$ standing for $(\gamma_1,\gamma_2,\gamma_3)$.

The action is relativistic invariant in the sense that it is invariant under the transformations:

$$\left(\psi_\Lambda\right)_\alpha(t,x)=\sum_\beta \Lambda^{-1}_{\alpha\beta}\psi_\beta(t',x'),\ \left(\bar{\psi}_\Lambda\right)_\alpha(t,x)=\sum_\beta \bar{\psi}_\beta(t',x')\Lambda_{\beta\alpha}$$

where the matrix $\boldsymbol{\Lambda}$ belongs to the spin group $\mathrm{Spin}(1,3)$ (see section A11.1.4 adapted to real time) and $(t',x')=\mathbf{R}(t,x)$, where $\mathbf{R}(\Lambda)$ is the corresponding 4×4 matrix, element of the group $SO(1,3)$ isomorphic to a subgroup of $\mathrm{Spin}(1,3)$.

The group $SO(1,3)$ preserves the metric tensor $\mathbf{g}$, corresponding to the diagonal matrix with coefficients $(+1,-1,-1,-1)$:

$$\mathbf{R}\,\mathbf{g}\,\mathbf{R}^T = \mathbf{g} \tag{11.9}$$

($\mathbf{R}^T$ denotes the transpose of the matrix $\mathbf{R}$ of elements $R_{\mu\nu}$.) and $\det \mathbf{R} = 1$.

After the linear change of variables $\mathbf{R}(t,x) \mapsto (t',x')$, the invariance of the action relies on the identity

$$\sum_{\nu=0}^{3} R_{\mu\nu} \Lambda \tilde{\gamma}_\nu \Lambda^{-1} = \tilde{\gamma}_\mu \text{ with } \tilde{\gamma}_0 = \gamma_0\,, \tilde{\boldsymbol{\gamma}} = i\boldsymbol{\gamma}\,.$$

In relativistic conventions, one introduces a field $\bar{\psi}$ that is not the conjugate of ψ (at variance with our usual notation), because its transformation properties under the spin group makes the action explicitly covariant.

Indeed, matrices Λ belonging to the fundamental representation of the spin group $\mathrm{Spin}(1,3)$, satisfy $\gamma_0 \Lambda^\dagger \gamma_0 = \Lambda^{-1}$, as one verifies by adapting the expressions of appendix A11 to real time.

However, to identify the action (11.8) with an action of the form (10.65b) resulting from a Hamiltonian formalism, it is necessary to express $\bar{\psi}$ in terms of the conjugate field $\psi^\dagger$, a notation we use for relativistic fermions. With our conventions

$$\psi^\dagger = -\bar{\psi}\gamma_0 \,. \tag{11.10}$$

Then,

$$\mathcal{A}_0(\psi,\psi^\dagger) = \int \mathrm{d}t\, \mathrm{d}^3x\, \psi^\dagger(t,x)\left[i\partial_t - \gamma_0(\boldsymbol{\gamma}\cdot\nabla_x + m)\right]\psi(t,x)\,.$$

Since the γ matrices are hermitian, we find $\mathcal{A}_0 = \mathcal{A}_0{}^\dagger$ and thus the corresponding Hamiltonian is hermitian.

To diagonalize the quadratic form, we proceed by Fourier transformation, setting

$$\psi(t,x) = \int \mathrm{d}^3\hat{p}\ \mathrm{e}^{i\hat{p}x}\,\tilde{\psi}(t,\hat{p}), \quad \psi^\dagger(t,x) = \int \mathrm{d}^3\hat{p}\ \mathrm{e}^{-i\hat{p}x}\,\tilde{\psi}^\dagger(t,\hat{p}).$$

The free action becomes

$$\mathcal{A}_0(\psi,\psi^\dagger)=(2\pi)^3\int \mathrm{d}t\,\mathrm{d}^3\hat{p}\,\tilde{\psi}^\dagger(t,\hat{p})\,[i\partial_t+h(\hat{p})]\,\tilde{\psi}(t,\hat{p}),$$

where $h(\hat{p})=h^\dagger(\hat{p})$ is the hermitian matrix

$$h(\hat{p})=-\gamma_0\,(i\boldsymbol{\gamma}\cdot\hat{p}+m)\,. \tag{11.11}$$

In contrast with the scalar case, due to the spin structure the Hamiltonian is not completely diagonalized, but the diagonalization has been reduced to a simple matrix problem. One verifies that

$$h^2(\hat{p})=\omega^2(\hat{p}),\quad \omega(\hat{p})=\sqrt{\hat{p}^2+m^2}\,.$$

It is convenient to introduce the matrix

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4\ \Rightarrow \gamma_5=\gamma_5^\dagger\,,\ \gamma_5^2=1\,.$$

The matrix h has two eigenvalues $\pm\omega(\hat{p})$. Since

$$\gamma_0\gamma_5 h(\hat{p})\gamma_5\gamma_0 = -h(\hat{p}),$$

if a spinor $u(\hat{p})$ is an eigenvector with eigenvalue $\omega(\hat{p})$, $\gamma_0\gamma_5 u(\hat{p})$ ($\gamma_0\gamma_5$ is associated with time reversal) is an eigenvector with eigenvalue $-\omega(\hat{p})$. The two corresponding subspaces have equal dimensions.

The two orthogonal, hermitian projectors $P_\pm$ on the positive and negative energy sectors are

$$P_\pm = \tfrac{1}{2}\left[1 \pm h(\hat{p})/\omega(\hat{p})\right], \quad \Rightarrow \ P_+ + P_- = 1\,, \quad P_\pm^2 = P_\pm\,, \quad P_+P_- = 0\,. \tag{11.12}$$

We note that with the simplest assignment of ψ as the Grassmann field associated with creation operators, states can be created with both positive and negative energies. This means that we have misidentified the vacuum state, which must be the ground state.

As we have shown in section 10.14.4, by exchanging the role of conjugate fields we change the sign of the one-particle energy (a property specific to fermions).

In contrast to what one might naively have guessed, as in the scalar case the fields ψ and $\psi^\dagger$ must be decomposed into a sum of analytic and anti-analytic components, to ensure that one-particle states have positive energy.

We thus define

$$\chi^*_-(t,\hat p) = [2\omega(\hat p)]^{1/2} P_- \tilde\psi(t,\hat p), \qquad \varphi_+(t,\hat p) = [2\omega(\hat p)]^{1/2} P_+ \tilde\psi(t,\hat p) \quad (11.13a)$$

$$\varphi^\dagger_+(t,\hat p) = [2\omega(\hat p)]^{1/2} \tilde\psi^\dagger(t,\hat p) P_+\,, \qquad \chi^T_-(t,\hat p) = [2\omega(\hat p)]^{1/2} \tilde\psi^\dagger(t,\hat p) P_- \,. \quad (11.13b)$$

Conversely,

$$\tilde\psi(t,\hat p) = \frac{1}{\sqrt{2\omega(\hat p)}} \left(\chi^*_-(t,\hat p) + \varphi_+(t,\hat p)\right) \qquad (11.14a)$$

$$\tilde\psi^\dagger(t,\hat p) = \frac{1}{\sqrt{2\omega(\hat p)}} \left(\chi^T_-(t,\hat p) + \varphi^\dagger_+(t,\hat p)\right)\,. \qquad (11.14b)$$

The action becomes

$$\mathcal{A}_0(\chi_-, \varphi_+) = (2\pi)^3 \int \mathrm{d}t\, \frac{\mathrm{d}^3\hat{p}}{2\omega(\hat{p})} \Big[\varphi_+^\dagger(t,\hat{p})\big(i\partial_t + \omega(\hat{p})\big)\varphi_+(t,\hat{p})$$
$$+\chi_-^\dagger(t,\hat{p})\big(i\partial_t + \omega(\hat{p})\big)\chi_-(t,\hat{p})\Big]\,.$$

The spectrum thus contains two 2-component particles of mass m, φ_+ transforming under the fundamental representation of the static spin group $\mathrm{Spin}(3) \equiv SU(2)$, χ_- under the conjugated and equivalent representation. To verify the equivalence, we set

$$\varphi_-(t,\hat{p}) = C^\dagger \chi_-(t,\hat{p}), \tag{11.15}$$

where C is a unitary matrix, $C^\dagger C = 1$. The field χ_- satisfies $P_-^T\chi_- = \chi_-$ (equation (11.13b)).

If we can find a matrix C such that

$$C^{\dagger} P_{-}^{T} C = P_{+} \,,$$

then $P_{+}\varphi_{-} = \varphi_{-}$, and the two fields $\varphi_{\pm}$ have the same transformation properties (see equation (11.13a)). We use the explicit expressions (11.11), (11.12) of $P_{\pm}$. Since C is unitary, the equation reduces to

$$C^{\dagger} h^{T}(\hat{p}) C = -h(\hat{p}) \quad \Leftrightarrow \quad C^{\dagger}\left(i\boldsymbol{\gamma}^{T} \cdot \hat{p} + m\right)\gamma_{0}^{T} C = -\gamma_{0}\left(i\boldsymbol{\gamma} \cdot \hat{p} + m\right) .$$

One verifies that the matrix

$$C = \gamma_{1}\gamma_{3}$$

satisfies (equation (A11.23))

$$C^{-1}\gamma_{\mu}^{T} C = -\gamma_{\mu} \,.$$

and thus has the required property. (For details see appendix A11.2.5, where we discuss charge conjugation.)

The action can then be rewritten as

$$\mathcal{A}_0(\varphi_+, \varphi_-) = (2\pi)^3 \int \mathrm{d}t \, \frac{\mathrm{d}^3\hat{p}}{2\omega(\hat{p})} \sum_{\epsilon=\pm} \varphi_\epsilon^\dagger(t,\hat{p}) \big(i\partial_t + \omega(\hat{p})\big) \varphi_\epsilon(t,\hat{p}). \quad (11.16)$$

The final form of the action shows that the Dirac field carries two equal mass particles transforming under the fundamental representation of the static spin group $\mathrm{Spin}(3) \equiv SU(2)$, spin $1/2$ particles, related by charge conjugation (in the case of charged particles they have opposite charge).

In these variables, the free Hamiltonian reads

$$\mathbf{H}_0 = (2\pi)^3 \int \mathrm{d}^3\mathrm{d}\hat{p} \, \omega(\hat{p}) \left(\varphi_+(\hat{p}) \frac{\delta}{\delta\varphi_+(\hat{p})} - \frac{\delta}{\delta\varphi_-(\hat{p})} \varphi_-(\hat{p}) \right).$$

Written in normal order, the Hamiltonian is

$$\mathbf{H}_0 = (2\pi)^3 \int \mathrm{d}^3\mathrm{d}\hat{p} \, \omega(\hat{p}) \left(\varphi_+(\hat{p}) \frac{\delta}{\delta\varphi_+(\hat{p})} + \varphi_-(\hat{p}) \frac{\delta}{\delta\varphi_-(\hat{p})} \right) + E_0(\mathrm{Dirac}).$$

The ground state (vacuum) energy $E_0(\text{Dirac})$ is negative, and proportional to the free scalar vacuum energy (5.33):

$$E_0(\text{Dirac}) = -4E_0(\text{scalar}).$$

We are now in a situation analogous to the scalar case, and the non-relativistic Fermi gas of section 10.14.4. The generating functional of n-particle wave functions, which is an element of Fock's space, is a general Grassmann analytic function of $\varphi_\pm$.

The particle number operators for both particles commute with the Hamiltonian,

$$\mathbf{N}_\pm = (2\pi)^3 \int \mathrm{d}^3\hat{p}\, \varphi_\pm(\hat{p}) \frac{\delta}{\delta\varphi_\pm(\hat{p})}, \quad [\mathbf{N}_\pm, \mathbf{H}_0] = 0\,,$$

a property that no longer holds in general for a local interacting theory.

Remarks.

(i) By adding to one Dirac fermion 4 scalar bosons of the same mass m, one obtains a theory with zero vacuum energy. One can show that this boson–fermion free theory then has a special fermion-type symmetry called supersymmetry.

(ii) The problem of the negative energy states can only be solved because fermions anti-commute. This is the reflection of the connection between spin and statistics, a property specific to local relativistic quantum field theory.

11.4.2 Space reflections

A reflection along the $i = 1, 2, 3$ axis corresponds to the transformation Π_i:

$$\Pi_i : \ \psi_{\Pi_i}(\mathbf{x}) = \gamma_5 \gamma_i \psi(\tilde{\mathbf{x}}), \ \bar{\psi}_{\Pi_i}(\mathbf{x}) = \bar{\psi}(\tilde{\mathbf{x}}) \gamma_i \gamma_5$$

with

$$\tilde{x}_\mu = x_\mu \text{ for } \mu \neq i \text{ and } \tilde{x}_i = -x_i \, .$$

The mass term in action (11.8) is clearly invariant. In the term $\not\partial$ the space reflection changes ∂_i in $-\partial_i$, but then $\gamma_5 \gamma_i$ anti-commutes with γ_i and commutes with all other γ_μ matrices. The total action (11.8) is thus invariant.

The reflection matrices acting on coordinates (x_0, x_1, x_2, x_3) are matrices of determinant -1. The product of the $SO(1, 3)$ group and reflections thus generates the whole group $O(1, 3)$, still defined by the condition (11.9) but with $\det \mathbf{R} = \pm 1$. The corresponding spin group contains $O(1, 3)$ as a subgroup.

11.5 Chiral symmetry and massless fermions

Chiral symmetry. We consider the $U(1)$ group of chiral transformations

$$\psi(t,x) \mapsto \mathrm{e}^{i\theta\gamma_5}\,\psi(t,x), \quad \bar{\psi}(t,x) \mapsto \bar{\psi}(t,x)\,\mathrm{e}^{i\theta\gamma_5}, \;\; \theta \in \mathbb{R}. \tag{11.17}$$

The transformation of $\bar{\psi}$ is consistent with hermitian conjugation because $\psi^{\dagger} = -\bar{\psi}\gamma_0$.

In expression (11.8) the kinetic part

$$\bar{\psi}(t,x)\,(-i\gamma_0\partial_t + \boldsymbol{\gamma}\cdot\nabla_x)\,\psi(t,x)$$

is invariant while the mass term $m\bar{\psi}\psi$ changes sign.

Therefore, the condition of chiral symmetry leads to a massless fermion theory. Such a symmetry has deep consequences, in particular, implying an important difference between scalar bosons and Dirac fermions. In contrast with bosons, the condition for fermions to be massless can be enforced by a symmetry of the action.

Chiral components. We show in section A11.1.4 that the set of the six matrices

$$\tilde{\sigma}_{\mu\nu} = \tfrac{1}{2}[\gamma_\mu, \gamma_\nu],\ 0 \le \mu < \nu \le 3\,,$$

generates the spin group $\mathrm{Spin}(4)$ but then also $\mathrm{Spin}(1,3)$. These matrices commute with γ_5: $[\gamma_5, \tilde{\sigma}_{\mu\nu}] = 0$. Thus, as further discussed in section A11.2.4, the spinor representation can be reduced.

One can define chiral components $\psi_\pm$ of the fermion field

$$\psi_\pm(x) = \tfrac{1}{2}(1 \pm \gamma_5)\psi(x)\,.$$

and correspondingly $\bar{\psi}_\pm(x)$:

$$\bar{\psi}_\pm(x) = \bar{\psi}(x)\tfrac{1}{2}\,(1 \pm \gamma_5)\,,$$

often denoted by $\psi_{\mathrm{R}}(x)$, $\psi_{\mathrm{L}}(x)$, $\bar{\psi}_{\mathrm{R}}(x)$, $\bar{\psi}_{\mathrm{L}}(x)$ for right and left components, by reference to the propagation in time.

However, with two of these spinors it is possible to construct only a massless theory:

$$\mathcal{S}_0(\bar{\psi}_-, \psi_+) = -\int \mathrm{d}^4x\, \bar{\psi}_-(x)\not{\partial}\psi_+(x), \qquad (11.18)$$

because $\bar{\psi}_-\psi_+ = \bar{\psi}_+\psi_- = 0$. To construct an action for a massive propagating fermion the four spinors are needed.

