Lecture 4: BOSONS IN THE HOLOMORPHIC REPRESENTATION

In this lecture, we introduce a description of quantum mechanics in terms of a Hilbert space of square integrable analytic functions, called the holomorphic representation. We then construct the corresponding path integral representation of the statistical operator and partition function.

The holomorphic formalism is specially well adapted to a study of the harmonic oscillator and, more generally, of perturbed harmonic oscillators. Nevertheless, one may wonder whether it is really useful to construct another representation of the harmonic oscillator, which many other methods, including the usual path integral formalism, already allows discussing quite thoroughly.

Actually, the main motivation comes from quantum N-particle problems (one speaks also of N-body problems). It is based on a characteristic property of the harmonic oscillator, the additivity of its spectrum: the energy of the level N of the harmonic oscillator is the sum of the ground state energy and N times the splitting between two neighbouring levels.

This energy thus can also be considered as the sum of a vacuum energy and the total energy of N identical and independent particles of boson type. The perturbations to the harmonic oscillator then correspond to interactions between particles.

This gives an intuitive understanding of the role of generalized holomorphic path integrals (field integrals) in representations of the partition function of the quantum Bose gas and, in quantum field theory, of boson scattering or S-matrices, which we will discuss in next lecture.

The holomorphic formalism is based on the properties of a certain type of integrals over complex variables, which we first recall.

4.1 Formal complex integration and Wick's theorem

Let us consider an integral over the plane,

$$I = \int_{\mathbb{R}^2} \mathrm{d}p\, \mathrm{d}q\, f(p, q),$$

in which the integrand f is an entire function of p and q. Then, one can embed the $\mathbb{R}^2$ plane into the complex $\mathbb{C}^2$ space, and consider the variables p, q as complex variables, the initial domain of integration being defined by $\mathrm{Im}\, p = \mathrm{Im}\, q = 0$. In $\mathbb{C}^2$, it may sometimes be convenient to introduce another pair of variables $(z, \bar{z})$ related to (p, q) by the unitary transformation,

$$z = (p + iq)/\sqrt{2}\,, \;\; \bar{z} = (p - iq)/\sqrt{2} \;\Rightarrow\; \mathrm{d}\bar{z}\, \mathrm{d}z = i \mathrm{d}p\, \mathrm{d}q\,. \tag{4.1}$$

In the new variables, the domain of integration is then defined by $\bar{z} = z^*$, where z^* denotes the complex conjugate of z, and the integral becomes

$$I = -i \int_{\bar{z}=z^*} \mathrm{d}z\, \mathrm{d}\bar{z}\, f\big(p(z, \bar{z}), q(z, \bar{z})\big).$$

In particular, these new variables are useful when rotations are involved since a rotation of angle θ acting on the vector (p, q) is represented by the $U(1)$ transformations $z \mapsto z\,\mathrm{e}^{i\theta}$, $\bar{z} \mapsto \mathrm{e}^{-i\theta}\,\bar{z}$. A rotation invariant function is then a function only of the product $z\bar{z}$.

4.1.1 Formal complex conjugation

Above we have denoted by * the usual conjugation of complex numbers. We introduce now a slightly different conjugation: we define a formal complex conjugation where numbers are still replaced by complex conjugate but the two integration variables are exchanged, $z \leftrightarrow \bar{z}$.

We stress that z and $\bar{z}$ are independent integration variables and are complex conjugate only in a formal sense: the integration contours over the variables p and q can be deformed in complex space where p and q also take complex values. The symbol $\mathrm{d}z\mathrm{d}\bar{z}$ corresponds to an integration over a surface of real dimension 2, embedded in $\mathbb{C}^2$.

Applying the definition of formal complex conjugation to power series, $f \mapsto \bar{f}$, one obtains

$$f(z,\bar{z}) = \sum_{m,n} f_{mn} z^m \bar{z}^n, \quad f_{mn} \in \mathbb{C} \;\Rightarrow\; \overline{f(z,\bar{z})} \equiv \sum_{m,n} f^*_{mn} \bar{z}^m z^n,$$

where f^*_{mn} denotes the usual complex conjugate of f_{mn}.

A function such that $f = \bar{f}$ is called formally real.

4.1.2 Gaussian integrals

The simplest Gaussian integral that has an integrand invariant under rotations in the plane is

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}p \, \mathrm{d}q \; \mathrm{e}^{-a(p^2+q^2)/2} = \int \frac{\mathrm{d}z \mathrm{d}\bar{z}}{2i\pi} \, \mathrm{e}^{-a\bar{z}z} = \frac{1}{a}\,,$$

where, in the initial variables (p,q), one integrates over the real plane and $\mathrm{Re}\, a > 0$.

More generally, we consider Gaussian $2n$-dimensional integrals of the form

$$\mathcal{Z}_0(\mathbf{A}) = \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi} \right) \mathrm{e}^{-\mathcal{A}_0(\bar{\mathbf{z}},\mathbf{z})},$$

where $\mathcal{A}_0(\bar{\mathbf{z}}, \mathbf{z})$ is the quadratic form

$$\mathcal{A}_0(\bar{\mathbf{z}}, \mathbf{z}) = \sum_{i,j=1}^{n} \bar{z}_i A_{ij} z_j \,, \ \det \mathbf{A} \neq 0$$

and the matrix $\mathbf{A}$ with elements A_{ij} is complex. Because terms of the form zz and $\bar{z}\bar{z}$ are absent, the integral is left invariant by the $U(1)$ transformation

$$z_i \mapsto z_i \,\mathrm{e}^{i\theta}, \ \bar{z}_i \mapsto \bar{z}_i \,\mathrm{e}^{-i\theta} \,. \tag{4.2}$$

If $\mathbf{A}$ is a hermitian positive matrix, it can be diagonalized by a unitary transformation:

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger,$$

where $\mathbf{U}$ is a unitary matrix and $\mathbf{D}$ is the diagonal matrix with eigenvalues $a_i > 0$. The change of variables $(z, \bar{z}) \mapsto (z', \bar{z}')$ with

$$z_i = \sum_j U_{ij} z'_j\,, \quad \bar{z}_i = \sum_j U^*_{ij} \bar{z}'_j\,,$$

has a Jacobian $|\det \mathbf{U}|^2 = 1$ and, thus, the integral becomes

$$\mathcal{Z}_0(\mathbf{A}) = \prod_i \int \frac{\mathrm{d}z'_i \mathrm{d}\bar{z}'_i}{2i\pi}\, \mathrm{e}^{-a_i \bar{z}'_i z'_i} = \prod_i \frac{1}{a_i} = \frac{1}{\det \mathbf{A}}\,.$$

Remarks

(*i*) If the matrix $\mathbf{A}$ is Hermitian, the quadratic form $\mathcal{A}_0(\bar{\mathbf{z}}, \mathbf{z})$ is formally real. Indeed,

$$\overline{\mathcal{A}_0(\bar{\mathbf{z}}, \mathbf{z})} = \sum_{i,j}^{n} z_i A_{ij}^* \bar{z}_j = \sum_{i,j}^{n} \bar{z}_i [A^\dagger]_{ij} z_j = \sum_{i,j}^{n} \bar{z}_i A_{ij} z_j \,.$$

We note that when the quadratic form $\mathcal{A}_0(\bar{\mathbf{z}}, \mathbf{z})$ is formally real, the integral is real.

