Lecture 12: SYMMETRIES AND SYMMETRY BREAKING

Up to now, we have discussed quantum field theory mainly from the point of view of general algebraic properties, power counting, renormalization and renormalization group though, occasionally, we have made references to some symmetry properties.

In this chapter, we discuss the physical and algebraic consequences of some symmetries of action, in particular for what concerns renormalization.

We deal only with global linear continuous symmetries corresponding to compact Lie groups because they imply interesting formal properties; consequences of discrete symmetries can also be studied but with somewhat different methods. Also, we deal below only with infinitesimal group transformations and, therefore, topological properties of groups will play no role.

We consider explicitly dimension four, though most of the algebraic discussion can generalized to generic dimensions, except fermion chiral symmetry, which is special to even dimensions.

Renormalization and symmetries. When the action in the tree approximation has some symmetry properties, it can be expected that the renormalized action will not have the most general form allowed by pure power counting arguments but will, instead, keep some trace of the initial symmetry. Technically, this means that, as a consequence of the symmetry, the divergences generated in perturbation theory are not of generic form and, therefore, the renormalization constants are not all independent.

The analysis of this problem is based on the following strategy:

(i) We first introduce a regularization that preserves the symmetry.

(ii) We then prove identities, generally called Ward–Takahashi (WT) identities (for historical reasons), consequences of the symmetry of the action and satisfied by the generating functional of vertex functions.

(iii) These identities imply relations between the divergences of correlation functions and thus between the counter-terms that render the theory finite. From these relations we derive the generic form of counter-terms. Such an analysis is based on a perturbative loop expansion.

(iv) Finally, we read off the properties of the renormalized action.

More generally, some non-trivial relations survive when soft symmetry breaking terms are added to the action. We specifically consider the examples of linear symmetry breaking and the important limiting case of spontaneous symmetry breaking.

Finally, in section 12.5 we apply the formalism to the physical example of chiral symmetry breaking in low-energy effective models of hadrons.

12.1 Algebraic preliminaries

Before beginning the discussion, we describe our notation and conventions for group and Lie algebras. Quite generally, we use boldface for vectors or matrices when we want to emphasize the vector or matrix character without listing arguments, for example, $\boldsymbol{\phi}$ represents the vector of components ϕ_i.

12.1.1 Conventions and notation

We consider continuous symmetries corresponding to various Lie groups and algebras. In this context, we adopt the following set of conventions except if explicitly stated otherwise: for continuous symmetries we only consider compact Lie groups. Fields $\boldsymbol{\phi}$, from the group point of view, will be N-component vectors transforming linearly under an orthogonal or unitary representation $\mathcal{R}(G)$ of a Lie group G. Since unitary representations can always be rewritten as orthogonal representations, in the general discussion we consider only orthogonal representations.

The vector $\boldsymbol{\phi}$ then transforms like

$$\phi_i' = \sum_j R_{ij}(\mathbf{g})\phi_j \,, \quad \mathbf{g} \in G \,, \ \mathbf{R}\mathbf{R}^T = \mathbf{1} \,. \tag{12.1}$$

To the group corresponds a Lie algebra $\mathcal{L}(G)$ whose generators can be represented by $N \times N$ real antisymmetric matrices $\mathbf{t}^\alpha$. The trace of two antisymmetric matrices defines a scalar product. We can use it to normalize the matrices by

$$\mathrm{tr}\,\mathbf{t}^\alpha \mathbf{t}^\beta = -N\delta_{\alpha\beta} \,. \tag{12.2}$$

With this convention the structure constants $f_{\alpha\beta\gamma}$ of the Lie algebra defined by

$$[\mathbf{t}^\alpha, \mathbf{t}^\beta] = \sum_\gamma f_{\alpha\beta\gamma} \mathbf{t}^\gamma, \tag{12.3}$$

are completely antisymmetric in the three indices.

The basis of the Lie algebra is fixed up to an orthogonal transformation. In the special case of unitary groups, we also sometimes represent the generators by hermitian or anti-hermitian matrices (this is a matter of convenience) and then normalize them by (in the hermitian case),

$$\operatorname{tr} \mathbf{t}^{\alpha}\mathbf{t}^{\beta} = N\delta_{\alpha\beta}\,.$$

As a consequence, as in the orthogonal case, the structure constants defined by

$$[\mathbf{t}^{\alpha}, \mathbf{t}^{\beta}] = i\sum_{\gamma} f_{\alpha\beta\gamma}\mathbf{t}^{\gamma}$$

and, thus,

$$f_{\alpha\beta\gamma} = \frac{2}{N}\operatorname{Im}\operatorname{tr}\left(\mathbf{t}^{\alpha}\mathbf{t}^{\beta}\mathbf{t}^{\gamma}\right),$$

are also completely antisymmetric.

To a group element close to the identity $\mathbf{g} = \mathbf{1} + \sum_\alpha \omega_\alpha \mathbf{t}_\alpha + O|\boldsymbol{\omega}|^2$ corresponds a variation $\delta\phi_i = \phi'_i - \phi_i$ of the vector ϕ_i of the form

$$\delta\phi_i = \sum_{\alpha,j} t^\alpha_{ij} \phi_j \omega_\alpha \,. \tag{12.4}$$

12.1.2 Examples

For illustration purpose, describe two groups relevant to this lecture.

The $SU(2)$ and $SO(3)$ groups. To the group $SU(2)$ of 2×2 unitary matrices with determinant 1 corresponds the Lie algebra generated by the hermitian Pauli σ matrices, which satisfy (see also sections A11.1.4, A11.2)

$$\mathrm{tr}\, \sigma_i \sigma_j = 2\delta_{ij} \,, \quad [\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk} \sigma_k \,,$$

where ϵ_{ijk} is the completely antisymmetric symbol with $\epsilon_{123} = 1$.

The adjoint representation of $SU(2)$ is the group $SO(3)$ of 3×3 orthogonal matrices of determinant 1 (rotations in three dimensions). Indeed,

$$\sum_{i,j} R_{ij} x_j \sigma_i = \Lambda(\mathbf{x} \cdot \boldsymbol{\sigma}) \Lambda^\dagger \text{ with } \mathbf{x} \in \mathbb{R}^3,\ \Lambda \in SU(2),\ \mathbf{R} \in SO(3).$$

This implies the relation (see also equation ($A11.21$))

$$R_{ij} = \tfrac{1}{2} \operatorname{tr}[\sigma_i \Lambda \sigma_j \Lambda^\dagger].$$

A 2×2 unitary matrix can be parametrized as

$$\Lambda = \cos(\theta/2) + i \sin(\theta/2)\, \mathbf{n} \cdot \boldsymbol{\sigma}\,,\ \ \mathbf{n}^2 = 1\,. \tag{12.5}$$

Then,

$$R_{ij} = \cos\theta\, \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \sum_k \epsilon_{ijk} n_k\,.$$

This explicit expression shows that indeed $\mathbf{R}$ belongs to $SO(3)$ and proves that $SU(2)$ and $SO(3)$ have the same Lie algebra. In the adjoint representation, the generators are antisymmetric real matrices.

The groups $SU(2)\times SU(2)$ and $SO(4)$. These groups, euclidean continuations of relativistic groups, have already been discussed in appendix A11. Since chiral symmetry plays a role in this lecture, we only recall here the relation between the two groups.

A real four-vector $\mathbf{x}\equiv(x_1,x_2,x_3,x_4)$ can be represented by a 2×2 matrix, which expressed in terms of Pauli matrices, has the form

$$\mathbf{M}(\mathbf{x})=x_4\mathbf{1}+i\sum_{i=1}^{3}x_i\sigma_i \;\Rightarrow\; \mathbf{M}\mathbf{M}^\dagger=\mathbf{x}^2\mathbf{1}\,.$$

The matrix $\mathbf{M}$ is thus proportional to a unitary matrix.

We consider now the linear action of the group $SU(2)\times SU(2)$, with elements $(\mathbf{U}_1,\mathbf{U}_2)\in SU(2)$, on the vector $\mathbf{x}$ defined by

$$\mathbf{M}'(\mathbf{x})=\mathbf{U}_1\mathbf{M}(\mathbf{x})\mathbf{U}_2^\dagger.$$

The matrix $\mathbf{M}'(\mathbf{x})$ satisfies

$$\mathbf{M}'(\mathbf{x})\mathbf{M}'(\mathbf{x})^\dagger=\mathbf{U}_1\mathbf{M}(\mathbf{x})\mathbf{M}^\dagger(\mathbf{x})\mathbf{U}_1^\dagger=\mathbf{x}^2\mathbf{1}\,.$$

The four-dimensional real representation of the group $SU(2) \times SU(2)$ preserves the length of vectors and thus is a subgroup of $O(4)$, which we have shown in appendix A11 is the group $SO(4)$ of rotations in four dimensions.

12.1.3 Lie algebra and differential operators

A few algebraic remarks concerning the representation of the Lie algebra in terms of differential operators are useful, in particular, for the discussion of the renormalization of symmetries in complicated situations.

The variation of a differentiable function $\mathcal{S}(\phi)$ under an infinitesimal transformation (12.4) can be written as

$$\delta \mathcal{S}(\phi) = \sum_{\alpha,i,j} t^{\alpha}_{ij} \phi_j \frac{\partial \mathcal{S}}{\partial \phi_i} \omega_\alpha \,. \tag{12.6}$$

We introduce the differential operators

$$\mathcal{D}_\alpha = \sum_{i,j} t^{\alpha}_{ij} \phi_j \frac{\partial}{\partial \phi_i} \,.$$

The variation (12.6) can then be rewritten in a compact form as

$$\delta\mathcal{S}(\phi) = \sum_\alpha \omega_\alpha \mathcal{D}_\alpha \mathcal{S}\,.$$

In particular, an invariant function $\mathcal{S}(\phi)$ satisfies

$$\mathcal{D}_\alpha \mathcal{S} = 0\,. \tag{12.7}$$

The differential operators $\mathcal{D}_\alpha$ are generators of the Lie algebra of the group G realized as differential operators acting on functions of ϕ_i. The expected commutation relations

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = \sum_\gamma f_{\alpha\beta\gamma} \mathcal{D}_\gamma\,, \tag{12.8}$$

can be verified by direct calculation, using the commutation relations (12.3) of the generators $\mathbf{t}^\alpha$.

We shall mainly be concerned with situations in which $\boldsymbol{\phi}$ is a field depending on a space variable x, and $\mathcal{S}$ is the action, functional of $\boldsymbol{\phi}$. The operator $\mathcal{D}_\alpha$ then has the typical form

$$\mathcal{D}_\alpha = \int \mathrm{d}^d x \sum_{i,j} t^\alpha_{ij} \phi_j(x) \frac{\delta}{\delta \phi_i(x)}, \tag{12.9}$$

but the analysis is the same.

12.2 Linear global symmetries

Definition. We call global symmetry a symmetry which corresponds to a transformation of the fields whose parameters are space-independent. More precisely, let $\phi_i(x)$ be a set of fields transforming linearly under an orthogonal representation $\mathcal{R}(G)$ of a compact Lie group G:

$$\phi_i(x) \mapsto \phi'_i(x) = \sum_j R_{ij}(\mathbf{g}) \phi_j(x). \tag{12.10}$$

The transformation (12.10) is global if the group element $\mathbf{g}$ does not depend on the space variable x. Sometimes the expression rigid symmetry is also used to avoid confusions with 'global' in the sense of global topological properties of the symmetry group. In what follows we explore the consequences of invariance only under infinitesimal group transformations.

Infinitesimal group transformations. In the notation of section 12.1.1, we write the variation $\delta\phi$ of ϕ under transformation (12.10) as

$$\delta\phi_i(x) = \sum_{j,\alpha} t^{\alpha}_{ij}\phi_j(x)\omega_\alpha\,, \tag{12.11}$$

in which ω_α are the space-independent parameters of the transformation.

A classical action $\mathcal{S}(\phi)$ invariant under such a transformation then satisfies

$$\mathcal{D}_\alpha\mathcal{S}(\phi) \equiv \int \mathrm{d}^d x \sum_{i,j} t^{\alpha}_{ij}\phi_j(x)\frac{\delta\mathcal{S}}{\delta\phi_i(x)} = 0\,. \tag{12.12}$$

12.2.1 Regularization

In the case of linearly realized global symmetries, it is always possible to find a regularization which preserves the symmetry of the action. For scalar boson fields, one can use dimensional, lattice or momentum cut-off regularizations.

In the latter case, one modifies the propagator by adding to the tree action $\mathcal{S}(\phi)$ quadratic invariant terms involving higher order derivatives:

$$\boldsymbol{\phi}(x)\cdot(-\nabla_x^2+m^2)\boldsymbol{\phi}(x) \mapsto \boldsymbol{\phi}(x)\cdot(-\nabla_x^2+m^2)\prod_{r=1}^{r_{\max}}(1-\nabla_x^2/M_r^2)\boldsymbol{\phi}(x), \quad (12.13)$$

in which the masses $M_r > 0$ scale with the cut-off Λ. By choosing $r_{\max}$ large enough, it is always possible to render the theory finite. The regularization terms are obviously symmetric since they are invariant under arbitrary orthogonal transformations.

In four dimensions, in the case of massless chiral fermions, if the transformation involves the matrix γ_5, like

$$\delta\psi_i(x) = \gamma_5 \sum_{\alpha,j} t^{\alpha}_{ij}\psi_j(x)\omega_\alpha\,,$$

the substitution

$$\bar{\psi}(x)\not{\partial}\psi(x) \mapsto \bar{\psi}(x)\not{\partial}\prod_{r=1}^{r_{\max}}(1-\nabla_x^2/M_r^2)\psi(x),$$

preserves chiral symmetry.

12.2.2 WT identities: the example of scalar field theories

We consider the generating functional of correlation functions $\mathcal{Z}(J)$ corresponding to the symmetric action $\mathcal{S}(\phi)$ (equation(12.12)):

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi]\exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}x\,\mathbf{J}(x)\cdot\boldsymbol{\phi}(x)\right]. \qquad (12.14)$$

To derive the consequences of equation (12.12) for $\mathcal{Z}(J)$, we perform an infinitesimal change of variables of the form of a transformation (12.11), setting

$$\phi_i(x) = \phi'_i(x) + \sum_{j,\alpha} t^{\alpha}_{ij}\phi'_j(x)\omega_\alpha \,, \tag{12.15}$$

in the field integral (12.14). As a consequence of the symmetry as expressed by equation (12.12), the action $\mathcal{S}(\phi)$ and, therefore, the regularized action $\mathcal{S}_\Lambda(\phi)$ are left invariant under the transformation (12.15).

The measure of integration $[\mathrm{d}\phi_i]$ in the field integral (12.14) is the flat euclidean measure, which is invariant under all orthogonal transformation.