The action (11.18), in contrast with the action (11.8), is not invariant under reflection, since reflection exchanges chiral components:

$$\gamma_5\gamma_\mu\,(1+\gamma_5) = (1-\gamma_5)\,\gamma_5\gamma_\mu\,.$$

The representation of the semi-direct product of the spin group and space reflections is thus irreducible. This group has a subgroup isomorphic to $O(1,3)$.

11.6 Interacting theory and S-matrix

The expression of the S-matrix in an interacting theory follows from a simple extension of the method explained in the scalar case. One has only to be careful of signs. One verifies that the S-matrix is given by

$$S(\varphi,\bar{\varphi})=\int[\mathrm{d}\bar{\psi}\mathrm{d}\psi]\exp i\mathcal{A}(\bar{\psi}\sqrt{Z}+\bar{\psi}_c,\psi\sqrt{Z}+\psi_c),$$

that is, again a field integral in a background field, where the classical anti-commuting fields $\bar{\psi}_c,\psi_c$ are solutions to the free field equations, which can be parametrized in the form (11.14), (11.15):

$$\tilde{\psi}_c(\hat{p})=C^*\varphi_-(\hat{p})+\varphi_+(\hat{p}),\quad \tilde{\psi}_c^\dagger(\hat{p})=\varphi_-(\hat{p})+\bar{\varphi}_+(\hat{p})C^T\,,\quad \bar{\psi}_c=-\psi_c^\dagger\gamma_0\,,$$

and $P_+\varphi_\pm=\varphi_\pm$.

A renormalization constant Z is also required here, to obtain S-matrix elements with the proper normalization.

In the same notation, the unitarity of the S-matrix takes the form

$$\int [\mathrm{d}\bar{\varphi}'(\hat{p})\mathrm{d}\varphi'(\hat{p})]\mathcal{S}^*(\varphi', \bar{\varphi})\mathcal{S}(\varphi', \bar{\varphi}) \exp\left[(2\pi)^3 \int \frac{\mathrm{d}^3\hat{p}}{2\omega(\hat{p})} \sum_{\varepsilon=\pm} \bar{\varphi}'_\varepsilon(\hat{p})\varphi'_\varepsilon(\hat{p})\right]$$
$$= \exp\left[(2\pi)^3 \int \frac{\mathrm{d}^3\hat{p}}{2\omega(\hat{p})} \sum_{\varepsilon=\pm} \varphi_\varepsilon(\hat{p})\bar{\varphi}_\varepsilon(\hat{p})\right].$$

11.7 Massive fermions: the non-relativistic limit

In analogy with the discussion for scalar bosons of section 5.12, we show here how a non-relativistic quantum theory emerges as the low energy, low momentum limit of a relativistic theory involving massive fermions.

As an example, we consider self-interacting fermions with the the action

$$\mathcal{A}(\bar{\psi},\psi)=\int \mathrm{d}t\,\mathrm{d}^3x\left[\bar{\psi}\left(\frac{1}{i}\gamma_0\partial_t+\boldsymbol{\gamma}\cdot\nabla_x+m\right)\psi+\tfrac{1}{2}G\left(\bar{\psi}\psi\right)^2\right],$$

which we express in terms of $\psi,\psi^\dagger$ (equation (11.10)):

$$\mathcal{A}(\psi,\psi^\dagger)=\int \mathrm{d}t\,\mathrm{d}x\left[\psi^\dagger(t,x)\left[i\partial_t-\gamma_0(\boldsymbol{\gamma}\cdot\nabla_x+m)\right]\psi(t,x)+\tfrac{1}{2}G\left(\psi^\dagger\gamma_0\psi\right)^2\right].$$

Due to the spin structure and the linearity in ∇_x of the action, extracting the non-relativistic limit requires slightly more work than in the scalar case.

However, one can use the transformations that lead to the action (11.16). In the kinematic term one then expands the one-particle energy,

$$\omega(\hat{p}) = \sqrt{\hat{p}^2 + m^2} = m + \hat{p}^2/2m + O(m^{-3}).$$

In the interaction terms, one neglects possible momentum dependences relative to the mass.

The projectors $P_\pm$ defined by equation (11.12) reduce to $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$. The transformation between fields ψ and $\varphi_\pm$ becomes local. Choosing a different normalization, one finds

$$\begin{aligned} \varphi_-(t,x) &= P_+ C^\dagger \psi^*(t,x), \qquad & \varphi_+(t,x) &= P_+ \psi(t,x), \\ \varphi_+^\dagger(t,x) &= \psi^\dagger(t,x) P_+\,, & \varphi_-^\dagger(t,x) &= \psi^T(t,x) C P_+\,, \end{aligned}$$

or, conversely,

$$\psi(t,x) = \varphi_+(t,x) + C^* \varphi_-^*(t,x).$$

For γ matrices it is convenient to choose a basis in which γ_0 is diagonal and we restrict, below, the spinor indices to the non-vanishing components of $\varphi_\pm$. Using the relation,

$$-\bar{\psi}(t,x)\psi(t,x) = \psi^\dagger(t,x)\gamma_0\psi(t,x) = \varphi_+^\dagger(t,x)\varphi_+(t,x) + \varphi_-^\dagger(t,x)\varphi_-(t,x),$$

one obtains the action

$$\begin{aligned}\mathcal{A}(\varphi_\pm,\varphi_\pm^\dagger) = \int \mathrm{d}t\,\mathrm{d}^3x \Bigg[& \sum_{\epsilon=\pm} \varphi_\epsilon^\dagger(t,x)\left[i\partial_t + m - \nabla_x^2/2m\right]\varphi_\epsilon(t,x) \\ & + \tfrac{1}{2}G\Bigg(\sum_{\epsilon=\pm}\varphi_\epsilon^\dagger\varphi_\epsilon\Bigg)^2\Bigg].\end{aligned}$$

One then proceeds in analogy with the boson case. One translates the one-particle energy by the mass m, setting

$$\varphi_\pm(t,x) \mapsto \mathrm{e}^{imt}\,\varphi_\pm(t,x), \quad \varphi_\pm^\dagger(t,x) \mapsto \mathrm{e}^{-imt}\,\varphi_\pm^\dagger(t,x).$$

One neglects in the interaction all terms that depend explicitly on time. One finds

$$\mathcal{A}(\varphi_\pm,\varphi^\dagger_\pm)=\int \mathrm{d}t\,\mathrm{d}^3x\biggl[\sum_{\epsilon=\pm}\varphi^\dagger_\epsilon(t,x)\left[i\partial_t-\nabla_x^2/2m)\right]\varphi_\epsilon(t,x)$$
$$+\tfrac{1}{2}G\biggl(\sum_{\epsilon=\pm}\varphi^\dagger_\epsilon\varphi_\epsilon\biggr)^2\biggr].$$

This action describes a many-body theory with two fermions of the same mass and with spin, the spin playing the role of an external quantum number decoupled from space-time.

Borrowing the result of section 10.14.4, and comparing with the action (11.6), one infers the non-relativistic Hamiltonian $\mathbf{H}$, up to an infinite energy shift. One verifies that in the non-relativistic limit the number of particles is conserved and, therefore, sectors with different particle number decouple:

$$\mathbf{N}_{\pm\alpha}=\int \mathrm{d}x\,\varphi_{\pm\alpha}(x)\frac{\delta}{\delta\varphi_{\pm\alpha}(x)}\quad\Rightarrow\quad[\mathbf{N}_{\pm\alpha},\mathbf{H}]=0\,.$$

A general n-particle contribution to the generating functional of wave functions can be written as

$$\Phi(\varphi) = \frac{1}{n!}\int \mathrm{d}x_1 \dots \mathrm{d}x_n\, \phi_{\varepsilon_1\alpha_1,\varepsilon_2\alpha_2,\dots,\varepsilon_n\alpha_n}(x_1,\dots,x_n)\varphi_{\varepsilon_1\alpha_1}(x_1)$$
$$\times\, \varphi_{\varepsilon_2\alpha_2}(x_2)\dots\varphi_{\varepsilon_n\alpha_n}(x_n),$$

where $\phi_{\varepsilon_1\alpha_1,\varepsilon_2\alpha_2,\dots,\varepsilon_n\alpha_n}(x_1,\dots,x_n)$ is a completely antisymmetric wave function.

In the n-particle sector, the Hamiltonian reads

$$H_n = -\frac{1}{2m}\sum_{i=1}^{n}\nabla^2_{x_i} - G\sum_{i<j}\delta(x_i - x_j).$$

The fermions interact through a two-body δ-function pseudo-potential that, in the case of fermions can be repulsive or attractive. The spin acts only through the Pauli principle which dictates the possible symmetries of the wave function Φ.

Remark. In two dimensions (*i.e.*, one space dimension), the fermion theory we have considered here is equivalent to the well-known massive Thirring model.

The non-relativistic limit, the δ-function model, can be exactly solved by the Bethe ansatz, that is, a complete set of wave-functions is provided by a superposition of a finite number of plane waves in each of the $n!$ sectors corresponding to all possible ordering of particle positions.

Its relativistic generalization is also integrable, because particle production does not arise (a possibility restricted to two dimensions).

11.8 Free euclidean relativistic fermions

We now perform the analytic continuation to euclidean time. This allows discussing quantum statistics of relativistic fermions and, quite generally, simplifies perturbative calculations.

We first describe the symmetries of euclidean relativistic fermion actions, like invariance under the spin group, under other continuous symmetries like phase rotation or chiral transformations and under several discrete symmetries like hermiticity, reflection and charge conjugation, which determine the free action as well as the coupling to other fields.

Some of the discrete symmetries have a form somewhat different from what one is familiar with in real time quantum field theory. Indeed, after continuation to imaginary time, symmetries that involve a complex conjugation are no longer directly symmetries of the euclidean action.

For example, initial hermiticity translates in another symmetry and time reversal has a natural definition that makes it indistinguishable from space reflections.

The technical basis for the discussion, like properties of the spin group and the definition of γ matrices, as well as our conventions and notation, can be found in appendix A11.

Euclidean Dirac fermions. The free fermion action $\mathcal{S}_0(\bar\psi,\psi)$ for generalized massive Dirac fermions, continuation to imaginary time of the standard action for spinor fields (11.8), can be written as

$$\mathcal{S}_0(\bar\psi,\psi) = -\int \mathrm{d}^4 x \sum_{\alpha,\beta} \bar\psi_\alpha(x) \left[(\partial\!\!\!/)_{\alpha\beta} + m\delta_{\alpha\beta}\right] \psi_\beta(x), \tag{11.19}$$

where the fields $\bar\psi_\alpha(x), \psi_\alpha(x)$ are also generators of a Grassmann algebra. In expression (11.19), we have introduced the traditional notation

$$\partial\!\!\!/ \equiv \sum_{\mu=1,\ldots,4} \gamma_\mu \partial_\mu \,.$$

11.8.1 Hermitian conjugation

According to the discussion of section 10.13, hermiticity of the Hamiltonian is equivalent to invariance of the euclidean action under complex conjugation followed by euclidean time reversal.

However, one has here to take into account a peculiarity of the relativistic formalism, the hermitian conjugate of ψ is not $\bar{\psi}$ but instead (equation (11.10)) $\psi^\dagger = -\bar{\psi}\gamma_4$, where γ_4 is the γ matrix associated with the euclidean time component.

When one combines these two transformations, one verifies that they can be realized differently, in a way that no longer singles out the time variable. One defines $\bar{\psi}$ now as the hermitian conjugate of ψ, instead of $\psi^\dagger$, and after hermitian conjugation perform the transformation

$$\psi(\mathbf{x}) \mapsto \gamma_\mu \psi(\tilde{\mathbf{x}}), \quad \bar{\psi}(\mathbf{x}) \mapsto \bar{\psi}(\tilde{\mathbf{x}})\gamma_\mu\,, \quad \text{with } \tilde{\mathbf{x}} = P_\mu \mathbf{x}\,, \tag{11.20}$$

where P_μ is the space reflection along the μ axis.

Acting on a space vector $\mathbf{x}$, P_μ changes the sign of its component μ:

$$P_\mu \mathbf{x} = \tilde{\mathbf{x}} \quad \text{with} \quad \tilde{\mathbf{x}} : \begin{cases} \tilde{x}_\mu = -x_\mu , \\ \tilde{x}_\lambda = x_\lambda \ \text{for} \ \lambda \neq \mu \, . \end{cases} \tag{11.21}$$

We have chosen a generic component μ to emphasize that all euclidean components are equivalent. The symmetry corresponding to the product of these two transformations is called reflection hermiticity.

Applying the transformation on the action (11.19), after hermitian conjugation, one finds

$$\mathcal{S}_0^\dagger(\bar\psi, \psi) = -\int \mathrm{d}^4 x \, \bar\psi(x) \, (-\not\partial + m) \, \psi(x),$$

because ∂_μ is anti-hermitian. In the transformation (11.20), the mass term is invariant and in $\not\partial$ the contribution $\sum_{\lambda \neq \mu} \gamma_\lambda \partial_\lambda$ changes sign as a consequence of the anti-commutation with γ_μ while the remaining term changes sign from $\partial_\mu \mapsto -\partial_\mu$.

The determinant resulting from the integral over ψ and $\bar{\psi}$ in the functional integral is thus real. Eigenvalues of the operator $\not{\partial} + m$ are real or appear as complex conjugate pairs.

The action (11.18) has also reflection hermiticity as a symmetry. Indeed,

$$\mathcal{S}_0^\dagger(\bar{\psi}_-, \psi_+) = \int \mathrm{d}^4x\, \bar{\psi}_+(x)\not{\partial}\psi_-(x),$$

since γ_5 is hermitian. But then in the second transformation,

$$(1-\gamma_5)\gamma_\mu = \gamma_\mu(1+\gamma_5),$$

in such a way that the initial chirality is recovered.

11.8.2 Spin group and reflections

Both actions (11.18) and (11.19) are invariant under the transformations of the spin group $\mathrm{Spin}(4) \equiv SU(2) \times SU(2)$ (see section A11.2.6), analytic continuation of $\mathrm{Spin}(1,3)$. The spinors ψ and $\bar{\psi}$ transform as

$$(\psi_\Lambda)_\alpha (x) = \sum_\beta \Lambda^\dagger_{\alpha\beta} \psi_\beta(\mathbf{R}x), \quad (\bar{\psi}_\Lambda)_\alpha (x) = \sum_\beta \bar{\psi}_\beta(\mathbf{R}x)\Lambda_{\beta\alpha},$$

where Λ belong to the spin group and the matrix $\mathbf{R}(\Lambda)$ is the corresponding element of $SO(4)$ (equation (A11.20)). After the change of variables $\mathbf{R}x \mapsto x'$, the invariance of the action follows from the identity

$$\sum_\nu R_{\mu\nu} \Lambda \gamma_\nu \Lambda^\dagger = \gamma_\mu \,,$$

which is implied by equation (A11.20).

Reflections. Space and time reflections are now indistinguishable. A reflection along the $\mu = 1$ axis corresponds to the transformation Π_1 (section A11.2.4):

$$\Pi_1 : \ \psi_{\Pi_1}(\mathbf{x}) = \gamma_5\gamma_1\psi(\tilde{\mathbf{x}}), \ \bar{\psi}_{\Pi_1}(\mathbf{x}) = \bar{\psi}(\tilde{\mathbf{x}})\gamma_1\gamma_5 \text{ with } \tilde{\mathbf{x}} = (-x_1, x_2, x_3, x_4).$$

The mass term in action (11.19) is clearly invariant. In the term $\not\partial$ the space reflection changes ∂_1 in $-\partial_1$, but then $\gamma_5\gamma_1$ anti-commutes with γ_1 and commutes with all other γ_μ matrices. The total action (11.19) is thus invariant.

The action (11.18), in contrast with the action (11.19), is not invariant under reflection, since reflection exchanges chiral components.