(*ii*) In contrast with real Gaussian integrals, the result is a rational function of matrix elements and can thus be extended by analytic continuation to arbitrary complex matrices.

Actually, from a purely algebraic viewpoint, the result can be obtained directly by performing the change of variables $z_i \mapsto z'_i = \sum_j A_{ij} z_j$, $\bar{z}_i \mapsto \bar{z}_i$.

The Jacobian is $1/\det \mathbf{A}$ and $\mathcal{A}_0(\bar{\mathbf{z}},\mathbf{z})$ becomes $\sum_i \bar{z}_i z_i$. One then obtains

$$\mathcal{Z}_0(\mathbf{A}) = \frac{1}{\det \mathbf{A}} \prod_{i=1}^{n} \int \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi}\, \mathrm{e}^{-\bar{z}_i z_i} = \frac{1}{\det \mathbf{A}}. \tag{4.3}$$

4.1.3 Gaussian expectation values: generating function

We define expectation values with the Gaussian weight $\mathrm{e}^{-\mathcal{A}_0}/\mathcal{Z}_0(\mathbf{A})$ as

$$\left\langle \bar{z}_{i_1} \ldots \bar{z}_{i_p} z_{j_1} \ldots z_{j_q} \right\rangle_0 = \det \mathbf{A} \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi} \right) \bar{z}_{i_1} \ldots \bar{z}_{i_p} z_{j_1} \ldots z_{j_q}\, \mathrm{e}^{-\mathcal{A}_0(\bar{\mathbf{z}},\mathbf{z})}\,.$$

The invariance under the transformation (4.2), $z_i \mapsto z_i\,\mathrm{e}^{i\theta}$, $\bar{z}_i \mapsto \bar{z}_i\,\mathrm{e}^{-i\theta}$, of $\mathcal{A}_0$ implies that only the monomials with an equal number of z and $\bar{z}$ factors have a non-vanishing Gaussian expectation value. Indeed, after the change of variables $z_i \mapsto \mathrm{e}^{i\theta}\, z_i$, $\bar{z}_i \mapsto \mathrm{e}^{-i\theta}\, \bar{z}_i$ with Jacobian 1, one obtains,

$$\left\langle \bar{z}_{i_1} \ldots \bar{z}_{i_p} z_{j_1} \ldots z_{j_q} \right\rangle_0 = \mathrm{e}^{i(q-p)\theta} \left\langle \bar{z}_{i_1} \ldots \bar{z}_{i_p} z_{j_1} \ldots z_{j_q} \right\rangle_0 = 0 \quad \text{if} \quad p \neq q\,.$$

We consider now the more general Gaussian integral

$$\mathcal{Z}_0(\mathbf{A};\mathbf{b},\bar{\mathbf{b}}) = \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi}\right) \exp\left[-\mathcal{A}_0(\bar{\mathbf{z}},\mathbf{z}) + \bar{\mathbf{b}}\cdot\mathbf{z} + \bar{\mathbf{z}}\cdot\mathbf{b}\right], \tag{4.4}$$

where the $\mathbf{b}$ and $\bar{\mathbf{b}}$ are two independent sets of arguments.

The integral is a generating function of Gaussian expectation values, as one verifies by differentiating with respect to $\mathbf{b}$ and $\bar{\mathbf{b}}$ and, after differentiation, setting $\bar{\mathbf{b}} = \mathbf{b} = 0$:

$$\left\langle \bar{z}_{i_1} \ldots \bar{z}_{i_p} z_{j_1} \ldots z_{j_p} \right\rangle_0 = \det \mathbf{A}\, \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_p}} \frac{\partial}{\partial \bar{b}_{j_1}} \cdots \frac{\partial}{\partial \bar{b}_{j_p}} \mathcal{Z}_0(\mathbf{A};\mathbf{b},\bar{\mathbf{b}})\Big|_{b=\bar{b}=0}. \tag{4.5}$$

Explicit calculation. To eliminate the terms linear in $\mathbf{z}$ and $\bar{\mathbf{z}}$ in the integral (4.4), one first solves the equations

$$\frac{\partial}{\partial z_i}\left[\mathcal{A}_0(\bar{\mathbf{z}},\mathbf{z}) - \bar{\mathbf{b}}\cdot\mathbf{z} - \bar{\mathbf{z}}\cdot\mathbf{b}\right] = \sum_j \bar{z}_j A_{ji} - \bar{b}_i = 0 \;\Rightarrow\; \bar{z}_i = \sum_j \bar{b}_j \Delta_{ji}\,,$$

$$\frac{\partial}{\partial \bar{z}_i}\left[\mathcal{A}_0(\bar{\mathbf{z}},\mathbf{z}) - \bar{\mathbf{b}}\cdot\mathbf{z} - \bar{\mathbf{z}}\cdot\mathbf{b}\right] = \sum_j A_{ij} z_j = b_i = 0 \;\Rightarrow\; z_i = \sum_j \Delta_{ij} b_j$$

with the notation $\boldsymbol{\Delta} = \mathbf{A}^{-1}$. One then change variables $\{z_i, \bar{z}_i\} \mapsto \{v_i, \bar{v}_i\}$, shifting $\mathbf{z}$, $\bar{\mathbf{z}}$ by the solutions:

$$z_i = \sum_j \Delta_{ij} b_j + v_i\,,\;\; \bar{z}_i = \sum_j \bar{b}_j \Delta_{ji} + \bar{v}_i\,.$$

The resulting Gaussian integral over the variables $\mathbf{v}, \bar{\mathbf{v}}$ is simply the integral (4.3) and, thus,

$$\mathcal{Z}_0(\mathbf{A};\mathbf{b},\bar{\mathbf{b}}) = (\det \mathbf{A})^{-1} \exp\left(\sum_{i,j=1}^{n} \bar{b}_i \Delta_{ij} b_j\right). \tag{4.6}$$

4.1.4 Gaussian expectation values and Wick's theorem

This result allows calculating expectation values with the Gaussian weight $\mathrm{e}^{-\mathcal{A}_0}/\mathcal{Z}_0(\mathbf{A})$ by substituting the result (4.6) into the expression (4.5):

$$\left\langle \bar{z}_{i_1} z_{j_1} \ldots \bar{z}_{i_p} z_{j_p} \right\rangle_0 = \frac{\partial}{\partial b_{i_1}} \frac{\partial}{\partial \bar{b}_{j_1}} \cdots \frac{\partial}{\partial b_{i_p}} \frac{\partial}{\partial \bar{b}_{j_p}} \exp\left(\sum_{i,j=1}^{n} \bar{b}_i \Delta_{ij} b_j \right) \Bigg|_{b=\bar{b}=0} . \tag{4.7}$$

Second cumulant. A basic quantity for Gaussian expectation values is the second cumulant (here equal to the second moment), which is given by

$$\langle \bar{z}_k z_\ell \rangle_0 = \frac{\partial}{\partial b_k} \frac{\partial}{\partial \bar{b}_\ell} \exp\left(\sum_{i,j=1}^{n} \bar{b}_i \Delta_{ij} b_j \right) \Bigg|_{b=\bar{b}=0} = \Delta_{\ell k} .$$

Wick's theorem. More generally, one can perform explicitly the differentiations with respect to $\mathbf{b}$ and $\bar{\mathbf{b}}$ in expression (4.7).