The only variation comes from the non-symmetric source term. This implies

$$0 = \delta\mathcal{Z}(J) = \int [\mathrm{d}\phi'] \, \delta \, [\text{source term}] \exp\left[-\mathcal{S}(\phi') + \int \mathrm{d}x \sum_i J_i(x)\phi'_i(x)\right].$$

The variation of the source term is

$$\delta\,[\text{source term}] = \int \mathrm{d}x \sum_{i,j,\alpha} J_i(x) t^{\alpha}_{ij} \phi'_j(x) \omega_\alpha \,.$$

This leads to the equation

$$0 = \sum_{\alpha} \omega_\alpha \int [\mathrm{d}\phi] \int \mathrm{d}x \sum_{i,j} J_i(x) t^{\alpha}_{ij} \phi_j(x) \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}y\, \mathbf{J}(y)\cdot\boldsymbol{\phi}(y)\right]. \tag{12.16}$$

We have now renamed ϕ'_i, ϕ_i since ϕ' is a dummy integration variable.

Equation (12.16), being valid for any set of parameters ω_α, can be rewritten for each component α. Finally, we use the identity

$$\begin{aligned}
&\int [\mathrm{d}\phi]\, \phi_i(x) \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}y\, \mathbf{J}(y)\cdot\boldsymbol{\phi}(y)\right] \\
&\quad = \frac{\delta}{\delta J_i(x)} \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}y\, \mathbf{J}(y)\cdot\boldsymbol{\phi}(y)\right].
\end{aligned}$$

It allows us rewriting equation (12.16) as an equation for the functional $\mathcal{Z}(J)$:

$$\int \mathrm{d}x \sum_{i,j} t_{ij}^{\alpha} J_i(x) \frac{\delta \mathcal{Z}(J)}{\delta J_j(x)} \equiv -\mathcal{D}_{\alpha} \mathcal{Z}(J) = 0\,, \tag{12.17}$$

with the definition (12.9) and using $t_{ij}^{\alpha} = -t_{ji}^{\alpha}$. Equation (12.17) immediately implies an identical equation for the generating functional $\mathcal{W}(J) = \ln \mathcal{Z}(J)$ of connected correlation functions:

$$\mathcal{D}_{\alpha} \mathcal{W}(J) = 0\,. \tag{12.18}$$

Expanding equation (12.18) in a power series of the source $J(x)$, one obtains identities relating various connected correlation functions that characterize the physical implications of the symmetry of the action.

Remark. More general identities satisfied by the generating functional $\mathcal{Z}(J)$, like the quantum equation of motion, can also be obtained by performing infinitesimal changes of variables in the field integral.

Vertex functions. For renormalization purpose, it is necessary to also derive an equation for the generating functional $\Gamma(\varphi)$ of vertex functions, which is given by the Legendre transformation

$$\Gamma(\varphi) + \mathcal{W}(J) = \int \mathrm{d}x\, \mathbf{J}(x) \cdot \boldsymbol{\varphi}(x)$$

with

$$\varphi_i(x) = \frac{\delta \mathcal{W}}{\delta J_i(x)}.$$

Equation (12.18) expressed in terms of φ and Γ then becomes

$$\mathcal{D}_\alpha \Gamma(\varphi) = 0\,. \tag{12.19}$$

The equation implies that the regularized functional $\Gamma(\varphi)$ is invariant under the transformation (12.11) and, thus, symmetric. Equation (12.19), expanded in powers of φ, yields WT identities for vertex functions.

12.2.3 Renormalization

We assume now that a field theory is renormalizable by power counting. The issue then is to find out whether it can be renormalized in a way that preserves the symmetry of the initial action.

To investigate this question, we perform a loop expansion,

$$\Gamma(\varphi) = \sum_{\ell=0}^{\infty} \Gamma_\ell(\varphi) g^\ell,$$

where the parameter g is a coupling constant playing the formal role of $\hbar$ and introduced to organize the loop expansion. At leading order $\Gamma_0(\varphi) = \mathcal{S}(\varphi)$, where $\mathcal{S}(\varphi)$ is the regularized tree approximation to the renormalized action.

Since equation (12.19) is linear in $\Gamma(\varphi)$ and independent of g, all functionals $\Gamma_\ell(\varphi)$ also satisfy equation (12.19). The regularized one-loop functional $\Gamma_1(\varphi)$ thus satisfies

$$\mathcal{D}_\alpha \Gamma_1(\varphi) = 0 \,. \tag{12.20}$$

We then perform an asymptotic expansion of $\Gamma_1(\varphi)$ in terms of the regularizing parameter. For example, for the cut-off regularization, the divergent contributions will involve powers and logarithms of the cut-off, in dimensional regularization poles in the deviation from the initial dimension. Because equation (12.20) is valid for any value of the regularizing parameter, it is valid for each term in the expansion and thus for the sum $\Gamma_1^{\text{div.}}(\varphi)$ of the divergent contributions. Thus,

$$\mathcal{D}_\alpha \Gamma_1^{\text{div.}}(\varphi) = 0\,. \tag{12.21}$$

General renormalization theory tells us that $\Gamma_1^{\text{div.}}(\varphi)$ is a general local functional of the fields restricted only by power counting; equation (12.21) tells us in addition that it is symmetric. Adding $-\Gamma_1^{\text{div.}}(\varphi)$ (the one-loop counterterm) to the action renders the theory one-loop finite (an example of minimal subtraction).

The one-loop renormalized action is still symmetric and, therefore, the modified two-loop functional $\Gamma_2(\varphi)$ still satisfies equation (12.19). After one-loop renormalization $\Gamma_2(\varphi)$ has only local divergences. $\Gamma_2^{\mathrm{div.}}(\varphi)$ again defined by minimal subtraction satisfies equation (12.21) and all arguments can be repeated. It is clear that the arguments can be generalized to all orders.

We conclude that the renormalized action $\mathcal{S}_{\mathrm{r}}$, sum of the initial action and all counter-terms, is the most general local functional of the field ϕ_i compatible with power counting and symmetry.

A reader familiar with perturbative calculations will realize that this is a sophisticated derivation of a straightforward result. However, since the same strategy, suitably adapted, allows discussing more general situations, it has seemed useful to explain it first in a very simple example.

Finally, we have renormalized here the field theory using a minimal subtraction scheme. Additional finite renormalizations which are consistent with the symmetry can still be performed.

12.3 Linear symmetry breaking

For some applications (see for example section 12.5) it is useful to consider the following situation: the action $\mathcal{S}(\boldsymbol{\phi})$ is the sum of a G-symmetric part $\mathcal{S}_{\text{sym.}}(\boldsymbol{\phi})$, that is, invariant under the transformation (12.10), and a term linear in the fields breaking the symmetry :

$$\mathcal{S}(\boldsymbol{\phi}) = \mathcal{S}_{\text{sym}}(\boldsymbol{\phi}) - \int \mathbf{c} \cdot \boldsymbol{\phi}(x) \mathrm{d}x \,, \tag{12.22}$$

in which $\mathbf{c}$ is a constant vector.

An example of such a situation is provided by the action

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}x \left[\tfrac{1}{2} \left(\nabla_x \boldsymbol{\phi}(x) \right)^2 + \tfrac{1}{2} u \boldsymbol{\phi}^2(x) + \tfrac{1}{4!} g \left(\boldsymbol{\phi}^2(x) \right)^2 - \mathbf{c} \cdot \boldsymbol{\phi}(x) \right], \tag{12.23}$$

in which $\boldsymbol{\phi}(x)$ is an N-component vector and g, u are constants. The action $\mathcal{S}(\boldsymbol{\phi})$ is the sum of a $G \equiv O(N)$ invariant part and a linear symmetry breaking term.

An important role is played by the subgroup H of the group G that leaves the vector $\mathbf{c}$ invariant (sometimes called the little group). Since the complete action is then invariant under H, one already knows that the field theory can be renormalized while preserving the H-symmetry. In the preceding $O(N)$ example, the remaining symmetry corresponds to the subgroup $O(N-1)$.

Loop expansion. The loop expansion corresponding to action (12.23) is obtained by the following method: at leading order, one looks for a minimum of the classical action, which corresponds to a constant field $\mathbf{v}_0$ satisfying

$$\left.\frac{\delta\mathcal{S}}{\delta\phi_i(x)}\right|_{\boldsymbol{\phi}(x)=\mathbf{v}_0} = \left.\frac{\delta\mathcal{S}_{\mathrm{sym.}}}{\delta\phi_i(x)}\right|_{\boldsymbol{\phi}(x)=\mathbf{v}_0} - c_i = 0$$

with the condition

$$\left.\frac{\delta^2\mathcal{S}_{\mathrm{sym.}}}{\delta\phi_i(x)\delta\phi_j(y)}\right|_{\boldsymbol{\phi}(x)=\mathbf{v}_0} \geq 0\,,$$

in the kernel sense.

In the example (12.23), $\mathbf{v}_0$ satisfies the equation

$$\left(u + \frac{g}{6}\mathbf{v}_0^2\right)\mathbf{v}_0 = \mathbf{c}\,. \tag{12.24}$$

If the action has several minima, one is in general instructed to choose the absolute minimum of the action, though this is irrelevant from the point of view of formal perturbation theory. The quantity $\mathbf{v}_0$ is the expectation value (vacuum expectation value in the particle physics framework) of the field $\boldsymbol{\phi}$ in the tree approximation.

One then translates the field $\boldsymbol{\phi} \mapsto \boldsymbol{\chi}$, setting

$$\boldsymbol{\phi}(x) = \mathbf{v}_0 + \boldsymbol{\chi}(x).$$

After translation, the action no longer contains a linear term and the perturbative calculation proceeds in the standard manner.

However, the example (12.23) shows that after translation the mass term is no longer symmetric and a non-symmetric χ^3 interaction has been generated. Correlation functions will no longer be symmetric and the form of the UV divergences from the point of view of the symmetry is *a priori* unknown. It is thus important to understand whether the structure of the renormalized action reflects in some way the structure of the action (12.22).

The answer here follows from a simple argument. With obvious notation, we have

$$\mathcal{Z}(\mathbf{J}) = \mathcal{Z}_{\text{sym.}}(\mathbf{J} + \mathbf{c})$$

and, thus,

$$\mathcal{W}(\mathbf{J}) = \mathcal{W}_{\text{sym.}}(\mathbf{J} + \mathbf{c}).$$

Equation (12.18) then implies

$$\int \mathrm{d}x \sum_{i,j} \left[J_i(x) + c_i\right] t^{\alpha}_{ij} \frac{\delta \mathcal{W}(\mathbf{J})}{\delta J_j(x)} = 0\,.$$

Expanding in powers of $J_i(x)$, one obtains a set of relations (WT identities) between connected correlation functions which can be most conveniently expressed in the momentum representation:

$$\begin{aligned}&\sum_{i,j} c_i t^\alpha_{ij} \tilde{W}^{(n+1)}_{jk_1\ldots k_n}(0, p_1, \ldots, p_n)\\ &\quad + \sum_{r=1}^{n} \sum_{j} t^\alpha_{k_r j} \tilde{W}^{(n)}_{k_1\ldots k_{r-1} j k_{r+1}\ldots k_n}(p_1, \ldots, p_n) = 0\,. \end{aligned} \tag{12.25}$$

The 1PI functional Γ is given by the Legendre transformation

$$\begin{aligned}\Gamma(\boldsymbol{\varphi}) + \mathcal{W}(\mathbf{J}) &= \int \mathrm{d}x\, \mathbf{J}(x) \cdot \boldsymbol{\varphi}(x),\\ \varphi_i(x) = \frac{\delta \mathcal{W}}{\delta J_i(x)} &= \frac{\delta \mathcal{W}_{\text{sym.}}(\mathbf{J} + \mathbf{c})}{\delta J_i(x)}.\end{aligned} \tag{12.26}$$

In the symmetric situation these relations read instead

$$\begin{aligned}
\Gamma_{\text{sym.}}(\boldsymbol{\xi}) + \mathcal{W}_{\text{sym.}}(\mathbf{J}) &= \int \mathrm{d}x\, \mathbf{J}(x)\cdot\boldsymbol{\xi}(x), \\
\xi_i(x) &= \frac{\delta \mathcal{W}_{\text{sym.}}(\mathbf{J})}{\delta J_i(x)}\,.
\end{aligned} \tag{12.27}$$

Replacing $\mathbf{J}(x)$ by $\mathbf{J}(x)+\mathbf{c}$ in the relations (12.27), one obtains

$$\begin{aligned}
\Gamma_{\text{sym.}}(\boldsymbol{\varphi}) + \mathcal{W}_{\text{sym.}}(\mathbf{J}+\mathbf{c}) &= \int \mathrm{d}x\, \left(\mathbf{J}(x)+\mathbf{c}\right)\cdot\boldsymbol{\varphi}(x), \\
\varphi_i(x) &= \frac{\delta \mathcal{W}_{\text{sym.}}(\mathbf{J}+\mathbf{c})}{\delta J_i(x)}
\end{aligned} \tag{12.28}$$

and, therefore, comparing (12.26) with (12.28),

$$\Gamma(\boldsymbol{\varphi}) = \Gamma_{\text{sym.}}(\boldsymbol{\varphi}) - \int \mathrm{d}x\, \mathbf{c}\cdot\boldsymbol{\varphi}(x).$$

This identity proves that the divergences of both functionals $\Gamma(\boldsymbol{\varphi})$ and $\Gamma_{\text{sym.}}(\boldsymbol{\varphi})$ are identical. Therefore, if one replaces the regularized symmetric action by the renormalized symmetric action, the theory becomes finite for any vector $\mathbf{c}$. This is casually expressed by stating that the linear breaking term is not renormalized.

To generate vertex functions, one has to translate $\boldsymbol{\varphi}$ by the $\boldsymbol{\phi}$-field expectation value setting

$$\boldsymbol{\phi}(x) = \mathbf{v} + \boldsymbol{\chi}(x) \tag{12.29}$$

with

$$\left.\frac{\delta\Gamma}{\delta\varphi_i(x)}\right|_{\boldsymbol{\varphi}(x)=\mathbf{v}} = 0 \ \Leftrightarrow \ \left.\frac{\delta\Gamma_{\text{sym.}}}{\delta\varphi_i(x)}\right|_{\boldsymbol{\varphi}(x)=\mathbf{v}} = c_i\,, \tag{12.30}$$

and $\delta^2\Gamma(\mathbf{v})/\delta\phi_i\delta\phi_j \geq 0$.

Vertex functions are then the coefficients of the expansion of $\Gamma(\boldsymbol{\varphi})$ in powers of χ. In the tree approximation, one recovers $\mathbf{v} = \mathbf{v}_0$.