11.8.3 Charge conjugation, charge conservation

Charge conjugation. We introduce a unitary matrix C and transform spinors as

$$\psi_\alpha(x) = \sum_\beta \bar\psi'_\beta(x) C^\dagger_{\beta\alpha}\,, \quad \bar\psi_\alpha(x) = -\sum_\beta C_{\alpha\beta}\psi'_\beta(x). \tag{11.22}$$

In terms of the new fields ψ' and $\bar\psi'$ the action (11.19), after an integration by parts, becomes

$$\mathcal{S}_0(\bar\psi', \psi') = -\int \mathrm{d}^4x\, \bar\psi'(x) \left(-C^\dagger \not\partial^T C + m\right) \psi'(x).$$

The action (11.19) is thus invariant if the matrix C satisfies

$$C^\dagger \gamma_\mu^T C = -\gamma_\mu\,. \tag{11.23}$$

A solution for the charge conjugation matrix is $C = \gamma_1\gamma_3$ (see also section A11.2.5).

In the case of the action (11.18), the transformation (11.22) leads to ($\gamma_5 = \gamma_5^T$, $\gamma_5 C = C\gamma_5$)

$$\not\partial(1+\gamma_5) \mapsto -C^\dagger(1+\gamma_5)^T \not\partial^T C = -(1+\gamma_5)C^\dagger \not\partial^T C = (1+\gamma_5)\not\partial = \not\partial(1-\gamma_5)\,.$$

Since

$$\not\partial(1+\gamma_5) \mapsto \not\partial\,(1-\gamma_5)\,,$$

charge conjugation is not a symmetry. However, charge conjugation multiplied by space reflection, which exchanges chiral components, is a symmetry.

To justify the denomination charge conjugation we consider charged fields ψ and $\bar\psi$, with charges $\mp e$, coupled to an external electromagnetic field $A_\mu(x)$. The action then takes the form

$$\mathcal{S}(\bar\psi,\psi) = -\int \mathrm{d}^4x\,\bar\psi(x)\,(\not\partial + m + ie\not A)\,\psi(x).$$

After charge conjugation, as a consequence of equation (11.23) (see also equation (A11.23)) the sign of the charge e has changed.

Self-conjugate spinors. In dimension four, it is possible to construct a consistent theory for self-conjugate spinors, that is, that satisfy $\bar{\psi} = C\psi$. They correspond to neutral fermion fields, and are called Majorana spinors. The existence of Majorana neutrinos has been proposed for theoretical and experimental reasons but is not established.

Fermion number conservation. If we assign a fermion number $+1$ to ψ and -1 to $\bar{\psi}$ we see that the action (11.19) conserves fermion number. To fermion number conservation corresponds a $U(1)$ invariance of the action:

$$\psi_\theta(x) = \mathrm{e}^{i\theta}\,\psi(x), \qquad \bar{\psi}_\theta(x) = \mathrm{e}^{-i\theta}\,\bar{\psi}(x). \tag{11.24}$$

For charged fermions the fermion number is proportional to the electric charge.

The existence of Majorana particles leads to some form of fermion number violation.

11.9 Partition function. Correlations

In section 11.2 we have derived an expression for the statistical operator of a system of non-relativistic fermions in the form of a field integral. Here we generalize the expression to a relativistic quantum field theory, using the formalism of euclidean fermions introduced in section 11.8.

We define the partition function and introduce the generating functional of correlation functions, continuation to imaginary time of the Green's fermions which lead to the S-matrix. We show how to calculate them in a perturbative expansion.

11.9.1 The partition function

From the combined analyses of sections 10.13 and 11.8, we infer that the partition function for self-interacting massive Dirac fermions is given by an integral over Grassmann fields of the form ($t \equiv x_4$ is the euclidean time)

$$\begin{aligned}\mathcal{Z}(\beta) = \int [\mathrm{d}\psi(t,x)\mathrm{d}\bar{\psi}(t,x)] \\ \times \exp\left[-\mathcal{S}(\bar{\psi},\psi) + \mu \int_0^\beta \mathrm{d}t \int \mathrm{d}^3x\, \bar{\psi}(t,x)\gamma_4\psi(t,x)\right]\end{aligned} \tag{11.25}$$

with

$$\mathcal{S}(\bar{\psi},\psi) = -\int_0^\beta \mathrm{d}t \int \mathrm{d}^3x \left[\bar{\psi}(t,x)(\not{\partial} + m)\psi(t,x) + V\left(\bar{\psi}(t,x),\psi(t,x)\right)\right], \tag{11.26}$$

where $1/\beta$ is the temperature and μ the chemical potential. Fermion fields satisfy anti-periodic boundary conditions in the euclidean time direction:

$$\psi(t \equiv x_4 = 0, x) = -\psi(\beta, x), \quad \bar{\psi}(0,x) = -\bar{\psi}(\beta,x).$$

The term coupled to the chemical potential μ is hermitian, is proportional to the conserved fermion charge (section 12.1), and thus corresponds to a quantum operator that commutes with the Hamiltonian, and has the correct non-relativistic limit.

In what follows we specialize, for simplicity, to zero temperature and zero chemical potential. In this limit, the boundary conditions play no role, and we no longer distinguish between space and time; x denotes all four coordinates.

11.9.2 Correlation functions

We introduce Grassmann sources $\bar{\eta}, \eta$ and consider the generating functional of ψ, $\bar{\psi}$ field correlation functions,

$$\begin{aligned}\mathcal{Z}(\bar{\eta}, \eta) = \int & \left[\mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x)\right] \\ & \times \exp\left\{-\mathcal{S}(\bar{\psi}, \psi) + \int \mathrm{d}^4 x \left[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right]\right\}. \end{aligned} \tag{11.27}$$

Then,

$$\prod_{i=1}^{n} \frac{\delta}{\delta\eta(x_i)} \prod_{j=1}^{n} \frac{\delta}{\delta\bar{\eta}(y_j)} \mathcal{Z}(\eta,\bar{\eta})\bigg|_{\eta=\bar{\eta}=0}$$
$$= \mathcal{Z}(0)\left[(-1)^n \left\langle \bar{\psi}(x_1)\ldots\bar{\psi}(x_n)\psi(y_1)\ldots\psi(y_n)\right\rangle\right].$$

Because the sources $\eta(x), \bar{\eta}(x)$ are generators of a Grassmann algebra, correlation functions are antisymmetric in their arguments, in agreement with Fermi–Dirac statistics.

The Gaussian integral. The Gaussian integral with external sources reads

$$\mathcal{Z}_{\mathrm{G}}(\bar{\eta},\eta) = \int \left[\mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x)\right]$$
$$\times \exp\left[-\mathcal{S}_0(\bar{\psi},\psi) + \int \mathrm{d}^4x \left(\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right)\right], \quad (11.28)$$

where $\mathcal{S}_0(\bar{\psi},\psi)$ is the free action (11.19):

$$\mathcal{S}_0(\bar{\psi},\psi) = -\int \mathrm{d}^4x\, \bar{\psi}(x)\,(\not{\partial} + m)\,\psi(x).$$

As usual we shift variables $\psi \mapsto \psi'$ to eliminate linear terms, setting

$$\psi(x) + (\partial\!\!\!/ + m)^{-1}\,\eta(x) = \psi'(x), \quad \bar{\psi}(x) + \bar{\eta}(x)\,(\partial\!\!\!/ + m)^{-1} = \bar{\psi}'(x).$$

Normalizing the field integral (11.28) by $\mathcal{Z}(0,0) = 1$, we obtain

$$\mathcal{Z}_{\mathrm{G}}(\bar{\eta}, \eta) = \exp\left[-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\eta}(y)\Delta_{\mathrm{F}}(y,x)\eta(x)\right],$$

in which the fermion propagator Δ_{F} is given by

$$\Delta_{\mathrm{F}}(y,x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\,\mathrm{e}^{-ip(x-y)}\,\frac{(m - i p\!\!\!/)}{p^2 + m^2}\,. \tag{11.29}$$

One verifies that on mass-shell ($p^2 = -m^2$), $m - i p\!\!\!/$ is a projector on a two-dimensional space. This reflects the property that physical massive fermion states can be classified according to the static spin group $\mathrm{Spin}(3) \equiv SU(2)$, the subgroup of $\mathrm{Spin}(4)$ that leaves the momentum p invariant.

The fermion two-point correlation function in a free or Gaussian theory is then

$$\left\langle \bar{\psi}_{\alpha}(x)\psi_{\beta}(y)\right\rangle_{0} = -\frac{\delta}{\eta_{\alpha}(x)}\frac{\delta}{\bar{\eta}_{\beta}(y)}\mathcal{Z}_{\mathrm{G}}(\bar{\eta},\eta) = (\Delta_{\mathrm{F}})_{\beta\alpha}(y,x). \tag{11.30}$$

A generalization of equation (10.64) to the action (11.26) yields the identity

$$\mathcal{Z}(\bar{\eta},\eta) = \exp\left[\int \mathrm{d}^4x\, V\left(-\frac{\delta}{\delta\eta(x)},\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]\mathcal{Z}_{\mathrm{G}}(\bar{\eta},\eta).$$

The identity leads to the perturbative expansion of a field theory with self-interacting fermions. Alternatively, the field integral (11.27) can be expanded in powers of $V, \eta, \bar{\eta}$ and all terms evaluated with the corresponding Wick's theorem for fermion fields, a straightforward extension of the form (10.21).

Wick's theorem. Wick's theorem can be written in terms of free field expectation values as

$$\left\langle \prod_{i=1,n} \bar\psi_{\alpha_i}(x_i)\psi_{\beta_i}(y_i) \right\rangle_0 = \sum_{\substack{\text{permutations}\\ P \text{ of } \{1,2,\ldots,n\}}} \operatorname{sgn}(P) \prod_{i=1,n} \left\langle \bar\psi_{\alpha_{P(i)}}(x_{P(i)})\psi_{\beta_i}(y_i)\right\rangle_0 ,$$

where $\operatorname{sgn}(P)$ is the signature of the permutation P.

Theories with bosons and fermions. Many field theories involve both fermions and bosons. A simple example, which is studied later, corresponds to an action $\mathcal{S}(\bar\psi,\psi,\phi)$ of the general form

$$\begin{aligned}\mathcal{S}(\bar\psi,\psi,\phi) = \int \mathrm{d}^4x \Big\{ & -\bar\psi(x)\left[\not\partial + M + g\phi(x)\right]\psi(x) + \tfrac{1}{2}\left(\nabla_x\phi(x)\right)^2 \\ & + \tfrac{1}{2}m^2\phi^2(x) + \tfrac{1}{24}\lambda\phi^4(x)\Big\} ,\end{aligned} \tag{11.31}$$

in which ψ, $\bar\psi$ are Grassmann fields and ϕ a real field.

Correlation functions can the be derived from the generating functional

$$\mathcal{Z}(\bar\eta,\eta,J)=\int\left[\mathrm{d}\psi\,\mathrm{d}\bar\psi\,\mathrm{d}\phi\right]\exp\Bigl\{-\mathcal{S}(\bar\psi,\psi,\phi)
+\int\mathrm{d}^4x\,\left[\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)+J(x)\phi(x)\right]\Bigr\},\quad(11.32)$$

in which η, $\bar\eta$ are Grassmann fields and J is a real field. The functional $\mathcal{Z}(\bar\eta,\eta,J)$ generates both ϕ field and ψ, $\bar\psi$ field correlation functions. For free fields, it is given by

$$\mathcal{Z}_{\mathrm{G}}(\bar\eta,\eta,J)=\exp\left[\int\mathrm{d}^4x\,\mathrm{d}^4y\left(\tfrac{1}{2}J(x)\Delta(x,y)J(y)-\bar\eta(x)\Delta_{\mathrm{F}}(x,y)\eta(y)\right)\right].$$

In the example (11.31), the integral over fermions is Gaussian and can also be performed explicitly.

This leads to a scalar field theory with additional non-local interactions:

$$\int [\mathrm{d}\psi\, \mathrm{d}\bar{\psi}] \exp \int \mathrm{d}^4x \, \left[\bar{\psi}(x) \left(\not{\partial} + M + g\phi(x)\right) \psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right]$$
$$\propto \exp\left[-\mathcal{S}_{\mathrm{F}}(\phi, \eta, \bar{\eta})\right]$$

with

$$\mathcal{S}_{\mathrm{F}} = -\operatorname{tr}\ln\left[\not{\partial} + M + g\phi(x)\right] + \int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar{\eta}(y) \left[\not{\partial} + M + g\phi(\cdot)\right]^{-1}(y, x)\eta(x)\,.$$

The expansion of $\mathcal{S}_{\mathrm{F}}(\phi, 0, 0)$ in powers of ϕ, generates a set of one fermion loop Feynman diagrams (see section 6.4). A similar integral over boson fields would have generated a contribution of the form $+\operatorname{tr}\ln$. Hence, compared to boson loops, fermion loops are multiplied by an additional minus sign.

11.10 Generating functionals

We have discussed connected functions and proper vertices in the case of a boson field theory. We have briefly extended these notions to Grassmann variables in section 10.6.1. The extension to fermion fields is straightforward.

Let $\bar{\psi}, \psi$ be Dirac fermion fields and $\mathcal{S}(\bar{\psi}, \psi)$ the corresponding local action. We denote by $\bar{\eta}$ and η the sources for ψ and $\bar{\psi}$. Then, $\mathcal{W}(\eta, \bar{\eta}) = \ln \mathcal{Z}(\eta, \bar{\eta})$ is still the generating functional of connected correlation functions.

Following the conventions of section 11.9, if one writes the source terms in the field integral as $\bar{\eta}\psi + \bar{\psi}\eta$, then one defines the Legendre transform of $\mathcal{W}$ by

$$\Gamma(\bar{\psi}, \psi) + \mathcal{W}(\eta, \bar{\eta}) = \int \mathrm{d}x \, \left[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right], \qquad (11.33a)$$

$$\psi(x) = \frac{\delta \mathcal{W}}{\delta \bar{\eta}(x)}, \quad \bar{\psi}(x) = -\frac{\delta \mathcal{W}}{\delta \eta(x)}. \qquad (11.33b)$$

The equations $(11.33b)$ are equivalent to

$$\eta(x) = \frac{\delta\Gamma}{\delta\bar{\psi}(x)}, \quad \bar{\eta}(x) = -\frac{\delta\Gamma}{\delta\psi(x)} .$$

With these conventions, one easily verifies that in the tree approximation

$$\Gamma(\bar{\psi}, \psi) = \mathcal{S}(\bar{\psi}, \psi).$$

All the other algebraic properties derived for bosons generalize to the fermion case. However, we recall here that a Gaussian integration over fermion fields yields a determinant instead of the inverse of a determinant for a complex scalar. This implies that, in Feynman diagrams, fermion loops are affected by an additional minus sign compared to boson loops.

11.11 Connection between spin and statistics

In section 11.4.1, we have noticed that fields transforming under the fundamental (spin $1/2$) representation of the spin group could only be quantized as fermions. This connection between spin and statistics is a deep consequence of locality, hermiticity of the Hamiltonian and relativistic invariance: fermions transform under representations of odd degree of the spin group, while bosons transform under the $SO(4)$ group. This implies that in four dimensions bosons must have integer spin while fermions must have half-integer spin.

We illustrate here this property, which can be proven with a great deal of generality, by an explicit calculation.

It is simple to prove that, as a consequence of the hermiticity of the Hamiltonian, the two-point function has a spectral representation in terms of a positive measure. We have translated this result into the relativistic kinematics in section 5.11. All possible intermediate states contribute with the same sign. This result can easily be generalized to the discontinuity in the physical domain of diagonal scattering amplitudes. Let us then show that the sign of fermion loops implies a relation between spin and statistics.