The only contributions that do not vanish in the limit $\mathbf{b} = \bar{\mathbf{b}} = 0$, are obtained by pairing in all possible ways a differentiation with respect to b with a differentiation with respect to $\bar{b}$. From this observation, one infers Wick's theorem for this type of complex integrals:

$$\langle \bar{z}_{i_1} z_{j_1} \ldots \bar{z}_{i_p} z_{j_p} \rangle_0 = \sum_{\substack{\text{all permutations} \\ P \text{ of } \{j_1,\ldots,j_p\}}} \Delta_{j_{P_1} i_1} \Delta_{j_{P_2} i_2} \ldots \Delta_{j_{P_p} i_p} \tag{4.8a}$$

$$= \sum_{\substack{\text{all permutations} \\ P \text{ of } \{j_1,\ldots,j_p\}}} \langle \bar{z}_{i_1} z_{j_{P_1}} \rangle_0 \langle \bar{z}_{i_2} z_{j_{P_2}} \rangle_0 \cdots \langle \bar{z}_{i_p} z_{j_{P_p}} \rangle_0 \,. \tag{4.8b}$$

For example, $\langle \bar{z}_i z_j \bar{z}_k z_l \rangle_0 = \langle \bar{z}_i z_j \rangle_0 \langle \bar{z}_k z_l \rangle_0 + \langle \bar{z}_i z_l \rangle_0 \langle \bar{z}_k z_j \rangle_0$.

For a Gaussian distribution, all expectation values can be derived from the second moment using the appropriate Wick's theorem.

4.2 Perturbative expansion

Wick's theorem simplifies the calculation of perturbed Gaussian integrals by expanding around the Gaussian approximation. We consider here integrals that enumerate the number of Feynman diagrams of a field theory with a complex scalar field and a $\frac{1}{4}\lambda(\phi\phi^*)^2$ interaction.

4.2.1 The normalization integral

The integral

$$Z(\lambda) = \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi} \right) \exp\left[-\mathcal{A}(\bar{\mathbf{z}}, \mathbf{z})\right] \tag{4.9}$$

with

$$\mathcal{A}(\bar{\mathbf{z}}, \mathbf{z}) = \mathcal{A}_0(\bar{\mathbf{z}}, \mathbf{z}) + \tfrac{1}{4}\lambda \sum_{i=1}^{n} \bar{z}_i^2 z_i^2 ,$$

calculated in the form of an expansion in powers of the parameter λ, counts the number of vacuum diagrams. It can be , .

With the notation $\langle \bullet \rangle_0$ for Gaussian expectation values, using Wick's theorem, one finds at order λ^2,

$$\begin{aligned}
Z(\lambda)/Z(0) &= 1 - \frac{\lambda}{4} \sum_i \left\langle \bar{z}_i^2 z_i^2 \right\rangle_0 + \frac{\lambda^2}{2!\, 4^2} \sum_{i,j} \left\langle \bar{z}_i^2 z_i^2 \bar{z}_j^2 z_j^2 \right\rangle_0 + O(\lambda^3) \\
&= 1 - \frac{\lambda}{2} \sum_i \Delta_{ii}^2 + \lambda^2 \sum_{i,j} \left(\tfrac{1}{8} \Delta_{ij}^2 \Delta_{ji}^2 + \tfrac{1}{2} \Delta_{ii} \Delta_{ij} \Delta_{jj} \Delta_{ji} \right) \\
&\quad + \frac{\lambda^2}{2!\, 4^2} \sum_{i,j} \left\langle \bar{z}_i^2 z_i^2 \right\rangle_0 \left\langle \bar{z}_j^2 z_j^2 \right\rangle_0 + O(\lambda^3).
\end{aligned}$$

As implied by a general result, the last and non-connected (factorizable) contribution cancels in $\ln Z$ since, if we expand the term of order λ we generate

$$\ln\bigl(Z(\lambda)/Z(0)\bigr) = -\frac{\lambda}{4} \sum_i \left\langle \bar{z}_i^2 z_i^2 \right\rangle_0 - \frac{\lambda^2}{2 \times 4^2} \sum_{i,j} \left\langle \bar{z}_i^2 z_i^2 \right\rangle_0 \left\langle \bar{z}_j^2 z_j^2 \right\rangle_0 + \cdots \,.$$

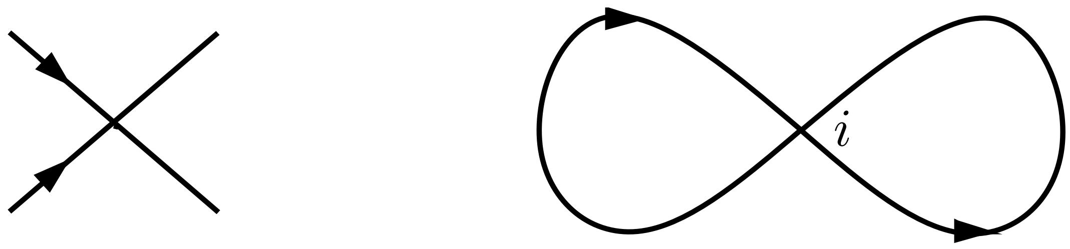

Fig. 4.1 – Feynman diagrams: vertex $(\bar{z}_i z_i)^2$ (left) and the contribution $\left\langle \bar{z}_i^2 z_i^2 \right\rangle_0$ of order λ (right) to the normalization $\mathcal{Z}(\lambda)$.

Note that, since $\langle \bar{z}_i z_j \rangle$ is not symmetric in ij, a faithful representation in terms of Feynman diagrams involves oriented lines, which go, for instance, from z to $\bar{z}$ (see Fig. 4.1).

4.2.2 The second moment

Another example, which counts the diagrams of the two-point function, is provided by the expansion of the expectation value $\langle \bar{z}_k z_\ell \rangle_\lambda$ corresponding to the normalized measure induced by the integrand (4.9).

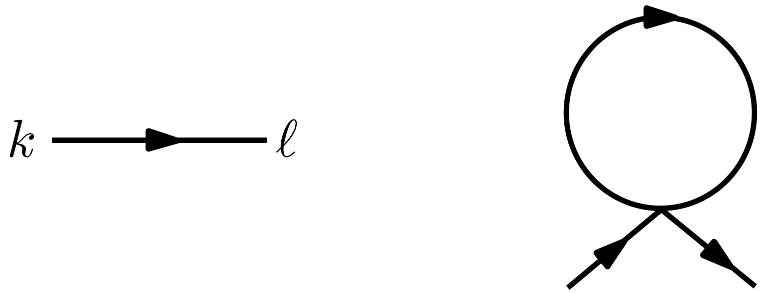

Fig. 4.2 – Feynman diagrams: contributions of order 1 and λ to $\langle \bar{z}_k z_\ell \rangle_\lambda$ to the second moment.

