Lecture 14: NON-ABELIAN GAUGE THEORIES: THE BASIS OF THE STANDARD MODEL

In chapter 13 we have described the structure and the formal properties of Abelian gauge theories, which provide a framework for the construction of Quantum Electrodynamics. However, to be able to describe other fundamental interactions, Weak and Strong Interactions, it is necessary to generalize the concept of gauge symmetry to non-Abelian groups. Therefore, in this chapter we construct a field theory invariant under local, that is, space-dependent, transformations of a general compact Lie group G. Inspired by the Abelian example of chapter 13, we immediately introduce the geometric concept of parallel transport. All the required mathematical quantities then follow quite naturally.

We quantize gauge theories and study some of the formal properties of the quantum theory like the BRS(T) symmetry, which is essential for renormalization. We show how perturbation theory can be regularized, a somewhat non-trivial problem. Finally, we discuss general aspects of the non-Abelian Higgs mechanism.

14.1 Parallel transport and gauge invariance

We consider a scalar field $\boldsymbol{\phi}(x)$ transforming under a linear unitary or orthogonal representation $\mathcal{R}(G)$ of a compact Lie group G. We want to construct a field theory that has a local G-symmetry, that is, a theory where the action is invariant under space-dependent group transformations, also called gauge transformations. Denoting by $\mathbf{g}$ a matrix belonging to the representation $\mathcal{R}(G)$, we write the $\boldsymbol{\phi}$-field transformation:

$$\boldsymbol{\phi}'(x) = \mathbf{g}(x)\boldsymbol{\phi}(x). \tag{14.1}$$

In the case of products of fields taken at the same point, global invariance ($\mathbf{g}$ constant) implies local invariance. However, in the case of global invariant functions involving field derivatives or, more generally, products of fields taken at different points this is no longer true. To solve the problem one introduces parallel transporters, which are curve-dependent elements of the representation $\mathcal{R}(G)$.

14.1.1 Parallel transport

Let C_{yx} be a continuous, piecewise differentiable oriented curve with origin a point x and end-point a point y. We define curve-dependent elements $\mathbf{U}(C)$ of the group $\mathcal{R}(G)$ that satisfy

$$\mathbf{U}(C_{zy} \cup C_{yx}) = \mathbf{U}(C_{zy})\mathbf{U}(C_{yx}) \;\Rightarrow\; \mathbf{U}(C_{xx} \equiv \text{point } x) = \mathbf{1}\,. \tag{14.2}$$

Moreover, denoting by C^{-1} the curve C with opposite orientation, one imposes

$$\mathbf{U}(C^{-1}) = \mathbf{U}^{-1}(C). \tag{14.3}$$

Gauge transformations. If $\mathbf{g}(x)$ is a space-dependent group element, we define the gauge transformation of $\mathbf{U}(C_{xy})$ by

$$\mathbf{U}^{\mathbf{g}}(C_{xy}) = \mathbf{g}(x)\mathbf{U}(C_{xy})\mathbf{g}^{-1}(y). \tag{14.4}$$

The gauge transformation is consistent with the rules (14.2, 14.3). Moreover, $\operatorname{tr}\mathbf{U}(C)$, where C is a closed curve, is gauge invariant.

Matter and gauge fields. The definition of parallel transport is such that the vector

$$\boldsymbol{\phi}(U,x) = \mathbf{U}(C_{xy})\boldsymbol{\phi}(y), \tag{14.5}$$

transforms by $\mathbf{g}(x)$ instead of $\mathbf{g}(y)$. Thus, the quantity $\boldsymbol{\phi}^{\dagger}(x)\mathbf{U}(C_{xy})\boldsymbol{\phi}(y)$ is gauge invariant.

We now assume that in the limit of an infinitesimal differentiable curve,

$$y_\mu = x_\mu + \mathrm{d}x_\mu\,, \tag{14.6}$$

$\mathbf{U}(C)$ is differentiable in x_μ. We can then parametrize it in terms of the connection $\mathbf{A}_\mu(x)$, which is a vector from the point of view of space transformations, and a matrix (antisymmetric or anti-hermitian) belonging to the representation of the Lie algebra of $\mathcal{R}(G)$:

$$\mathbf{U}(C) = \mathbf{1} + \sum_\mu \mathbf{A}_\mu(x)\mathrm{d}x_\mu + o\left(\|\mathrm{d}x_\mu\|\right).$$

The transformation properties of $\mathbf{A}_\mu(x)$ are obtained by expanding equation (14.4) to first order in $\mathrm{d}x_\mu$,

$$\mathbf{A}'_\mu(x) = \mathbf{g}(x)\mathbf{A}_\mu(x)\mathbf{g}^{-1}(x) + \mathbf{g}(x)\partial_\mu \mathbf{g}^{-1}(x). \tag{14.7}$$

From the point of view of global transformations ($\mathbf{g}(x)$ constant), the field $\mathbf{A}_\mu(x)$ transforms by the adjoint representation of the group G.

However, $\mathbf{A}_\mu(x)$, which is usually called the gauge field or Yang–Mills field, is not a tensor for gauge transformations, the transformation being affine.

14.1.2 Covariant derivative and curvature tensor

To the connection $\mathbf{A}_\mu(x)$ is associated a covariant derivative $\mathbf{D}_\mu$, whose explicit form depends on the tensor on which it is acting. To obtain its expression when acting on $\boldsymbol{\phi}(x)$ we consider in equation (14.5) the limit (14.6) of an infinitesimal curve. The equation (14.5) becomes

$$\begin{aligned}\boldsymbol{\phi}(U,x) &= \left(\mathbf{1}+\sum_\mu \mathbf{A}_\mu(x)\mathrm{d}x_\mu\right)\left(\boldsymbol{\phi}(x)+\sum_\mu \partial_\mu\boldsymbol{\phi}(x)\mathrm{d}x_\mu\right)+o\left(\|\mathrm{d}x_\mu\|\right)\\ &= \left(\mathbf{1}+\sum_\mu \mathrm{d}x_\mu\mathbf{D}_\mu\right)\boldsymbol{\phi}(x)+o\left(\|\mathrm{d}x_\mu\|\right)\end{aligned}$$

with

$$\mathbf{D}_\mu = \mathbf{1}\,\partial_\mu + \mathbf{A}_\mu\,. \tag{14.8}$$

$\mathbf{D}_\mu$ is both a differential operator acting on space variables and a matrix.

The identity

$$\mathbf{g}(x)\,(\mathbf{1}\,\partial_\mu + \mathbf{A}_\mu)\,\mathbf{g}^{-1}(x) = \mathbf{1}\,\partial_\mu + \mathbf{g}(x)\mathbf{A}_\mu(x)\mathbf{g}^{-1}(x) + \mathbf{g}(x)\partial_\mu\mathbf{g}^{-1}(x), \tag{14.9}$$

shows that $\mathbf{D}_\mu$ is a tensor, since $\mathbf{D}'_\mu$, the transform of $\mathbf{D}_\mu$ under the gauge transformation (14.7), is

$$\mathbf{D}'_\mu = \mathbf{g}(x)\mathbf{D}_\mu\,\mathbf{g}^{-1}(x). \tag{14.10}$$

In the equations (14.9, 14.10) the products have to be understood as products of differential and multiplicative operators.

Infinitesimal gauge transformations. Setting,

$$\mathbf{g}(x) = \mathbf{1} + \boldsymbol{\omega}(x) + o\left(\|\boldsymbol{\omega}\|\right),$$

in which $\boldsymbol{\omega}(x)$ is an element of the Lie algebra of $\mathcal{R}(G)$, we derive from equation (14.7) the form of the infinitesimal gauge transformation of the field $\mathbf{A}_\mu$,

$$-\delta\mathbf{A}_\mu(x) = \partial_\mu\boldsymbol{\omega} + [\mathbf{A}_\mu, \boldsymbol{\omega}] \equiv \mathbf{D}_\mu\boldsymbol{\omega}\,. \tag{14.11}$$

In equation (14.8) we have given the form of the covariant derivative corresponding to the representation $\mathcal{R}(G)$. The equation (14.11) yields the form of the covariant derivative in the adjoint representation. One verifies that

$$\partial_\mu\boldsymbol{\omega}' + \left[\mathbf{A}'_\mu, \boldsymbol{\omega}'\right] = \mathbf{g}(x)\left\{\partial_\mu\boldsymbol{\omega} + [\mathbf{A}_\mu, \boldsymbol{\omega}]\right\}\mathbf{g}^{-1}(x),$$

in which $\mathbf{A}'_\mu$ is given by equation (14.7) and $\boldsymbol{\omega}'$ by

$$\boldsymbol{\omega}'(x) = \mathbf{g}(x)\boldsymbol{\omega}(x)\mathbf{g}^{-1}(x).$$

Curvature tensor. The commutator of two covariant derivatives,

$$\mathbf{F}_{\mu\nu}(x) = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu] \,,$$

is no longer a differential operator. It is again an element of the Lie algebra of $\mathcal{R}(G)$ and transforms, as a consequence of equation (14.10), as

$$\mathbf{F}'_{\mu\nu}(x) = \mathbf{g}(x)\mathbf{F}_{\mu\nu}(x)\mathbf{g}^{-1}(x).$$

Thus, $\mathbf{F}_{\mu\nu}$ is a tensor, the curvature tensor, generalization of the electromagnetic field of QED. The curvature tensor is associated with parallel transport along an infinitesimal closed curve.