Boson contribution. We consider the leading order contribution from a scalar field ϕ to the two-point function of a coupled scalar field χ. The ϕ-field action is

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x \left[\tfrac{1}{2} \left(\nabla_x \phi(x)\right)^2 + \tfrac{1}{2} m^2 \phi^2(x) + \tfrac{1}{2} g \phi^2(x) \chi(x)\right].$$

The integration over ϕ yields a (non-local) contribution to the χ action ($\ln\det = \operatorname{tr}\ln$):

$$\delta\mathcal{S}(\chi) = \tfrac{1}{2} \operatorname{tr}\ln(-\nabla_x^2 + m^2 + g\chi).$$

If we expand this expression to order g^2 we find a term linear in χ that shifts the χ field expectation value and a quadratic term that modifies the two-point function in the Gaussian approximation. One verifies that the corresponding contribution to $\delta W^{(2)}_{\chi^2}$, the χ two-point function in Fourier variables, is

$$\delta W^{(2)}_{\chi^2} = \tfrac{1}{2} g^2 \Delta^2_\chi(p) B(p) + \text{const.}$$

with (Fig. 8.2)

$$B(p) = \frac{1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(q^2 + m^2)\left[(p+q)^2 + m^2\right]}.$$

Equation (9.12) gives the expression for the massless case in dimensional regularization:

$$B(p) = \frac{N_d}{\varepsilon} - N_d \ln p\,,$$

where N_d is the loop factor and $N_4 = 1/8\pi^2$. Setting $s = -p^2 > 0$, one infers

$$\frac{1}{2i}\left[B(p)|_{s+i\epsilon} - B(p)|_{s-i\epsilon}|_{\epsilon\to 0_+}\right] = \frac{1}{16\pi}\theta(s)\,.$$

A convenient integral representation of the massive diagram is obtained by introducing Feynman's parametrization 9.2.3, which indeed shows that the function has a cut for $s > (2m)^2$, the region of physical ϕ-particles emission, with a positive imaginary part. One finds

$$\frac{1}{2i}\left[B(p)|_{s+i\epsilon} - B(p)|_{s-i\epsilon}\right] = \frac{1}{16\pi}\theta(s-4m^2)\sqrt{\frac{s-4m^2}{s}}\,,$$

a result which is consistent with the representation (5.62) for $\delta W^{(2)}_{\chi^2}$.

By contrast, if one considers the contribution coming from scalar fermions, the $\operatorname{tr}\ln$ is replaced by $-\operatorname{tr}\ln$, and the contribution has the opposite sign, which is inconsistent with hermiticity.

Dirac fermions. We now calculate the contribution of spin $1/2$ Dirac fermions. After Gaussian integration, the contribution to the χ action is

$$\delta S(\chi) = -\operatorname{tr}\ln(\not{\partial} + m + g\chi),$$

which, expanded to order g^2, yields the contribution to the χ two-point function,

$$\begin{aligned}\delta W^{(2)}_{\chi^2}(p) &= -\frac{g^2}{(2\pi)^4}\operatorname{tr}\int \frac{\mathrm{d}^4 q\,(-i\not{q} + m)(-i\not{p} - i\not{q} + m)}{(q^2+m^2)\left[(p+q)^2+m^2\right]} \\ &= -\frac{4g^2}{(2\pi)^4}\int \frac{\mathrm{d}^4 q\,(m^2 - pq - p^2)}{(q^2+m^2)\left[(p+q)^2+m^2\right]}.\end{aligned}$$

We use the identity

$$m^2 - pq - q^2 = 2m^2 + \tfrac{1}{2}p^2 - \tfrac{1}{2}\left[(p+q)^2 + m^2 + q^2 + m^2\right].$$

The two terms inside the brackets cancel a denominator and thus yield a (divergent) constant contribution, which has no discontinuity. Then,

$$\delta W^{(2)}_{\chi^2}(p) = -4g^2(p^2 + 4m^2)\Delta^2_\chi(p)B(p) + \text{ const. } .$$

In the region of physical particle emission $s = -p^2 > (2m)^2$ the factor $(p^2 + 4m^2 = 4m^2 - s)$, which reflects the spin structure, is negative and cancels the negative sign due to the fermion loop.

11.12 The Gross–Neveu–Yukawa model: a Higgs-top toy model

Since the Higgs mass corresponds to a moderate self-coupling, even for a semi-quantitative understanding of RG trajectories, it is necessary to include the couplings at least to the top quark and vector bosons. Nevertheless, even when vector bosons are neglected, a general though more qualitative picture emerges. Therefore, we discuss here a model that tries to represent the Higgs–top physics, the Gross–Neveu–Yukawa (GNY) model.

In the GNY model, fermions receive masses by spontaneous chiral symmetry breaking. Even without gauge symmetry, no Goldstone boson is generated because the chiral symmetry is discrete.

The model is renormalizable in four dimensions. Using RG arguments one can show that, in the limit of large cut-off, the ratio of fermion and boson masses can be predicted as a consequence of IR freedom and the natural assumption that coupling constants have generic values at the cut-off scale.

More generally, the renormalization group flow can be studied as a function of the physical masses when the physical ratio differs from the limiting value.

11.12.1 The Gross–Neveu–Yukawa model

The GNY model involves a set of N massless Dirac fermions $\{\psi^i, \bar{\psi}^i\}$ and a scalar field H.

It has a discrete chiral $\mathbb{Z}_2$ symmetry under which the fields transform like

$$\boldsymbol{\psi} \mapsto \gamma_5 \boldsymbol{\psi}, \quad \bar{\boldsymbol{\psi}} \mapsto -\bar{\boldsymbol{\psi}} \gamma_5, \quad H \mapsto -H\,, \tag{11.34}$$

which prevents the addition of a fermion mass term to the action.

It has also a $U(N)$ symmetry that is implemented by the transformation

$$\psi \mapsto U\psi\,, \quad \bar{\psi} \mapsto U^\dagger \bar{\psi}\,.$$

The model exhibits a phase where $\langle H \rangle \neq 0$, which illustrates the physics of spontaneous chiral symmetry breaking and fermion mass generation.

A renormalizable symmetric action then takes the form (a cut-off Λ, consistent with the symmetries, is implied)

$$\mathcal{S}\left(\bar{\psi},\psi,H\right)=\int \mathrm{d}^4x\left[-\bar{\psi}\cdot\left(\not{\partial}+gH\right)\psi+\tfrac{1}{2}\left(\nabla_x H\right)^2+\tfrac{1}{2}u\Lambda^2H^2+\frac{\lambda}{4!}H^4\right].$$

Indeed, the scalar field H has dimension 1, the fermion field dimension $3/2$ in such a way that u and the coupling constants g and λ are dimensionless. Moreover, the action contains all possible terms consistent with renormalizability and symmetry.

The renormalized action. Calling μ the renormalization scale and $g_\mathrm{r},\lambda_\mathrm{r}$ the renormalized couplings, one can write the renormalized action as

$$\begin{aligned}\mathcal{S}_\mathrm{r}(H,\psi,\bar{\psi})=\int \mathrm{d}^4x\Big\{&-Z_\psi\left[\bar{\psi}(x)\cdot\left(\not{\partial}+g_\mathrm{r}Z_gZ_H^{1/2}H(x)\right)\psi(x)\right]\\&+\tfrac{1}{2}Z_H\left[\left(\nabla_xH(x)\right)^2+u_\mathrm{r}Z_mH^2(x)\right]+Z_\lambda\frac{\lambda_\mathrm{r}}{4!}Z_H^2H^4(x)\Big\}.\end{aligned}\qquad(11.35)$$

where $Z_\psi,Z_g,Z_H,Z_m,Z_\lambda$ are renormalization constants.

Remark. Note that if the H^4 interaction would have been omitted, renormalization theory and renormalization group tell us that the effective H^4 interaction can perhaps vanish at one scale but not at all scales.

In what follows we set

$$\Lambda^2 u = \Lambda^2 u_c + \tau\,,$$

where u_c is defined by the property that for $\tau = 0$ the physical masses of ψ and H vanish. The new parameter τ, in the framework of macroscopic phase transitions, plays the role of the deviation from the critical temperature.

The phase transition in the tree approximation. In the tree approximation,

$$S_{\text{tree}}\left(\bar{\psi},\psi,H\right) = \int \mathrm{d}^4x \left[-\bar{\psi}\cdot\not{\partial}\psi - g\,H\bar{\psi}\cdot\psi + \tfrac{1}{2}\left(\nabla_x H\right)^2 + \tfrac{1}{2}\tau H^2 + \frac{\lambda}{4!}H^4\right],$$

and a phase transition occurs, for $\tau = 0$, between a massless fermion symmetric phase for $\tau > 0$ and a phase for $\tau < 0$ where the chiral symmetry is spontaneously broken and a fermion mass is generated.

In the broken phase, the H expectation value

$$\langle H \rangle = \pm\sqrt{-6\tau/\lambda}\,,$$

gives a mass to the fermions by a mechanism reminiscent of the Standard Model of weak-electromagnetic interactions. The fermion and boson masses are then

$$m_\psi = g\,\langle H \rangle\,, \quad m_H = \sqrt{\frac{\lambda}{3}}\,\langle H \rangle \quad \Rightarrow\ \frac{m_H}{m_\psi} = \frac{1}{g}\sqrt{\frac{\lambda}{3}}\,.$$

The H expectation value cancels in the mass ratio.

11.12.2 RG equations: β-functions

Beyond the tree approximation, the model can be discussed, like the ϕ^4 theory, by RG techniques. This involves evaluating the divergent parts of the diagrams of Fig. 11.1–11.4.

For $|\tau| \ll \Lambda^2$ (this is the usual fine tuning problem), the corresponding renormalized (1PI) vertex functions of $2l$ fermion fields and n scalar fields satisfy the RG equations

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta_{g^2}\frac{\partial}{\partial g^2} + \beta_\lambda\frac{\partial}{\partial\lambda} - l\eta_\psi - \tfrac{1}{2}n\eta_H - \eta_m\tau\frac{\partial}{\partial\tau}\right]\Gamma^{(l,n)} = 0\,. \qquad (11.36)$$

At one-loop order, the RG β-functions are

$$\beta_\lambda = \frac{1}{8\pi^2}\left(a\lambda^2 + b\lambda g^2 + cg^4\right), \quad \beta_{g^2} = \frac{d}{8\pi^2}g^4$$

with

$$a = \tfrac{3}{2}, \quad b = 4N, \quad c = -24N, \quad d = 2N + 3\,.$$

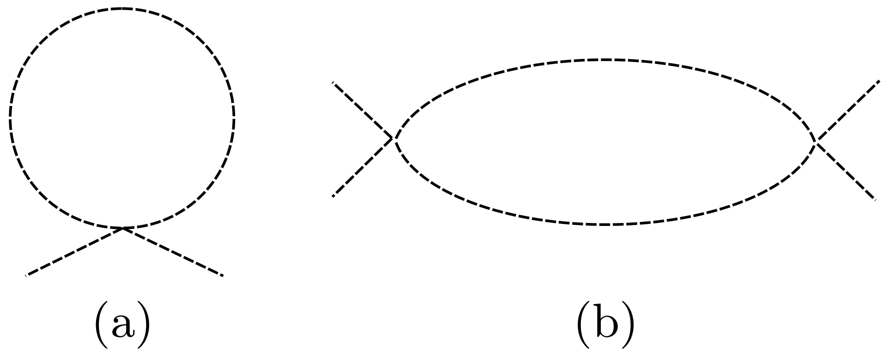

Fig. 11.1 – One-loop H^4 divergent contributions.

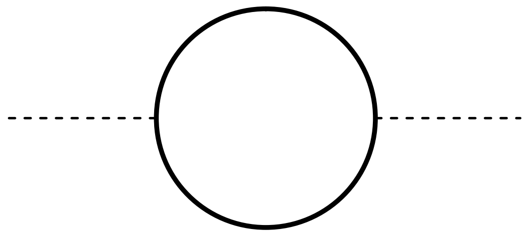

Fig. 11.2 – Boson two-point function: contribution from the fermion loop (the fermions and bosons correspond to continuous and dotted lines, respectively).

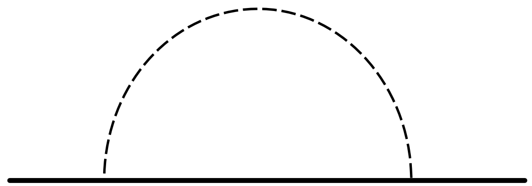

Fig. 11.3 – Fermion two-point function at one-loop.

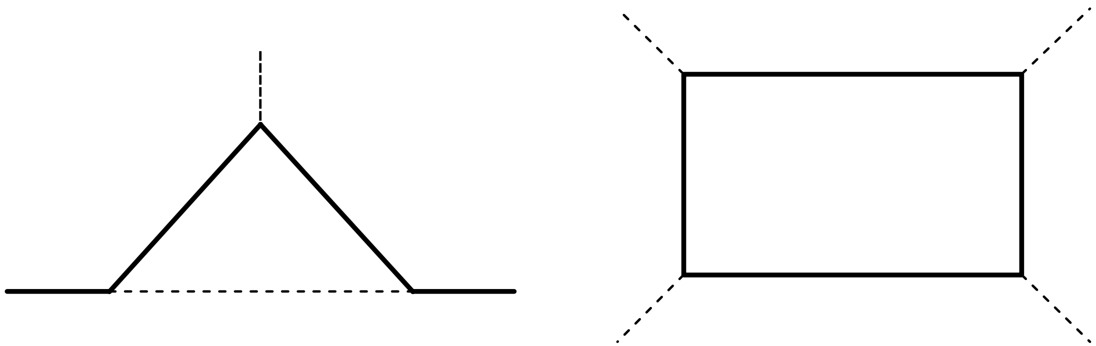

Fig. 11.4 – Three and four-point functions: other divergent one-loop diagrams.

11.12.3 IR freedom and mass ratio

One easily verifies that the origin $\lambda = g^2 = 0$ is IR stable. The model GNY, like the ϕ^4 field theory, is thus trivial or IR free, that is, Gaussian up to logarithmic corrections that vanish for infinite cut-off.

We assume that the dimensionless couplings $\lambda(\Lambda)$ and $g(\Lambda)$ are generic (*i.e.*, of order 1, numerically $8\pi^2$, which is the loop factor) at the cut-off scale Λ. Solving the RG equations, one infers that the coupling constants at a scale $\mu \ll \Lambda$ decrease like

$$g^2(\mu) \sim \frac{8\pi^2}{(2N+3)\ln(\Lambda/\mu)}, \quad \lambda(\mu) \sim \frac{8\pi^2 R_*(N)}{(2N+3)\ln(\Lambda/\mu)}$$

with

$$R_*(N) = \tfrac{1}{3}\left[-(2N-3) + \sqrt{4N^2+132N+9}\right].$$

In particular, choosing $\mu \sim \langle H \rangle$ and if the mass scale $\langle H \rangle \ll \Lambda$, one concludes that the ratio of H and fermion masses goes to the limit

$$\frac{m_H^2}{m_\psi^2} = \frac{\lambda(\langle H \rangle)}{3g^2(\langle H \rangle)} = \tfrac{1}{3} R_*(N) = \tfrac{1}{9}\left(-(2N-3) + \sqrt{4N^2 + 132N + 9}\right). \tag{11.37}$$

As a function of N, when N varies between 1 and ∞, the ratio m_H/m_ψ varies from about 1.20 to 2, which corresponds to the $\bar\psi\psi$ threshold and the large N limit.

IR freedom of the theory, the assumptions that the couplings are generic at the cut-off scale and that the cut-off scale is large enough, fix the ratio between the masses of the top and Higgs particles.

11.12.4 The general renormalization group flow at one-loop

Identifying the boson with the Higgs field and the fermion with the top field, one can put numbers on the vacuum expectation value and effective couplings at physical scale $\mu = O(\langle H \rangle)$. From,

$$\langle H \rangle = 246.\,\mathrm{Gev}\,, \quad m_\psi = 173.2\,\mathrm{Gev}\,, \quad m_H = 125.\,\mathrm{Gev}\,,$$

one infers $\lambda(\mu) = 0.775$, $g^2(\mu) = 0.496$. The main neglected contributions correspond to Higgs couplings to W and Z vector bosons and, therefore, the picture can only be semi-quantitative but the analysis is here much simplified and, thus, more transparent.

More generally, two-dimensional RG flows can be easily studied because RG trajectories can only meet at fixed points, here $g = \lambda = 0$.

One verifies immediately that the lines $g = 0$ and $\lambda = R_* g^2$ are fixed trajectories and thus cannot be crossed. By contrast, the line $\lambda = 0$ can be formally crossed and the RG trajectories then enter an unphysical region.