To order λ^2, the expansion is (see diagrams of Figs. 4.2 with oriented lines, up to one-loop)

$$\langle \bar{z}_k z_\ell \rangle_\lambda = \frac{Z(0)}{Z(\lambda)} \left[\Delta_{\ell k} - \frac{\lambda}{4} \sum_i \left\langle \bar{z}_k z_\ell \bar{z}_i^2 z_i^2 \right\rangle_0 + \frac{\lambda^2}{32} \sum_{i,j} \left\langle \bar{z}_k z_\ell \bar{z}_i^2 z_i^2 \bar{z}_j^2 z_j^2 \right\rangle_0 \right] + O(\lambda^3)$$

Using Wick's theorem, one obtains

$$\langle \bar{z}_k z_\ell \rangle_\lambda = \Delta_{\ell k} - \lambda \sum_i \Delta_{\ell i}\Delta_{ii}\Delta_{ik} + \lambda^2 \sum_{i,j} (\Delta_{\ell j}\Delta_{jj}\Delta_{ji}\Delta_{ii}\Delta_{ik}$$
$$+\Delta_{\ell i}\Delta_{ij}\Delta_{jj}\Delta_{ji}\Delta_{ik} + \tfrac{1}{2}\Delta_{\ell j}\Delta_{ji}\Delta_{ij}\Delta_{ji}\Delta_{ik}\big) + O\left(\lambda^3\right).$$

Note that all non connected contributions cancel after division by $Z(\lambda)$.

4.3 Quantum mechanics in the holomorphic representation

The holomorphic formalism originates from the idea of associating (complex) classical variables $z, \bar{z}$ to the creation and annihilation operators $a^\dagger, a$ that, for example, are introduced in the simplest algebraic solution of the harmonic oscillator.

4.3.1 Hilbert space of analytic functions

One considers the complex vector space of analytic entire functions endowed with a scalar product: the scalar product of two entire functions $f(z)$ and $g(z)$ is defined by

$$(g, f) = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2i\pi}\,\mathrm{e}^{-z\bar{z}}\;\overline{g(z)}\;f(z), \tag{4.10}$$

where $\bar{z}$ is the variable formally complex conjugate to z and the integration is defined as in section 4.1.

As we show below, the scalar product (f, f) defines a positive norm $\|f\| = (f, f)^{1/2}$. Entire functions with a finite norm form a Hilbert space, which we denote below by $\mathfrak{H}$.

One can then define operators acting on this Hilbert space, and, thus, construct a representation of quantum mechanics, called the holomorphic representation.

Orthonormal basis. The monomials $z^n/\sqrt{n!}$ form an orthonormal basis of the Hilbert space $\mathfrak{H}$ since

$$\int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi}\, \mathrm{e}^{-z\bar{z}}\, \bar{z}^n z^m = n!\, \delta_{mn}\,. \tag{4.11}$$

This result is an immediate consequence of Wick's theorem (4.8).

We can expand functions on this basis:

$$f(z) = \sum_{n \geq 0} f_n \frac{z^n}{\sqrt{n!}}\,, \quad g(z) = \sum_{n \geq 0} g_n \frac{z^n}{\sqrt{n!}}\,,$$

Then, the scalar product of these two functions is simply the generalization of the scalar product of complex vectors:

$$(g,f)=\sum_n g_n^* f_n\,.$$

In particular, the norm of a function is finite if

$$(f,f)=\sum_n |f_n|^2<\infty\,, \tag{4.12}$$

confirming that these functions belong to a subclass of entire functions.

δ-function. In the holomorphic formalism, the role of Dirac's δ-function is played by

$$\delta(z)\equiv\frac{1}{2i\pi}\int \mathrm{d}\bar z\,\mathrm{e}^{-\bar z z}\ . \tag{4.13}$$

Indeed, as a consequence of the orthogonality relations,

$$\int\frac{\mathrm{d}z\mathrm{d}\bar z}{2i\pi}\,\mathrm{e}^{-z\bar z}\,f(z)=\int\frac{\mathrm{d}z\mathrm{d}\bar z}{2i\pi}\,\mathrm{e}^{-z\bar z}\sum_{n=0}\frac{z^n}{n!}f^{(n)}(0)=f(0)\,.$$

4.3.2 Operators in the holomorphic representation

We now construct a representation of the operators of quantum mechanics acting on the Hilbert space $\mathfrak{H}$. The construction is based on two operators z and $\mathrm{d}/\mathrm{d}z$, acting by multiplication and differentiation on holomorphic functions: (in this lecture $\hbar = 1$ except when stated otherwise)

$$f(z) \mapsto zf(z), \quad f(z) \mapsto f'(z).$$

We immediately note the commutation relation

$$[\mathrm{d}/\mathrm{d}z, z] = 1\,. \tag{4.14}$$

Then, we consider the scalar product of a function $g(z)$ by a function $f(z)$ on which acts the operator $\mathrm{d}/\mathrm{d}z$:

$$(g, \mathrm{d}f/\mathrm{d}z) = \int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi}\, \mathrm{e}^{-z\bar{z}}\, \overline{g(z)}\, \frac{\mathrm{d}}{\mathrm{d}z} f(z).$$

After integration by parts, one finds

$$
\begin{aligned}
(g, \mathrm{d}f/\mathrm{d}z) &= -\int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi} f(z)\overline{g(z)}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{-z\bar{z}} \\
&= \int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi}\,\mathrm{e}^{-z\bar{z}}\,\overline{zg(z)}f(z).
\end{aligned}
$$

This identity proves that z and $\mathrm{d}/\mathrm{d}z$ are hermitian conjugate operators with respect to the scalar product (4.10) in the Hilbert space.

Note that by using the commutation relation (4.14), one can expand any operator on a basis of monomials written in a canonical form with all operators z on the left of all operators $\mathrm{d}/\mathrm{d}z$,

$$
z^m \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n ,
$$

a form called normal order.

4.4 Bosons in the second quantization formalism

As we show below, the holomorphic representation is a convenient framework to discuss the quantum harmonic oscillator. However, our main purpose here is to describe bosons in the so-called second quantization formalism.

We consider a system with only one quantum state that can be occupied by an arbitrary number of bosons. The function z^n corresponding to n bosons occupying the state and n is the occupation number. The constant 1 corresponds to the empty state or vacuum.

The hermitian operator

$$N = z\frac{\mathrm{d}}{\mathrm{d}z}$$

preserves the occupation number. Its eigenvalues are the occupation numbers since

$$Nz^n \equiv z\frac{\mathrm{d}}{\mathrm{d}z}z^n = nz^n\,,\ n \geq 0\,.$$

Thus, N is the operator number of particles or occupation number operator.

We have already shown that the eigenvectors z^n are orthogonal (equation (4.11)), a property consistent with the hermiticity of the operator N.

In the holomorphic framework, one defines a Hilbert space that contains linear combinations of vectors with different number of particles and this explains the denomination second quantization.