14.1.3 Expressions in component form

In many situations it is useful to write previous expressions in component form. We expand $\mathbf{A}_\mu(x)$ on a basis $\{\mathbf{t}^a\}$ of generators of the Lie algebra in the representation $\mathcal{R}$. The generators satisfy the commutation relations,

$$[\mathbf{t}^b, \mathbf{t}^c] = \sum_a f_{bca}\mathbf{t}^a,$$

where f_{bca} are the structure constants. The gauge field can be expanded on the basis and we define,

$$\mathbf{A}_\mu(x) = \sum_a A^a_\mu(x)\mathbf{t}^a.$$

The covariant derivative (14.8) then reads,

$$(\mathbf{D}_\mu)_{ij} = \partial_\mu \delta_{ij} + \sum_a A^a_\mu(x) t^a_{ij} \,. \tag{14.12}$$

The equation (14.11) involves the structure constants f_{abc} of the Lie algebra,

$$-\delta A_\mu^a(x) = \partial_\mu \omega_a(x) + \sum_{b,c} f_{bca} A_\mu^b(x)\omega_c(x).$$

This equation yields also the form of the covariant derivative in the adjoint representation.

Finally, the curvature tensor can also be expanded on the basis:

$$\mathbf{F}_{\mu\nu}(x) = \sum_a F_{\mu\nu}^a(x)\mathbf{t}^a$$

and, therefore,

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + \sum_{b,c} f_{bca} A_\mu^b(x) A_\nu^c(x). \qquad (14.13)$$

This last expression is independent of the group representation.

14.2 Gauge invariant action

Gauge field. The simplest gauge invariant action $\mathcal{S}(\mathbf{A})$ function of the gauge field $\mathbf{A}_\mu$, and generalization of the Abelian action in (13.28), has the form

$$\mathcal{S}(\mathbf{A}) = -\frac{1}{4e^2}\int \mathrm{d}^4x \sum_{\mu,\nu} \operatorname{tr} \mathbf{F}^2_{\mu\nu}(x). \tag{14.14}$$

It is associated with the parallel transport along an infinitesimal closed curve.

We have not added a mass term for the gauge field as in section 13.1. Indeed, one can show that in the non-Abelian case the zero mass limit is singular.

We have also chosen the normalization of the gauge field, in such a way that all geometric quantities become independent of the gauge coupling constant. The sign in front of the action takes into account that, with our definition, the matrix $\mathbf{F}_{\mu\nu}$ is anti-hermitian or antisymmetric.

Two remarks are immediately in order:

(i) In contrast with the Abelian case, because the gauge field transforms non-trivially under the group, as equation (14.7) shows (the gauge field is 'charged'), the curvature tensor $\mathbf{F}_{\mu\nu}$ is not gauge invariant, and thus not directly associated with a physical observable.

The action (14.14) is no longer a free field action; the gauge field has self-interactions and even the spectrum of the pure gauge action is non-perturbative (some analytic results can be obtained in dimension two).

We indicate in chapter 16 how lattice gauge theory provides a framework for non-perturbative investigations.

(ii) As in the Abelian case, the action, because it is gauge invariant, does not provide a dynamics to the degrees of freedom of the gauge field which correspond to gauge transformations and, therefore, some gauge fixing is required.

Matter fields. For boson fields transforming by (14.1), and taking into account the transformation (14.10) of the covariant derivatives, one verifies that the action

$$\mathcal{S}_{\mathrm{B}}(\boldsymbol{\phi}) = \int \mathrm{d}^4 x \left[\sum_\mu \bigl(\mathbf{D}_\mu \boldsymbol{\phi}(x)\bigr)^\dagger \mathbf{D}_\mu \boldsymbol{\phi}(x) + V\bigl(\boldsymbol{\phi}(x)\bigr)\right],$$

is gauge invariant if $V(\boldsymbol{\phi})$ is a group invariant function of the scalar field $\boldsymbol{\phi}$.

Similarly, for fermions transforming by $\mathcal{R}(G)$, the action

$$\mathcal{S}_{\mathrm{F}}(\bar{\psi}, \psi) = -\int \mathrm{d}^4 x\, \bar{\psi}(x)\, (\not{\mathbf{D}} + M)\, \psi(x),$$

is gauge invariant.

14.3 Hamiltonian formalism. Quantization

In this section we discuss only the quantization of a gauge theory without matter corresponding to the action (14.14), because the extension of all arguments to a general gauge invariant theory is straightforward.

We first show that non-Abelian gauge theories can be quantized, using a simple Hamiltonian formalism, by the method explained in the Abelian case in section 13.4. This leads to a field theory that, at least at the formal level, is unitary because it corresponds to a hermitian Hamiltonian.

Classical field equations. We first consider real time field theory, we denote by $t \equiv x_0 = ix_4$ time and the corresponding field component by $\mathbf{A}_0 = -i\mathbf{A}_4$. We use the notation $\dot{Q}$ for the time derivative of Q. Space components will carry Roman indices ($\mathbf{A}_i, x_i$) and four-dimensional components Greek indices.

To the continuation to real time of the action (14.14) corresponds the classical field equation

$$\sum_{\mu} \mathbf{D}_{\mu} \mathbf{F}^{\mu\nu}(x) = 0 \,, \tag{14.15}$$

in which the explicit form of $\mathbf{D}_{\mu}$ is given by equation (14.11). The action (14.14) does not lead to a standard quantization because, as in the Abelian case, it does not depend on $\dot{\mathbf{A}}_0$, the time derivative of $\mathbf{A}_0$. Thus, here also $\mathbf{A}_0$ is not a dynamical variable, the $\mathbf{A}_0$ field equation is a constraint equation that can be used to eliminate $\mathbf{A}_0$ from the action.

However, in the absence of a mass term, the reduced action does not depend on all space components of the gauge field. Only the combination $\left[\delta_{ij} - \mathrm{D}_i (\mathrm{D}_{\perp}^2)^{-1} \mathrm{D}_j\right] \dot{\mathbf{A}}_j$ appears ($\mathrm{D}_{\perp}^2$ is the covariant space Laplacian).

But in contrast with the Abelian case the projector acting on $\mathbf{A}_i$ depends on the field itself, and, therefore, the procedure which led to Coulomb's gauge does not work here, at least in its simplest form.

14.3.1 Temporal (Weyl) gauge: classical field equations

Therefore, we discuss the quantization in the simpler temporal gauge with a straightforward generalization of the method used in the Abelian case.

We first note that if $\mathbf{A}_\mu(t,x)$ is a solution of equation (14.15), any gauge transform of $\mathbf{A}_\mu(t,x)$ is also a solution. All solutions can thus be obtained from a solution belonging to a gauge section, a section in the space of all gauge fields which intersects once all gauge orbits, and a gauge transformation. One gauge section is well-suited to the construction of a Hamiltonian formalism, the temporal gauge which is defined by the equation

$$\mathbf{A}_0(t,x) = 0\,. \tag{14.16}$$

The gauge condition (14.16) is left invariant by time-independent gauge transformations, which thus form a symmetry group of the gauge fixed Lagrangian.

In the field equation (14.15), we separate the space and time components. The space components yield

$$\sum_{l=1}^{3} \mathbf{D}_l \mathbf{F}_{l0} = 0\,, \quad \mathbf{D}_0 \mathbf{F}_{0k} - \sum_{l=1}^{3} \mathbf{D}_l \mathbf{F}_{lk} = 0\,, \quad k \leq 3\,. \tag{14.17}$$

The indices k, l correspond to space components. For $A_0 = 0$, the equations simplify and become

$$\dot{\mathbf{E}}_k = \sum_l \mathbf{D}_l \mathbf{F}_{lk}, \tag{14.18}$$

$$\sum_l \mathbf{D}_l \mathbf{E}_l = 0 \tag{14.19}$$

with

$$\mathbf{E}_k = -\dot{\mathbf{A}}_k / e^2. \tag{14.20}$$

The equation (14.18) is a dynamical equation that can be directly derived from the initial Lagrangian in which the condition (14.16) has been used:

$$\mathcal{L}(\mathbf{A}) = -\operatorname{tr}\int \mathrm{d}^3x \left[\frac{1}{2e^2}\sum_k \dot{\mathbf{A}}_k^2(t,x) - \frac{1}{4e^2}\sum_{k,l}\mathbf{F}_{kl}^2(t,x)\right]. \tag{14.21}$$

The expression (14.21) defines a standard Lagrangian for the space components of the gauge field: $\mathbf{E}_k$ is the conjugated momentum of $\mathbf{A}_k$ (equation (14.20)) and the corresponding Hamiltonian takes the form

$$\mathcal{H}(\mathbf{E},\mathbf{A}) = -\operatorname{tr}\int \mathrm{d}^3x \left(\frac{e^2}{2}\sum_k \mathbf{E}_k^2(x) + \frac{1}{4e^2}\sum_{k,l}\mathbf{F}_{kl}^2(x)\right). \tag{14.22}$$

The equation (14.19) is a constraint, non-Abelian generalization of Gauss's law. The only relevant solutions of the field equations are those that satisfy the constraint. The constraint is compatible with the classical motion because the Poisson brackets of the constraint and the Hamiltonian vanish.