The WT identities satisfied by $\Gamma(\boldsymbol{\varphi})$ can be inferred from the identity (12.19) for $\Gamma_{\text{sym.}}$:

$$\int \mathrm{d}x \sum_{i,j} t^{\alpha}_{ij} \left[\frac{\delta\Gamma}{\delta\varphi_i(x)} + c_i\right] \varphi_j(x) = 0\,,$$

which after the translation (12.29) becomes

$$\int \mathrm{d}x \sum_{i,j} t^{\alpha}_{ij} \left[\frac{\delta\Gamma}{\delta\chi_i}(\boldsymbol{\chi}+\mathbf{v}) + c_i\right] (\chi_j + v_j) = 0\,. \tag{12.31}$$

Application. We now show that this identity leads to some non-trivial relations between the vertex functions. Setting $\boldsymbol{\chi}=0$, we obtain

$$\sum_{i,j} t^{\alpha}_{ij} c_i v_j = 0\,, \tag{12.32}$$

which implies that the breaking vector $\mathbf{c}$ and the expectation value $\mathbf{v}$ are left invariant by the subgroup H of G. In the example of the $SO(N)$ symmetry, equation (12.32) implies that the vector $\mathbf{v}$ is proportional to the vector $\mathbf{c}$.

Differentiating once with respect to $\chi_k(y)$ and then setting $\chi = 0$, we relate the one- and two-point functions:

$$\int \mathrm{d}x \sum_{i,j} v_j t^\alpha_{ij} \Gamma^{(2)}_{ik}(x-y) + \sum_i t^\alpha_{ik} c_i = 0 \tag{12.33}$$

with

$$\Gamma^{(2)}_{ij}(x-y) = \left. \frac{\delta^2 \Gamma(\chi + v)}{\delta\chi_i(x)\delta\chi_j(y)} \right|_{\chi=0} .$$

In terms of the Fourier transform $\tilde{\Gamma}^{(2)}_{ij}(p)$ of the two-point function,

$$\Gamma^{(2)}_{ij}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, \mathrm{e}^{-ip(x-y)} \, \tilde{\Gamma}^{(2)}_{ij}(p),$$

equation (12.33) becomes

$$\sum_{i,j} v_j t^\alpha_{ji} \tilde{\Gamma}^{(2)}_{ik}(0) + \sum_i t^\alpha_{ki} c_i = 0 \, . \tag{12.34}$$

This equation determines the geometrical structure of the zero momentum propagator in the presence of the linear symmetry breaking term.

In the $O(N)$ example, the identity (12.34) yields the value of the propagator $\tilde{\Gamma}^{(2)}_{\mathrm{T}}$ of the components of the field orthogonal to the vector $\mathbf{c}$, at zero momentum:

$$\tilde{\Gamma}^{(2)}_{\mathrm{T}}(0) = c/v\,.$$

Equation (12.34) is the last equation that involves $\mathbf{c}$ explicitly. The terms of higher degree in χ are functions only of the expectation value $\mathbf{v}$. By identifying the coefficient of degree $(n+1)$ in χ, one obtains a relation between the Fourier transform of the $(n+1)$-point function $\tilde{\Gamma}^{(n+1)}$ with one momentum set to zero and the n-point function $\tilde{\Gamma}^{(n)}$:

$$\sum_{j,k} v_j t^{\alpha}_{jk} \tilde{\Gamma}^{(n+1)}_{k i_1 \ldots i_n}(0, p_1, \ldots, p_n) + \sum_{r=1}^{n} \sum_{i} t^{\alpha}_{i_r k} \tilde{\Gamma}^{(n)}_{i_1 \ldots i_{r-1} k i_{r+1} \ldots i_n}(p_1, \ldots, p_n) = 0\,. \tag{12.35}$$

For example, the equation for $n = 2$ reads

$$\sum_{i,j} v_j t^{\alpha}_{ji} \tilde{\Gamma}^{(3)}_{ikl}(0, p, -p) + \sum_i \left(t^{\alpha}_{li} \tilde{\Gamma}^{(2)}_{ik}(p) + t^{\alpha}_{ki} \tilde{\Gamma}^{(2)}_{il}(p) \right) = 0 \,.$$

If we choose to renormalize by fixing the value of the primitively divergent correlation functions at some given point in momentum space, then the set of WT identities implies relations between the different parameters. Apart from the vector $\mathbf{v}$, the non-symmetric theory depends on the same number of independent parameters as the symmetric theory. In the example of the $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory in four dimensions, it is possible to impose one arbitrary renormalization condition on $\tilde{\Gamma}^{(4)}_{1111}(p_i)$ and two conditions on $\tilde{\Gamma}^{(2)}_{11}(p)$. All others are given by the WT identities (12.35) used for $n = 1$ to 4.

12.4 Spontaneous symmetry breaking

Spontaneous symmetry breaking (SSB) is a possible limit of linear symmetry breaking when the breaking parameter goes to zero. In this limit, the action becomes symmetric, but depending on the values of other parameters (in our examples the coefficient of $\phi^2(x)$ in the action), the physics may or may not become symmetric.

Many physical models in particle physics are based on the concept of SSB. The reason is that the mechanism of SSB allows devising models with broken symmetries which depend on no more parameters than the symmetric models. The appearance of massless particles (Goldstone modes) is, in general, the most characteristic feature of such models (except for gauge symmetries).

The study of SSB plays also an essential role in critical phenomena.

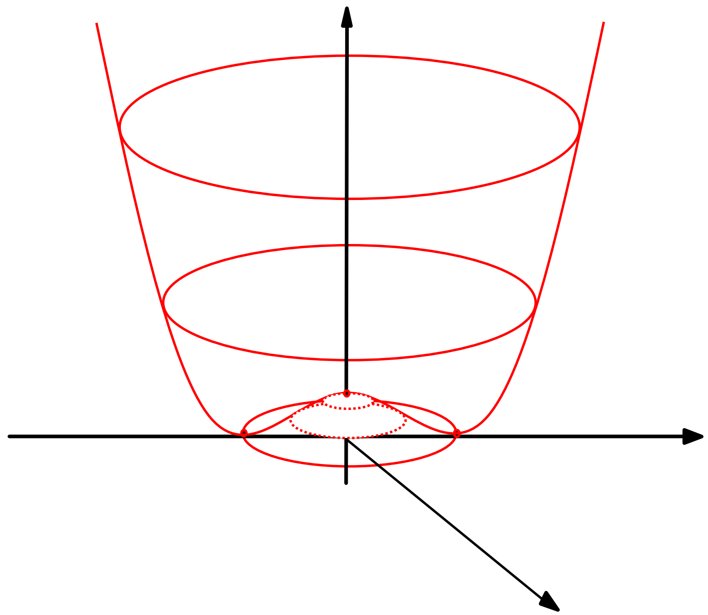

Fig. 12.1 – Spontaneous symmetry breaking: an $O(2)$-symmetric potential.

SSB, in the perturbative framework, is associated with degenerate classical minima (see Fig. 12.1). Each minimum is the starting point of a perturbative expansion. A problem then immediately arises: should one choose only one minimum or sum over the contributions of all minima?

In fact the correct procedure depends on the real physical situation beyond perturbation theory, SSB or restored symmetry.

In the absence of phase transitions, one must sum over all minima. Quantum fluctuations restore the symmetry broken in the classical approximation and the true quantum ground state is unique.

For group invariant correlation functions, all minima give the same contribution and the summation yields only a global normalization factor.

For non-invariant correlation functions a summation over all minima is equivalent to a group average and projects onto invariant functions: the exact correlation functions are invariant and the field has no expectation value.

By contrast, when a phase transition occurs, in the several phase region SSB is related to a breaking of ergodicity and one must choose one specific minimum. Correspondingly, the quantum ground state is degenerate.

Below, we assume that the latter situation is realized.

12.4.1 Classical analysis: the $O(N)$ example

As an example, we consider the $O(N)$ model with the the action (12.23) and discuss the expectation value of the field in the tree or classical approximation. The action density for a constant field $\boldsymbol{\phi}$ is

$$\mathcal{E}(\boldsymbol{\phi}) \equiv \mathcal{S}(\boldsymbol{\phi}) \,/\text{volume} = \tfrac{1}{2}u\boldsymbol{\phi}^2 + \tfrac{1}{4!}g(\boldsymbol{\phi}^2)^2 - \mathbf{c}\cdot\boldsymbol{\phi}\,.$$

As long as $\mathbf{c}$ does not vanish, it is possible to pass continuously from a situation in which the parameter u is positive to a situation in which u is negative without encountering any singularity. For instance, the expectation value $\mathbf{v}$ is, at $\mathbf{c}$ fixed, a regular function of u at $u = 0$. By contrast, if $\mathbf{c}$ vanishes, the expectation value $\mathbf{v}$ vanishes identically for $u > 0$ and takes a non-trivial value for $u < 0$ such that

$$|\mathbf{v}| = \sqrt{-6u/g}\,. \tag{12.36}$$

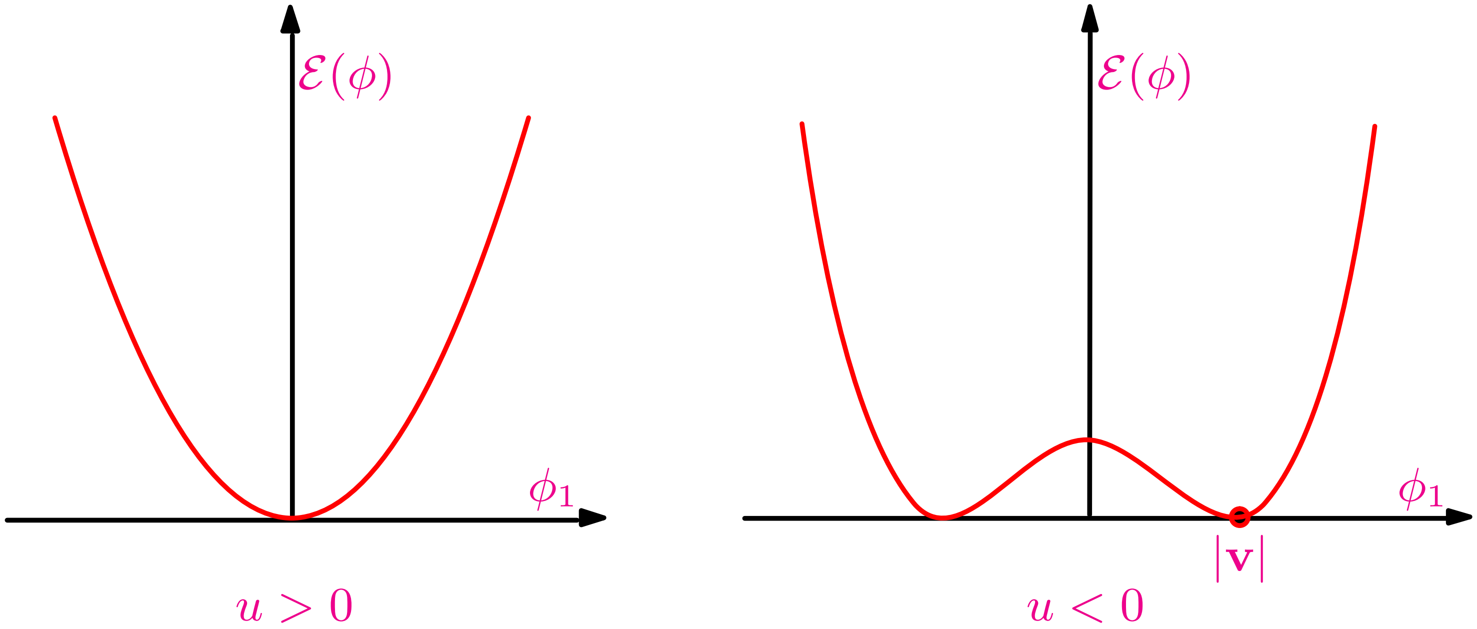

Fig. 12.2 – Section of the ϕ-action density.

This can be easily understood by displaying the action density for both situations (see Fig. 12.2). In the second case, the classical minimum of the action density is degenerate and the minima are located on a sphere with a radius given by equation (12.36).

Starting from a given minimum, it is possible to describe all other minima by acting on the vector $\mathbf{v}$ with the symmetry group , here the $O(N)$ group.

Assuming a situation of SSB, we construct a perturbation theory around one minimum $\mathbf{v}$ that, at leading order, is the field expectation value. We then shift the field:

$$\boldsymbol{\phi}(x) = \mathbf{v} + \boldsymbol{\chi}(x)\,.$$

The $\boldsymbol{\chi}$-field mass matrix is obtained by calculating the second derivatives of the action density at the minimum. Using equation (12.36), one finds

$$\left.\frac{\partial \mathcal{E}}{\partial \phi_i \partial \phi_j}\right|_{\boldsymbol{\phi}=\mathbf{v}} = \left(u + \tfrac{1}{6} g v^2\right) \delta_{ij} + \tfrac{1}{3} g v_i v_j = \tfrac{1}{3} g\, v_i v_j\,.$$

The matrix has $(N-1)$ zero eigenvalues corresponding to eigenvectors orthogonal to $\mathbf{v}$. This is not surprising since the potential $\mathcal{E}(\boldsymbol{\phi})$ is flat along a group orbit.

The physical consequence is that spontaneous breaking of a continuous symmetry implies the existence of Goldstone modes, from the point of view of particle physics massless scalar particles called Goldstone bosons.

12.4.2 General continuous symmetry group

We now examine the situation of a general group G and a symmetric action that has degenerate minima. We denote by $\mathbf{v}$ the location of the minimum chosen to expand perturbation theory, and thus the field expectation value at leading order.

We introduce the subgroup H of G, little group (stabilizer) of the vector $\mathbf{v}$, that is, the subgroup of G that leaves the vector $\mathbf{v}$ invariant. By definition, the p generators of the Lie algebra $\mathcal{L}(H)$ of H satisfy

$$\mathcal{L}(H): \quad 1 \le \alpha \le p \quad \Rightarrow \quad \sum_j t^{\alpha}_{ij} v_j = 0\,.$$

We denote by $\mathcal{L}(G/H)$ the vector space (it is not an algebra!) generated by the complementary set in the Lie algebra $\mathcal{L}(G)$ of G. The set $\mathcal{L}(G/H)$ is characterized by

$$\sum_{\alpha>p}\sum_{j}\omega_\alpha t^\alpha_{ij}v_j=0 \quad\Rightarrow\quad \omega_\alpha=0 \quad \text{for all } \alpha\,.$$

For $\alpha>p$, the vectors $(v^\alpha)_i=\sum_j t^\alpha_{ij}v_j$ are thus linearly independent. We then parametrize the field $\boldsymbol{\phi}$ in the form of a group element acting on a vector:

$$\boldsymbol{\phi}(x)=\exp\left(\sum_{\alpha>p}\mathbf{t}^\alpha\xi^\alpha(x)\right)\left(\mathbf{v}+\boldsymbol{\rho}(x)\right)=\mathbf{v}+\sum_{\alpha}\xi^\alpha(x)\mathbf{t}^\alpha\mathbf{v}+\boldsymbol{\rho}(x)+\cdots\ ,$$

in which $\boldsymbol{\rho}(x)$ has components only in the subspace orthogonal to all vectors $\mathbf{t}^\alpha\mathbf{v}$. In the $O(N)$ example, $\boldsymbol{\rho}$ has only one component along $\mathbf{v}$.