4.4.1 Harmonic oscillator and holomorphic representation

One easily verifies that linear combinations of z and $\mathrm{d}/\mathrm{d}z$ are unbounded from below. The first possible hermitian Hamiltonian (up to a constant shift) is

$$H_0 = \tfrac{1}{2}\omega \left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z} z \right) = \omega \left(N + \tfrac{1}{2} \right) , \quad \omega > 0 \,. \tag{4.15}$$

The eigenvectors of H_0 are again the monomials z^n and

$$H_0 z^n = \left(n + \tfrac{1}{2} \right) \omega z^n \,, \quad n \ge 0 \,.$$

We recognize that H_0 is the Hamiltonian of the harmonic oscillator and that the holomorphic formalism is equivalent to the formalism of creation and annihilation operators $a^\dagger, a$, which can be used to determine algebraically the spectrum of H_0:

$$a^\dagger \mapsto z\,, \quad a \mapsto \mathrm{d}/\mathrm{d}z\,, \quad H_0 = \tfrac{1}{2}\omega\left(a^\dagger a + a a^\dagger\right). \tag{4.16}$$

However, the energy spectrum of the harmonic oscillator has also another interpretation in terms of bosons that can occupy only one state of energy ω. In this interpretation, the Hamiltonian conserves the number of particles since obviously $[N, H_0] = 0$.

The energy corresponding to the vector z^n is the sum of the vacuum energy, $\frac{1}{2}\omega$, and the energy of n bosons occupying the state of energy ω.

Before working out this interpretation in more details, we construct a path integral representation of the statistical operator.

4.5 Operators: kernel representation

To operators $\mathcal{O}$ that, in the basis of the eigenvectors of the harmonic oscillator, have matrix elements $\mathcal{O}_{mn}$, we associate the kernels

$$\mathcal{O}(z,\bar{z}) = \sum_{m,n} \mathcal{O}_{mn} \frac{z^m}{\sqrt{m!}} \frac{\bar{z}^n}{\sqrt{n!}}, \tag{4.17}$$

where we recall that the functions $\{z^n/\sqrt{n!}\}$ are the normalized eigenvectors of the harmonic oscillator in the holomorphic representation (equation (4.11)).

Notation. To give to kernels a more suggestive and convenient form, we will also use a formal notation of matrix elements with bras and kets:

$$\mathcal{O}(z,\bar{z}) \equiv \langle z|\, \mathcal{O}\, |\bar{z}\rangle\,,$$

without trying to define very precisely the corresponding vectors.

Examples. The kernel associated to the identity operator is

$$\mathcal{I}(z,\bar{z}) = \sum_{n=0} \frac{z^n}{\sqrt{n!}} \frac{\bar{z}^n}{\sqrt{n!}} = \mathrm{e}^{z\bar{z}} \,. \tag{4.18}$$

The kernel associated to the number operator N is

$$N(z,\bar{z}) = \sum_n \frac{n(z\bar{z})^n}{n!} = z\bar{z}\,\mathrm{e}^{z\bar{z}} = z\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{I}(z,\bar{z}),$$

and, thus, the Hamiltonian (4.15) of the harmonic oscillator is represented by

$$H_0(z,\bar{z}) = \omega\left(z\bar{z} + \tfrac{1}{2}\right)\mathrm{e}^{z\bar{z}} \,. \tag{4.19}$$

Finally, for the statistical operator $U_0(\beta) = \mathrm{e}^{-\beta H_0}$, one finds

$$\langle z|\, U_0(\beta)\, |\bar{z}\rangle = \sum_n \frac{\mathrm{e}^{-\beta\omega(n+1/2)}(z\bar{z})^n}{n!} = \mathrm{e}^{-\beta\omega/2}\,\mathrm{e}^{z\bar{z}\,\mathrm{e}^{-\beta\omega}} \,. \tag{4.20}$$

4.5.1 Action of a kernel on a function

The action of a kernel $\mathcal{O}(z,\bar{z})$, associated to an operator $\mathcal{O}$, on a function $f(z)$ is given by

$$(\mathcal{O}f)(z) = \int \frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{2i\pi} \mathcal{O}(z,\bar{z}')\,\mathrm{e}^{-z'\bar{z}'}\,f(z'). \tag{4.21}$$

Indeed, expanding $\mathcal{O}(z,\bar{z})$, and $f(z)$ in the form

$$f(z) = \sum_n f_n \frac{z^n}{\sqrt{n!}},$$

one finds

$$(\mathcal{O}f)(z) = \sum_{m,n} \mathcal{O}_{mn} f_n \frac{z^m}{\sqrt{m!}},$$

as expected.

4.5.2 Differential operators and kernels

It follows from the representation (4.21) that by applying any operator $\mathcal{O}$ written in normal form as

$$\mathcal{O} = \sum_{m,n} \mathbf{O}_{mn} z^m (\mathrm{d}/\mathrm{d}z)^n \,,$$

on both sides, $\mathcal{O}$ is represented by the kernel

$$\mathcal{O}(z,\bar{z}) = \sum_{m,n} \mathbf{O}_{mn} z^m \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \mathcal{I}(z,\bar{z}) = O(z,\bar{z})\, \mathrm{e}^{z\bar{z}}$$

with

$$O(z,\bar{z}) = \sum_{m,n} \mathbf{O}_{mn} z^m \bar{z}^n . \tag{4.22}$$

4.5.3 Product of operators

The kernel associated with the product of two operators is given by

$$\int \frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{2i\pi} \langle z|\, \mathcal{O}_2\, |\bar{z}'\rangle\, \mathrm{e}^{-z'\bar{z}'} \langle z'|\, \mathcal{O}_1\, |\bar{z}\rangle = \langle z|\, \mathcal{O}_2\mathcal{O}_1\, |\bar{z}\rangle\,. \tag{4.23}$$

Indeed, inserting the explicit expressions (4.17),

$$\mathcal{O}_i(z,\bar{z}) = \sum_{mn} [\mathcal{O}_i]_{mn} \frac{z^m}{\sqrt{m!}} \frac{\bar{z}^n}{\sqrt{n!}},$$

one obtains

$$[\mathcal{O}_2\mathcal{O}_1](z,\bar{z}) = \int \frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{2i\pi} \sum_{m,n,m',n'} [\mathcal{O}_2]_{mn} \frac{z^m}{\sqrt{m!}} \frac{\bar{z}'^n}{\sqrt{n!}}\, \mathrm{e}^{-z'\bar{z}'} [\mathcal{O}_1]_{m'n'} \frac{z'^{m'}}{\sqrt{m'!}} \frac{\bar{z}^{n'}}{\sqrt{n'!}}.$$

The integral follows from the orthogonality condition (4.11),

$$\int \frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{2i\pi} \frac{\bar{z}'^n}{\sqrt{n!}}\, \mathrm{e}^{-z'\bar{z}'} \frac{z'^{m'}}{\sqrt{m'!}} = \delta_{nm'}\,.$$

Thus,

$$\langle z|\,\mathcal{O}_2\mathcal{O}_1\,|\bar{z}\rangle = \sum_{m,n,n'} \frac{z^m}{\sqrt{m!}}[\mathcal{O}_2]_{mn}[\mathcal{O}_1]_{nn'}\frac{\bar{z}^{n'}}{\sqrt{n'!}}.$$

The definition (4.23) of the product of operators is thus consistent with the definition of the product in the harmonic oscillator basis.