This is a direct consequence of the symmetry under time-independent gauge transformations of the Lagrangian (14.21) and thus of the Hamiltonian (14.22).

The quantities $\sum_l \mathbf{D}_l \mathbf{E}_l$ are the generators, in the sense of Poisson brackets, of the symmetry group.

14.3.2 Temporal gauge: quantum theory

Quantization in the temporal gauge, as in the Abelian case, then is standard. In the euclidean formalism, the partition function can be written as

$$\mathcal{Z} = \int [\mathrm{d}\mathbf{A}_\mu]\, \delta(\mathbf{A}_4) \exp\left[\frac{1}{4e^2} \int \mathrm{d}^4 x \, \mathrm{tr} \sum_{\mu,\nu} \mathbf{F}^2_{\mu\nu}(x)\right]. \tag{14.23}$$

The constraint. In the quantum theory, the quantum operators $\sum_l \mathbf{D}_l \mathbf{E}_l$ generators of a symmetry group, commute with the Hamiltonian.

The space of admissible physical states $\Psi(\mathbf{A})$ is restricted by the quantum generalization of Gauss's law:

$$\sum_l \mathbf{D}_l \mathbf{E}_l \Psi(\mathbf{A}) \equiv \sum_l \mathbf{D}_l \frac{1}{i} \frac{\delta}{\delta \mathbf{A}_l(x)} \Psi(\mathbf{A}) = 0 \,.$$

The equation implies that physical states are gauge invariant, that is, belong to the invariant sector of the symmetry group, a subspace which is left invariant by quantum evolution.

Note that at zero temperature the perturbative vacuum is automatically gauge invariant and Gauss's law plays no role. This is no longer the case at finite temperature.

Remarks. The theory we have constructed is not explicitly space time covariant and this is the source of serious difficulties as we have already pointed out in the Abelian case (see section 13.4).

In particular, in the temporal gauge the theory is not renormalizable in the sense of power counting. Indeed the propagator in this gauge

$$W_{ij}^{(2)}(\mathbf{k}_\perp, k_4) = \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}_\perp^2}\right) + \frac{1}{k_4^2}\frac{k_i k_j}{\mathbf{k}_\perp^2},$$

in which $\mathbf{k}_\perp$ is the 'space' part of $\mathbf{k}$, does not decrease at k_4 fixed for large spatial momenta $|\mathbf{k}_\perp|$.

These problems are solved by showing that gauge invariant observables can alternatively be calculated from another quantum action which leads to a theory which is explicitly covariant and renormalizable by power counting.

14.3.3 Covariant gauges

It is generally more convenient to work with a covariant gauge rather than with the temporal gauge (14.23). We want to implement the covariant gauge condition,

$$\sum_\mu \partial_\mu \mathbf{A}_\mu(x) - \boldsymbol{\nu}(x) = 0\,, \tag{14.24}$$

where $\boldsymbol{\nu}(x)$ is a given external field which belongs to the Lie algebra of $\mathcal{R}(G)$. This can be achieved by starting from the equation

$$E(\mathbf{g}, x) \equiv \sum_\mu \partial_\mu \mathbf{A}^g_\mu(x) - \boldsymbol{\nu}(x) = 0 \tag{14.25}$$

for the space-dependent group element $\mathbf{g}(x)$, $\mathbf{A}^g_\mu$ being the gauge transform of $\mathbf{A}_\mu$ (equation (14.7)) by $\mathbf{g}$. If $\boldsymbol{\nu}(x)$ is a stochastic field, the equation (14.25) gives to $\mathbf{g}(x)$ a stochastic distribution.

We assume that the equation (14.25) has a unique solution, which is equivalent to assert that in the space of gauge fields the surface defined by equation (14.24) intersects once and only once all gauge orbits.

This condition is satisfied for small fields and, thus, in perturbation theory but not necessarily beyond (see the remark at the end of the section).

We then use the identity

$$1 = \int [\mathrm{d}g]\delta\big(E(\mathbf{g}, x)\big) \det \mathbf{M}(\mathbf{g}), \tag{14.26}$$

where $\mathrm{d}\mathbf{g}$ is the group invariant measure and $\det \mathbf{M}$ the Jacobian of the transformation $\mathbf{E} \mapsto \mathbf{g}$.

The operator $\mathbf{M}$ is of the form $\delta\mathbf{E}/\delta\mathbf{g}$. We thus need the variation of $\mathbf{E}$ induced by an infinitesimal variation of $\mathbf{g}$, which can be parametrized as $\delta\mathbf{g}(x) = \boldsymbol{\omega}(x)\mathbf{g}(x)$, $\boldsymbol{\omega}(x)$ belonging to the Lie algebra.

The variation of the $\delta\mathbf{E}$ is

$$\delta E(\mathbf{g},x) = -\sum_\mu \partial_\mu \mathbf{D}_\mu(\mathbf{A}^g)\boldsymbol{\omega}(x) \ \Rightarrow\ \mathbf{M} = \sum_\mu \partial_\mu \mathbf{D}_\mu(\mathbf{A}^g).$$

We now introduce a boson field $\boldsymbol{\lambda}$ and spinless fermions $\bar{\mathbf{C}}$ and $\mathbf{C}$, the Faddeev–Popov 'ghosts', and all transforming under the adjoint representation. We replace the functional δ-function by its the Fourier representation,

$$\delta\big(\mathbf{E}(\mathbf{g},x)\big) = \int [\mathrm{d}\boldsymbol{\lambda}] \exp\left[-\operatorname{tr}\int \mathrm{d}^d x\, \mathbf{E}(\mathbf{g},x)\boldsymbol{\lambda}(x)\right].$$

We express $\det\mathbf{M}$ as an integral over the fermions $\bar{\mathbf{C}}$ and $\mathbf{C}$ as

$$\det\mathbf{M} = \int [\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}] \exp\left[-\operatorname{tr}\int \mathrm{d}^d x \sum_\mu \mathbf{C}(x)\partial_\mu \mathbf{D}_\mu(\mathbf{A})\bar{\mathbf{C}}(x)\right].$$

These two identities allow rewriting the identity (14.26) as

$$1=\int[\mathrm{d}\mathbf{g}\,\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}\,\mathrm{d}\boldsymbol{\lambda}]\exp\left[-\mathcal{S}_{\text{gauge}}(\mathbf{A}^{g},\bar{\mathbf{C}},\mathbf{C},\boldsymbol{\lambda},\boldsymbol{\nu})\right], \tag{14.27}$$

with

$$\begin{aligned}\mathcal{S}_{\text{gauge}}(\mathbf{A},\bar{\mathbf{C}},\mathbf{C},\boldsymbol{\lambda},\boldsymbol{\nu})=\int\mathrm{d}^{4}x\ \mathrm{tr}\Bigg\{\boldsymbol{\lambda}(x)\Bigg[\sum_{\mu}\partial_{\mu}\mathbf{A}_{\mu}(x)-\boldsymbol{\nu}(x)\Bigg]\\ +\sum_{\mu}\mathbf{C}(x)\partial_{\mu}\mathbf{D}_{\mu}(\mathbf{A})\bar{\mathbf{C}}(x)\Bigg\}.\end{aligned} \tag{14.28}$$

We introduce the identity (14.27) in the representation (14.23) of the partition function in the temporal gauge and obtain

$$\begin{aligned}\mathcal{Z}=\int[\mathrm{d}\mathbf{g}\,\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}\,\mathrm{d}\boldsymbol{\lambda}\,\mathrm{d}\mathbf{A}_{\mu}]\delta(\mathbf{A}_{4})\\ \times\exp\left[\frac{1}{4e^{2}}\int\mathrm{d}^{4}x\ \mathrm{tr}\,\mathbf{F}_{\mu\nu}^{2}(x)-\mathcal{S}_{\text{gauge}}(\mathbf{A}^{g},\bar{\mathbf{C}},\mathbf{C},\boldsymbol{\lambda},\boldsymbol{\nu})\right].\end{aligned}$$