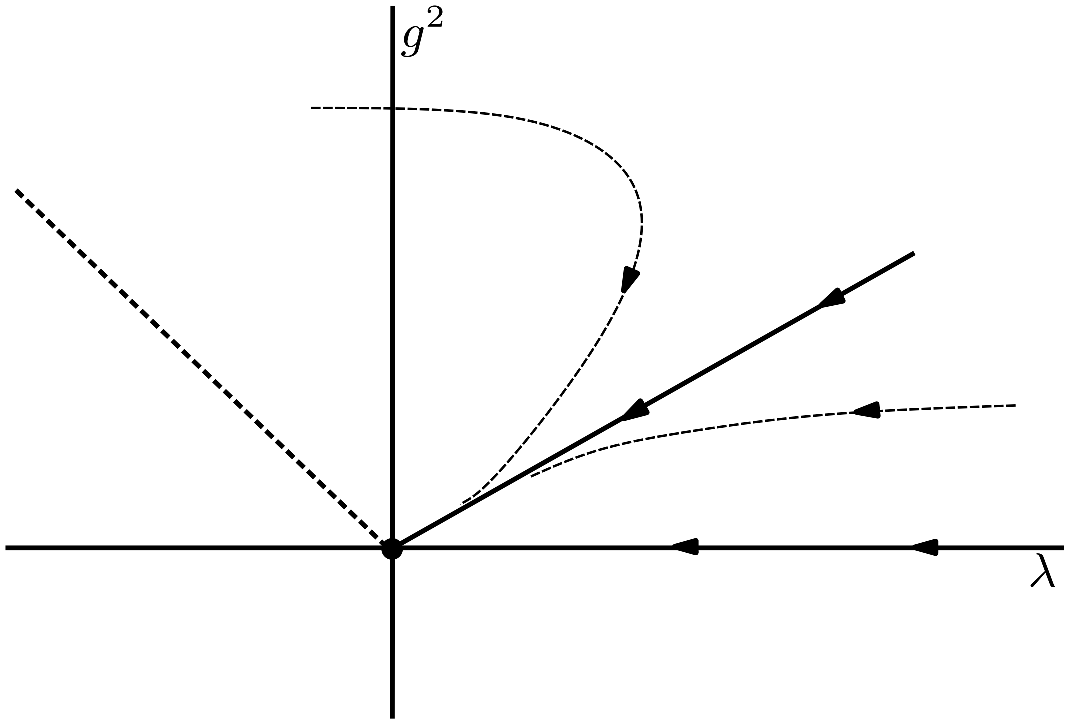

Fig. 11.5 – RG flow: the dotted line on the left is an unphysical fixed line.

We note that the coupling constants at physical scale (or renormalized) are small. This justifies using perturbation theory and indicates that IR freedom is relevant since it predicts small renormalized couplings when the initial couplings at a large momentum cut-off scale are of order 1.

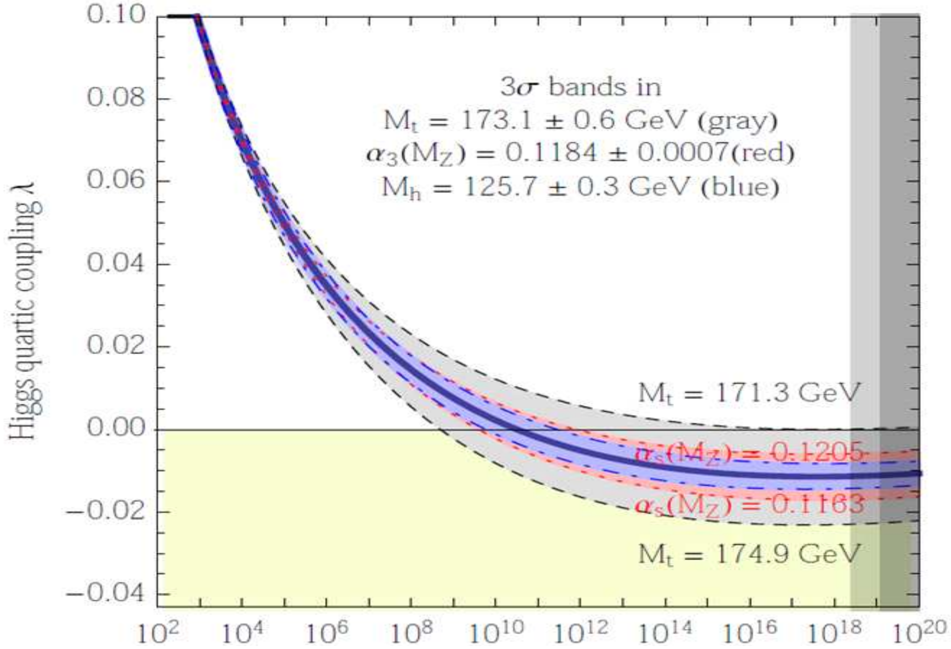

Fig. 11.6 – Precise RG flow (Degrassi et *et al* 2012).

However, the physical ratio $R = \lambda/3g^2 \approx 0.52$ is smaller than what is predicted by the model. More realistic RG calculations, including vector bosons, have been performed.

They seem to indicate that the physical Higgs mass is very close to a fixed RG trajectory. Depending on the precise top mass, deviations appear but at a very high energy scale, at least 10^9 GeV.

This RG result is puzzling because it suggests that the Standard Model could be valid up to such a high scale. However, then the problem of the fine-tuning of the Higgs mass, which is of the order of $(\Lambda/m_H)^2$ (Λ is the scale of new physics), which had been disregarded until recently, becomes extremely severe.

11.13 An effective field theory: the Gross–Neveu model

The Gross–Neveu (GN) model is also described in terms of an $U(N)$ symmetric action for a set of $N > 1$ massless fermions $\{\psi^i, \bar{\psi}^i\}$, but interacting through an attractive four-fermion self-interaction,

$$\mathcal{S}(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}) = -\int \mathrm{d}^d x \left[\bar{\boldsymbol{\psi}}(x) \cdot \not{\partial} \boldsymbol{\psi}(x) + \tfrac{1}{2} \Lambda^{2-d} G \left(\bar{\boldsymbol{\psi}}(x) \cdot \boldsymbol{\psi}(x)\right)^2\right], \; G > 0 \,, \tag{11.38}$$

reminiscent of the Fermi–Feynman–Gell-Mann model of weak interactions. Like the GNY model, in addition to the $U(N)$ symmetry, in even dimensions the model has a discrete chiral $\mathbb{Z}_2$ symmetry, which in four dimensions corresponds to the transformation

$$\boldsymbol{\psi} \mapsto \gamma_5 \boldsymbol{\psi}, \quad \bar{\boldsymbol{\psi}} \mapsto -\bar{\boldsymbol{\psi}} \gamma_5 \,. \tag{11.39}$$

(In two dimensions γ_5 is replaced by σ_3.) This symmetry prevents the addition of a fermion mass term.

The model has the same fermion content and symmetries as the GNY model but the perturbative GN model displays only the symmetric massless phase. This is a situation reminiscent of the correspondence between the $(\phi^2)^2$ and the non-linear σ-model. The main difference is that the role of the spontaneously broken and the explicitly symmetric phase are interchanged. Indeed, it is always the massless phase that is unstable in low dimensions.

The model requires a symmetric UV cut-off, for example,

$$\not\partial \mapsto \not\partial(1 - \nabla_x^2/\Lambda^2 + \cdots),$$

and for $d = 2$ is affected by IR divergences which require an IR cut-off (like a fermion mass, which breaks explicitly the chiral symmetry).

In four dimensions, the GN model is not renormalizable and provides an interesting example of an effective field theory, similar in structure to the Fermi model of weak interactions, except that the interaction is scalar-scalar.

The analysis of the model at and near dimension two as well as in the large N limit, strongly suggests that the GN model has the same physics as the GNY model: at a value G_c of the fermion self-coupling G, a phase transition occurs and above G_c a phase with symmetry breaking, spontaneous fermion mass generation and a scalar bound state appears.

From an RG viewpoint, the value G_c corresponds to an UV fixed point, which is repulsive for large distance physics. Above G_c, the condition of small masses implies a fine-tuning of G such that $G - G_c \ll 1$.

The existence of a UV fixed point realizes the condition of asymptotic safety. In the limit $G - G_c \ll 1$, below dimension 4 it allows defining a field theory that has both universal large distance and short distance physics and thus defining a renormalized theory consistent on all scales.

In particular, the GN model is asymptotically free in two dimensions ($G_c = 0$), the dimension in which it is renormalizable, where it displays only spontaneous symmetry breaking and a massive phase.

11.13.1 Four dimensions: an effective field theory.

The action (11.38) then reads

$$\mathcal{S}(\bar{\psi},\psi) = -\int \mathrm{d}^4x \left[\bar{\psi}(x)\cdot \partial\!\!\!/ \psi(x) + \frac{G}{2\Lambda^2}\left(\bar{\psi}(x)\cdot\psi(x)\right)^2\right], \ G>0\,, \tag{11.40}$$

where a momentum cut-off Λ is implicitly assumed. The fermion field has dimension $3/2$, the coupling constant G is attractive and dimensionless. The model is non-renormalizable and dimensional analysis implies that the interaction is suppressed by a factor $1/\Lambda^2$. In the spirit of effective field theory, other local interactions could be added to the action but they are suppressed at least by a factor $1/\Lambda^4$.

Though the interaction is small and the tree approximation is expected to give the leading contribution, the theory is not renormalizable and divergences appear at higher orders in the perturbative expansion, which may cancel the $1/\Lambda^2$ factors. This problem has to be analysed.

One-loop contributions to vertex functions. At one-loop order, the contributions to the two, four, six and eight-point functions are divergent.

The contributions to the two-point function are constants that cancel due to chiral symmetry.

To evaluate the four-point function, which has a quadratic divergence, we use the propagator (11.29) for $m = 0$ and the expression (10.27) with the correspondence

$$\sum_i \mapsto \sum_{i,\alpha} \int \mathrm{d}^4 x\,, \ \Delta_{ji} \mapsto \Delta_{ij}^{\alpha\beta}(y,x) = -i\frac{\delta_{ij}}{(2\pi)^4} \int \mathrm{d}^4 p\ \mathrm{e}^{-ip(x-y)}\,\frac{\not{p}_{\alpha\beta}}{p^2}$$

$$V_{ij} \mapsto \delta^{(4)}(x-y).$$

The first diagram of Fig. 10.4 is proportional to

$$\begin{aligned} N\frac{G^2}{\Lambda^4}\int \mathrm{d}^4 q\,\frac{\operatorname{tr}\not{q}(\not{q}+\not{p})}{q^2(p+q)^2} &= 4N\frac{G^2}{\Lambda^4}\int \mathrm{d}^4 q\,\frac{(q^2+pq)}{q^2(p+q)^2} \\ &= 2N\frac{G^2}{\Lambda^4}\left(2\int\frac{\mathrm{d}^4 q}{q^2} - \int \mathrm{d}^4 q\,\frac{p^2}{q^2(p+q)^2}\right). \end{aligned}$$

The first integral is proportional to Λ^2 and acts like a renormalization, finite but dependent on the cut-off function, to the coupling constant G. The second integral contains a local logarithmically divergent contribution equivalent to an interaction of the form

$$\frac{\ln \Lambda}{\Lambda^4} \int \mathrm{d}^4 x \, \bar{\boldsymbol{\psi}}(x) \cdot \boldsymbol{\psi}(x) \nabla_x^2 \big(\bar{\boldsymbol{\psi}}(x) \cdot \boldsymbol{\psi}(x) \big)$$

and a cut-off independent finite contribution of order G^2/Λ^4.

The other diagrams again generate constant contributions of order $1/\Lambda^2$ that renormalize G, various contributions of the form of chiral invariant local interactions with two derivatives of order $\ln \Lambda/\Lambda^4$ and, finally, contributions that is short distance insensitive of order $1/\Lambda^4$.

The six-point function generates a $(\bar{\psi}\psi)^2 \bar{\psi} \not{\partial} \psi$ local term of order $\ln \Lambda/\Lambda^6$ as well as finite contribution multiplied by $1/\Lambda^6$. The eight-point function generates a local $(\bar{\psi}\psi)^4$ term of order $\ln \Lambda/\Lambda^8$ and a finite contribution multiplied by $1/\Lambda^8$.

Higher orders. Quite generally, at higher orders, divergent contributions generate chiral invariant local interactions multiplied by powers of Λ determined by dimensional analysis ($\bar{\psi}\psi$ has dimension three), multiplied by possible powers of $\ln \Lambda$. In addition, finite contributions are also generated, suppressed by increasing powers of Λ.

From this analysis, it would seem that physics is entirely governed by the first orders of perturbation theory. The only possible trouble could come from the increasing powers of $\ln \Lambda$ that are generated at higher orders and which could sum to powers of Λ, invalidating the perturbative power counting.

11.13.2 The Gross–Neveu model in two dimensions

In perturbation theory the GN model is renormalizable in two dimensions. The $(\bar{\psi}\psi)^2$ interaction can be multiplicatively renormalized. As a function of the cut-off Λ, vertex functions satisfy the RG equations

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(G)\frac{\partial}{\partial G} - \frac{n}{2}\eta_\psi(G)\right]\tilde{\Gamma}^{(n)}(p_i; G, \Lambda) = 0\,. \tag{11.41}$$

A direct calculation of the RG β-function yields

$$\beta(G) = -(N-1)\frac{G^2}{\pi} + (N-1)\frac{G^3}{2\pi^2} + O\left(G^4\right). \tag{11.42}$$

The field renormalization RG function $\eta_\psi(G)$ is

$$\eta_\psi(G) = \frac{2N-1}{8\pi^2}G^2 + O\left(G^3\right). \tag{11.43}$$

The negative sign of the leading term of the β-function shows that the model is asymptotically free: the Gaussian fixed point $G = 0$ is repulsive at low momentum and attractive for high momenta.

The spectrum is non-perturbative and a number of arguments (in particular the $1/N$ expansion that is studied in section 11.13.4) lead to the conclusion that the chiral symmetry is always spontaneously broken and a fermion mass generated.

S-matrix considerations have then led to the conjecture that, for N finite, the exact spectrum is given by

$$m_n = M \frac{2(N-1)}{\pi} \sin\left(\frac{n\pi}{2(N-1)}\right), \quad n = 1, 2, \ldots < N, \ N > 2\,,$$

where the parameter M is a mass-scale with a dependence on the coupling constant given by RG arguments. Since it is an RG invariant, it satisfies

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(G) \frac{\partial}{\partial G}\right) M(\Lambda, G) = 0\,.$$

For dimensional reasons, $M = \Lambda F(G)$. The RG equation reduces to

$$F(G) + \beta(G) F'(G) = 0\,. \tag{11.44}$$

Thus,

$$M = \Lambda F(G) \propto \Lambda \exp\left(-\int^{G} \frac{\mathrm{d}g'}{\beta(g')}\right) \propto \Lambda G^{1/2(N-1)}\, \mathrm{e}^{-\pi/(N-1)G},$$

the coefficient being non-perturbative.

In particular the ratio between the masses of the fundamental fermion and the lowest lying boson is

$$\frac{m_\sigma}{m_\psi} = 2\cos\left(\frac{\pi}{2(N-1)}\right). \tag{11.45}$$

11.13.3 $d = 2 + \varepsilon$ *dimensions*

The perturbative expansion has a dimensional continuation. For $d = 2 + \varepsilon$ dimensions, $\varepsilon > 0$, in the framework of a double expansion in powers of G and ε, the β-function is given by

$$\beta(G) = \varepsilon G - (N-1)\frac{G^2}{\pi} + (N-1)\frac{G^3}{2\pi^2} + O\left(G^4\right).$$

The origin $G = 0$ is now an IR fixed point and for G small, the physics is perturbative with chiral symmetry and massless fermions. However, for ε small, the β-function has another zero,

$$G_c = \frac{\pi\varepsilon}{N-1} + O(\varepsilon^2) \;\Rightarrow\; \beta'(G_c) = -\varepsilon + O(\varepsilon)^2 .$$

This zero corresponds to a UV fixed point relevant for large momentum behaviour, realizing the condition of asymptotic safety. Continuity with $d = 2$ suggests that it is also a transition point between the small G symmetric, perturbative phase and a phase with spontaneous chiral symmetry breaking and massive particles. The conjecture is comforted by the results of the large N expansion.