Finally, one verifies that the trace of the operator $\mathcal{O}$ is given by

$$\operatorname{tr}\mathcal{O} = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2i\pi}\,\mathrm{e}^{-z\bar{z}}\,\mathcal{O}(z,\bar{z}) = \sum_n \mathcal{O}_{nn}\,, \qquad (4.24)$$

by again expanding $\mathcal{O}(z,\bar{z})$. Using the relation (4.23), one verifies directly that the definition satisfies the cyclic condition $\operatorname{tr}\mathcal{O}_1\mathcal{O}_2 = \operatorname{tr}\mathcal{O}_2\mathcal{O}_1$.

Remarks.

(i) From the property that z and $\mathrm{d}/\mathrm{d}z$ are hermitian conjugate, follows that hermitian conjugation acting on kernels is represented by formal complex conjugation:

$$\mathcal{O} \mapsto \mathcal{O}^\dagger \;\Rightarrow\; \mathcal{O}(z,\bar{z}) \;\mapsto\; \overline{\mathcal{O}(z,\bar{z})}, \tag{4.25}$$

as one verifies also directly by comparing the scalar products $(f,\mathcal{O}g)$ and $(\mathcal{O}^\dagger f,g)$.

(ii) Matrix elements of operators in the holomorphic representation are analogous to matrix elements in the mixed position–momentum representation. The latter can be obtained from the matrix elements $\langle q|\,\mathcal{O}\,|q'\rangle$ of an operator in the position basis by a partial Fourier transformation on the right:

$$\langle q|\,\mathcal{O}\,|\tilde{p}\rangle = \int \mathrm{d}q'\;\mathrm{e}^{ipq'/\hbar}\,\langle q|\,\mathcal{O}\,|q'\rangle\,.$$

In the classical limit, the complex variables $(z, \bar{z})$ provide a parametrization of phase space alternative to the parametrization in terms of the real momentum and position variables (p, q).

4.5.4 Hamiltonian and statistical operator

The matrix elements of the Hamiltonian (4.15) of the harmonic oscillator are (equation (4.19))

$$\langle z| H_0 |\bar{z}\rangle = \omega \left(\tfrac{1}{2} + z\bar{z}\right) \mathrm{e}^{z\bar{z}} \,.$$

The matrix elements of the statistical operator $U_0(\beta) = \mathrm{e}^{-\beta H_0}$, (proportional to the matrix density at thermal equilibrium at temperature $1/\beta$), is given by equation (4.20),

$$\langle z| U_0(\beta) |\bar{z}\rangle = \mathrm{e}^{-\omega\beta/2 + z\bar{z}\,\mathrm{e}^{-\omega\beta}} \,. \tag{4.26}$$

One verifies that it satisfies the differential equation

$$\frac{\partial}{\partial\beta} \langle z|\, U_0(\beta)\, |\bar{z}\rangle = -\langle z|\, H_0 U_0(\beta)\, |\bar{z}\rangle = -\omega \left(\tfrac{1}{2} + z\frac{\partial}{\partial z}\right) \langle z|\, U_0(\beta)\, |\bar{z}\rangle$$

with the boundary condition

$$\langle z|\, U_0(0)\, |\bar{z}\rangle = \mathcal{I}(z,\bar{z}) = \mathrm{e}^{\bar{z}z}\,.$$

Note that the action of the quantum statistical operator on a function $f(z)$ is given by

$$[U_0 f](z) = \mathrm{e}^{-\omega\beta/2} \int \frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{2i\pi}\, \mathrm{e}^{z\bar{z}'\,\mathrm{e}^{-\omega\beta}}\, \mathrm{e}^{-z'\bar{z}'}\, f(z') = \mathrm{e}^{-\omega\beta/2}\, f(\mathrm{e}^{-\omega\beta}\, z), \tag{4.27}$$

where the representation (4.13) of the δ-function has been used.

Finally, using the property (4.25), one verifies directly that H_0 and $U_0(\beta)$ are hermitian.

The partition function $\mathcal{Z}_0(\beta) = \operatorname{tr} \mathrm{e}^{-\beta H_0}$ is the trace of $U_0(\beta)$. Using equations (4.24, 4.26), one obtains

$$\mathcal{Z}_0(\beta) = \operatorname{tr} U_0(\beta) = \mathrm{e}^{-\omega\beta/2} \int \frac{\mathrm{d}\bar{z}\mathrm{d}z}{2i\pi} \, \mathrm{e}^{-z\bar{z}} \, \mathrm{e}^{z\bar{z}\,\mathrm{e}^{-\omega\beta}} = \frac{\mathrm{e}^{-\omega\beta/2}}{1-\mathrm{e}^{-\omega\beta}} \,, \qquad (4.28)$$

which is the partition function of a standard harmonic oscillator with eigenvalues $(n+\frac{1}{2})\omega$, $n \ge 0$.

4.6 Path integral: the harmonic oscillator

Once the product of operators in kernel form has been defined, it is possible to construct a path integral representation of the quantum statistical operator (or the evolution operator), based on the holomorphic formalism. The strategy is the same as in the real coordinate case. One reduces the problem to the evaluation of the kernel for short time intervals. However, while the algebra is similar, the question of convergence is more subtle.

Path integral. The matrix elements $\langle z \left| U_0(t) \right| \bar{z} \rangle$ of the statistical operator $U_0(t) = \mathrm{e}^{-tH_0}$, where H_0 is the Hamiltonian of the harmonic oscillator, are given in (4.26). At first order in t (omitting now the vacuum energy),

$$\langle z \left| U_0(t) \right| \bar{z} \rangle = \exp \left[\bar{z}z(1-\omega t) + O\left(t^2\right) \right] . \tag{4.29}$$

The semi-group property $U_0(t) = [U_0(t/n)]^n$, expressed in terms of matrix elements, then allows calculating $U_0(t \equiv t'' - t')$ for finite time.

In the formal limit $n \to \infty$, one obtains the path integral representation

$$\langle z'' | U_0(t'', t') | \bar{z}' \rangle = \int \left[\frac{\mathrm{d}\bar{z}(t) \mathrm{d}z(t)}{2i\pi} \right] \mathrm{e}^{\bar{z}(t')z(t')} \exp\left[-\mathcal{S}_0(z, \bar{z}) \right], \quad (4.30)$$

$$\mathcal{S}_0(z, \bar{z}) = \int_{t'}^{t''} \mathrm{d}t \, \bar{z}(t) \left[-\dot{z}(t) + \omega z(t) \right]$$

with the asymmetric boundary conditions $z(t'') = z''$, $\bar{z}(t') = \bar{z}'$. The symmetry of the action between initial and final times, is not explicit, but can be verified by an integration by parts of the term $\bar{z}\dot{z}$:

$$\mathcal{S}_0(z, \bar{z}) - \bar{z}(t')z(t') = -\bar{z}(t'')z(t'') + \int_{t'}^{t''} \mathrm{d}t \, z(t) \left[\dot{\bar{z}}(t) + \omega \bar{z}(t) \right].$$

Let us point out, however, that the validity of this integration within the path integral assumes that differentiation and expectation value for equal time products commute and thus relies on the convention $\mathrm{sgn}(0) = 0$.