We then change variables $\mathbf{A}_\mu^g \mapsto \mathbf{A}_\mu$. The classical gauge action (14.14) is gauge invariant. Only the gauge condition $\delta(\mathbf{A}_4)$ is affected. Changing $\mathbf{g}$ into $\mathbf{g}^{-1}$, one finds

$$\mathcal{Z} = \int [\mathrm{d}\mathbf{g}\,\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}\,\mathrm{d}\boldsymbol{\lambda}\,\mathrm{d}\mathbf{A}_\mu]\delta(\mathbf{A}_4^g)\exp\left[-\mathcal{S}(\mathbf{A},\bar{\mathbf{C}},\mathbf{C},\boldsymbol{\lambda},\boldsymbol{\nu})\right]$$

with

$$\mathcal{S}(\mathbf{A},\bar{\mathbf{C}},\mathbf{C},\boldsymbol{\lambda},\boldsymbol{\nu}) = -\frac{1}{4e^2}\int \mathrm{d}^4x\ \mathrm{tr}\sum_{\mu,\nu}\mathbf{F}_{\mu\nu}^2(x) + \mathcal{S}_{\mathrm{gauge}}(\mathbf{A},\bar{\mathbf{C}},\mathbf{C},\boldsymbol{\lambda},\boldsymbol{\nu}). \tag{14.29}$$

We integrate over the group field $\mathbf{g}(x)$. The result of the integral $\int [\mathrm{d}\mathbf{g}]\,\delta(\mathbf{A}_4^g)$ is gauge invariant. We can thus calculate it only for fields satisfying the gauge condition $\mathbf{A}_4 = 0$. Then,

$$\mathbf{A}_4^g(x) = \mathbf{g}(x)\partial_4\mathbf{g}^{-1}(x),$$

and the integral yields a constant independent of $\mathbf{A}$.

This leads to the field integral representation

$$\mathcal{Z} = \int [\mathrm{d}\mathbf{A}_\mu \, \mathrm{d}\bar{\mathbf{C}} \, \mathrm{d}\mathbf{C} \, \mathrm{d}\boldsymbol{\lambda}] \exp\left[-\mathcal{S}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu})\right],$$

where the quantum action is given by

$$\mathcal{S}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = -\frac{1}{4e^2} \int \mathrm{d}^4 x \ \mathrm{tr} \sum_{\mu,\nu} \mathbf{F}^2_{\mu\nu}(x) + \mathcal{S}_{\mathrm{gauge}}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu}).$$

Since the partition function is independent of $\boldsymbol{\nu}$, one can average over the 'noise' field $\boldsymbol{\nu}(x)$ with the Gaussian distribution

$$[\mathrm{d}\rho(\boldsymbol{\nu})] = [\mathrm{d}\boldsymbol{\nu}] \exp\left[\frac{1}{2\xi e^2} \int \mathrm{d}^4 x \ \mathrm{tr}\, \boldsymbol{\nu}^2(x)\right]. \qquad (14.30)$$

The averaged partition function $\mathcal{Z}$ then reads

$$\mathcal{Z} = \int \left[\mathrm{d}\mathbf{A}_\mu \, \mathrm{d}\bar{\mathbf{C}} \, \mathrm{d}\mathbf{C} \, \mathrm{d}\boldsymbol{\lambda}\right] \exp\left[-\mathcal{S}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda})\right], \tag{14.31}$$

where

$$\mathcal{S}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}) = -\frac{1}{4e^2} \int \mathrm{d}^4 x \ \mathrm{tr} \sum_{\mu,\nu} \mathbf{F}^2_{\mu\nu}(x) + \mathcal{S}_{\text{gauge}}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}), \tag{14.32}$$

$$\mathcal{S}_{\text{gauge}} = \int \mathrm{d}^4 x \ \mathrm{tr} \left[\frac{\xi e^2 \boldsymbol{\lambda}^2(x)}{2} + \sum_\mu \left(\boldsymbol{\lambda}(x) \partial_\mu \mathbf{A}_\mu(x) + \mathbf{C}(x) \partial_\mu \mathbf{D}_\mu \bar{\mathbf{C}}(x)\right)\right]. \tag{14.33}$$

Except in the $\xi = 0$ limit, it is also possible to integrate over $\boldsymbol{\lambda}(x)$. This amounts to setting

$$\boldsymbol{\lambda}(x) = -\frac{1}{\xi e^2} \partial_\mu \mathbf{A}_\mu(x).$$

One obtains the new quantum action better suited for perturbative calculations,

$$\mathcal{S}(\mathbf{A},\bar{\mathbf{C}},\mathbf{C})=\int \mathrm{d}^4x\ \mathrm{tr}\left\{-\frac{1}{e^2}\left[\tfrac{1}{4}\sum_{\mu,\nu}\mathbf{F}_{\mu\nu}^2+\frac{1}{2\xi}\left(\sum_{\mu}\partial_\mu\mathbf{A}_\mu\right)^2\right]\right.$$
$$\left.+\sum_{\mu}\mathbf{C}(x)\partial_\mu\mathbf{D}_\mu\bar{\mathbf{C}}(x)\right\}. \qquad (14.34)$$

However, geometric properties of the action are more apparent in expression (14.32), in particular the BRS symmetry.

The obvious drawback of the covariant gauge, which leads to a covariant, local and renormalizable theory, is the lack of explicit unitarity. In particular, Faddeev–Popov fermions being spinless do not obey to the spin–statistics connection and are, thus, unphysical.

The method used here to establish the formal equivalence between the two gauges (14.31) and (14.23) can be generalized to other gauges.

Gribov's ambiguity. Gribov has pointed out that, in contrast with the Abelian case, depending on the value of the gauge field $\mathbf{A}_\mu(x)$, the gauge condition (14.25) has not always a unique solution in $\mathbf{g}(x)$, a problem called Gribov's ambiguity. When two solutions merge, the operator $\sum_\mu \partial_\mu \mathbf{D}_\mu(\mathbf{A})$ has zero eigenvalues. This implies that the representation (14.31) is not meaningful beyond perturbation theory. The same ambiguity has been shown to arise for a large class of gauge conditions.

14.3.4 BRS(T) symmetry

After quantization, the action (14.34) is no longer gauge invariant. However, it follows from a general analysis of constraint equations inserted in integrals (see appendix A14.1) that the action now has a BRS symmetry due to the gauge fixing contraint.

To explain the form of the BRS transformations in the gauge theory context, we parametrize the gauge field as

$$\mathbf{A}_\mu(x) = \mathbf{B}_\mu^g(x),$$

where $\mathbf{B}_\mu(x)$ satisfies the gauge condition (14.25): $\sum_\mu \partial_\mu \mathbf{B}_\mu(x) = \boldsymbol{\nu}(x)$. BRS transformations in the case of group manifolds take the form

$$\begin{cases} \delta\mathbf{g}(x) = \varepsilon\bar{\mathbf{C}}(x)\mathbf{g}(x), & \delta\bar{\mathbf{C}}(x) = \varepsilon\bar{\mathbf{C}}^2(x), \\ \delta\mathbf{C}(x) = \varepsilon\boldsymbol{\lambda}(x), & \delta\boldsymbol{\lambda}(x) = 0\,, \end{cases} \tag{14.35}$$

where ε is an additional Grassmann variable (see equation (A14.13)).

The field $\mathbf{B}_\mu(x)$ has a dynamics provided by the gauge action and is invariant under BRS transformations, $\delta\mathbf{B}_\mu(x) = 0$, as well as any gauge invariant quantity.

The transformations (14.35) correspond for $\mathbf{g}(x)$ to an infinitesimal gauge transformation.

The variation of the field $\mathbf{A}_\mu$ comes from the variation of $\mathbf{g}(x)$ and thus

$$\delta\mathbf{A}_\mu(x) = \delta\mathbf{B}^g_\mu(x) = -\varepsilon\mathbf{D}_\mu\bar{\mathbf{C}}(x)\,. \tag{14.36}$$

The action of a BRS transformation on a function of $\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}$ can be reproduced by the action of the BRS differential operator

$$\mathcal{D} = \int \mathrm{d}^4x \ \mathrm{tr}\left[-\sum_\mu \mathbf{D}_\mu\bar{\mathbf{C}}(x)\frac{\delta}{\delta\mathbf{A}_\mu(x)} + \bar{\mathbf{C}}^2(x)\frac{\delta}{\delta\bar{\mathbf{C}}(x)} + \lambda(x)\frac{\delta}{\delta\mathbf{C}(x)}\right]. \tag{14.37}$$

In particular, we can express the BRS symmetry of the quantized action by the equation

$$\mathcal{D}\mathcal{S}(\mathbf{A}, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}) = 0\,. \tag{14.38}$$

Moreover, one verifies quite generally that $\mathcal{D}$ has the form of a cohomology operator:

$$\mathcal{D}^2 = 0\,.$$

Thus, in the cohomology terminology, equation (14.38) implies that $\mathcal{S}$ is BRS closed. Moreover, $\mathcal{S}_{\text{gauge}}$ is BRS exact. More explicitly, for expression (14.28)

$$\mathcal{S}_{\text{gauge}} = \mathcal{D} \int \mathrm{d}^4 x \ \mathrm{tr}\, \mathbf{C}(x) \left[\partial_\mu \mathbf{A}_\mu(x) - \boldsymbol{\nu}(x)\right],$$

or after integration over the $\boldsymbol{\nu}$ field (equation (14.33)),

$$\mathcal{S}_{\text{gauge}} = \mathcal{D} \int \mathrm{d}^4 x \ \mathrm{tr}\, \mathbf{C}(x) \left[\partial_\mu \mathbf{A}_\mu(x) + \tfrac{1}{2}\xi e^2 \boldsymbol{\lambda}(x)\right]. \tag{14.39}$$

WT identities associated with the BRS symmetry (14.35, 14.36) imply the structural stability of the quantum action (14.32) under renormalization.