Near G_c, equation (11.44) then leads to

$$M = \propto \Lambda \exp\left(-\int^G \frac{\mathrm{d}g'}{\beta(g')}\right) \propto \Lambda |G - G_c|^{-1/\beta'(G_c)} .$$

Therefore, for $|G-G_c| \ll 1$ a new mass scale is generated, much smaller than the cut-off. For $G < G_c$, it is a crossover scale between the low momentum perturbative behaviour and a large momentum critical behaviour (governed by critical exponents). For $G > G_c$, it provides a scale to the fermion and scalar physical masses.

11.13.4 The large N expansion

One can solve the GN model for N large. To generate a systematic $1/N$ expansion, we introduce a scalar field H and write the GN action in the equivalent form

$$\mathcal{S}(\bar{\psi}, \psi, H) = \int \mathrm{d}^4 x \left[-\bar{\psi}(x) \cdot (\not{\partial} + H(x))\psi(x) + \frac{\Lambda^2}{2G} H^2(x)\right] . \quad (11.46)$$

Indeed, the Gaussian H integration amounts to replacing H by the solution of the H field equation:

$$\frac{\delta \mathcal{S}}{\delta H(x)} = \frac{\Lambda^2}{G} H(x) - \bar{\boldsymbol{\psi}}(x) \cdot \boldsymbol{\psi}(x) = 0 \,.$$

After this substitution, the initial GN action is recovered.

Integrating over $(N-1)$ components of the field $\boldsymbol{\psi}$, one obtains $(\psi \equiv \psi_1)$

$$\mathcal{S}\left(\bar{\psi}, \psi, H\right) = \int \mathrm{d}^4 x \left[-\bar{\psi}\left(\not{\partial} + H\right)\psi + \frac{\Lambda^2}{2G} H^2 \right] - (N-1) \ln \det(\not{\partial} + H). \tag{11.47}$$

The large N limit is taken with H finite and $G = O(1/N)$. Then, the action is proportional to N for N large, and in the large N limit it can be calculated by the steepest descent method: the integral is dominated by space-independent fields H that minimize the action (11.47) (with $\bar{\psi}, \psi = 0$).

Since one looks only for space-independent fields $H(x)$, one needs only the action for $H(x) = H$, where H is the field expectation value,

$$\frac{\mathcal{S}\,(0,0,H)}{N \times \text{volume}} \sim \frac{\Lambda^2}{2NG} H^2 - \frac{1}{\text{volume}} \ln \det(\not{\partial} + H).$$

First, in the four dimensional space corresponding to γ matrices, the determinant satisfies

$$\det{}_\gamma(\not{\partial} + H) = \det{}_\gamma \gamma_5(\not{\partial} + H)\gamma_5 = \det{}_\gamma(-\not{\partial} + H) = [\det{}_\gamma(-\nabla_x^2 + H^2)]^2.$$

Then, the determinant in position space is given by

$$\begin{aligned}\ln\det(-\nabla_x^2 + H^2) &= \operatorname{tr}\ln(-\nabla_x^2 + H^2)\\ &= \frac{1}{\text{volume}} \int \mathrm{d}^4x \, \langle x| \ln(-\nabla_x^2 + H^2)|x\rangle = \frac{1}{(2\pi)^4} \int \mathrm{d}^4p \, \ln(p^2 + H^2).\end{aligned}$$

Thus,

$$\frac{\mathcal{S}\,(0,0,H)}{N \times \text{volume}} \sim \frac{\Lambda^2}{2NG} H^2 - \frac{2}{(2\pi)^4} \int \mathrm{d}^4 p \, \ln(p^2 + H^2).$$

Differentiating with respect to H, one obtains an equation for the expectation value of H for N large, generally called a gap equation,

$$\frac{\Lambda^2 H}{NG} - \frac{4H}{(2\pi)^4} \int^{\Lambda} \frac{\mathrm{d}^4 q}{q^2 + H^2} = 0\,, \tag{11.48}$$

(a cut-off Λ is implied). This equation has always the trivial solution $H = 0$ corresponding to a chiral-symmetric phase.

However, for $G > G_c$ where G_c is given by

$$\frac{1}{G_c} = \frac{4N}{(2\pi)^4} \frac{1}{\Lambda^2} \int^{\Lambda} \frac{\mathrm{d}^4 q}{q^2}, \tag{11.49}$$

another solution is found with $H \neq 0$, which gives the lowest action and which corresponds to spontaneous breaking of chiral symmetry.

Note that G_c has an infinite cut-off limit but which depends on the specific cut-off function.

Using equation (11.49) inside (11.48), one obtains an equation for the expectation value of H in the broken phase, as a function of G. Setting

$$\upsilon = \Lambda^2 \frac{(G - G_c)}{4NGG_c},$$

one finds

$$\upsilon = \frac{H^2}{(2\pi)^4} \int^{\Lambda} \frac{\mathrm{d}^4 q}{q^2\,(q^2 + H^2)} \sim \frac{1}{8\pi^2} H^2 \ln(\Lambda/H)\,. \qquad (11.50)$$

Returning to the expression (11.47), one sees that, at this order, the fermion mass is given by

$$m_\psi = H\,.$$

It is small with respect to the cut-off only for $\upsilon \ll \Lambda^2$ and thus G very close to G_c; this implies a fine tuning of the coupling constant.

Moreover, at v fixed, the fermion mass decreases in the large cut-off limit as

$$m_\psi^2 \sim \frac{8\pi^2 v}{\ln(\Lambda/\sqrt{v})},$$

a result reminiscent of the GNY model.

Finally, from the large N action one can derive the inverse H-propagator at leading order by differentiating the action twice with respect to $H(x)$. Using the gap equation, one finds in the massive phase

$$\Delta_H^{-1}(p) = \frac{4N}{2(2\pi)^4}\left(p^2 + 4H^2\right)\int^\Lambda \frac{\mathrm{d}^4 q}{(q^2+H^2)\left[(p+q)^2+H^2\right]}. \tag{11.51}$$

The propagator vanishes for $p^2 = -4H^2$, which corresponds to the $\bar\psi\psi$ threshold, showing that the GN model generates a scalar bound state, which, in this limit, has a mass $m_H = 2m_\psi$, result consistent with the large N limit of equation (11.45) and of equation (11.37) in the GNY model.

A systematic comparison between the GN and the GNY models then shows that, in the large N limit, the two models differ only by their parametrization. This observation generalizes to all orders in $1/N$ for large cut-off. Within the large N expansion, the GN and GNY models are equivalent.

11.13.5 A few concluding remarks

The study of the GN model leads to a few remarks. First, even when one probes only a region of small momenta and small masses compared to the cut-off, one encounters a value G_c of the dimensionless coupling G (which is not universal but does not scale with the ratio between microscopic and physical scale) at which the concept of perturbative effective field theory looses it relevance. Near and above G_c, physics becomes non-perturbative.

At leading order, for N large, the RG β-function can be calculated and one finds

$$\beta(G) = 2G - 2G^2/G_c + O\left(1/N\right).$$

Near G_c a non-trivial large momentum scale (but small compared with the cut-off) called crossover scale, is generated, which allows to define a renormalized field theory consistent for all scales for dimensions $2 \le d < 4$.

Above G_c, chiral symmetry is spontaneously broken, a fermion mass is generated and a scalar bound state appears. However, the fine-tuning problem that is typical of theories with scalar particles is not solved because masses remain small with respect to the cut-off only for $G - G_c \ll 1$.

Finally, the analysis of the large N expansion to all orders strongly indicates that the GN and GNY models are equivalent.

APPENDIX A11
EUCLIDEAN DIRAC FERMIONS, SPIN GROUP AND γ MATRICES

The appendix is somewhat mathematical. The convention of summation over repeated indices are used, except when stated explicitly otherwise.

We describe the formalism of euclidean fermions, analytic continuation to imaginary time of relativistic fermions with spin. In this continuation, the relativistic pseudo-orthogonal group $SO(1,3)$ transforms into the orthogonal group $SO(4)$ (rotations in four dimensions). Correspondingly, euclidean fermions transform under the spin group $\mathrm{Spin}(4) \equiv SU(2) \times SU(2)$, which is locally isomorphic to the group $SO(4)$.

We first define an abstract Clifford algebra, show that it is invariant under $SO(4)$ transformations, and use it to construct the spin group $\mathrm{Spin}(4)$.

We then exhibit a realization of the Clifford algebra in terms of hermitian matrices, Dirac γ matrices in a specific basis. A unitary representation of the euclidean spin group follows.

Finally, we evaluate traces of products of γ-matrices, quantities that are relevant to perturbative calculations, and define the Fierz transformation.

A11.1 Orthogonal and spin groups

In this section, we define a Clifford algebra and exhibit its relation with the $O(4)$ orthogonal group. We then construct the corresponding spin group.

A11.1.1 Clifford algebra

Let γ_μ, $1 \le \mu \le 4$, be the generators of an associative algebra on $\mathbb{R}$, satisfying the commutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\delta_{\mu\nu} \mathbf{1}\,, \tag{A11.1}$$

where $\mathbf{1}$ is the unit element.

They generate an algebra called Clifford algebra $\mathcal{C}(4)$ isomorphic to the algebra generated by the operators $(\theta_i + \partial/\partial\theta_i)$, in the notation of section 10.2.1, acting on Grassmann algebras.

It follows from the relations $(A11.1)$ that the elements of $\mathcal{C}(4)$ form a real vector space of dimension 16, spanned by $\mathbf{1}$ and the products $\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_p}$, with $\mu_1 < \mu_2 < \cdots < \mu_p$.

Automorphism: the element γ_5. In $\mathcal{C}(4)$, the product of all generators

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \frac{1}{4!}\sum_{\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\,, \tag{A11.2}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon_{1234} = 1$, anti-commutes with all generators. One verifies,

$$\gamma_5^2 = \mathbf{1}\,. \tag{A11.3}$$

It allows defining an automorphism P in $\mathcal{C}(4)$ by

$$\mathrm{P}(\gamma_\mu) \equiv \gamma_5\gamma_\mu\gamma_5 = -\gamma_\mu\,. \tag{A11.4}$$

It splits $\mathcal{C}(4)$ into two vector spaces, $\mathcal{C}_-(4)$ and $\mathcal{C}_+(4)$, containing odd and even elements, respectively,

$$\mathrm{P}(\mathcal{C}_\pm) = \pm\mathcal{C}_\pm \,,$$

where only $\mathcal{C}_+(4)$ is a sub-algebra.

Finally the relations $(A11.1)$, $(A11.3)$ and $(A11.4)$ can be summarized by

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\,\delta_{ij}\,, \quad \text{for } 1 \le i, j \le 5\,. \tag{A11.5}$$

We will use Greek letters μ, ν... to indicate that we exclude the value 5 for the index.

Centre of the algebra. One verifies that the centre of the algebra, that is, the set of elements which commute with all $\mathcal{C}(4)$, reduces to $r\mathbf{1}$, $r \in \mathbb{R}$ (multiples of the unit).

A11.1.2 Clifford algebra and orthogonal group

We consider linear transformations acting on the generators $\boldsymbol{\gamma} \mapsto \boldsymbol{\gamma}'$,

$$\gamma'_\mu = \sum_\nu R_{\mu\nu}\gamma_\nu\,, \quad \det \mathbf{R} \neq 0\,, \tag{A11.6}$$

where $\mathbf{R}$ of elements $R_{\mu\nu}$ is a real matrix, which preserve the relations $(A11.1)$. Then,

$$\gamma'_\mu\gamma'_\nu + \gamma'_\nu\gamma'_\mu = \sum_{\rho,\sigma} R_{\mu\rho}R_{\nu\sigma}\left(\gamma_\rho\gamma_\sigma + \gamma_\sigma\gamma_\rho\right) = 2\sum_\rho R_{\mu\rho}R_{\nu\rho}\mathbf{1} = 2\delta_{\mu\nu}\mathbf{1}\,.$$

Therefore, the matrix $\mathbf{R}$ is orthogonal since

$$\sum_\rho R_{\mu\rho}R_{\nu\rho} = \delta_{\mu\nu}\,.$$

The relations $(A11.1)$ are invariant under the orthogonal group $O(4)$ (rotations-reflections in four-dimensional euclidean space).

The element γ_5. An orthogonal transformation of matrix $\mathbf{R}$ acting on the generators transforms γ_5 into $\gamma_5 \det \mathbf{R} = \pm\gamma_5$. Thus, γ_5 is invariant under the subgroup $SO(4)$ (rotations) for which $\det \mathbf{R} = 1$ and changes sign under reflections for which $\det \mathbf{R} = 1$.

Remark. If in the right hand side of equation $(A11.1)$ the tensor $\delta_{\mu\nu}$ is replaced by another metric tensor $g_{\mu\nu}$, the symmetry group becomes the group that leaves the metric $g_{\mu\nu}$ invariant. In the case of the diagonal metric $+1, -1, -1, -1$ one obtains the relativistic group $O(1,3)$.

A11.1.3 Clifford algebra and spin group

We consider the set of invertible elements Λ of $\mathcal{C}(4)$ that satisfy

$$\Lambda^{-1}\gamma_\mu\Lambda = \sum_\nu R_{\mu\nu}(\Lambda)\gamma_\nu\,, \tag{A11.7}$$

where the elements $R_{\mu\nu}$ of the matrix $\mathbf{R}$ are real. The elements Λ form a group that we denote by $\mathfrak{G}(4)$ (the unit and the inverse belong to the set).

The relation $(A11.7)$ induces a homomorphism of groups. Indeed, if

$$\Lambda_1 \mapsto \mathbf{R}_1\,, \quad \Lambda_2 \mapsto \mathbf{R}_2\,,$$

to the product $\Lambda_1\Lambda_2$ corresponds the product of real matrices $\mathbf{R}_1\mathbf{R}_2$.

The relation $(A11.7)$ implies

$$\Lambda^{-1}\gamma_\mu\Lambda\Lambda^{-1}\gamma_\rho\Lambda = \Lambda^{-1}\gamma_\mu\gamma_\rho\Lambda = \sum_{\nu,\sigma} R_{\mu\nu}R_{\rho\sigma}\gamma_\nu\gamma_\sigma\,.$$

Adding the symmetric relation $\mu \leftrightarrow \rho$, and using the relations $(A11.1)$, we obtain

$$\sum_\nu R_{\mu\nu}R_{\rho\nu} = \delta_{\mu\rho}\,.$$

Therefore, the real matrices $\mathbf{R}$ form a group, subgroup of the orthogonal group $O(4)$ (rotations–reflections) and the transformations $(A11.7)$ realize a subgroup of the transformations $(A11.6)$.

Remarks.

(*i*) If Λ corresponds to $\mathbf{R}$ and $\lambda \neq 0$ a non-vanishing real number, then $\lambda\Lambda$ corresponds the same matrix $\mathbf{R}$.

(*ii*) If Λ belongs to the group $\mathfrak{G}(4)$ and is associated with the matrix $\mathbf{R}$, then a short calculation shows that

$$\Lambda^{-1}\gamma_5\Lambda = \det \mathbf{R}\,\gamma_5 \;\;\Leftrightarrow\;\; \gamma_5\Lambda = \det \mathbf{R}\,\Lambda\gamma_5\,. \tag{A11.8}$$

From this commutation relation, we conclude:

Even elements of $\mathfrak{G}(4)$ are associated to orthogonal matrices with determinant 1, that is, belonging to the subgroup $SO(4)$ (rotations) of $O(4)$. By contrast, odd elements are associated with orthogonal matrices with determinant -1.

(*iii*) For example, the generators γ_ρ are invertible and they satisfy the relation $(A11.7)$ with matrices $\mathbf{R}^\rho$ with elements

$$R^\rho_{\mu\nu} = 2\delta_{\rho\mu}\delta_{\rho\nu} - \delta_{\mu\nu}\,. \tag{A11.9}$$

Thus, they belong to the group $\mathfrak{G}(4)$. One verifies that indeed $\mathbf{R}^\rho$ is an orthogonal matrix with determinant -1.