This parametrization is such that the mapping of fields $\{\boldsymbol{\rho}(x),\xi^\alpha(x)\}\mapsto \boldsymbol{\phi}(x)-\mathbf{v}$ can be inverted for small fields.

This property ensures that if the fluctuations of the field $\boldsymbol{\phi}$ around its expectation value are in some sense small, perturbation theory is at least qualitatively sensible.

Inserting the parametrization into the action we note the following: the contributions to the action which are derivative-free depend only on $\boldsymbol{\rho}(x)$ because they are G-invariant. The dependence in the fields $\xi^{\alpha}(x)$ is entirely contained in the terms with derivatives, therefore, these fields are massless.

We conclude that spontaneous breaking of symmetry of a group G to a subgroup H, the group which leaves the field expectation value invariant, yields a number of massless Goldstone bosons equal to the number of generators of G that do not belong to H. This result is valid in the classical approximation. We now generalize it to the full quantum theory.

12.4.3 WT identities and spontaneous symmetry breaking

To connect continuously the two phases, symmetric and with SSB, without encountering any singularity, we start from the situation $u > 0$, $\mathbf{c} = 0$; we give to $\mathbf{c}$ a non-vanishing value, perform the continuation from $u > 0$ to $u < 0$, and again take the vanishing $\mathbf{c}$ limit. We then assume the existence of non-trivial solutions to the equation

$$\left.\frac{\delta\Gamma}{\delta\varphi_i(x)}\right|_{\boldsymbol{\varphi}(x)=\mathbf{v}} = 0\,.$$

Since the WT identities (12.31) hold for any value of the parameters and we have proceeded by analytic continuation, in the $u < 0$, $\mathbf{c} = 0$ limit the equation

$$\int \mathrm{d}x \sum_{i,j} t^{\alpha}_{ij} \frac{\delta\Gamma(\chi + v)}{\delta\chi_i(x)} (\chi_j + v_j) = 0\,,$$

still holds, the direction of v_i being fixed by equation (12.32).

Goldstone modes. One important consequence of WT identities is obtained by taking the $\mathbf{c} = \mathbf{0}$ limit in equation (12.34):

$$\sum_{i,j} v_j t^{\alpha}_{ji} \tilde{\Gamma}^{(2)}_{ik}(0) + \sum_{i} t^{\alpha}_{ki} c_i = 0 \underset{\mathbf{c}=0}{\Rightarrow} \sum_{i,j} v_j t^{\alpha}_{ji} \tilde{\Gamma}^{(2)}_{ik}(0) = 0\,. \tag{12.37}$$

To give an interpretation of the equation, as in the classical analysis we introduce the subgroup H of G, little group (stabilizer) of the vector $\mathbf{v}$. Since for $\alpha > p$, the vectors $(v^{\alpha})_i = \sum_j t^{\alpha}_{ij} v_j$ are linearly independent, equation (12.37) implies that the real symmetric matrix $\tilde{\Gamma}_{ij}(0)$ has as many eigenvectors with eigenvalue zero as there are generators in $\mathcal{L}(G/H)$, confirming the classical analysis. The corresponding components of the field are Nambu–Goldstone modes, massless scalar particles associated with the spontaneous breaking of the G-symmetry.

Discrete symmetries. Discrete symmetries do not lead to WT identities and the preceding analysis does not apply.

However, it can be easily proved that when the initial unrenormalized action is symmetric, the renormalized action remains symmetric.

Correlation functions in the presence of an additional linear symmetry breaking term can be expanded in power series of the breaking parameter. The coefficients are symmetric correlation functions. Therefore, it remains true that the counter-terms which render the symmetric theory finite renormalizes the theory with linear symmetry breaking.

12.5 Approximate chiral symmetry in hadron physics

One of the striking feature of Strong Interactions in low energy Particle Physics is the observation of approximate spontaneously broken $SU(N) \times SU(N)$ chiral symmetries, which manifest themselves, in particular, in the small masses of the pseudoscalar mesons. In particular, the π-meson is especially light, an indication that the explicit breaking of the $SU(2) \times SU(2)$ symmetry is small.

Within the framework of the Standard Model of particle physics, this property is a consequence of the small masses of the **u** and **d** quarks (see for example section 11.8.3) and the vector-like coupling of quarks to gluons. The mass of the **s** quark and thus the explicit breaking of the $SU(3) \times SU(3)$ symmetry are larger as can be seen from the masses of the K and η pseudoscalar mesons.

Note that, according to the power counting analysis, since a fermion mass operator in a renormalizable field theory in four dimensions has dimension 3, the concept of a symmetry broken by fermion mass terms is indeed meaningful.

The search for analytic methods to derive low energy properties of hadrons directly from Quantum ChromoDynamics (QCD) , the fundamental theory of quarks and gluons, has up to now proved elusive.

Most direct results are thus obtained from computer intensive studies of discretized lattice versions of QCD (see lecture 15). For years, progress has been slow but a number of precise results have now been obtained. The most serious numerical difficulties were related to the dynamics of quarks, in particular, from the point of view of chiral properties.

Here instead, we describe how effective low energy theories based on observed hadrons like protons, neutrons, π-mesons... can be constructed.

In such theories the chiral symmetry is explicitly broken by terms linear in some scalar fields, which have the transformation properties of fermion mass terms and which, together with the pseudoscalars, transform under representations of the chiral group. Therefore, we face the situation we have discussed at some length in section 12.3.

12.5.1 The chiral symmetry: general structure

The action for N free massless Dirac fermions in four dimensions reads

$$\mathcal{S}(\psi,\bar\psi)=-\int \mathrm{d}^4x \sum_i \bar\psi_i(x)\not{\partial}\psi_i(x).$$

It has a $U(N)\times U(N)$ chiral symmetry corresponding to the transformations

$$\boldsymbol{\psi}'=\left[\tfrac{1}{2}(\mathbf{1}+\gamma_5)\mathbf{U}_+ +\tfrac{1}{2}(\mathbf{1}-\gamma_5)\mathbf{U}_-\right]\boldsymbol{\psi}\,, \tag{12.38}$$

$$\bar{\boldsymbol{\psi}}'=\bar{\boldsymbol{\psi}}\left[\tfrac{1}{2}(\mathbf{1}+\gamma_5)\mathbf{U}_-^\dagger +\tfrac{1}{2}(\mathbf{1}-\gamma_5)\mathbf{U}_+^\dagger\right], \tag{12.39}$$

where $\mathbf{U}_\pm$ are two $N\times N$ unitary matrices corresponding to the two $U(N)$ groups.

We couple the fermions to spinless bosons forming a $N \times N$ complex matrix $\mathbf{M}(x)$. One verifies that the interaction term,

$$-g \int \mathrm{d}^4 x \sum_{i,j} \bar{\psi}_i \left[\tfrac{1}{2}(\mathbf{1} + \gamma_5) M_{ij} + \tfrac{1}{2}(\mathbf{1} - \gamma_5) M^\dagger_{ij} \right] \psi_j \,,$$

is invariant under the transformations (12.38,12.39) provided the matrix $\mathbf{M}$ transforms like

$$\mathbf{M}' = \mathbf{U}_- \mathbf{M} \mathbf{U}_+^\dagger \,. \tag{12.40}$$

The total action also satisfies reflection hermiticity as defined in section 11.8.1. Under a charge conjugation $\mathcal{C}$, $\mathbf{M}$ transforms like

$$\mathbf{M}_{\mathcal{C}} = \mathbf{M}^T \,. \tag{12.41}$$

It can be made invariant under a space reflection P (section 11.8.2) if $\mathbf{M}$ transforms like

$$\mathbf{M}_P(x) = \mathbf{M}^\dagger(\tilde{x}), \tag{12.42}$$

in which $\tilde{x}$ is obtained from x by changing the sign of one component.

Therefore, the matrix $\mathbf{\Sigma} = (\mathbf{M} + \mathbf{M}^\dagger)/\sqrt{2}$ represents a set of scalar fields and $\mathbf{\Pi} = (\mathbf{M} - \mathbf{M}^\dagger)/\sqrt{2}$ a set of pseudoscalar fields. A possible action for the boson fields symmetric under $U(N) \times U(N)$ transformations is then

$$\mathcal{S}(\mathbf{M}) = \int \mathrm{d}^4x \ \mathrm{tr}\left(\nabla_x \mathbf{M} \nabla_x \mathbf{M}^\dagger + V(\mathbf{M}\mathbf{M}^\dagger)\right),$$

where $V(\varphi)$ is a polynomial of the matrix φ. The addition of a term proportional to $\det \mathbf{M} + \det \mathbf{M}^\dagger$, reduces the symmetry to $SU(N) \times SU(N) \times U(1)$ (the factor $U(1)$ corresponds to the baryonic charge).

The most general symmetry breaking term linear in the boson fields, consistent with the discrete symmetries (12.42) and (12.41), is

$$\mathcal{S}_B(\mathbf{M}) = -\frac{1}{\sqrt{2}} \int \mathrm{d}^4x \ \mathrm{tr}\, \mathbf{C} \left(\mathbf{M} + \mathbf{M}^\dagger\right),$$

in which $\mathbf{C}$ is a hermitian matrix:

$$\mathbf{C} = \mathbf{C}^\dagger .$$

To the transformations (12.38,12.39) and (12.40) correspond two currents (for more details see Appendix A12.1). It is convenient to consider the vector current $\mathbf{V}_\mu(x)$, which is associated with the Lie algebra of the diagonal subgroup $U(N)$ of $U(N)\times U(N)$ ($\mathbf{U}_+ = \mathbf{U}_-$) that conserves parity:

$$-iV_\mu^\alpha(x) = -\bar{\boldsymbol{\psi}} t^\alpha \gamma_\mu \boldsymbol{\psi} + \operatorname{tr} t^\alpha \left\{ \left[\partial_\mu \mathbf{M}^\dagger, \mathbf{M}\right] + \left[\partial_\mu \mathbf{M}, \mathbf{M}^\dagger\right] \right\},$$

and the axial current $\mathbf{A}_\mu(x)$ associated with the complementary set of generators in the Lie algebra, that is, $\mathcal{L}\left(U(N)\times U(N)/U(N)\right)$:

$$-iA_\mu^\alpha(x) = -\bar{\boldsymbol{\psi}} t^\alpha \gamma_5 \gamma_\mu \boldsymbol{\psi} + \operatorname{tr} t^\alpha \left\{ \left[\partial_\mu \mathbf{M}^\dagger, \mathbf{M}\right]_+ + \left[\partial_\mu \mathbf{M}, \mathbf{M}^\dagger\right]_+ \right\}. \quad (12.43)$$

The $+$ index means that the expression between brackets is an anticommutator.

If the matrix $\mathbf{C}$ is proportional to the identity, the chiral symmetry is broken, but the diagonal symmetry remains and the vector current is conserved.

The axial current is conserved only if $\mathbf{C}$ vanishes:

$$\sum_\mu \partial_\mu V_\mu^\alpha(x) = -i\,\mathrm{tr}\left\{[t^\alpha, \mathbf{C}]\,\boldsymbol{\Sigma}\right\},$$

$$\sum_\mu \partial_\mu A_\mu^\alpha(x) = \mathrm{tr}\left\{[t^\alpha, \mathbf{C}]_+\,\boldsymbol{\Pi}\right\}.$$

12.6 An example: the linear σ-model

The case $N = 2$ is of particular interest because the pion mass is especially small and, thus, the explicit breaking of chiral symmetry is small.

Previous analysis leads to a theory with eight real boson fields. However, the group $SU(2)$ (but not the group $U(2)$) has the property that a representation and its complex conjugate are equivalent:

$$\mathbf{U} = \tau_2 \mathbf{U}^* \tau_2 \ , \quad \forall\ \mathbf{U} \in SU(2),$$

in which τ_2 is the usual Pauli matrix (we denote in this section the Pauli matrices by τ_i rather than σ_i, as in appendix A11.2, to eliminate possible confusion with the traditional notation for fields). Therefore, $\mathbf{M}$ and $\tau_2 \mathbf{M}^* \tau_2$ have the same transformation law. The representation can be reduced and the matrix $\mathbf{M}$ parametrized in terms of two fields $\sigma(x)$ and $\boldsymbol{\pi}(x)$ in the form

$$\mathbf{M} = \tau_2 \mathbf{M}^* \tau_2 \equiv \frac{1}{\sqrt{2}} (\sigma + i \boldsymbol{\tau} \cdot \boldsymbol{\pi}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sigma + i\pi_0 & \pi_2 + i\pi_1 \\ -\pi_2 + i\pi_1 & \sigma - i\pi_0 \end{bmatrix}.$$

The group $SU(2) \times SU(2)$ is the covering group of $O(4)$ which is also the symmetry group of the boson sector of the action. A breaking of the $O(4)$ symmetry by a term linear in the boson fields singles out one direction in the 4-dimensional space and, therefore, reduces the $O(4)$ symmetry to a residual $O(3)$ symmetry. We assume without loss of generality that the linear breaking term is proportional to $\sigma(x)$.

The action can then be written as

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^4x \Big\{ & -\bar{\mathcal{N}}(x) \left[\not{\partial} + g(\sigma + i\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}) \right] \mathcal{N}(x) \\ & + \tfrac{1}{2} \left[\left(\nabla_x \sigma(x) \right)^2 + \left(\nabla_x \boldsymbol{\pi}(x) \right)^2 \right] + V\left(\sigma^2(x) + \boldsymbol{\pi}^2(x) \right) - c\sigma(x) \Big\}\end{aligned} \qquad (12.44)$$

with

$$V(\rho) = \tfrac{1}{2} u\rho + \tfrac{1}{4!} \lambda \rho^2 .$$

The fermion doublet $\mathcal{N}(x)$ is identified with the two nucleon fields associated to protons and neutrons.

The action (12.44) has an exact $SU(2) \times U(1)$ symmetry, to which corresponds the conservation of the vector current, and implements the idea of partially conserved axial current (PCAC) for $SU(2)$. In the standard normalization, which differs by a factor 2 from the definition (12.43) (see equations (12.50,12.60)),

$$\sum_{\mu} \partial_\mu \mathbf{A}_\mu(x) = c\boldsymbol{\pi}(x). \tag{12.45}$$

Finally, it follows from equation (12.42) that $\sigma(x)$ is a neutral scalar field and $\boldsymbol{\pi}(x)$ a pseudoscalar field (associated to the pi-meson), π_0 being the neutral component, while the combinations

$$\pi_{\pm} = (\pi_1 \pm i\pi_2)/\sqrt{2}\,,$$

correspond to charged mesons, as charge conjugation shows.