The simplest and most general proof relies on the Zinn-Justin equation.

14.4 Perturbation theory, regularization

Compared with the Abelian case, the new features of the action (14.34) are the presence of gauge field self-interactions and ghost terms. Let us first write the different terms of the gauge action in component form to establish conventions and normalizations. The gauge action takes the form:

$$\mathcal{S}(A_\mu^a) = \frac{1}{4e^2} \int \mathrm{d}^4 x \sum_{a,\mu,\nu} [F_{\mu\nu}^a(x)]^2 ,$$

where the curvature tensor is given by equation (14.13) and the trace of the unit matrix has been swallowed into a redefinition of the coupling constant. In the covariant gauge of section 14.3.3, the gauge field propagator is (equation (13.17))

$$[\Delta_\xi]_{\mu\nu}^{ab}(k) = e^2 \delta_{ab} \left(\frac{\delta_{\mu\nu}}{k^2} + (\xi - 1) \frac{k_\mu k_\nu}{(k^2)^2} \right) .$$

In four dimensions, as in the Abelian case, the gauge field has dimension 1. The ghost field action becomes

$$\mathcal{S}_{\text{ghost}} = \int \mathrm{d}^4 x \sum_{a,c,\mu} C^a(x)\partial_\mu \left[\partial_\mu \delta_{ac} + \textstyle\sum_b f_{bca} A^b_\mu(x)\right] \bar{C}^c(x).$$

The ghost fields thus have a simple δ_{ab}/k^2 propagator and canonical dimension 1 in four dimensions. The interaction terms have all dimension 4 and, therefore, the theory is renormalizable by power counting in four dimensions. The power counting for matter fields is the same as in the Abelian case.

Compared with the Abelian case, the non-Abelian theory has three new vertices coming from the gauge field self-interactions and the interaction with the ghost fields. The gauge field three-point function at leading order is

$$[\tilde{\Gamma}^{(3)}]^{abc}_{\mu\nu\rho}(p,q,r) = \frac{i}{e^2} f_{abc} \left[(r-q)_\mu \delta_{\nu\rho} + (p-r)_\nu \delta_{\rho\mu} + (q-p)_\rho \delta_{\mu\nu}\right] . \tag{14.40}$$

The gauge field four-point function is given by

$$[\tilde{\Gamma}^{(4)}]^{abcd}_{\mu\nu\rho\sigma} = \frac{1}{e^2} \sum_e \left[f_{eab} f_{ecd} \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + f_{eac} f_{ebd} \left(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) \right.$$
$$\left. + f_{ead} f_{ecb} \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\rho}\right)\right] .$$

All terms are obtained from the first by exchanging the indices to make the correlation function totally symmetric. Finally, the ghost gauge field vertex is

$$\left\langle C^a(p) \bar{C}^b(q) A^c_\mu(r) \right\rangle = -i f_{abc} p_\mu .$$

Notice that in a generic gauge the two ghost fields play a different role. In a graphic representation of Feynman diagrams ghost propagator lines are oriented. However, in the special case $\xi = 0$ corresponding to Landau's gauge, because $\sum_\mu \partial_\mu \mathbf{A}_\mu$ vanishes, the vertex can be antisymmetrized and a symmetry between ghost fields is established.

Matter fields. The coupling to matter fields differs from the Abelian case only by some geometric factors corresponding to group indices. For example, the coupling to fermions generated by the covariant derivative (14.12) is simply $\gamma_\mu t^a_{ij}$.

Infrared divergences. In the covariant gauge, and in the absence of a Higgs mechanism which provides a mass to gauge fields, only the gauge $\xi = 1$, Feynman's gauge , leads to a theory which is obviously IR finite. In contrast to the Abelian case, it is impossible to give an explicit mass to the gauge field and to then construct a theory which is both unitary and renormalizable.

On the other hand, we want eventually to prove the gauge independence of the theory and therefore we must be able to define it for more than one gauge. One way to introduce an IR regulator is to consider the theory in a finite volume.

Regularization. The problem of regularization in non-Abelian gauge theories has many features in common with the Abelian case, as well as with the non-linear σ-model. We review the three regularization methods that we have always considered in this work. Dimensional regularization is the most convenient for practical calculations and works in the absence of chiral fermions. Lattice regularization, which is also relevant for non-perturbative calculations can be used generally (see lecture 16 for details) since a (non-trivial) method for handling chiral fermions has been found (related to Ginsparg–Wilson's relation). Finally, momentum or Pauli–Villars's type regularizations work partially in geometric models, in the sense that they regularize all diagrams except one-loop diagrams.

We verify this property again here.

The regularized gauge action takes the form:

$$\mathcal{S}(\mathbf{A}) = \int \mathrm{d}^4 x \ \mathrm{tr} \sum_{\mu,\nu} \mathbf{F}_{\mu\nu} P(\mathbf{D}^2/\Lambda^2) \mathbf{F}_{\mu\nu} \,,$$

in which P is a polynomial of arbitrary degree. In the same way the gauge function $\sum_\mu \partial_\mu \mathbf{A}_\mu$ is changed into

$$\sum_\mu \partial_\mu \mathbf{A}_\mu \longmapsto Q(\partial^2/\Lambda^2) \sum_\mu \partial_\mu \mathbf{A}_\mu \,,$$

in which Q is a polynomial of same degree as P. As a consequence both the gauge field propagator and the ghost propagator can be made arbitrarily convergent. However, as in the Abelian case, the covariant derivatives generate new interactions which are more singular.

It is easy to verify that the power counting of one-loop diagrams is unchanged while higher order diagrams can be made convergent by taking the degrees of P and Q large enough.

For matter fields the situation is the same as in the Abelian case, for example, massive fermions contributions can be regularized by adding a set of regulator fields, massive fermions and bosons with spin.

Again in the case of chiral fermions, global chiral properties can be preserved, but problems arise with local chiral transformations. However, the problem of the compatibility between the gauge symmetry and the quantum corrections is reduced to an explicit verification of the WT identities for the one-loop diagrams. Note that the preservation of gauge symmetry is necessary for the cancellation of unphysical states in physical amplitudes, and thus essential to the physical consistency of the quantum field theory.

WT identities and renormalization. From the BRS symmetry corresponding to the transformations (14.35–14.36) follow WT identities. Their form is somewhat complicated and we postpone the discussion to lecture ?.? , where we derive the form of the renormalized action for a general gauge theory. We give here the result only in the example of the pure gauge action in the covariant gauge. We can assume that the gauge group G is simple. Then the renormalized form of the action (14.34) is given by the substitution:

$$\begin{cases} e^2 \longmapsto Z_e e^2 \,, & \mathbf{A}_\mu \longmapsto Z_A^{1/2} \mathbf{A}_\mu \,, \\ \xi \longmapsto Z_A Z_e^{-1} \xi \,, & \mathbf{C}\bar{\mathbf{C}} \longmapsto Z_C \mathbf{C}\bar{\mathbf{C}} \,. \end{cases}$$

This result has a simple interpretation: the gauge structure (14.34) is preserved and the coefficient of $(\sum_\mu \partial_\mu \mathbf{A}_\mu)^2$ is unrenormalized exactly as in the Abelian case. However, unlike the Abelian case, the gauge transformation of the gauge field and, more generally the form of the covariant derivative, are modified by the gauge field renormalization.

14.5 The non-Abelian Higgs mechanism

We have already discussed the Higgs mechanism in the Abelian case. The basic idea is the same in non-Abelian gauge theories: the spontaneous breaking of a global symmetry associated with a gauge invariance leads to masses for gauge fields without generating massless Goldstone particles. Simply, because the group structure is richer, a number of different situations may arise.

We consider a classical gauge invariant action, here expressed in terms of real fields, for a gauge field coupled to a scalar boson $\boldsymbol{\phi}$ transforming under an orthogonal representation of the symmetry group (the generators $\mathbf{t}^\alpha$ are antisymmetric matrices):

$$\mathcal{S}(\mathbf{A},\boldsymbol{\phi}) = \int \mathrm{d}^4x \left[-\frac{1}{4e^2}\,\mathrm{tr}\sum_{\mu,\nu}\mathbf{F}^2_{\mu\nu}(x) + \frac{1}{2}\sum_{\mu}\mathbf{D}_\mu\boldsymbol{\phi}(x)\cdot\mathbf{D}_\mu\boldsymbol{\phi}(x) + V\big(\boldsymbol{\phi}(x)\big)\right].$$

We assume that the symmetric potential $V(\boldsymbol{\phi})$ has non-symmetric minima.