(vi) The element γ_5 also belongs to the group $\mathfrak{G}(4)$ and corresponds to the rotation matrix $-\mathbf{1}$, an element of $SO(4)$ since

$$(\gamma_5)^{-1}\gamma_\mu\,\gamma_5 = -\gamma_\mu\,. \tag{A11.10}$$

Finally, since γ_μ is associated with the orthogonal matrix $(A11.9)$, the product

$$\Pi_\mu = \gamma_5\gamma_\mu\,, \tag{A11.11}$$

corresponds to a reflection P_μ along the μ axis, $x_\mu \mapsto -x_\mu$ (as defined by (11.21)). Note that

$$P_\mu^2 = \mathbf{1} \quad \text{but} \quad (\Pi_\mu)^2 = -\mathbf{1}\,.$$

A11.1.4 Spin group and Lie algebra

We now show by explicit construction that to a subgroup of $\mathfrak{G}(4)$, the spin group $\mathrm{Spin}(4)$, obtained by dividing $\mathfrak{G}(4)$ by the Abelian factor, is associated to the whole group $SO(4)$.

One can define a topology in the Clifford algebra since it has the form of a finite-dimensional vector space. With such a topology the groups $\mathfrak{G}(4)$ or $\mathrm{Spin}(4)$ are Lie groups and we can discuss their Lie algebras.

We consider the elements

$$\tilde{\boldsymbol{\sigma}}_{\mu\nu} = \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]. \tag{A11.12}$$

Only six elements $\tilde{\boldsymbol{\sigma}}_{\mu\nu}$ are linearly independent; a basis is, for example, $\tilde{\boldsymbol{\sigma}}_{\mu\nu}$ with $\mu < \nu$. Using the relations ($A11.1$), one verifies that for $\mu \neq \nu$ (no summation over μ, ν being implied)

$$\tilde{\boldsymbol{\sigma}}_{\mu\nu}^2 = (\gamma_\mu \gamma_\nu)^2 = -\mathbf{1}\,.$$

To $\tilde{\boldsymbol{\sigma}}_{\mu\nu}$ $(\mu \neq \nu)$ we associate the elements

$$\begin{aligned}\Lambda(\theta) &= \exp[-\tfrac{1}{2}\theta\tilde{\boldsymbol{\sigma}}_{\mu\nu}]\,, \quad \theta \in \mathbb{R} \\ &= \cos(\theta/2)\,\mathbf{1} - \sin(\theta/2)\,\tilde{\boldsymbol{\sigma}}_{\mu\nu}\,.\end{aligned} \tag{A11.13}$$

The elements $\Lambda(\theta)$ for a given matrix $\tilde{\boldsymbol{\sigma}}_{\mu\nu}$ generate a group isomorphic to the Abelian groups $U(1)$ or $SO(2)$ and correspond to rotations of angle $\theta/2$.

A straightforward calculation leads to

$$\Lambda^{-1}(\theta)\gamma_\rho\Lambda(\theta) = \begin{cases} \gamma_\rho & \text{for } \rho \neq \mu \text{ and } \rho \neq \nu, \\ \cos\theta\gamma_\mu - \sin\theta\gamma_\nu & \text{for } \rho = \mu, \\ \cos\theta\gamma_\nu + \sin\theta\gamma_\mu & \text{for } \rho = \nu. \end{cases}$$

Therefore, $\Lambda(\theta)$ is an element of $\mathfrak{G}(4)$ and the corresponding orthogonal matrix $\mathbf{R}$ represents a rotation of angle θ leaving the space orthogonal to the (μ, ν) plane invariant.

The whole group $SO(4)$ can be generated by a product of such rotations.

Lie algebras and groups. The generators of the Lie algebra of the group $SO(4)$ in the defining representation are 4×4 antisymmetric matrices $\mathbf{T}^{\rho\sigma}$ with elements

$$(\mathbf{T}^{\rho\sigma})_{\alpha\beta} = \delta_{\rho\alpha}\delta_{\sigma\beta} - \delta_{\rho\beta}\delta_{\sigma\alpha}\,,$$

where only six are independent. If $\theta_{\rho\sigma}$ is an arbitrary antisymmetric matrix, with this normalization

$$\tfrac{1}{2}\sum_{\rho,\sigma}(\mathbf{T}^{\rho\sigma})_{\alpha\beta}\,\theta_{\rho\sigma} = \theta_{\alpha\beta}\,.$$

A general element $\mathbf{R}$ of $SO(4)$ can thus be written as

$$\mathbf{R} = \mathrm{e}^{\boldsymbol{\theta}} = \exp\left[\tfrac{1}{2}\sum_{\rho,\sigma}\mathbf{T}^{\rho\sigma}\theta_{\rho\sigma}\right]. \qquad (A11.14)$$

Spin group. It follows from the homomorphism between groups that the matrices $\tilde{\boldsymbol{\sigma}}_{\mu\nu}/2$ satisfy the commutation relations of the generators $\mathbf{T}^{\mu\nu}$ of the Lie algebra of the group $SO(4)$.

Exponentiating one finds a general representation of the elements of the spin group Spin(4), subgroup of the group $\mathfrak{G}(4)$,

$$\Lambda = \exp\left[\frac{1}{4}\sum_{\mu,\nu}\theta_{\mu\nu}\tilde{\boldsymbol{\sigma}}_{\mu\nu}\right]. \qquad (A11.15)$$

The centre of the spin group contains only one non-trivial element, $-\mathbf{1}$, which corresponds in $(A11.13)$ to $\theta = 2\pi$. Therefore, the spin group Spin(4) and $SO(4)$ are not isomorphic since the two elements $\pm\Lambda$ of the spin group correspond to the same rotation matrix.

Finally, as we have shown above, the addition of one reflection $\Pi_\mu = \gamma_5\gamma_\mu$, an odd element of the group $\mathfrak{G}(4)$, allows generating the whole $O(4)$ group.

Therefore, the group of transformations $(A11.7)$ contains as subgroup the whole group of transformations $(A11.6)$.

A11.2 The γ matrices: a hermitian representation

We now construct an explicit representation of the Clifford algebra $\mathcal{C}(4)$ generated by hermitian (and thus unitary) matrices of minimal size. We use for the matrices representing the generators the same notation γ_μ, $1 \le \mu \le 4$.

Since the dimension of $\mathcal{C}(4)$ is 16, the Clifford algebra cannot be represented by matrices of dimension smaller than 4. We now give an inductive construction $(2 \mapsto 4)$ of hermitian matrices γ_μ satisfying the defining relations $(A11.1)$, which can be generalized to all even dimensions.

A11.2.1 The dimension two

For $d = 2$, the standard Pauli matrices play the role of γ matrices and realize the algebra $\mathcal{C}(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{A11.16}$$

The third Pauli matrix σ_3 plays the role of the matrix γ_5 (equation ($A11.2$)):

$$\sigma_3 = i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A11.17}$$

The three matrices are hermitian. The matrices σ_1 and σ_3 are symmetric, and σ_2 is antisymmetric.

A11.2.2 Dimension four

We now consider the matrices

$$\gamma_i = \sigma_1 \otimes \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad 1 \leq i \leq 3,$$
$$\gamma_4 = \sigma_2 \otimes \mathbf{1}_2 = \begin{pmatrix} 0 & -i\mathbf{1}_2 \\ i\mathbf{1}_2 & 0 \end{pmatrix},$$

in which $\mathbf{1}_2$ is the unit matrix in two dimensions.

The matrices γ_i are tensor products involving the matrices σ_i and $\mathbf{1}_2$. A straightforward calculation shows that if the matrices σ_i satisfy relations $(A11.1)$, the matrices γ_i, $i = 1, \dots, 4$, satisfy the same relations.

Since $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, one finds

$$\gamma_5 = \sigma_3 \otimes \mathbf{1}_2 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}.$$

By inspection, we see that the γ matrices are all hermitian. In addition

$$\gamma_i^T = \begin{pmatrix} 0 & \sigma_i^T \\ \sigma_i^T & 0 \end{pmatrix}, \quad 1 \le i \le 3\,.$$

Therefore, if σ_i is symmetric or antisymmetric, γ_i has the same property. The matrix γ_4 is antisymmetric, and γ_5 is symmetric. It follows immediately that, in this representation, all γ matrices with odd index are symmetric, all matrices with even index are antisymmetric:

$$\gamma_i^T = (-1)^{i+1}\gamma_i\,, \quad 1 \le i \le 5\,.$$

A11.2.3 Spin group: a unitary representation

When the generators of the Clifford algebra are represented by hermitian matrices, the generators $\tilde{\boldsymbol{\sigma}}_{\mu\nu}$ of the spin group are represented by anti-hermitian, traceless, matrices (for which we use below the same notation). The complex vectors on which the representation acts are called spinors.

Instead of the anti-hermitian matrices $\tilde{\boldsymbol{\sigma}}_{\mu\nu}$, one often uses the hermitian matrices

$$\boldsymbol{\sigma}_{\mu\nu} = \frac{1}{i}\tilde{\boldsymbol{\sigma}}_{\mu\nu} = \frac{1}{2i}\left[\gamma_\mu, \gamma_\nu\right]. \qquad (A11.18)$$

Then for $\mu \neq \nu$ the matrices $\boldsymbol{\sigma}_{\mu\nu}$ have the property

$$\boldsymbol{\sigma}^2_{\mu\nu} = \mathbf{1}\,.$$

The matrices belonging to the representation of the spin group $\mathrm{Spin}(4)$ can be written as

$$\Lambda = \exp\!\left(\tfrac{i}{4}\sum_{\mu,\nu}\boldsymbol{\sigma}_{\mu\nu}\theta_{\mu\nu}\right), \qquad (A11.19)$$

where $\theta_{\mu\nu}$ is a real antisymmetric matrix. Since Λ is the exponential of an anti-hermitian traceless matrix, it is a unitary matrix of determinant one: the representation of the spin group $\mathrm{Spin}(4)$ is a unitary group, subgroup of the unitary group $SU(4)$.

The relation $(A11.7)$ can be written as

$$\Lambda\gamma_\mu\Lambda^\dagger = \sum_\nu \gamma_\nu R_{\nu\mu}\,. \tag{A11.20}$$

Introducing a space vector p_μ, we can write the equation in an equivalent form:

$$\Lambda \sum_\mu p_\mu\gamma_\mu\Lambda^\dagger = \sum_\mu (p_R)_\mu\gamma_\mu \quad \text{with} \quad (p_R)_\mu = \sum_\nu R_{\mu\nu}p_\nu\,.$$

In this form the equation shows explicitly that the group $SO(4)$ is isomorphic to the adjoint representation of the spin group $\mathrm{Spin}(4)$.

As we have seen, the spin group and $SO(4)$ have the same Lie algebra but are not isomorphic because $\pm\Lambda$ correspond to the same rotation matrix.

The matrix $R_{\mu\nu}$ can be calculated explicitly from equation $(A11.20)$ in terms of Λ by taking a trace:

$$R_{\mu\nu} = \operatorname{tr}\left(\Lambda^{\dagger}\gamma_{\mu}\Lambda\gamma_{\nu}\right) / \operatorname{tr}\mathbf{1}\,. \qquad (A11.21)$$

Spin groups in dimensions two to four. For $d = 2$, the spin group is isomorphic to a group $SO(2)$, but as we have seen, a rotation of angle $\theta/2$ in the spin group corresponds to a rotation of angle θ in the adjoint representation which is also isomorphic to $SO(2)$, a peculiarity of Abelian groups.

For $d = 3$, the spin group is $SU(2)$ and is associated to the group $SO(3)$. For $d = 4$, the spin group $SU(2) \times SU(2)$ and is associated with$SO(4)$.

A11.2.4 Reflections and chiral components

Reducibility. In four dimensions, since the elements of the spin group are even elements of the Clifford algebra, γ_5 commutes with the whole spin group:

$$[\Lambda, \gamma_5] = 0\,.$$

Therefore, the unitary representation of Spin(4) is reducible. The representation is reduced by projecting spinors ψ onto the two eigenspaces of γ_5 using the projectors $(\mathbf{1} \pm \gamma_5)/2$. This defines two spinors $\psi_\pm$, the chiral components of the spinor ψ:

$$\psi_\pm = \tfrac{1}{2}(\mathbf{1} \pm \gamma_5)\psi\,. \qquad (A11.22)$$

Space reflections and chiral components. To generate the full orthogonal group we have still to represent reflections. The elements $\pm\Pi_\mu$, ($\Pi_\mu = \gamma_5\gamma_\mu$, see equation ($A11.11$)) correspond to reflections P_μ (equation (11.21)) that act on a position vector $\mathbf{x}$ by changing the sign of the μ component. The anti-commutation relation,

$$\gamma_5\Pi_\mu = -\gamma_5\Pi_\mu \,,$$

implies

$$\Pi_\mu \, \tfrac{1}{2}\left(1+\gamma_5\right) = \tfrac{1}{2}\left(1-\gamma_5\right)\Pi_\mu .$$

A reflection exchanges chiral components. The representation of the spin group associated with the group $O(4)$ is thus irreducible.

A11.2.5 Charge conjugation

We exhibit unitary matrices C such that

$$C^{\dagger}\,\gamma_{\mu}^{T}\,C=-\gamma_{\mu}\quad\Leftrightarrow\quad C\,\gamma_{\mu}\,C^{\dagger}=-\gamma_{\mu}^{T}\,. \tag{A11.23}$$

Since the matrices γ_{μ} are hermitian $\gamma_{\mu}^{T}=\gamma_{\mu}^{*}$.

In the representation of section A11.2, we can take

$$C=\pm\gamma_{1}\gamma_{3}\quad\Rightarrow C^{\dagger}C=\mathbf{1}\,.$$

(In two dimensions, σ_2 corresponds to charge conjugation.)

The unitary matrix

$$\tilde{C}=C\gamma_{5}\,, \tag{A11.24}$$

then satisfies

$$\begin{aligned}\tilde{C}^{\dagger}\,\gamma_{\mu}^{T}\,\tilde{C}&=\hat{\gamma}^{\dagger}C^{\dagger}\,\gamma_{\mu}^{T}\,C\hat{\gamma}=-\hat{\gamma}^{\dagger}\gamma_{\mu}\hat{\gamma}\\&=\gamma_{\mu}\,.\end{aligned} \tag{A11.25}$$

The matrix γ_5 is symmetric. Under C or $\tilde{C}$, it transforms like

$$C^\dagger \gamma_5^T C = \gamma_5 \,.$$

Spin group: conjugate representation. We now apply the transformation ($A11.23$) to an element Λ of the unitary representation of the spin group, using the form ($A11.19$),

$$C \exp\left(\tfrac{1}{8} \sum_{\mu,\nu} \theta_{\mu\nu} [\gamma_\mu, \gamma_\nu]\right) C^\dagger = \exp\left(\tfrac{1}{8} \sum_{\mu,\nu} \theta_{\mu\nu} [\gamma_\mu^*, \gamma_\nu^*]\right),$$

and thus

$$\Lambda^* = C \Lambda C^\dagger .$$

This identity shows that the unitary representation and the representation obtained by complex conjugation are equivalent. The same property holds for the matrix $\tilde{C}$.

Finally,

$$C\gamma_5 C^\dagger = \gamma_5^*,$$

and this extends the property to reflections.

For reasons which will become clear later we call these transformations charge conjugation.

Charge conjugation and chiral components. The transformation properties of γ_5 imply

$$C^\dagger(1+\gamma_5^T)C = (1+\gamma_5).$$

In dimension four, C respects chirality. Charge conjugation multiplied by a reflection like $C\Pi_\mu$ has the opposite property.

A11.2.6 Summary

With the conventions of section A11.2, the 4×4 γ-matrices take the form

$$\gamma_{i=1,2,3} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -i\mathbf{1}_2 \\ i\mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}.$$

The matrices $\boldsymbol{\sigma}_{\mu\nu}$ then become

$$\boldsymbol{\sigma}_{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \text{ for } i, j, k \le 3\,, \qquad \boldsymbol{\sigma}_{i4} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \text{ for } i \le 3\,.$$

We recognize in the matrices

$$\boldsymbol{\sigma}_i^{\pm} = \tfrac{1}{4} \sum_{j,k} \epsilon_{ijk} \boldsymbol{\sigma}_{jk} \pm \tfrac{1}{2} \boldsymbol{\sigma}_{i4}\,,$$

the generators of the group $SU(2) \times SU(2)$. The projectors $\frac{1}{2}(1 \pm \gamma_5)$ decompose a Dirac spinor into the sum of two vectors transforming as the $(1/2, 0)$ and $(0, 1/2)$ representations of the group (Weyl spinors). A reflection exchanges the two vectors (as expected since the representation then is no longer reducible). In terms of Weyl spinors the construction of invariants with respect to the spinor group reduces to considerations about $SU(2)$.