12.6.1 Tree approximation

Boson sector. We discuss the pattern of symmetry breaking in the classical approximation. Furthermore, we consider only the case $N = 2$ because it is the simplest and physically the most relevant. In the absence of fermions we simply have the $(\phi^2)^2$ field theory with $O(4)$ symmetry. Equation (12.24) gives the relation between the expectation value v of the field σ and the symmetry breaking parameter c in the classical approximation:

$$v(u + \lambda v^2/6) = c\,.$$

Setting

$$\sigma(x) = v + s(x),$$

in action (12.44), we read off the masses of the $\boldsymbol{\pi}$ and σ particles at the tree order:

$$m_\pi^2 = u + \lambda v^2/6\,, \quad m_\sigma^2 = u + \lambda v^2/2\,. \tag{12.46}$$

The assumption that accounts for the success of PCAC phenomenology is that the explicit symmetry breaking term is small and one is close to a situation of SSB. For the model (12.44) this means in particular that m_π is small compared to m_σ. With this assumption it is possible to predict some general features of low energy π–π scattering. Introducing the standard invariant variables

$$s = -(p_1 + p_2)^2 \ , \quad t = -(p_1 + p_3)^2 \ , \quad u = -(p_1 + p_4)^2,$$

we can write the connected amputated π-field four-point function at this order as

$$\begin{aligned}\left[W^{(4)}_{ijkl}\right]_{\mathrm{amp}} = {} & \frac{s - m_\pi^2}{v^2} \frac{m_\sigma^2 - m_\pi^2}{m_\sigma^2 - s} \delta_{ij}\delta_{kl} + \frac{t - m_\pi^2}{v^2} \frac{m_\sigma^2 - m_\pi^2}{m_\sigma^2 - t} \delta_{ik}\delta_{jl} \\ & + \frac{u - m_\pi^2}{v^2} \frac{m_\sigma^2 - m_\pi^2}{m_\sigma^2 - u} \delta_{il}\delta_{jk} \, . \end{aligned} \tag{12.47}$$

We have used the relations (12.46) to eliminate m and λ. The physical scattering amplitude is obtained by setting all momenta on the mass shell: $p_i^2 = -m_\pi^2$ and then $s + t + u = 4m_\pi^2$.

The expectation value v is experimentally accessible from the weak π-meson decay as a consequence of relation (12.45) and is denoted traditionally by f_π. Since m_σ is supposed to be large compared to m_π, the expression (12.47) makes quantitative predictions for s, t, u of order m_π^2 or smaller, that is, at low energy. Values corresponding to infinite σ-mass are often quoted. Although the π–π scattering amplitude, of course, cannot be measured directly, indirect methods provide an experimental confirmation of the resulting pattern.

Fermion sector. In the symmetric phase, the mass of the fermion vanishes for $c = 0$. The largest contribution to the fermion mass m_N is thus generated by the Yukawa coupling and the σ expectation value

$$m_N = gv\,.$$

The Yukawa coupling constant g is arbitrary in the model and must be extracted from some experimental information: at this order, the parameter g can be identified with the coupling constant $g_{\pi NN}$ which governs the long range part of the N–N potential due to π exchange. We then have the relation between physical quantities:

$$g_{\pi NN} = m_N / f_\pi \, . \tag{12.48}$$

This is the Goldberger–Treiman relation in the tree approximation. It agrees semi-quantitatively with experiment since

$$g_{\pi NN} = 13.6 \, , \quad \frac{m_N}{f_\pi} \simeq \frac{939.}{93.3} = 10. \, . \tag{12.49}$$

Then all parameters but m_σ are fixed. The low energy π–N scattering amplitude, for example, can be calculated. A definite prediction can be made only for m_σ infinite; it agrees reasonably well with experimental data.

12.6.2 *Beyond leading order approximation*

Since the field theory model is renormalizable, it is possible to calculate loop corrections. Then, several problems arise. First, there is a matter of principle. The $(\phi^2)^2$ field theory, as well as the theory (12.44) with fermions, is IR free in four dimensions and thus physically acceptable only for a limited range of energies or momenta.

More precisely, although the theory is renormalizable in perturbation theory, it is impossible to send the cut-off Λ to infinity: the effective (renormalized) couplings at a mass scale $\mu \ll \Lambda$ are bounded by const.$/ \ln(\Lambda/\mu)$—this is the triviality issue. Therefore, the addition of loop corrections is meaningful only if the momenta and the coupling constants are small enough (in a correlated way as stated above). A Landau 'ghost' is typically a manifestation of this problem. Still the loop corrections may be useful to improve the tree level amplitudes from the point of view of unitarity at low energy.

Second, from the computational point of view several difficulties are encountered.

(i) Loop corrections become large at moderate energies. For example in π–π scattering one encounters the ρ resonance. Then it becomes necessary to apply a summation method to the perturbation series. Calculations have been performed using the summation method of Padé approximants.

(ii) Since the σ mass is larger than $2m_\pi$, the σ particle is unstable (it is a resonance) because it can decay into two pions. In the exact π–π scattering amplitude, the resonance leads to singularities in the second sheet of the unitarity cut in the complex s-plane.

However, at any finite order in perturbation theory, the singularities associated with the σ-particle are on the real axis since the width of the particle is a non-perturbative effect. Fits of experimental data seem to impose a rather small σ mass. Therefore, loop corrections are affected by unphysical singularities even at rather low energy.

This problem of the perturbative treatment of fields corresponding to unstable particles remains to a large extent unsolved. One possible idea is to make a systematic large m_σ expansion, but the validity of the expansion is then limited to energies smaller than $4m_\sigma^2$, that is, rather low energies.

(*iii*) Finally, perturbative corrections to the nucleon mass are large, and this also adversely affects the position of singularities in scattering amplitudes involving fermions.

Therefore, although much effort has gone into the study of the model (12.44), only limited results have been obtained beyond the simple predictions which rely on the geometry of the model and are, therefore, mostly contained in WT identities as we explain below.

12.6.3 The RG viewpoint

From the RG viewpoint, in four dimensions, the coefficient of $\sigma(x)$, the linear symmetry breaking term in the action (12.44), has dimension three and thus scales as Λ^3, up to logarithmic corrections, where Λ is the cut-off. Therefore, it is of order 1 at the physical scale only it is extremely small at the cut-off scale. By contrast with spontaneously symmetry breaking, this implies another fine-tuning, in addition to the fine-tuning of the coefficient of $\sigma^2(x)$.

Therefore, one may wonder why the preceding analysis is relevant for the linear σ-model. There seems to be two reasons. First, the ratio between the QCD scale and the π mass is not very large. Moreover, the masses of the $\mathbf{u}$ and $\mathbf{d}$ quarks, responsible for the $SU(2) \times SU(2)$ symmetry breaking, are extremely small, especially compared to the more natural top quark mass, another mystery in the Standard Model of fundamental interactions.

12.7 Ward–Takahashi identities

We have described the difficulties one encounters when one tries to derive consequences from a phenomenological chiral action. However, some relations are valid beyond perturbation theory: the WT identities which are direct consequences of the broken symmetry.

Unfortunately, equation (12.25) shows that the WT identities always involve correlation functions with one π-field at zero momentum. Therefore, they would lead to relations between observables only if the π-meson were massless, that is, if the symmetry were spontaneously broken.

In reality, it is necessary to extrapolate from zero momentum to the pion mass-shell. This extrapolation is model-dependent, and the results are only reliable if the predictions at zero pion mass are already in qualitative agreement with experiment.

12.7.1 Boson sector

First, we discuss again the boson sector. The interesting part of the WT identities corresponds to the transformations

$$\delta\boldsymbol{\pi}(x) = -\boldsymbol{\omega}\sigma(x)\,, \quad \delta\sigma(x) = \boldsymbol{\omega}\cdot\boldsymbol{\pi}(x). \tag{12.50}$$

Calling $\mathbf{J}(x)$ the source for the π-field and $K(x)$ the source for the σ-field, we can write the WT identities for the generating functional of connected correlation functions $\mathcal{W}(\mathbf{J}, K)$ as

$$\int \mathrm{d}x \left[J_i(x)\frac{\delta}{\delta K(x)} - \big(c + K(x)\big)\frac{\delta}{\delta J_i(x)}\right]\mathcal{W} = 0\,. \tag{12.51}$$

It is convenient to introduce some additional notation to take into account the residual $O(3)$ symmetry. We set

$$\begin{aligned}
\tilde{W}^{(2)}_{ij}(p) &= \delta_{ij} D_\pi(p), \\
\tilde{W}^{(2)}(p) &= D_\sigma(p), \\
\tilde{W}^{(3)}_{ij}(p_1, p_2; p_3) &= \delta_{ij} D_\pi(p_1) D_\pi(p_2) D_\sigma(p_3) C(p_1, p_2; p_3), \\
\Big[\tilde{W}^{(4)}_{ijkl}(p_1, p_2, p_3, p_4)\Big]_{\text{amp}} &= \delta_{ij}\delta_{kl} A(p_1, p_2, p_3, p_4) + \delta_{ik}\delta_{jl} A(p_1, p_3, p_2, p_4) \\
&\quad + \delta_{il}\delta_{kj} A(p_1, p_4, p_3, p_2),
\end{aligned}$$

with the conventions that indices correspond to π-fields, and in mixed π–σ correlation functions the arguments of the π-fields are written first.

Differentiating equation (12.51) with respect to J_j, and setting the sources to zero, we obtain the equivalent of equation (12.34):

$$v = \langle \sigma \rangle = c D_\pi(0) \equiv c/\mu^2, \tag{12.52}$$

where $\mu^2 = 1/D_\pi(0)$ is now different from the pion mass squared m_π^2.

Differentiating once with respect to J_j and K, we obtain

$$\delta_{ij} \tilde{W}^{(2)}(p) - \tilde{W}_{ij}^{(2)}(p) = c \tilde{W}_{ij}^{(3)}(0, p; -p),$$

and thus using equation (12.52),

$$D_\pi^{-1}(p) - D_\sigma^{-1}(p) = v C(0, p; -p). \tag{12.53}$$

Setting $p = 0$ and defining $m_\sigma^2 = D_\sigma^{-1}(0)$ we get, in particular,

$$\mu^2 - m_\sigma^2 = v C(0, 0; 0).$$

Differentiating thrice with respect to J, we obtain a relation between three- and four-point correlation functions:

$$c\tilde{W}^{(4)}_{ijkl}(0,p_2,p_3,p_4) = \delta_{ij}\tilde{W}^{(3)}_{kl}(p_3,p_4;p_2) + 2 \text{ terms}\,. \tag{12.54}$$

It follows that

$$vA(0,p_2,p_3,p_4) = C(p_3,p_4;p_2)D_\sigma(p_2)D_\pi^{-1}(p_2). \tag{12.55}$$

First, for $p_2^2 = -m_\pi^2$, the equation reduces to Adler's consistency condition

$$A(0,p_2(p_2^2=-m_\pi^2),p_3,p_4) = 0\,. \tag{12.56}$$

Moreover, setting $p_3 = 0$ in equation (12.55) and eliminating the function C between (12.53) and (12.55), one finds

$$v^2A(0,p,0,-p) = D_\pi^{-1}(p)\left[D_\sigma(p)D_\pi^{-1}(p) - 1\right]. \tag{12.57}$$

The first term in the right hand side has a double zero at the pion mass. Therefore, taking the derivative with respect to p^2, we recover Weinberg's relation:

$$v^2 \frac{\partial}{\partial p^2}\left(A(0,p,0,-p)+D_\pi^{-1}(p)\right)\Big|_{p^2=-m_\pi^2}=0\,. \tag{12.58}$$

These equations yield model- and parameter-independent constraints on the π–π scattering amplitude, which unfortunately is slightly off-shell because at least one of the π momenta vanishes. One verifies that the function A in the tree approximation (12.47) satisfies both conditions (12.56, 12.58).

Another constraint on the π–π scattering amplitude is obtained, for example, by setting all momenta to zero in (12.57):

$$v^2 A(0,0,0,0)=\mu^2(\mu^2/m_\sigma^2-1). \tag{12.59}$$

However, this equation involves an independent free parameter m_σ. Again one verifies that expression (12.47) satisfies equation (12.59) in the tree approximation.

12.7.2 Fermion sector

The infinitesimal transformations of the fermion fields, corresponding to the equations (12.50) are

$$\delta\psi = \tfrac{1}{2} i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\omega}\psi\,, \qquad \delta\bar{\psi} = \tfrac{1}{2} i\bar{\psi}\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\omega}\,. \tag{12.60}$$

We denote by $\bar{\eta}$ and η the sources for the fermion fields. The generating functional $\mathcal{W}(\eta,\bar{\eta},J,K)$ of connected correlation functions then satisfies the WT identity:

$$\int \mathrm{d}^4 x \left\{ \frac{i}{2}\left[\bar{\eta}(x)\gamma_5\boldsymbol{\tau}\frac{\delta}{\delta\bar{\eta}(x)} - \eta(x)\gamma_5\boldsymbol{\tau}\frac{\delta}{\delta\eta(x)}\right] - \mathbf{J}(x)\frac{\delta}{\delta K(x)} \right.$$
$$\left. +(K(x)+c)\frac{\delta}{\delta \mathbf{J}(x)} \right\} \mathcal{W}(\eta,\bar{\eta},J,K) = 0\,.$$

The relations relevant for particle physics correspond to $K=0$.

The Goldberger–Treiman relation. The simplest and best known identity is obtained by differentiating with respect to η and $\bar{\eta}$ and setting all sources to zero. It is, actually, most conveniently written in terms of vertex functions:

$$v\tilde{\Gamma}^{(3)}_{\pi NN}(0;p,-p) = \frac{i\boldsymbol{\tau}}{2}\left\{\gamma_5, \tilde{\Gamma}^{(2)}_{NN}(p)\right\}_+ .$$

The index $+$ in the right hand side means anticommutator in the space of γ matrices. We have explicitly taken into account the property that the fermion propagator is proportional to the identity in the group indices. This relation between the inverse nucleon propagator and the πNN vertex generalizes the relation (12.48).

The right hand side is known when the nucleons are on mass-shell. The left hand side can be approximately related to the nucleon weak β-decay, which involves the matrix element of the axial current at zero momentum between nucleon states since the pion has the quantum numbers of the divergence of the axial current, as can be seen in equation (12.45).