In the absence of gauge symmetry, this is the situation which we have already analysed in section 12.4. Since the spectrum in the classical limit depends on the group structure and the representation content of the field $\boldsymbol{\phi}$, we consider here only two families of examples.

14.5.1 Simple Lie groups

We first assume that G, the symmetry group of the action, is simple and is thus also the gauge group. Moreover, we assume for simplicity that the field $\boldsymbol{\phi}$ belongs to an irreducible representation. We assume that $\boldsymbol{\phi}$ has an non-vanishing expectation value, $\langle \boldsymbol{\phi} \rangle = \mathbf{v}$, corresponding to a degnerate minimum of the potential and call H the subgroup of G that leaves the vector $\mathbf{v}$ invariant. We separate the generators $\mathbf{t}^{\alpha}$ of G in the matrix representation into two subsets $\alpha \le p$ corresponding to the Lie algebra $\mathcal{L}(H)$ of the subgroup H, and the complementary set $\mathcal{L}(G/H)$.

We parametrize the scalar field $\boldsymbol{\phi}(x)$ as

$$\boldsymbol{\phi}(x) = \exp\left[\sum_{\alpha>p} t^{\alpha}\theta^{\alpha}(x)\right]\bigl(\mathbf{v} + \boldsymbol{\rho}(x)\bigr), \tag{14.41}$$

in which the vectors $\boldsymbol{\rho}$ and $\{\ t^{\alpha}_{ij}v_j\}$ span two orthogonal subspaces. The transformation

$$\boldsymbol{\phi}(x) \longmapsto \{\theta^{\alpha}(x), \quad \boldsymbol{\rho}(x)\},$$

is such that the new fields $\boldsymbol{\rho}(x)$ and $\theta^{\alpha}(x)$ can be expanded in powers of $\boldsymbol{\phi}(x) - \mathbf{v}$. In the absence of gauge fields, we have used the representation (14.41) to show that the fields $\theta^{\alpha}(x)$ correspond to massless Goldstone modes induced by the spontaneous breaking of the G-symmetry.

Here the equation (14.41) can also be viewed as a local group transformation relating the two fields $\boldsymbol{\phi}$ and $\boldsymbol{\rho} + \mathbf{v}$. If we perform on the field $\mathbf{A}_\mu$ a gauge transformation of the form (14.7) with

$$\mathbf{g}(x) = \exp\left[\sum_{\alpha>p} \mathbf{t}^\alpha \theta^\alpha(x)\right], \tag{14.42}$$

we eliminate the fields θ^α from the action completely. In fact we have fixed (at least partially) the gauge. If we now examine the scalar field contribution to the action, we see that for $\boldsymbol{\rho} = 0$ it reduces to a mass term for the gauge field:

$$\tfrac{1}{2}\sum_\mu \mathbf{D}_\mu\boldsymbol{\phi} \cdot \mathbf{D}_\mu\boldsymbol{\phi}|_{\boldsymbol{\rho},\theta^\alpha=0} = \tfrac{1}{2}\sum_{\alpha,\beta,\mu} \sigma_{\alpha\beta} A^\alpha_\mu A^\beta_\mu,$$

with the mass matrix

$$e^2\sigma_{\alpha\beta} = e^2 \sum_{i,j,k} t^\alpha_{ij} v_j t^\beta_{ik} v_k \,.$$

The matrix $\sigma_{\alpha\beta}$ is positive and has a rank equal to the number of generators of $\mathcal{L}(G/H)$, which is also the number of fields θ^α, that is, the number of would-be Goldstone bosons. We conclude that the spontaneous breaking of the G-symmetry generates no Goldstone bosons but instead gives masses to all gauge fields except those which are associated with the unbroken subgroup H. In particular, when the symmetry is completely broken, all components of the gauge field acquire a mass.

If one considers directly the classical action obtained after the gauge transformation associated with group element (14.42), the set of massive vector fields can be quantized in a completely standard way. However, as in the Abelian case the quantized theory is not renormalizable.

14.5.2 The $G \times G$ symmetry

Another possibility is that the symmetry group of the action is the direct product of the gauge group G by another group G'. We here consider only the simplest example where the symmetry group is $G \times G$ and G is simple. We assume that the scalar boson field $\boldsymbol{\phi}$ is a matrix transforming under $G \times G$ by

$$\boldsymbol{\phi}' = \mathbf{g}_1 \boldsymbol{\phi} \mathbf{g}_2^{-1},$$

in which $\mathbf{g}_1$ and $\mathbf{g}_2$ are two elements of G in a matrix representation and only the group acting on the left is gauged:

$$\mathbf{D}_\mu \boldsymbol{\phi} = \left(\mathbf{1}\partial_\mu + \mathbf{A}_\mu\right) \boldsymbol{\phi}\,.$$

We further assume that one minimum of the potential is proportional to the unit matrix $\boldsymbol{\phi} = v\mathbf{1}$ in such a way that the subgroup H is isomorphic to G with elements of the form $(\mathbf{g}, \mathbf{g})$.

As before the would-be Goldstone bosons correspond to gauge transformations and can thus be eliminated from the action. In this example all components of the gauge field acquire the same mass m_A because the symmetry corresponding to the gauge field is completely broken and a remaining diagonal G symmetry survives:

$$\mathrm{tr}\sum_{\mu}\left(\mathbf{D}_{\mu}^{T}\boldsymbol{\phi}\right)\left(\mathbf{D}_{\mu}\boldsymbol{\phi}\right)|_{\boldsymbol{\phi}=v\mathbf{1}} = -v^2\,\mathrm{tr}\sum_{\mu}\mathbf{A}_{\mu}^{2}\ \Rightarrow\ m_A = |ev|.$$

The action obtained after the gauge transformation specified by equation (14.42) contains only physical degrees of freedom and the quantization of all vector fields is straightforward, hence the name of unitary gauge. From the point of view of the initial theory the gauge has been completely fixed. We have constructed an action for massive vector fields transforming under the adjoint representation of a symmetry group G. However, the corresponding field theory is not renormalizable by power counting.

The difference with the massive vector field theory is that the suitable addition of some scalar fields makes this theory equivalent, at least for physical observables, with a renormalizable theory with additional unphysical degrees of freedom.

Remark. If we formally take the non-linear model limit, that is, send the masses of all remaining scalar fields towards infinity at v fixed, we obtain an action for a self-interacting massive vector field.

14.5.3 The $SU(2) \times SU(2)$ example

We discuss more specifically the important example of the $SU(2)$ group because it can be considered as a simplified version of the Standard Model of weak-electromagnetic interactions that will be described in section 15.1. We choose for scalar field $\boldsymbol{\phi}$ a 2×2 complex matrix transforming under the $(1/2,1/2)$ representation of $SU(2) \times SU(2)$.

We know that the representation can be reduced and we restrict the matrix to the form (see section 12.5):

$$\boldsymbol{\phi} = \tfrac{1}{2}\left(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right), \tag{14.43}$$

in which σ and $\boldsymbol{\pi}$ are real fields and the $\boldsymbol{\tau}$ matrices are identical to the $\boldsymbol{\sigma}$ Pauli matrices defined in section A11.2.

We represent the gauge field as a three-component real vector $\mathbf{A}_\mu$ and the covariant derivative acts like

$$\mathbf{D}_\mu\boldsymbol{\phi} = \left(\partial_\mu + \tfrac{1}{2}i\mathbf{A}_\mu\cdot\boldsymbol{\tau}\right)\boldsymbol{\phi}\,.$$

We define the curvature tensor $\mathbf{F}_{\mu\nu}$ by

$$[\mathbf{D}_\mu,\mathbf{D}_\nu] = \tfrac{1}{2}i\mathbf{F}_{\mu\nu}\cdot\boldsymbol{\tau} \;\Rightarrow\; \mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - \mathbf{A}_\mu\times\mathbf{A}_\nu\,.$$

The simplest action can then be written as

$$\mathcal{S}(\mathbf{A},\boldsymbol{\phi}) = \int \mathrm{d}^4x\left[\frac{1}{4e^2}\sum_{\mu,\nu}\mathbf{F}^2_{\mu\nu} + \mathrm{tr}\sum_\mu\left(\mathbf{D}_\mu\boldsymbol{\phi}\right)^\dagger\mathbf{D}_\mu\boldsymbol{\phi} + r\,\mathrm{tr}\,\boldsymbol{\phi}\boldsymbol{\phi}^\dagger + \frac{\lambda}{6}\left(\mathrm{tr}\,\boldsymbol{\phi}\boldsymbol{\phi}^\dagger\right)^2\right]. \tag{14.44}$$