A useful remark in this context, is that the representation and its complex conjugate are equivalent since

$$U^* = \sigma_2 U \sigma_2 \ \forall\, U \in SU(2)$$

(see also section A11.2.5) and thus if φ and χ are two $SU(2)$ spinors the combination

$$\sum_{\alpha,\beta} \varphi_\alpha \,(\sigma_2)_{\alpha\beta}\, \chi_\beta = -i \sum_{\alpha,\beta} \epsilon_{\alpha\beta} \varphi_\alpha \chi_\beta\,,$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor ($\epsilon_{12} = 1$), is an $SU(2)$ invariant.

We recall that for charge conjugation we can take $C = \gamma_1\gamma_3 = \mathbf{1}_2 \otimes \sigma_2$.

A11.2.7 Traces of products of γ matrices

Perturbative calculations involving relativistic fermions often require the calculation of traces of products of γ matrices, which we, therefore, explain in detail. It is possible to calculate traces within an explicit matrix representation, but here we define the trace as a linear mapping of the Clifford algebra ($A11.1$) to real or complex numbers that satisfies the cyclic condition. We normalize the trace by the value of the trace of the unit matrix, the only quantity that depends explicitly on the representation. We thus set $\mathrm{tr}\,\mathbf{1} = 4$.

The trace of any element of $\mathcal{C}(4)$ is a linear combination of traces of products of generators. Then, to evaluate the traces, one needs only the cyclic property and the relations ($A11.5$).

One finds that the only non-vanishing traces contains as factors only even powers of each generator.

Traces of odd elements. Therefore, all products of an odd number of generators vanish. For example,

$$\operatorname{tr} \gamma_\mu = 0 \,, \quad \operatorname{tr}[\gamma_\mu \gamma_\nu \gamma_\rho] = 0 \,, \ldots \,.$$

Also from the definition of γ_5, we conclude

$$\operatorname{tr} \gamma_5 \gamma_\mu = 0 \,, \quad \operatorname{tr} \gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho = 0 \,, \ldots \,.$$

The direct proof is simple. Denoting by Γ_A such a product, we obtain the chain of identities

$$\operatorname{tr} \Gamma_A = \operatorname{tr} \Gamma_A \gamma_5 \gamma_5 = \operatorname{tr} \gamma_5 \Gamma_A \gamma_5 = -\operatorname{tr} \Gamma_A \,.$$

Product of even numbers of generators γ_μ. First, taking the trace of the defining relation ($A11.1$), we obtain

$$\operatorname{tr} \gamma_\mu \gamma_\nu = 4 \, \delta_{\mu\nu} \,.$$

To calculate the trace of the product of an even number $2n$ of generators, $\operatorname{tr}\gamma_{\mu_1}\dots\gamma_{\mu_{2n}}$, we successively commute $\gamma_{\mu_{2n}}$ through all other factors $\gamma_{\mu_1},\dots,\gamma_{\mu_{2n-1}}$, using the commutation relations $(A11.1)$. We then generate a linear combination of traces of the products of $(2n-2)$ generators. At each commutation the sign changes. After all commutations, as a consequence of the cyclic property of the trace, we recover the opposite of the initial expression. As a consequence, we find

$$\begin{aligned}\operatorname{tr}\gamma_{\mu_1}\dots\gamma_{\mu_{2n}} &= \delta_{\mu_1\mu_{2n}}\operatorname{tr}\left(\gamma_{\mu_2}\dots\gamma_{\mu_{2n-1}}\right)-\delta_{\mu_2\mu_{2n}}\\ &\quad\times\operatorname{tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\dots\gamma_{\mu_{2n-1}}\right)+\dots+\delta_{\mu_{2n-1}\mu_{2n}}\operatorname{tr}\left(\gamma_{\mu_1}\dots\gamma_{\mu_{2n-2}}\right).\end{aligned}$$

Therefore,we prove by induction a kind of Wick's theorem for the trace of a product of an even number of generators γ_μ:

$$\operatorname{tr}\gamma_{\mu_1}\dots\gamma_{\mu_{2n}}=4\sum_{\substack{\text{all possible pairings}\\ \text{of }(1,2,\dots,2n)}}\operatorname{sgn}(P)\delta_{\mu_{P_1}\mu_{P_2}}\dots\delta_{\mu_{P_{2n-1}}\mu_{P_{2n}}},$$

in which $\mathrm{sgn}(P)$ is the signature of the permutation P when $P_{2m-1} < P_{2m}$ for $1 \leq m \leq n$.

The element γ_5*.* Of special interest are the traces of products of the form $\mathrm{tr}\,\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_{2n}}$. From Wick's theorem, it follows immediately that

$$\mathrm{tr}\,\gamma_5\gamma_\mu\gamma_\nu = 0\,.$$

Then, in terms of the completely antisymmetric tensor $\epsilon_{\mu_1\ldots\mu_4}$ normalized by $\epsilon_{1234} = 1$, one obtains

$$\mathrm{tr}\,\gamma_5\gamma_{\mu_1}\ldots\gamma_{\mu_4} = 4\,\epsilon_{\mu_1\ldots\mu_4}\,. \qquad (A11.26)$$

We shall see later that relation $(A11.26)$, which is specific to dimension four, has deep consequences. In particular, dimensional regularization does not preserve this relation and this is the source of possible anomalies ($i.e$, obstacles to a quantization preserving symmetries) in field theories that are chiral invariant in the classical approximation.

A11.3 The Fierz transformation

Within the algebra of γ matrices it is possible to define a basis of 16 hermitian matrices orthogonal by the trace. We denote these matrices by Γ^A. Then,

$$\operatorname{tr} \Gamma^A \Gamma^B = 4\delta_{AB} .$$

Any fermion two-point correlation function can then be expanded on such a basis. A four-point fermion correlation function can be expanded on a basis formed by the tensor products of these matrices.

However, in this case one has to first separate the four fermion fields into two pairs of fields and there are three ways of doing it. A connection between these different bases can be found through a Fierz transformation.

We first express that any 4×4 matrix $\mathbf{X}$ can be expanded on the Γ basis:

$$X_{ab} = \tfrac{1}{4} \sum_A \operatorname{tr} \left(\mathbf{X} \Gamma^A \right) \Gamma^A_{ab} . \qquad (A11.27)$$

We choose a matrix $\mathbf{X}$ of the form

$$X_{ab} = \Gamma^B_{cb}\Gamma^C_{ad}\,.$$

Identity $(A11.27)$ becomes

$$\Gamma^B_{cb}\Gamma^C_{ad} = \tfrac{1}{4}\sum_A \left(\Gamma^B\Gamma^A\Gamma^C\right)_{cd}\Gamma^A_{ab}.$$

An expansion of the product $\Gamma^B\Gamma^A\Gamma^C$ on the basis of Γ matrices, yields a decomposition of any element of one basis onto another:

$$\Gamma^B\Gamma^A\Gamma^C = 4\sum_D M^{BC}_{AD}\Gamma^D \;\Rightarrow\; \Gamma^B_{cb}\Gamma^C_{ad} = \sum_{A,D} M^{BC}_{AD}\Gamma^D_{cd}\Gamma^A_{ab}.$$

Examples

For $d = 4$ a basis is

$$\mathbf{1},\; \gamma_\mu,\; \gamma_5,\; i\gamma_5\gamma_\mu,\; \boldsymbol{\sigma}_{\mu\nu}\,.$$

We leave as an exercise to verify that the subset

$$\mathbf{1} \otimes \mathbf{1},\ \gamma_\mu \otimes \gamma_\mu,\ \gamma_5 \otimes \gamma_5,\ i\gamma_5\gamma_\mu \otimes i\gamma_5\gamma_\mu,\ \boldsymbol{\sigma}_{\mu\nu} \otimes \boldsymbol{\sigma}_{\mu\nu}$$

transforms into itself with a matrix $\mathbf{M}_4$ of square $\mathbf{1}$:

$$\mathbf{M}_4 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & -4 & 2 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 4 & 2 & -4 & -2 & 0 \\ 6 & 0 & 6 & 0 & -2 \end{pmatrix}.$$

Exercises

Recall that in these lectures we use a Euclidean (or imaginary time) notation, in particular for fermions, $\not\partial = \gamma_\mu \partial_\mu$, ($\mathbf{1}$ is the unit element)

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\delta_{\mu\nu} \mathbf{1}\,,$$
$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \ \Rightarrow\ \gamma_5^2 = \mathbf{1}\,.$$

The modified minimal subtraction scheme ($\overline{\mathrm{MS}}$). In the calculation of low order Feynman diagrams, a factor

$$N_d = \frac{\text{area of the sphere } S_{d-1}}{(2\pi)^d} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}, \qquad (A11.28)$$

L being the number of loops of the diagram, is generated naturally. To avoid expanding N_d in $\varepsilon = 4 - d$, it is convenient to rescale the loop expansion parameter to suppress this factor, for instance, by multiplying each Feynman diagram by a factor $(N_4/N_d)^L$, where L is the number of loops.

Exercise 11.1

Calculation of RG β-functions of the GNY model. Determine the RG functions of the GNY model, in particular, verifying the expressions for the two β-functions, using dimensional regularization and working in the $\overline{MS}$ scheme. This involves a determination of the divergent part at one-loop of various two, three and four point functions.

We give below some elements of the calculation of the divergent parts.

The boson diagrams: one-loop divergences

Fig. 11.1 displays the two one-loop divergent diagrams generated by the H^4 interaction.

In the massless theory, the contribution to the H two-point function,

$$\Omega_0 = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{q^2},$$

vanishes in dimensional regularization.

The second diagram contributes to the H four-point function. In the massless limit,

$$B_d(p) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{q^2 (p-q)^2}$$
$$= -\frac{\pi}{\sin(\pi d/2)} \frac{\Gamma^2(d/2)}{\Gamma(d-1)} N_d\, p^{4-d} \equiv N_d\, b(d) p^{-\varepsilon} \qquad (A11.29)$$

with

$$b(d) = -\frac{\pi}{\sin(\pi d/2)} \frac{\Gamma^2(d/2)}{\Gamma(d-1)} = \frac{1}{\varepsilon}\left(1 + \tfrac{1}{2}\varepsilon + O\left(\varepsilon^2\right)\right).$$

Then,

$$\langle H_1 H_2 H_3 H_4 \rangle_{1\mathrm{PI}}\big|_{g=0} = \lambda - \tfrac{1}{2}\lambda^2 \left[B_d(p_{12}) + B_d(p_{13}) + B_d(p_{14})\right] + O(\lambda^3),$$

where $H_i \equiv H(p_i)$ and $p_{ij} \equiv p_i + p_j$. Expanding for $\varepsilon \to 0$, one finds the divergent contribution

$$\langle H_1 H_2 H_3 H_4 \rangle_{1\mathrm{PI,div.}}\Big|_{g=0} = -\frac{3N_d}{2\varepsilon}\lambda^2. \qquad (A11.30)$$

Diagrams involving fermion propagators: One-loop divergences

We evaluate below only the one-loop divergent parts of the additional diagrams involving fermion propagators.

Boson two-point function. The diagram of figure 11.2 has a factor Ng^2, the sign coming from the Legendre transformation cancelling the sign coming from the fermion loop. It is then multiplied by

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d q \frac{\operatorname{tr} i\not{q} i\left(\not{q}+\not{p}\right)}{q^2(p+q)^2}=2p^2 B_d(p)=N_d\frac{2}{\varepsilon}\left(1+\tfrac{1}{2}\varepsilon+O\left(\varepsilon^2\right)\right)p^{2-\varepsilon}, \tag{A11.31}$$

where we have used the identity

$$2(q^2+p\cdot q)=(p+q)^2+q^2-p^2.$$

The contributions of the two first terms then vanishes in dimensional regularization. Therefore, the divergent part is

$$\langle HH\rangle_{\mathrm{1PI,1\,loop,\,div.}}=N_d\frac{2N}{\varepsilon}g^2.$$

Fermion two-point function. Taking into account the sign coming from the Legendre transformation, the diagram of figure 11.3 has a factor $-g^2$. Moreover, the diagram is proportional to $\not{p}$:

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{(p-q)^2}\frac{i\not{q}}{q^2} = i\not{p}X(p)$$

and thus multiplying both sides with $\not{p}$ and taking the trace, one infers

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{(p-q)^2}\frac{i\not{q}}{q^2} = \frac{i\not{p}}{p^2}\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q\, p\cdot q}{(p-q)^2 q^2} = \tfrac{1}{2} i\not{p} B_d(p)\,, \qquad (A11.32)$$

where the identity $2p\cdot q = p^2 + q^2 - (p-q)^2$ has been used. The divergent part is

$$\langle\bar{\psi}\psi\rangle_{1\mathrm{PI},1\,\mathrm{loop,\,div.}} = -g^2 N_d \frac{i\not{p}}{2\varepsilon}\,.$$

The $\langle\bar{\psi}\psi H\rangle$ vertex function. Figure 11.4 displays the remaining two Feynman diagrams. The diagram on the left has a factor $-g^2$ multiplied by

$$\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d q}{(p_1-q)^2}\frac{i\not{q}\, i\,(\not{q}-\not{p}_1-\not{p}_2)}{q^2(q-p_1-p_2)^2}.$$

Evaluated at $p_1=-p_2=p$ (zero boson momentum), the diagram reduces to

$$\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d q(i\not{q})^2}{q^4(p-q)^2}=-B_d(p)=-\frac{N_d}{\varepsilon}\left(1+\tfrac{1}{2}\varepsilon+O\left(\varepsilon^2\right)\right)p^{-\varepsilon}. \quad (A11.33)$$

Its divergent part is

$$\left\langle\bar{\psi}\psi H\right\rangle_{\text{1PI,1 loop, div.}}=g^3\frac{N_d}{\varepsilon}\,.$$

The $\langle HHHH\rangle$ vertex function. The diagram in the right of figure 11.4 has a factor Ng^4, the sign of the Legendre transformation cancelling the sign of the fermion loop. It is multiplied by

$$SQ(p_1,p_2,p_3,p_4) = \frac{1}{(2\pi)^d}\int \mathrm{d}^d q \frac{\mathrm{tr}\left[i\not{q}\, i\left(\not{q}+\not{p}_1\right) i\left(\not{q}+\not{p}_1+\not{p}_2\right) i\left(\not{q}-\not{p}_4\right)\right]}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2}$$

to which five diagrams corresponding to permutations of $\{p_2,p_3,p_4\}$ have to be added.

Evaluating the diagram for vanishing opposite momenta, for instance for $p_2=p_4=0$, $p_1=-p_3=p$, one simply finds the contribution

$$\begin{aligned} SQ(p,0,-p,0) &= \frac{1}{(2\pi)^d}\,\mathrm{tr}\int \mathrm{d}^d q \left(\frac{i\not{q}}{q^2}\right)^2 \left(\frac{i(\not{q}+\not{p})}{(p+q)^2}\right)^2 \\ &= 4B_d(p) = \frac{4N_d}{\varepsilon}\left(1+\tfrac{1}{2}\varepsilon+O\left(\varepsilon^2\right)\right)p^{-\varepsilon}. \quad (A11.34) \end{aligned}$$

The total one-loop divergence coming from the six diagrams, to which the contribution $(A11.30)$ has to be added, is then

$$\langle HHHH \rangle_{\text{1PI,1 loop, div.}} = \left(-\tfrac{3}{2}\lambda^2 + 24Ng^4\right)\frac{N_d}{\varepsilon}.$$

Exercise 11.2

RG equations. Determine the RG flow for the GNY model explicitly by solving the RG equations at one-loop order and discuss the solution. It will be convenient to parametrize the scale parameter as e^t.