Thus, one contribution to this matrix element has the pion pole. In the strict chiral limit with zero mass pions, this would be the only contribution. One assumes that since the pion mass is small, the chiral limit is a good approximation. The relation then takes the traditional form

$$\frac{G_A}{G_V} \simeq g_{\pi NN} \frac{f_\pi}{m_N},$$

called the Goldberger–Treiman relation.

Replacing by experimental numbers one finds 1.22 for the left hand side and 1.36 for the right hand side, a notable improvement over the tree approximation (12.49).

Other low energy relations. More generally, one can set K to zero, differentiate once with respect to η and $\bar{\eta}$, and an arbitrary number of times with respect to $\mathbf{J}$ and, finally, set all momenta on mass-shell. One then obtains model-independent relations (generalizing equation (12.56)) involving amplitudes for the emission of one unphysical pion at zero momentum.

However, to determine completely the low energy π–N scattering amplitudes, it is necessary to introduce also the $NN\sigma$ vertex and the result then depends on the σ-mass. The predictions for the π–N scattering lengths in the infinite σ-mass limit agree well with experimental data.

One popular way to go beyond these geometrical considerations in the low momentum, low quark mass regime, and in the spirit of effective field theories, is chiral perturbation theory.

12.8 $(\phi^2)^2$ field theory and non-linear σ-model

We have seen that, from the phenomenological viewpoint, chiral models are only predictive in the limit where the mass of the σ field, the component of the scalar field along the direction of $O(4)$ symmetry breaking, is large. This suggests integrating out the massive component to generate an effective field theory. However, this leads to a difficulty since the field with large mass is a member of a multiplet and its elimination may lead to an explicit breaking of the $O(4)$ symmetry.

12.8.1 The $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory in the broken phase

We consider the $O(N)$ symmetric action for an N-component field $\boldsymbol{\phi}$,

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}^4 x \left[\tfrac{1}{2}\left(\nabla_x \boldsymbol{\phi}(x)\right)^2 + \tfrac{1}{2} u \boldsymbol{\phi}^2(x) + \tfrac{1}{4!} g \left(\boldsymbol{\phi}^2(x)\right)^2\right]. \qquad (12.61)$$

In the spontaneously broken phase ($u < u_c$), the $O(N)$ symmetry is broken down to an $O(N-1)$ symmetry and this generates $(N-1)$ Goldstone modes associated with massless scalar particles. At low momentum or large distance, the physics is entirely dominated by the interaction between Goldstone modes. It is thus natural to try to integrate out the last massive component of the N-component field. However, the problem is to avoid loosing the information about the initial $O(N)$ symmetry of the model.

The problem can be solved by an appropriate parametrization of the field $\boldsymbol{\phi}$ and the integration over the massive mode then leads to the non-linear σ-model.

The integration. We consider the partition function

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}] \exp[-\mathcal{S}(\boldsymbol{\phi})].$$

We change variables in the field integral, setting (meaningful only if $\langle \rho \rangle > 0$)

$$\boldsymbol{\phi}(x) = \rho(x)\hat{\boldsymbol{\phi}}(x) \text{ with } \hat{\boldsymbol{\phi}}^2(x) = 1\,.$$

The field integral becomes (assuming a lattice or in the continuum a dimensional regularization to deal with the non-trivial ρ measure):

$$\mathcal{Z} = \int \left[\rho^{N-1}(x)\mathrm{d}\rho(x)\right] \left[\mathrm{d}\Omega(\hat{\boldsymbol{\phi}}(x))\right] \exp\left[-\mathcal{S}(\rho, \hat{\boldsymbol{\phi}})\right],$$

where $\Omega(\hat{\boldsymbol{\phi}})$ is the invariant measure on the sphere S_{N-1} and

$$\begin{aligned}\mathcal{S}(\rho, \hat{\boldsymbol{\phi}}) = \int \mathrm{d}^4 x \Big\{ & \tfrac{1}{2}\rho^2(x) \left(\nabla_x \hat{\boldsymbol{\phi}}(x)\right)^2 + \tfrac{1}{2}\left[\nabla_x \rho(x)\right]^2 + \tfrac{1}{2} u \rho^2(x) \\ & + \tfrac{1}{4!} g \rho^4(x) \Big\}. \end{aligned} \tag{12.62}$$

In the broken phase, the field $\rho(x)$ has a non-zero expectation value $\langle\rho\rangle > 0$ and is massive. If one interested only in momenta much smaller than the ρ-mass or distances much larger than the corresponding correlation length, one can integrate out perturbatively the ρ-field.

The integration over the $\rho(x)$ field generates an effective action $\mathcal{S}_{\text{eff.}}(\hat{\phi})$ for the field $\hat{\phi}$:

$$\exp\left[-\mathcal{S}_{\text{eff.}}(\hat{\phi})\right] \propto \int \left[\rho^{N-1}(x)\mathrm{d}\rho(x)\right] \exp\left[-\mathcal{S}(\rho,\hat{\phi})\right]. \tag{12.63}$$

To calculate the integral, we first set

$$\mathcal{S}(\rho) = \int \mathrm{d}^4x \left[\tfrac{1}{2}\left(\nabla_x \rho(x)\right)^2 + \tfrac{1}{2}u\rho^2(x) + \tfrac{1}{4!}g\rho^4(x)\right].$$

We define

$$\mathcal{Z}(J) = \int \left[\rho^{N-1}(x)\mathrm{d}\rho(x)\right] \exp\left[-\mathcal{S}(\rho) - \tfrac{1}{2}\int \mathrm{d}^4x\, J(x)\rho^2(x)\right]$$

with $\mathcal{Z}(0) = 1$. Then,

$$\mathcal{S}_{\text{eff.}}(\hat{\phi}) = -\ln \mathcal{Z}\left[J(x) = \left(\nabla_x \hat{\phi}(x)\right)^2\right].$$

We calculate $\mathcal{Z}(J)$ by first expanding in powers of J.

Denoting by $\langle \bullet \rangle$ an expectation value with respect to the normalized measure corresponding to $\mathrm{e}^{-\mathcal{S}(\rho)}$, one finds

$$\begin{aligned}\ln \mathcal{Z}(J) = &-\tfrac{1}{2}\left\langle \rho^2 \right\rangle \int \mathrm{d}^4x\, J(x) \\ &+ \tfrac{1}{8}\int \mathrm{d}^4x\,\mathrm{d}^4y\, \left\langle \rho^2(x)\rho^2(y)\right\rangle_{\mathrm{conn.}} J(x)J(y) + O(J^3).\end{aligned}$$

For large ρ mass M, all successive contributions become local. The first term,

$$\mathcal{S}_{\mathrm{eff.}}(\hat{\phi}) = \tfrac{1}{2}\left\langle \rho \right\rangle^2 \int \mathrm{d}^4x \left(\nabla_x \hat{\phi}(x)\right)^2, \tag{12.64}$$

has only two derivatives and thus governs the large distance behaviour. In the expression (12.64) one recognizes the action of the non-linear σ-model. It is the only action with two derivatives consistent with $O(N)$ symmetry.

The second term has four derivatives and will be suppressed by a factor $1/M^2$ and is subleading at large distance.

The perturbative ρ-integration. Each expectation value can be calculated perturbatively. We denote by v the expectation value of $\rho(x)$: $v = \langle\rho(x)\rangle$. We set

$$\rho(x) = v + \chi(x) \ \Rightarrow\ \langle\chi(x)\rangle = 0\,,$$

where v in the tree approximation is given by $v^2 = -6u/g$.

In terms of χ, the action (12.62) becomes

$$\mathcal{S}(\chi) = \int \mathrm{d}^4 x \left[\tfrac{1}{2}\big(\nabla_x \chi(x)\big)^2 + \tfrac{1}{6} g v^2 \chi^2 + \tfrac{1}{6} g v \chi^3 + \tfrac{1}{24} g \chi^4\right].$$

Neglecting all fluctuations of the field χ, one recovers the action (12.64) at leading order:

$$\mathcal{S}^{(0)}_{\text{eff.}}(\hat{\phi}) = \tfrac{1}{2} v^2 \int \mathrm{d}^4 x \left(\nabla_x \hat{\phi}(x)\right)^2 . \qquad (12.65)$$

Loop corrections are calculated with the χ propagator

$$\tilde{\Delta}_\chi(p) = \frac{1}{p^2 + M^2}, \quad M^2 = \tfrac{1}{3} g v^2,$$

and have then to be expanded for M large.

The contributions coming from the integration over χ first renormalize the coefficient v^2 of the term with only two derivatives in (12.65.)

The second term is proportional to the connected part of $\langle \rho^2(x)\rho^2(y) \rangle$,

$$\begin{aligned}\langle \rho^2(x)\rho^2(y)\rangle - \langle \rho^2(x)\rangle \langle \rho^2(y)\rangle &= \langle \rho^2(x)\rho^2(y)\rangle - \langle \rho^2(x)\rangle \langle \rho^2(y)\rangle \\ &= \langle \chi^2(x)\chi^2(y)\rangle - \langle \chi^2(x)\rangle \langle \chi^2(y)\rangle \\ &\quad + 4v^2 \langle \chi(x)\chi(y)\rangle \\ &= \Delta_\chi^2(x-y) + 4v^2\Delta(x-y).\end{aligned}$$

The first term has been evaluated in section 8.1.1 (equation (8.2)). It has a local expansion, derived from equation (8.3), where the leading contribution is proportional to $\ln(\Lambda/M)\delta(x-y)$, where Λ is a cut-off, the next contribution proportional to $\nabla_x^2\delta(x-y)/M^2$... .The second term has a leading order contribution that is simply $4v^2\delta(x-y)/M^2$.

Therefore, all loop contributions, except for a renormalization of the leading term, yield additional, irrelevant, ϕ-interactions with more derivatives, the leading one being proportional to $\int \mathrm{d}^4x((\nabla_x\hat{\phi})^2)^2$, and counterterms for the non-linear σ-model, which in perturbation theory is more divergent that the initial $(\phi^2)^2$ field theory.

Domain of validity. From the way the effective action has been derived, it is expected that its perturbative domain of validity is limited to the broken phase u fixed, $u < u_c$. In this regime, the non-linear σ-model (12.64) completely describes the long distance properties of the $(\phi^2)^2$ field theory.

For u close to u_c or $u > u_c$, the perturbative expansion of the non-linear σ is no longer useful. Physics becomes non-perturbative and the configurations for which ρ vanishes become important. The interpretation of the non-linear σ model as a perturbative effective theory is no longer valid, as we verify now more directly.

12.9 Non-linear σ-model: perturbation theory, power counting

The non-linear σ is defined as the effective action resulting from the χ integration restricted to the leading term with two derivatives. It is convenient to parametrize the action in d dimensions as

$$\mathcal{S}(\hat{\phi}) = \frac{\Lambda^{d-2}}{2g} \int \mathrm{d}^d x \left(\nabla_x \hat{\phi}(x)\right)^2, \tag{12.66}$$

where Λ is a cut-off of the order of the mass of massive component and g a dimensionless coupling constant, which also plays the formal role of $\hbar$ and thus orders the loop expansion.

The partition function is given by

$$\mathcal{Z} = \int [\mathrm{d}\Omega(\hat{\phi})]\, \mathrm{e}^{-\mathcal{S}(\hat{\phi})}. \tag{12.67}$$

For $g \to 0$ the field integral is dominated by the minima of the classical action,

$$\left|\nabla_x \hat{\phi}(x)\right| = 0 \quad \Rightarrow \quad \hat{\phi}(x) = \hat{\phi}_0\,, \quad \hat{\phi}_0^2 = 1\,,$$

where $\hat{\phi}_0$ is an arbitrary constant unit vector.

The action (12.66) has a continuous set of degenerate and equivalent minima which are related by $O(N)$ transformations. Each minimum is the starting point of a perturbative expansion. As we have already discussed in section 12.4, if the symmetry is unbroken one should sum over the contributions of all minima while if the symmetry is spontaneously broken one should select the minimum that corresponds to the field expectation value. This is a non-perturbative issue and we postpone its discussion. Here, we choose one minimum and calculate its contribution in the form of a perturbative expansion.

12.9.1 Formal perturbation theory

After a choice of the direction of spontaneous breaking, and we choose the vector $\mathbf{u} = (1, 0, \ldots, 0)$, the explicit calculation of the field integral (12.67) still depends on a choice of a local parametrization of the sphere near $\mathbf{u}$.

Local parametrization. We choose here the components of $\hat{\boldsymbol{\phi}}$,

$$\hat{\boldsymbol{\phi}} \equiv (\sigma(x), \boldsymbol{\pi}(x))$$

with

$$\sigma(x) = \sqrt{1 - \boldsymbol{\pi}^2(x)}. \tag{12.68}$$

With this parametrization, the infinitesimal group transformations that do not belong to the unbroken $O(N-1)$ subgroup take the non-linear form

$$\delta\boldsymbol{\pi} = \boldsymbol{\varepsilon}\sigma(x),$$

which implies

$$\delta\sigma(x) = -\boldsymbol{\varepsilon} \cdot \boldsymbol{\pi}\,.$$

The $O(4)$ example in four dimensions. In the case of the $O(4)$ symmetry, another popular choice is the following: vectors can be represented as 2×2 matrices in the form

$$\mathbf{M}=\sigma+i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,,$$

and the group $O(4)$ by $SU(2)\times SU(2)$ matrices acting by right and left multiplication, respectively. A vector with unit length can then be represented by an $SU(2)$ matrix of the form $\mathrm{e}^{i\boldsymbol{\tau}\cdot\boldsymbol{\pi}}$.

The action then is proportional to

$$-\int \mathrm{d}^4x\ \mathrm{tr}\left[\mathrm{e}^{-i\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)}\,\nabla_x\,\mathrm{e}^{i\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)}\right]^2,$$

and the coupling to fermions to

$$\int \mathrm{d}^4x\,\bar{\mathcal{N}}(x)\,\mathrm{e}^{i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)}\,\mathcal{N}(x).$$

The measure is formally the product for all space points of the $SU(2)$ invariant measure.

The action. With the parametrization (12.68), the action (12.66) becomes

$$\mathcal{S}(\boldsymbol{\pi}) = \frac{\Lambda^{d-2}}{2g} \int \mathrm{d}^d x \left[\left(\nabla_x \boldsymbol{\pi}(x)\right)^2 + \frac{\left(\boldsymbol{\pi}(x) \cdot \nabla_x \boldsymbol{\pi}(x)\right)^2}{1 - \boldsymbol{\pi}^2(x)} \right] .$$

For g small, the fields $\boldsymbol{\pi}(x)$ that contribute to the field integral, then, must be such that $|\nabla_x \boldsymbol{\pi}(x)| \sim \sqrt{g}$ and, since we expand around $\boldsymbol{\pi}(x) = 0$, the field itself must satisfy

$$|\boldsymbol{\pi}(x)| \sim \sqrt{g} \ .$$

The action is no longer a polynomial in the fields but since $\boldsymbol{\pi}$ is of order $g^{1/2}$, the action can be expanded in powers of $\boldsymbol{\pi}$ generating an infinite number of interactions with arbitrary even powers of $\boldsymbol{\pi}$ and two derivatives. One verifies that at finite loop order, to a given correlation function only a finite number of terms contributes.