In the $(\sigma, \boldsymbol{\pi})$ parametrization, infinitesimal gauge transformations take the form

$$\begin{cases} \delta \mathbf{A}_\mu = \partial_\mu \boldsymbol{\omega} - \mathbf{A}_\mu \times \boldsymbol{\omega}\,, \\ \quad \delta\sigma = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\pi}\,, \\ \quad \delta\boldsymbol{\pi} = -\frac{1}{2}\sigma\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\pi}\,. \end{cases}$$

The scalar field action in these variables becomes

$$\mathcal{S}_{\text{scalar}} = \int \mathrm{d}^4x \Big[\frac{1}{2}\sum_\mu \left(\partial_\mu\sigma - \frac{1}{2}\boldsymbol{\pi}\cdot\mathbf{A}_\mu\right)^2 + \frac{1}{2}\sum_\mu\left(\partial_\mu\boldsymbol{\pi} + \frac{1}{2}\sigma\mathbf{A}_\mu - \frac{1}{2}\mathbf{A}_\mu\times\boldsymbol{\pi}\right)^2 + \tilde{V}(\sigma^2+\boldsymbol{\pi}^2)\Big] \tag{14.45}$$

with

$$\tilde{V}(s) = \frac{1}{2}rs + \frac{1}{24}\lambda s^2.$$

Note that for the potential $\tilde{V}$, $SU(2)\times SU(2)$ symmetry implies $O(4)$ symmetry. The two groups are locally isomorphic.

As we have already discussed in section 12.5, if the potential $\tilde{V}$ has degenerate classical minima, the field $\boldsymbol{\phi}$ has a non-zero expectation value. Without loss of generality we choose the expectation value of $\boldsymbol{\phi}$ to be proportional to the unit matrix and, thus, the component σ to have a non-zero expectation value:

$$\langle \sigma \rangle = v > 0\,.$$

Then the symmetry $SU(2) \times SU(2)$ is broken down to the diagonal $SU(2)$ group. In the absence of gauge fields, the $\boldsymbol{\pi}$-field becomes a massless Goldstone boson. Here the $\boldsymbol{\pi}$-field can be eliminated by a gauge transformation, in such a way that the total action written in the unitary gauge becomes

$$\mathcal{S}(\mathbf{A}, \sigma) = \int \mathrm{d}^4 x \left[\frac{1}{4e^2} \sum_{\mu,\nu} \mathbf{F}^2_{\mu\nu} + \frac{1}{2} \sum_{\mu} (\partial_\mu \sigma)^2 + \frac{1}{8} \sum_{\mu} \sigma^2 \mathbf{A}^2_\mu + \frac{r}{2} \sigma^2 + \frac{\lambda}{24} \sigma^4 \right].$$

This action has an $O(3)$ symmetry. From the point of view of the $O(3)$ group the gauge field $\mathbf{A}$ is a three-vector and the field σ a scalar. In the classical approximation, the gauge field mass m_A is given in terms of the σ-field expectation value v by

$$m_A = |ev|/2\,,\ \ m_\sigma = \sqrt{\lambda/3}\,v \text{ and thus } \ m_\sigma/m_A = \sqrt{4\lambda/3}/|e|\,. \qquad (14.46)$$

14.5.4 Gauge fixing of the Higgs model in a covariant gauge

If we consider the contribution (14.45) to the action, we see that when the σ-field has an expectation value, a term of the form $\sum_\mu \partial_\mu \boldsymbol{\pi}\cdot\mathbf{A}_\mu$ is generated which introduces a mixing between the would-be Goldstone boson $\boldsymbol{\pi}$ and the longitudinal part of the vector field. This is a feature already encountered in the Abelian case (section 13.12). As suggested by 't Hooft, it is possible to use the gauge function to eliminate such a term. In the $SU(2)$ example we can take as gauge function

$$F(\mathbf{A},\boldsymbol{\pi}) = \sum_\mu \partial_\mu \mathbf{A}_\mu + \tfrac{1}{2}\lambda\xi\boldsymbol{\pi}\,.$$

After a Gaussian integration, the corresponding contribution to the action is

$$\mathcal{S}_F = \mathcal{S}_{\text{gauge}} + \mathcal{S}_{\text{ghost}}$$

with

$$\mathcal{S}_{\text{gauge}} = \frac{1}{2\xi e^2} \int \mathrm{d}^4 x \left(\sum_\mu \partial_\mu \mathbf{A}_\mu + \tfrac{1}{2} \lambda \xi \boldsymbol{\pi} \right)^2$$

and

$$\mathcal{S}_{\text{ghost}} = \int \mathrm{d}^4 x \left[\sum_\mu \partial_\mu \mathbf{C} \cdot (\partial_\mu \bar{\mathbf{C}} - \mathbf{A}_\mu \times \bar{\mathbf{C}}) + \frac{\lambda \xi}{4} \mathbf{C} \left(\sigma \bar{\mathbf{C}} + \boldsymbol{\pi} \times \bar{\mathbf{C}} \right) \right].$$

At leading order the term $\sum_\mu \partial_\mu \boldsymbol{\pi} \cdot \mathbf{A}_\mu$ is eliminated by the choice

$$\lambda = e^2 v \,.$$

This gauge has two advantages: it decouples the gauge field from the would-be Goldstone field and, therefore, simplifies the propagators; by explicitly breaking the global $SU(2) \times SU(2)$-symmetry, it generates a mass for the $\boldsymbol{\pi}$-field which is no longer a Goldstone boson. In this gauge the propagators (equations (13.75)) have no poles at zero momentum and no IR problems are encountered:

$$W^{(2)}_{\mu\nu} = \frac{e^2 \delta_{\mu\nu}}{k^2 + m_A^2} + \frac{e^2(\xi - 1)k_\mu k_\nu}{(k^2 + m_A^2)(k^2 + \xi m_A^2)},$$
$$W^{2)}_{\boldsymbol{\pi}\boldsymbol{\pi}} = \frac{1}{k^2 + \xi m_A^2}, \quad W^{(2)}_{\mathbf{C}\bar{C}} = \frac{1}{k^2 + \xi m_A^2},$$

in which m_A is the mass of $\mathbf{A}_\mu$ in the classical approximation (equation (14.46)). Furthermore, all unphysical states have a mass which explicitly depends on the gauge parameter ξ.

Unitarity. This property can be used to prove unitarity of the physical S-matrix: the S-matrix satisfies a generalized unitarity relation in which in the intermediate states one must include all particles both physical and unphysical. By showing that the S-matrix does not depend on the gauge, one proves simultaneously that the contributions of unphysical states cancels in the intermediate states and thus the S-matrix is unitary in the physical subspace.

APPENDIX A14
BRST SYMMETRY

A14.1 BRST symmetry: the origin

We first explain the origin of BRST symmetry in a rather general context. In particular, we show that BRST symmetry in gauge theories owe less to gauge symmetry than to gauge fixing.

For notational simplicity, we write the variables and the equations with a discrete index α, but the generalization to differential or functional equations is straightforward, summation over α being replaced by integration and summation, and differentiation by functional differentiation.

Equations and integral representations. Let φ^α be a set of dynamical variables satisfying a system of equations,

$$E_\alpha(\varphi) = 0\,. \tag{A14.1}$$

We assume that the functions $E_\alpha(\varphi)$ are smooth and that $E_\alpha = E_\alpha(\varphi)$ is a one-to-one mapping in some neighbourhood of $E_\alpha = 0$, which can be inverted in $\varphi^\alpha = \varphi^\alpha(E)$. In particular, this implies that the equation has a unique solution $\varphi^\alpha_\mathrm{s} \equiv \varphi^\alpha(0)$.

Moreover, in the neighbourhood of φ_s, the determinant $\det \mathbf{E}$ of the matrix $\mathbf{E}$ with elements

$$E_{\alpha\beta} \equiv \partial_\beta E_\alpha$$

does not vanish and thus one can choose $E_\alpha(\varphi)$ such that it is positive.

For any function $F(\varphi)$, one can derive a simple formal expression for $F(\varphi_{\mathrm{s}})$ that does not involve solving the equation explicitly. One starts from the obvious identity

$$F(\varphi_{\mathrm{s}}) = \int \left\{ \prod_{\alpha} \mathrm{d}E^{\alpha}\, \delta(E_{\alpha}) \right\} F\big(\varphi(E)\big),$$

where $\delta(E)$ is Dirac's δ-function. One then changes variables $E \mapsto \varphi$. This generates the Jacobian $\det \mathbf{E} > 0$. Thus,

$$F(\varphi_{\mathrm{s}}) = \int \left\{ \prod_{\alpha} \mathrm{d}\varphi^{\alpha}\, \delta\left[E_{\alpha}(\varphi)\right] \right\} \det \mathbf{E}(\varphi)\, F(\varphi). \qquad (A14.2)$$

In the context of non-Abelian gauge theories, $\det \mathbf{E}$ is the Faddeev–Popov determinant.