Finally, in the case of the holomorphic path integral, the discussion of the existence of a continuum limit, beyond the formal argument, is complicated, the nature of the problem being analogous to what is encountered in integrals over phase space.

4.6.1 General Gaussian integral

Expression (4.30) can be generalized to a system coupled linearly to external sources $b(t)$ and $\bar{b}(t)$, which are assumed to be independent conjugate functions, like $z(t)$, $\bar{z}(t)$. The corresponding action is

$$\mathcal{S}_{\mathrm{G}}(z,\bar{z}) = \mathcal{S}_0(z,\bar{z}) - \int_{t'}^{t''} \mathrm{d}t\, \left[\bar{z}(t)b(t) + \bar{b}(t)z(t)\right], \tag{4.31}$$

and the matrix elements of U_{G} are given by

$$\langle z''|\, U_{\mathrm{G}}(b,\bar{b}\,;t'',t')\,|\bar{z}'\rangle = \int \left[\frac{\mathrm{d}\bar{z}(t)\mathrm{d}z(t)}{2i\pi}\right] \mathrm{e}^{\bar{z}(t')z(t')} \exp\left[-\mathcal{S}_{\mathrm{G}}(z,\bar{z})\right]. \tag{4.32}$$

The quantity $\langle z''|\, U_{\mathrm{G}}(b,\bar b\,;t'',t')\,|\bar z'\rangle$ is a generating functional of $z(t)$, $\bar z(t)$ correlation functions, which can be all derived from the two-point function.

Explicit calculation of the path integral. One first solves the classical equations obtained by varying $\bar z(t)$ and $z(t)$:

$$-\dot z(t)+\omega z(t)-b(t)=0\,,\quad \dot{\bar z}(t)+\omega\bar z(t)-\bar b(t)=0\,.$$

The solutions that satisfy the boundary conditions are

$$z_c(t)=z''\,\mathrm{e}^{-\omega(t''-t)}+\int_t^{t''}\mathrm{e}^{-\omega(u-t)}\,b(u)\mathrm{d}u\,,$$
$$\bar z_c(t)=\bar z'\,\mathrm{e}^{-\omega(t-t')}+\int_{t'}^{t}\mathrm{e}^{-\omega(t-u)}\,\bar b(u)\mathrm{d}u\,.$$

Shifting integration variables by the solution of the classical equations, $z(t)\mapsto z_c(t)+v(t)$, $\bar z(t)\mapsto\bar z_c(t)+\bar v(t)$, one obtains a Gaussian integral that does not depend on the external sources and boundary conditions any more.

After integration, the result takes the form

$$\langle z''|\, U_{\mathrm{G}}(b,\bar{b}\,;t'',t')\,|\bar{z}'\rangle = \mathcal{N}\big(\omega(t''-t')\big)\,\mathrm{e}^{\bar{z}(t')z(t')-\mathcal{S}_{\mathrm{G}}(z,\bar{z})}, \tag{4.33}$$

where

$$\begin{aligned}
\bar{z}_c(t')z_c(t') &= \bar{z}'z''\,\mathrm{e}^{-\omega(t''-t')} + \bar{z}'\int_{t'}^{t''}\mathrm{d}t\ \mathrm{e}^{-\omega(t-t')}\,b(t),\\
-\,\mathcal{S}_{\mathrm{G}}(z_c,\bar{z}_c) &= \int_{t'}^{t''}\mathrm{d}t\,\bar{b}(t)z_c(t)\\
&= \int_{t'}^{t''}\mathrm{d}t\,\bar{b}(t)\,\mathrm{e}^{-\omega(t''-t)}\,z'' + \int_{t'\le t,u\le t''}\mathrm{d}u\,\mathrm{d}t\,\bar{b}(u)\theta(t-u)\,\mathrm{e}^{-\omega(t-u)}\,b(t),
\end{aligned}$$

$\theta(t)$ being the step function: $\theta(t)=1$ for $t>0$, $\theta(t)=0$ for $t<0$.

As in the case of the usual path integral, the normalization $\mathcal{N}$ can be determined entirely only by a comparison with a reference path integral.

4.6.2 Gaussian correlation functions

Taking the trace (equation (4.24)) of expression (4.33), after a straightforward calculation one finds

$$\mathcal{Z}_{\mathrm{G}}(b,\bar{b};\beta)=\mathcal{Z}_0(\beta)\exp\left[\int_{-\beta/2\le t,u\le\beta/2}\mathrm{d}u\,\mathrm{d}t\,\bar{b}(u)\Delta(t-u)b(t)\right], \tag{4.34}$$

where the normalization $\mathcal{Z}_0(\beta)$ is determined by setting $b=\bar{b}=0$ and, thus,

$$\mathcal{Z}_0(\beta)=\operatorname{tr}\mathrm{e}^{-\beta H_0}\,,$$

and $\Delta(t)$ is the Gaussian two-point function or propagator:

$$\Delta(t)=\tfrac{1}{2}\,\mathrm{e}^{-\omega t}\left[\operatorname{sgn}(t)+1/\tanh(\omega\beta/2)\right]=\mathrm{e}^{-\omega t}\left(\theta(t)+\frac{1}{\mathrm{e}^{\omega\beta}-1}\right) \tag{4.35}$$

with $\operatorname{sgn}(t)=1$ for $t>0$, $\operatorname{sgn}(t)=-1$ for $t<0$ and $\theta(t)=(1+\operatorname{sgn}(t))/2$ is the step function.

One notes that $\Delta(t)$ satisfies the periodic boundary condition $\Delta(-\beta/2) = \Delta(\beta/2)$.

The trace of the path integral (4.32) properly normalized, thus yields a generating functional of correlation functions of $z, \bar{z}$ with the Gaussian weight $\mathrm{e}^{-\mathcal{S}_0}/\mathcal{Z}_0$ and periodic boundary conditions.

One verifies also that $\Delta(t)$ is the solution of the differential equation

$$\dot{\Delta}(t) + \omega\Delta(t) = \delta(t), \tag{4.36}$$

with periodic boundary conditions on the interval $[-\beta/2, \beta/2]$ (we recall that, in the sense of distributions, $\mathrm{d}\,\mathrm{sgn}(t)/\mathrm{d}t = 2\delta(t)$).

In the zero temperature limit $\beta \to \infty$, $\Delta(t)$ reduces to

$$\Delta(t) = \theta(t)\,\mathrm{e}^{-\omega t}\,. \tag{4.37}$$

This form implies a time-ordering or causal property in perturbative calculations.