Perturbative expansion. Values of $\boldsymbol{\pi}(x)$ of order 1 give exponentially small contributions to the field integral (of order $\exp(-\text{const.}/g)$), which are negligible to any finite order of perturbation theory.

This has two implications:

(*i*) The restriction imposed by the parametrization (12.68), $\sigma(x) > 0$, is irrelevant in perturbation theory.

(*ii*) Moreover, in the field integral, one can freely integrate over $\boldsymbol{\pi}(x)$-field components from $+\infty$ to $-\infty$, disregarding the constraint

$$|\boldsymbol{\pi}(x)| \leq 1 \,.$$

Perturbation theory, then, again relies on the evaluation of simple Gaussian expectation values.

The integration measure. For each space point x, the $\boldsymbol{\pi}$ integration measure is $\mathrm{d}\boldsymbol{\pi}/\sqrt{1-\boldsymbol{\pi}^2}$. However, the product for all x of $\sqrt{1-\boldsymbol{\pi}^2(x)}$ is not defined. Formally, it yields a 'quantum' addition to the action of the form

$$\tfrac{1}{2}\delta^{(d)}(0)\int \mathrm{d}^d x\ \ln\bigl(1-\boldsymbol{\pi}^2(x)\bigr).$$

Since the measure term has no $1/g$ factor and starts contributing only at one-loop order and yields additional vertices without derivatives.

This undefined infinite product is a reflection of the problem of ordering in products of quantum operators. In the straightforward interpretation of the field integral formalism, which involves only classical quantities, the information about ordering is lost. The solution of this problem requires a specific regularization.

Regularizations. Dimensional regularization provides a purely perturbative regularization in which the measure term can be omitted.

Lattice regularization provides both a perturbative and non-perturbative regularization.

The addition of quadratic terms with higher order derivatives of the field $\hat{\phi}$ preserves the $O(N)$ symmetry and improves the behaviour of the propagator but generates new interactions.

One can show that all diagrams except one-loop diagrams can be regularized. The remaining one-loop divergences are unavoidable because they have to cancel the divergent contributions coming from the measure (in a lattice regularization).

12.9.2 *Propagator and power counting*

The propagator $\tilde{\Delta}_{ij}(p)$ of the $\boldsymbol{\pi}$-field is

$$\tilde{\Delta}_{ij}(p) = \delta_{ij} \frac{g\Lambda^{2-d}}{p^2}.$$

In the tree approximation, the $\boldsymbol{\pi}$-field is massless. Returning to the analysis of section 12.4, we conclude that, at leading order in perturbation theory, the non-linear σ-model automatically realizes the $O(N)$ symmetry in the phase of spontaneous symmetry breaking, the massless $\boldsymbol{\pi}$-field corresponding to the Goldstone modes. The massive partner of the $\boldsymbol{\pi}$-field in the linear realization, the σ component, has been eliminated by the constraint (12.68). This constraint formally freezes the fluctuations of $\boldsymbol{\phi}^2(x)$ and sends, in the classical limit, the σ mass to infinity.

Note that these properties are independent of the specific parametrization (12.68) of $\hat{\boldsymbol{\phi}}(x)$.

Power counting. The form of the propagator shows that the dimension $[\boldsymbol{\pi}]$ of the $\boldsymbol{\pi}$-field, in the sense of power counting, is

$$[\boldsymbol{\pi}] = \tfrac{1}{2}(d-2).$$

The dimension of a vertex containing $2n$ $\boldsymbol{\pi}$-fields thus is

$$\left[\partial^2 \boldsymbol{\pi}^{2n}\right] = n(d-2) + 2\,.$$

As a consequence,

(i) for $d = 2$, the non-linear σ model is renormalizable;
(ii) for $d > 2$, the model is not renormalizable.

Therefore, we first discuss the model in dimension $d = 2$. A peculiarity of dimension $d = 2$ is that although the theory is renormalizable by power counting, any local monomial in the field containing at most two derivatives and an arbitrary power of $\boldsymbol{\pi}$ can *a priori* appear as a counter-term.

The symmetry $O(N-1)$, which is linearly realized, only restricts the counter-terms to be of the general form

$$(\partial_\mu \boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 \left(\boldsymbol{\pi}^2\right)^n, \qquad (\partial_\mu \boldsymbol{\pi})^2 \left(\boldsymbol{\pi}^2\right)^n, \qquad \left(\boldsymbol{\pi}^2\right)^n.$$

However, it can be proved, using the non-linear part of the $O(N)$ symmetry, that, up to a renormalization of the field and a global normalization of the action, the renormalized action is unique. The proof is based on a set of WT identities that we derive now.

12.9.3 WT identities

To generate both $\boldsymbol{\pi}$ and $\sigma = \sqrt{1-\boldsymbol{\pi}^2}$ correlation functions, we consider the generating functional $\mathcal{Z}(\mathbf{J}, K)$ corresponding to the action with external sources

$$\mathcal{S}(\boldsymbol{\pi}, \mathbf{J}, H) = \mathcal{S}(\boldsymbol{\pi}) - \frac{\Lambda^{d-2}}{g} \int \mathrm{d}^d x \, [\mathbf{J}(x) \cdot \boldsymbol{\pi}(x) + K(x)\sigma(x)] \,.$$

In a change of variables of the form of the infinitesimal non-linear transformations

$$\delta\boldsymbol{\pi} = \boldsymbol{\varepsilon}\sigma(x), \tag{12.69}$$

the variation of the source terms is

$$\delta\,[\mathbf{J}(x) \cdot \boldsymbol{\pi}(x) + K(x)\sigma(x)] = \boldsymbol{\varepsilon} \cdot [\mathbf{J}(x)\sigma(x) - K(x)\boldsymbol{\pi}(x)] \,.$$

The invariance of the integral leads to the identity

$$\int \mathrm{d}^d x \left(\mathbf{J}(x) \frac{\delta}{\delta K(x)} - K(x) \frac{\delta}{\delta \mathbf{J}(x)} \right) \mathcal{Z}(\mathbf{J}, K) = 0 \,.$$

Thus, $\mathcal{W} = \ln \mathcal{Z}$, the generating functional of connected correlation functions, also satisfies

$$\int \mathrm{d}^d x \left(\mathbf{J}(x) \frac{\delta}{\delta K(x)} - K(x) \frac{\delta}{\delta \mathbf{J}(x)} \right) \mathcal{W}(\mathbf{J}, K) = 0 \,.$$

In a Legendre transformation,

$$\frac{\delta}{\delta K(x)} \mathcal{W}(\mathbf{J}, K) = -\frac{\delta}{\delta K(x)} \Gamma(\boldsymbol{\pi}, K).$$

The WT identities for the generating functional of vertex functions follow:

$$\int \mathrm{d}^d x \left(\frac{\delta \Gamma}{\delta \boldsymbol{\pi}(x)} \frac{\delta \Gamma}{\delta K(x)} + K(x) \boldsymbol{\pi}(x) \right) = 0 \,.$$

This equation is the basis of the proof that in two dimensions the renormalized action remains $O(N)$ invariant. It also allows to discuss the effect of symmetry breaking $c\sigma(x)$ since this amounts to expanding the WT identities around $K(x) = \text{constant} \neq 0$.

12.10 RG analysis at and above two dimensions

We now discuss the non-linear σ-model from the point of renormalization group. We first consider the dimension two where the model is renormalizable.

12.10.1 Two dimensions: RG equations

Perturbative expansions in two dimensions does not exist due to IR divergences. This divergences have a physical meaning: spontaneous symmetry breaking of a continuous symmetry with short range interactions is impossible in two dimensions. It is necessary to introduce an explicit symmetry breaking. The shift $K(x) \mapsto m^2 + K(x)$ breaks the $O(N)$ symmetry and leads to the propagator

$$\tilde{\Delta}_{ij}(p) = \delta_{ij} \frac{g}{p^2 + m^2}.$$

With this propagator, the perturbative expansion is defined and one can renormalize the non-linear σ-model. RG equations follow. For the connected correlation functions, they can be written as

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\zeta(g) + \left(\frac{\zeta(g)}{4} + \frac{\beta(g)}{2g}\right) m \frac{\partial}{\partial m}\right] \tilde{W}^{(n)}(p_i; g, m, \Lambda) = 0\,. \tag{12.70}$$

The two RG functions at leading order are

$$\beta(g) = -\frac{(N-2)}{2\pi} g^2 + O(g^3), \tag{12.71a}$$

$$\zeta(g) = \frac{(N-1)}{2\pi} g + O(g^2). \tag{12.71b}$$

The sign for $N > 2$ of the first coefficient of the β-function shows that the model is asymptotically free. Moreover, in the limit $m \to 0$, the spectrum is non-perturbative as expected since spontaneous symmetry breaking is impossible. In fact, from other considerations, it is known that $O(N)$ symmetry is restored and the spectrum contains a massive boson.

Its physical mass M is an RG invariant and thus has the general form

$$M \propto \Lambda \exp\left[-\int^{g} \frac{\mathrm{d}g'}{\beta(g')}\right] \propto \Lambda\, \mathrm{e}^{-(2\pi)/(N-2)g} .$$

The case $N = 2$ is peculiar because $SO(2)$ is an Abelian group. The content of the model depends on the regularization. In the simplest form it is equivalent to a free field theory. In the lattice regularization, it leads to the famous Kosterlitz–Thouless phase transition.

12.10.2 Dimension $2 + \varepsilon$

Above dimension two, the model can be studied in the form of a double series expansion in g and $\varepsilon = (d-2)$-expansion. The β-function becomes

$$\beta(g) = \varepsilon g - \frac{(N-2)}{2\pi} g^2 + O\left(g^3, g^2\varepsilon\right) . \tag{12.72}$$

The slope at the Gaussian fixed point $g = 0$ is now positive and, for g small, the large distance behaviour is governed by the Gaussian fixed point.

The physics of the model is thus perturbative and dominated by the massless Goldstone modes.

However, one finds in addition a fixed point with a negative slope,

$$g^* = \frac{2\pi\varepsilon}{N-2} + O(\varepsilon^2), \quad \beta'(g^*) = -\varepsilon + O(\varepsilon^2)\,,$$

an UV fixed point since it governs the universal large momentum behaviour for momenta $|p| \ll \Lambda$.

Solving the RG equations, one finds that at $g = g^*$, the field expectation value vanishes like

$$\langle\sigma\rangle \propto (g^* - g)^{-\zeta(g^*)/2\beta'(g^*)}\,, \quad -\frac{\zeta(g^*)}{2\beta'(g^*)} = \frac{N-1}{2(N-2)} + O(\varepsilon).$$

This result suggests that g^* corresponds to a phase transition between the spontaneously broken phase and a $g > g^*$ non-perturbative phase with symmetry restoration.

The perturbative expansion and the interpretation of the non-linear σ-model as an effective perturbative theory is thus valid only for $g < g^*$.

These conclusions are comforted by the large N-expansion, which indicates that beyond perturbation theory the non-linear σ-model and the corresponding $(\phi^2)^2$ field theory have the same large distance physics.

APPENDIX A12
CURRENTS AND NOETHER'S THEOREM

For completeness, in the appendix, we describe some properties of field equations and currents. We first discuss classical currents adopting the covariant notation of real time relativistic field theory with a metric tensor $g_{\mu\nu}$ with signature $(+---)$.

After a formal transition between real time and euclidean field theory, setting $x_d = ix_0 \equiv it$, we examine the properties of quantum currents.

Finally, we define the energy-momentum tensor, discuss its properties both in classical and quantum field theory and its relation to conformal invariance in massless theories.

In the appendix, summation over repeated indices is implied, except when stated otherwise.

A12.1 Currents in classical field theory

If the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ depends only on the field $\phi(x)$ and its derivatives $\partial_\mu \phi(x)$, the classical equation of motion obtained by varying the action

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x\, \mathcal{L}\big(\phi(x), \partial_\mu \phi(x)\big), \qquad (A12.1)$$

is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi(x)]} - \frac{\partial \mathcal{L}}{\partial \phi(x)} = 0$$

(in this notation $\phi(x)$ and $\partial_\mu \phi(x)$ are considered as independent variables).

We perform on $\phi(x)$ a space-dependent group transformation parametrized by a field $\Lambda(x)$,

$$\phi(x) \mapsto \phi_\Lambda(x)\,.$$

As a consequence of the equation of motion, the action is also stationary with respect to variations of $\Lambda(x)$ at ϕ fixed:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \Lambda(x)]} - \frac{\partial \mathcal{L}}{\partial \Lambda(x)} = 0\,. \qquad (A12.2)$$

We define a current $J^\mu(x)$, functional of $\phi(x)$, by

$$J^\mu(x) = \left.\frac{\partial \mathcal{L}}{\partial[\partial_\mu \Lambda(x)]}\right|_{\Lambda(x)=0}\,, \qquad (A12.3)$$

where we have assumed that $\Lambda(x) = 0$ corresponds in the group to the identity.

By construction, currents are directly associated with the generators of the Lie algebra of the symmetry group.

We can then rewrite equation $(A12.2)$ as (Noether's theorem)

$$\partial_\mu J^\mu(x) = \frac{\partial \mathcal{L}}{\partial \Lambda(x)}.$$

In addition, if the Lagrangian is invariant under space-independent group transformations, $\partial\mathcal{L}/\partial\Lambda$ vanishes and thus the current J_μ is conserved:

$$\partial_\mu J^\mu(x) = 0 \,. \tag{A12.4}$$

In classical field theory, the space integral of the time-component of the current is a charge:

$$\mathcal{Q}^\alpha(t \equiv x_0) = \int \mathrm{d}^3 x \, J_0^\alpha(x).$$

By differentiating with respect to t and using the current conservation equation ($A12.4$), one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}^\alpha(t) = \int \mathrm{d}^3 x \sum_{\mu=1}^{3} \partial_\mu J_\mu(x) = 0 \,.$$

The charges $\mathcal{Q}^\alpha(t)$ are constants of the classical motion.

Example. If the Lagrangian density has the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - V\left[\phi(x)\right] \qquad (A12.5)$$

(in real time covariant notation) and if the infinitesimal group transformations are

$$\delta\phi_i(x) = t^\alpha_{ij}\Lambda^\alpha(x)\phi_j(x), \qquad (A12.6)$$

the current $J^\alpha_\mu(x)$ is given by

$$J^\alpha_\mu(x) = t^\alpha_{ij}\partial_\mu\phi_i(x)\phi_j(x). \qquad (A12.7)$$

A12.2 Euclidean quantum field theory

We have already examined the consequences of symmetries for field theories and derived WT identities. These identities can also be derived in the operator formalism of quantum mechanics and in this case currents and charges, considered as quantum operators, play an important role. In our formulation, currents will appear either in the coupling at leading order of matter to gauge fields (see lectures 13, 14) or as polynomials in the fields satisfying some specific identities that lead to special renormalization properties.