A14.1.1 An invariant measure: ST identities

A precursor of BRST symmetry is a simple property of the integration measure in equation $(A14.2)$.

The integration measure $\prod_\alpha \mathrm{d}E_\alpha$ is an invariant measure for the group of translations $E_\alpha \mapsto E_\alpha + \nu_\alpha$, ν_α constant. It follows that the measure

$$\mathrm{d}\rho(\varphi) = \det \mathbf{E}(\varphi) \prod_\alpha \mathrm{d}\varphi^\alpha \tag{A14.3}$$

is an invariant measure for the translation group realized non-linearly on the new coordinates φ_α (provided ν_α is small enough):

$$\varphi^\alpha \mapsto \varphi'^\alpha \quad \text{with} \quad E_\alpha(\varphi') - \nu_\alpha = E_\alpha(\varphi). \tag{A14.4}$$

The infinitesimal form of the transformation can be written more explicitly as

$$\delta\varphi^\alpha = \sum_\beta [E^{-1}(\varphi)]^{\alpha\beta} \nu_\beta \,. \tag{A14.5}$$

This rather straightforward property of the measure and the corresponding infinitesimal transformations $(A14.5)$ have been used in the context of non-Abelian gauge theories by Slavnov and, independently, Taylor to derive a set of important identities satisfied by Green's functions, thus called Slavnov–Taylor identities.

These identities form the basis of the first proof by Lee and Zinn-Justin of the renormalizability of non-Abelian gauge theories in the broken phase and, therefore, that the weak-electromagnetic theory had a consistent perturbative expansion.

Reciprocal property. Conversely, one can characterize the general form of non-linear representations of the translation group. One recovers the form of the previous measure.

A14.1.2 BRST symmetry

In quantum field theory, the non-linear and non-local character of the transformations ($A14.5$) is the source of technical complications. Remarkably enough, the invariance under the infinitesimal transformations ($A14.5$) can be replaced by an invariance under linear anticommuting-type transformations at the expense of introducing additional variables.

One again starts from the identity ($A14.2$) and first replaces the δ-function by its Fourier representation,

$$\prod_{\alpha} \delta\left[E_{\alpha}(\varphi)\right] = \int \prod_{\alpha} \frac{\mathrm{d}\bar{\lambda}^{\alpha}}{2i\pi} \exp\left[\sum_{\alpha} -\bar{\lambda}^{\alpha} E_{\alpha}(\varphi)\right],$$

where the $\bar{\lambda}$ integration runs along the imaginary axis.

The determinant can be written as an integral over Grassmann variables $\bar{c}^{\alpha}$ and c^{α}:

$$\det \mathbf{E} = \int \prod_{\alpha} (\mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha}) \exp\Big(\sum_{\alpha,\beta} c^{\alpha} E_{\alpha\beta} \bar{c}^{\beta}\Big).$$

In the context of non-Abelian gauge theories, $\mathbf{c}$ and $\bar{\mathbf{c}}$ are the Faddeev–Popov ghosts.

The expression $(A14.2)$ then becomes

$$\mathcal{F}(\varphi_{\mathrm{s}}) = \mathcal{N} \int \prod_{\alpha} \left(\mathrm{d}\varphi^{\alpha} \mathrm{d}\bar{\lambda}^{\alpha} \mathrm{d}\bar{c}^{\alpha} \mathrm{d}c^{\alpha}\right) F(\varphi) \exp\left[-S(\varphi, \bar{\lambda}, c, \bar{c})\right], \tag{A14.6}$$

in which $\mathcal{N}$ is a constant normalization factor such that for $F(\varphi) \equiv 1$, $F(\varphi_{\mathrm{s}}) = 1$, and $S(\varphi, \bar{\lambda}, c, \bar{c})$ the function (and element of the Grassmann algebra)

$$S(\varphi, \bar{\lambda}, c, \bar{c}) = \sum_{\alpha} \bar{\lambda}^{\alpha} E_{\alpha}(\varphi) - \sum_{\alpha,\beta} c^{\alpha} E_{\alpha\beta}(\varphi) \bar{c}^{\beta}. \tag{A14.7}$$

A14.1.3 BRST symmetry

Somewhat surprisingly, the function S has a new type of symmetry, the BRST symmetry, extension to Grassmann variables of the invariance of the measure $\det \mathbf{E}(\varphi)\mathrm{d}\boldsymbol{\varphi}$ under the infinitesimal transformations ($A14.5$).

The BRST symmetry, first discovered in the context of quantized gauge theories by Becchi, Rouet, Stora, and Tyutin, is a Grassmann symmetry in the sense that the parameter ε of the transformation is an anticommuting constant, an additional generator of the Grassmann algebra. The variations of the various integration variables are

$$\delta\varphi^\alpha = \varepsilon\bar{c}^\alpha\,, \qquad \delta\bar{c}^\alpha = 0\,,$$
$$\delta c^\alpha = \varepsilon\bar{\lambda}^\alpha\,, \qquad \delta\bar{\lambda}^\alpha = 0$$

with

$$\varepsilon^2 = 0\,, \qquad \varepsilon\bar{c}^\alpha + \bar{c}^\alpha\varepsilon = 0\,, \qquad \varepsilon c^\alpha + \varepsilon c^\alpha = 0\,.$$

A14.1.4 BRST transformations: Differential operator representation

The BRST transformation, when acting on functions of $\{\varphi, \bar{\lambda}, c, \bar{c}\}$, can be represented by the Grassmann differential operator

$$\mathcal{D} \equiv \sum_{\alpha} \left(\bar{c}^{\alpha} \frac{\partial}{\partial \varphi^{\alpha}} + \bar{\lambda}^{\alpha} \frac{\partial}{\partial c^{\alpha}} \right). \qquad (A14.8)$$

The BRST operator satisfies the identity

$$\mathcal{D}^2 = 0 \,. \qquad (A14.9)$$

The differential operator $\mathcal{D}$ has the form of a cohomology operator, generalization of the exterior differentiation of differential forms. In particular, the first term $\sum_{\alpha} \bar{c}^{\alpha} \partial / \partial \varphi_{\alpha}$ in the BRST operator is identical to the differentiation of forms in a formalism in which the Grassmann variables $\bar{c}^{\alpha}$ generate the corresponding exterior algebra.

A BRST invariant function, which thus satisfies

$$\mathcal{D}S = 0\,, \tag{A14.10}$$

is called BRST closed.

The equation $\mathcal{D}^2 = 0$ implies that all quantities of the form $\mathcal{D}Q(\varphi, \bar{\lambda}, c, \bar{c})$, quantities called BRST exact, are BRST invariant (BRST closed).

One immediately verifies that the function S defined by equation (A14.7) is BRST exact since

$$S = \mathcal{D} \sum_{\alpha} c^{\alpha} E_{\alpha}(\varphi). \tag{A14.11}$$

The reciprocal property, the meaning and implications of the BRST symmetry rely on considerations of BRST cohomology.

These properties play an important role, in particular, in the discussion of the renormalization of non-Abelian gauge theories, including gauge invariant operators.

A14.1.5 BRST symmetry and group manifolds

When the variables φ^α parametrize an element $\mathbf{g}(\varphi)$ of a Lie group in some matrix representation, it is convenient to express BRST transformations on $\mathbf{g}(\varphi)$ directly and to parametrize the variation of $\mathbf{g}$ in terms of a Grassmann matrix $\bar{\mathbf{C}}$ belonging to the Lie algebra of the group:

$$\delta\mathbf{g} = \varepsilon\bar{\mathbf{C}}\mathbf{g}\,. \tag{A14.12}$$

Thus,

$$\bar{\mathbf{C}} = \sum_\alpha \bar{\mathbf{c}}_\alpha \frac{\partial\mathbf{g}}{\partial\varphi_\alpha}\mathbf{g}^{-1}.$$

A short calculation show that the variation of $\bar{\mathbf{C}}$ then is

$$\delta\bar{\mathbf{C}} = -\sum_{\alpha,\beta} \bar{\mathbf{c}}_\alpha \frac{\partial\mathbf{g}}{\partial\varphi_\alpha}\mathbf{g}^{-1}\varepsilon\bar{\mathbf{c}}_\beta \frac{\partial\mathbf{g}}{\partial\varphi_\beta}\mathbf{g}^{-1} = \varepsilon\bar{\mathbf{C}}^2.$$

The set of BRST transformations then becomes

$$\delta\mathbf{g} = \varepsilon\bar{\mathbf{C}}\mathbf{g}\,,\ \delta\bar{\mathbf{C}} = \varepsilon\bar{\mathbf{C}}^2, \quad \delta\mathbf{C} = \varepsilon\bar{\boldsymbol{\lambda}}\,,\ \delta\bar{\boldsymbol{\lambda}} = 0\,. \tag{A14.13}$$

One recognizes expressions that appears in non-Abelian gauge theories, the group element being there associated with gauge transformations and the transformation ($A14.13$) being an infinitesimal gauge transformation.