In what follows dimensional regularization is assumed.

We consider the generating functional

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + \int J_i(x)\phi_i(x)\mathrm{d}x\right], \tag{A12.8}$$

in which the action is invariant under group transformations whose infinitesimal form is given by equation $(A12.6)$ when $\Lambda(x)$ is a constant.

We perform a change of variables of the form of a transformation (12.11) in the integral (A12.8). We define the euclidean current $J^{\alpha}_{\mu}(x)$ by equation (A12.3) in terms of the euclidean action density. If $\mathcal{S}(\phi)$ is symmetric, the variation of the action reads

$$\delta\mathcal{S}(\phi) = \int \partial_{\mu}\Lambda^{\alpha}(x) J^{\alpha}_{\mu}(x) \mathrm{d}x \,.$$

Identifying the coefficient of $\Lambda^{\alpha}(x)$, we obtain

$$\int [\mathrm{d}\phi] \left[\partial_{\mu} J^{\alpha}_{\mu}(x) - J_i(x) t^{\alpha}_{ij} \phi_j(x)\right] \exp\left[-\mathcal{S}(\phi) + \int J_i(x)\phi_i(x)\mathrm{d}x\right] = 0 \,.$$

The identity can be written as

$$\partial^{x}_{\mu} \mathcal{Z}_{J^{\alpha}_{\mu}(x)} = J_i(x) t^{\alpha}_{ij} \frac{\delta \mathcal{Z}}{\delta J_j(x)}, \qquad (A12.9)$$

where $\mathcal{Z}_{J^{\alpha}_{\mu}(x)}$ is the generating functional of correlation functions with a $J^{\alpha}_{\mu}(x)$ operator insertion.

The same equation is valid for connected correlation functions. After Legendre transformation, one finds

$$\partial_\mu^x \Gamma_{J_\mu^\alpha(x)} = -\frac{\delta\Gamma}{\delta\varphi_i(x)} t_{ij}^\alpha \varphi_j(x). \qquad (A12.10)$$

Equations ($A12.9$,$A12.10$) are the analogues for correlation functions of the current conservation equation ($A12.4$). Integrated over all space, they yield, not surprisingly, equations (12.17–12.19), that is, the WT identities consequences of the symmetry.

From the point of view of renormalization, equation ($A12.10$) implies that the insertion of $\partial_\mu J_\mu^\alpha(x)$ in a renormalized correlation function is finite.

In a simple renormalizable $(\phi^2)^2_4$-like field theory, covariance then implies that the same must be true for the current $J_\mu^\alpha(x)$. This result is non-trivial since from expression ($A12.7$) we see that $J_\mu^\alpha(x)$ is an operator of dimension 3. A further consequence is that the insertion of a conserved current in a correlation function does not modify the form of the RG equations.

A12.3 The energy–momentum tensor

If the action is translation invariant, the substitution $\phi(x) \mapsto \phi(x+\varepsilon)$, in which ε is a constant, leaves the action invariant. In the spirit of section A12.1, we perform a space-dependent translation, which in fact coincides with a general change of variables. We thus substitute in the action $\phi(x) \mapsto \phi(x+\varepsilon(x))$. If $\phi(x)$ satisfies the equation of motion, the variation of the action $(A12.1)$ at first order in ε vanishes. In the substitution, the derivatives transform like

$$\partial_\mu \phi(x) \mapsto \partial_\mu \phi(x+\varepsilon) + \partial_\mu \varepsilon^\nu \partial_\nu \phi(x+\varepsilon).$$

To calculate the variation we then change variables $x+\varepsilon = y$. Translation invariance implies that the action density depends on x only through the field ϕ. Therefore, the only new effect is to change the measure of integration:

$$\mathrm{d}y^\mu = \mathrm{d}x^\mu + \partial_\nu \varepsilon^\mu \mathrm{d}x^\nu\,.$$

If we now compare the new action with the initial one $(A12.1)$ we see that the modifications come only from the derivatives and the integration measure (y is a dummy integration variable). Collecting the terms of order ε and integrating by parts, we obtain the identity

$$\partial_\mu T^\mu_\nu(x) = 0\,,$$

in which the energy–momentum tensor $T^\mu_\nu(x)$ is defined by

$$T^\mu_\nu(x) = \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi(x)]}\partial_\nu \phi(x) - \delta^\mu_\nu \mathcal{L}\left[\phi(x)\right].$$

It is convenient to also introduce the tensor

$$T_{\mu\nu}(x) = g_{\mu\lambda} T^\lambda_\nu(x),$$

in which $g_{\mu\nu}$ is the Minkowski metric tensor. In the example of the Lagrangian $(A12.5)$, $T_{\mu\nu}$ is symmetric since

$$T_{\mu\nu}(x) = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\tfrac{1}{2}(\partial_\rho\phi)(\partial^\rho\phi) - V(\phi)\right].$$

To the tensor $T_{\mu\nu}(x)$ correspond constants of the classical motion P_μ, energy and momentum, obtained by integrating the time components (with respect to one index) of $T_{\mu\nu}$ over space:

$$P_\mu(t \equiv x_0) = \int \mathrm{d}^3x\, T_{0\mu}(x).$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\mu = 0\,.$$

We noted that a space–time-dependent change of variables on x^μ is an arbitrary change of coordinates. This explains that the tensor $T_{\mu\nu}$ appears in the coupling of matter field to the metric tensor in General Relativity.

Also any current associated with an additional space-time symmetry of the action can be related to $T_{\mu\nu}$.

For instance, the $O(1,3)$ pseudo-orthogonal transformations whose infinitesimal form is

$$\delta x^\mu = \Lambda^\mu_\nu(x) x^\nu,$$

correspond to the choice

$$\varepsilon^\mu = \Lambda^\mu_\nu x^\nu.$$

The corresponding currents $M^{\mu\nu\rho}$ are then

$$M^{\mu\nu\rho}(x) = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu.$$

Dilatation invariance. We again consider, as an example, the ϕ^4 field theory in four dimensions:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) - \tfrac{1}{4!}g\phi^4(x).$$

In the absence of the mass term, the action is scale-invariant, that is, invariant in the substitution

$$\phi(x) \mapsto \phi_\lambda(x) = \lambda\phi(\lambda x).$$

For what concerns the variation of the argument, dilatation corresponds to taking ε^μ of the form

$$\varepsilon^\mu = x^\mu \lambda(x).$$

We thus expect the dilatation current S^μ to involve $x^\nu T^\mu_\nu$. A short calculation leads to

$$S^\mu(x) = x^\nu \left[T^\mu_\nu(x) + \frac{1}{6} \left(\partial^2 \delta^\mu_\nu - \partial_\mu \partial^\nu \right) \phi^2(x) \right]. \qquad (A12.11)$$

In the presence of a mass term, the current $S^\mu(x)$ is no longer conserved. Instead,

$$\partial_\mu S^\mu(x) = m^2 \phi^2(x).$$

We now introduce the tensor

$$\tilde{T}_{\mu\nu}(x) = T_{\mu\nu}(x) + \frac{1}{6} \left(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu \right) \phi^2(x).$$

The tensor $\tilde{T}_{\mu\nu}$ can be used as energy–momentum tensor instead of $T_{\mu\nu}$: it is a polynomial in the field, symmetric as a tensor, and satisfies the conservation equation

$$\partial_\mu \tilde{T}^{\mu}_{\nu} = 0\,.$$

In terms of $\tilde{T}^{\mu}_{\nu}(x)$, equation $(A12.11)$ then reads

$$S^{\mu}(x) = x^{\nu} \tilde{T}^{\mu}_{\nu}(x),$$

and the divergence of the dilatation current is

$$\partial_\mu S^{\mu} = \tilde{T}^{\mu}_{\mu}\,.$$

In dilatation-invariant theories, the trace of the ‘improved’ energy–momentum tensor $\tilde{T}^{\mu}_{\nu}$ vanishes.

A12.4 Energy–momentum tensor and euclidean field theory

Performing the infinitesimal change of variables

$$\phi(x) = \phi'\big(x + \varepsilon(x)\big),$$

in the functional integral, one can derive WT identities for the insertion of the energy–momentum tensor (also called the stress tensor). The variation of the action with a source is

$$\delta\left[\int J\phi\,\mathrm{d}x - \mathcal{S}(\phi)\right] = \varepsilon_\nu(x)\left[J(x)\partial_\nu\phi(x) + \partial_\mu T_{\mu\nu}(x)\right].$$

It follows that

$$\partial^x_\mu \mathcal{Z}_{T_{\mu\nu}(x)} + J(x)\partial^x_\nu \frac{\delta\mathcal{Z}}{\delta J(x)} = 0\,.$$

Integrating this identity over space yields

$$\int \mathrm{d}x\, J(x)\partial_\nu \frac{\delta\mathcal{Z}}{\delta J(x)} = 0\,,$$

which expresses the translation invariance of correlation functions.

After Legendre transformation, one finds

$$\partial_\mu^x \Gamma_{T_{\mu\nu}(x)} + \frac{\delta\Gamma}{\delta\varphi(x)}\partial_\nu\varphi(x) = 0\,.$$

Again, we conclude that the insertion of the operator $\partial_\mu T_{\mu\nu}(x)$ in a renormalized correlation function is finite. However, this does not imply that the insertion of $T_{\mu\nu}$ itself is finite. In the ϕ^4_4 field theory for example, $T_{\mu\nu}$ has dimension 4. The quantity $(\delta_{\mu\nu}\nabla^2 - \partial_\mu\partial_\nu)\phi^2$ is also a symmetric tensor of dimension 4 whose divergence vanishes. Therefore, it can appear as an additive counter-term in the renormalization of $T_{\mu\nu}$:

$$(T_{\mu\nu})_{\mathrm{r}} = T_{\mu\nu} + A(\delta_{\mu\nu}\nabla^2 - \partial_\mu\partial_\nu)(\phi^2)_{\mathrm{r}}\,.$$

Note that the renormalized energy–momentum tensor automatically has a non-vanishing trace, and it can no longer be improved since the coefficient A is divergent.

The dilatation current is not conserved but this should have been expected since it is impossible to regularize the theory without breaking the classical dilatation invariance, either by introducing a cut-off, or by changing the dimension. Nevertheless, it is possible to derive WT identities involving the divergence of the dilatation current. By integrating them over space, one obtains CS equations.

A12.5 Dilatation and conformal invariance

We now consider a general euclidean action, $\mathcal{S}$, invariant under translation, rotation and dilatation. We perform the infinitesimal change of variables

$$x_\mu \longmapsto x_\mu + \varepsilon_\mu(x).$$

Translation invariance implies that the variation of the action involves only the partial derivatives of $\varepsilon_\mu(x)$:

$$\delta\mathcal{S} = \int \mathrm{d}^4x\, T_{\mu\nu}(x)\partial_\mu\varepsilon_\nu(x).$$

Rotation invariance implies that $\delta\mathcal{S}$ vanishes for

$$\varepsilon_\mu = \Lambda_{\mu\nu} x_\nu \,,$$

in which $\Lambda_{\mu\nu}$ is an arbitrary antisymmetric matrix. Therefore, the integral of the stress tensor must be symmetric:

$$\int \mathrm{d}^4 x \, (T_{\mu\nu} - T_{\nu\mu}) = 0 \,.$$

Dilatation invariance corresponds to

$$\varepsilon_\mu = \lambda x_\mu \,,$$

and implies the vanishing of the integral of the trace of the stress tensor:

$$\int \mathrm{d}^4 x \, T_{\mu\mu} = 0 \,.$$

For the simplest class of theories, like scalar field theories with an action $\mathcal{S}(\phi)$ depending only on the field $\phi(x)$ and its first partial derivatives, the two integral conditions imply the existence of a symmetric, traceless stress–energy tensor:

$$T_{\mu\nu} = T_{\nu\mu}\,, \quad T_{\mu\mu} = 0\,.$$

It then follows that the variation of the action also vanishes for any function ε_μ which satisfies

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - 2\delta_{\mu\nu}\partial\cdot\varepsilon = 0\,. \tag{A12.12}$$

The group of transformations which satisfy equation $(A12.12)$ is larger than the product of transformations which we have considered so far: it is the whole conformal group. Indeed, let us calculate the variation of a line element of the form

$$(\mathrm{d}s)^2 = g(x)\mathrm{d}x_\mu \mathrm{d}x_\mu\,, \tag{A12.13}$$

which corresponds to a conformally flat metric.

We find

$$\delta\left[(\mathrm{d}s)^2\right] = \mathrm{d}x_\mu \mathrm{d}x_\mu \partial_\rho g(x)\varepsilon_\rho + \mathrm{d}x_\mu \mathrm{d}x_\nu \left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) g(x). \qquad (A12.14)$$

We now see that equation $(A12.12)$ is the necessary and sufficient condition for the line element to retain the form $(A12.13)$. By definition, the transformations which preserve the form of the metric $(A12.13)$ are conformal transformations.

From equation $(A12.12)$ it follows that

$$\left[\delta_{\mu\nu}\nabla^2 + 2\partial_\mu\partial_\nu\right]\partial\cdot\varepsilon = 0\,.$$

The equation implies that all second derivatives of $\partial\cdot\varepsilon$ vanish. Returning then to equation $(A12.12)$ one shows easily that all third derivatives of ε_μ also vanish. Solutions of degree 0 correspond to translations. Solutions of degree 1 correspond to rotations and dilatations.

The additional solutions of equation $(A12.12)$ are second degree polynomials of the form

$$\varepsilon_\mu = a_\mu x^2 - 2x_\mu a \cdot x \, .$$

They correspond to special conformal transformations. The integrated form of these transformations is

$$x'_\mu = \frac{x_\mu + a_\mu x^2}{1 + 2a \cdot x + a^2 x^2} .$$

The conformal group is isomorphic to $SO(5,1)$. Imposing conformal invariance on correlation functions determines, in particular, two- and three-point functions.

Of course, scale invariance of the classical theory is broken at the quantum level. However, there exist situations in which the RG β-function vanishes, at least for some values of the coupling constants. Then both the dilatation invariance and, therefore, the conformal invariance are restored.

Remark. The condition that the action should depend only on the field and its first derivatives can be illustrated by a simple counter-example. Consider the free action $\mathcal{S}(\phi)$

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x \big(\nabla^2 \phi(x)\big)^2 .$$

The propagator in Fourier space is $1/p^4$. The theory is obviously translation, rotation and scale invariant. However, one verifies that it is not conformal invariant.