Periodic boundary conditions. As suggested by the property that $\Delta(t)$ satisfies periodic boundary conditions, one verifies that the result (4.34) can also recovered more directly from a path integral with periodic boundary conditions:

$$\mathcal{Z}_{\mathrm{G}}(b,\bar{b};\beta)=\int\left[\frac{\mathrm{d}\bar{z}(t)\mathrm{d}z(t)}{2i\pi}\right]\exp\left[-\mathcal{S}_{\mathrm{G}}(z,\bar{z})\right] \tag{4.38}$$

with $z(-\beta/2)=z(\beta/2)$, $\bar{z}(-\beta/2)=\bar{z}(\beta/2)$ and

$$\mathcal{S}_{\mathrm{G}}(z,\bar{z})=\int_{-\beta/2}^{\beta/2}\mathrm{d}t\,\left\{\bar{z}(t)\left[-\dot{z}(t)+\omega z(t)\right]-\bar{z}(t)b(t)-\bar{b}(t)z(t)\right\}.$$

Generating functional of correlation functions. The generating functional of correlation functions with periodic boundary conditions, fully determined by the condition $\langle 1 \rangle = 1$, is then

$$\mathcal{Z}_{\mathrm{G}}(b, \bar{b}; \beta)/\mathcal{Z}_0(\beta) = \exp\left[\int \mathrm{d}u\, \mathrm{d}t\, \bar{b}(u)\Delta(t-u)b(t)\right].$$

The initial expression (4.32) shows that the derivative with respect to b and $\bar{b}$, in the limit $b = \bar{b} = 0$, is the two-point function and, thus,

$$\langle \bar{z}(t) z(u) \rangle = \Delta(t-u). \tag{4.39}$$

Fourier representation. The periodic boundary conditions on the finite interval $[-\beta/2, +\beta/2]$ imply that one can expand $z(t)$, $\bar{z}(t)$ in a Fourier series,

$$z(t) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{2i\pi nt/\beta}\, \tilde{z}_n\,, \quad \bar{z}(t) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-2i\pi nt/\beta}\, \tilde{\bar{z}}_n\,.$$

Inverting the Fourier transformation, one finds

$$\tilde{z}_n = \frac{1}{\beta}\int \mathrm{d}t\ \mathrm{e}^{-2i\pi nt/\beta}\, z(t), \quad \tilde{\bar{z}}_n = \frac{1}{\beta}\int \mathrm{d}t\ \mathrm{e}^{2i\pi nt/\beta}\, \bar{z}(t).$$

Applying this representation to the two-point function (4.39) and using the explicit expression (4.35), one obtains

$$\langle \tilde{\bar{z}}_m \tilde{z}_n \rangle = \frac{\delta_{mn}}{\beta}\int_{-\beta/2}^{\beta/2} \mathrm{d}t\ \mathrm{e}^{2i\pi nt/\beta}\, \Delta(t) = \delta_{mn}\frac{1}{\omega\beta - 2i\pi n}\,. \tag{4.40}$$

4.6.3 Partition function

The derivative of the path integral (4.38) with respect to ω, taken for $b = \bar{b} = 0$, is the derivative of the partition function and, thus,

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln \mathcal{Z}_0(\beta) = -\int_{-\beta/2}^{\beta/2} \mathrm{d}t \, \langle \bar{z}(t) z(t) \rangle = -\beta \Delta(0).$$

Since

$$\Delta(t) = \tfrac{1}{2} \, \mathrm{e}^{-\omega t} \left[\mathrm{sgn}(t) + 1/\tanh(\omega\beta/2) \right]$$

formally,

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln \mathcal{Z}_0(\beta) = -\frac{\beta}{2} \left[\mathrm{sgn}(0) + \frac{\cosh(\omega\beta/2)}{\sinh(\omega\beta/2)} \right].$$

The result involves $\mathrm{sgn}(0)$ and, thus, is clearly not defined. This reflects a weakness of the path integral: since it is expressed only in terms of classical quantities, it looses the information about operator ordering in products.

The ambiguity, which is signalled by the appearance of the quantity $\operatorname{sgn}(0)$, is related to the choice of the order of operators in products. Different choices correspond to different choices of quantization.

It can be resolved only by regularizing the path integral, either by returning to discrete times or by adding higher order time derivatives to the action, which leads to adding higher powers of n in the denominator of (4.40).

Integrating formally and using the property that the ground state is not degenerate, which determines the normalization, one formally obtains

$$\mathcal{Z}_0(\beta) = \frac{\mathrm{e}^{-\beta\omega(1+\operatorname{sgn}(0))/2}}{1-\mathrm{e}^{-\omega\beta}} = \sum_{k=0} \mathrm{e}^{-\beta E_k} \,. \tag{4.41}$$

Expanding the denominator for $\beta \to \infty$, one finds the eigenvalues of H_0:

$$E_k = E_0 + \omega k \,, \quad E_0 = \omega(1+\operatorname{sgn}(0))/2 \,.$$

Changing the value of $\mathrm{sgn}(0)$ leads to a global shift of the whole spectrum, which indeed corresponds to different orderings of the product of the operators z and $\mathrm{d}/\mathrm{d}z$ in the Hamiltonian.

The choice $\mathrm{sgn}(0) = 0$ corresponds to $E_0 = \omega/2$ and thus to the standard symmetric Hamiltonian

$$H_0 = \tfrac{1}{2}\omega[z\mathrm{d}/\mathrm{d}z + (\mathrm{d}/\mathrm{d}z)z].$$

This choice allows exchanging path integration and time differentiation. A different choice, for instance, $\Delta(0) = \Delta(0_-)$ and, thus, $\mathrm{sgn}(0) = -1$ corresponds to normal order and leads to

$$\mathcal{Z}_0(\beta) = \frac{1}{1 - \mathrm{e}^{-\omega\beta}}.$$

We have already pointed out the merit of the time-symmetric choice $\operatorname{sgn}(0) = 0$. We illustrate it again here. Using the explicit form (4.35) of the two-point function, one infers the expectation value

$$\langle [\bar{z}(t+\delta) - \bar{z}(t)][z(t+\delta) - z(t)] \rangle = 2\Delta(0) - \Delta(\delta) - \Delta(-\delta).$$

Only the symmetric choice

$$\Delta(0) = \lim_{\delta \to 0} \tfrac{1}{2}\big(\Delta(\delta) + \Delta(-\delta)\big) = \frac{\cosh \omega(\beta/2)}{2 \sinh(\omega\beta/2)},$$

(equation (4.35)), and thus $\operatorname{sgn}(0) = 0$, ensures continuity when δ goes to zero. Then, the generic values of $[z(t+\delta) - z(t)][\bar{z}(t+\delta) - \bar{z}(t)]$ are of order $|\delta|$, in analogy with the behaviour of Brownian motion.

4.7 Path integral: general Hamiltonians

One can derive a path integral representation, based on the holomorphic formalism, for the matrix element of the statistical operator for a general class of Hamiltonians.

The matrix elements of the statistical operator that satisfy the equation

$$\frac{\partial}{\partial t} \langle z| U(t,t') |\bar{z}\rangle = -H\left(z, \partial/\partial z; t\right) \langle z| U(t,t') |\bar{z}\rangle \tag{4.42}$$

with $\langle z| U(t',t') |\bar{z}\rangle = \mathrm{e}^{\bar{z}z}$, are given by the path integral,

$$\langle z''| U(t'',t') |\bar{z}'\rangle = \int \left[\frac{\mathrm{d}\bar{z}(t)\mathrm{d}z(t)}{2i\pi}\right] \mathrm{e}^{\bar{z}(t')z(t')} \exp\left[-\mathcal{S}(z,\bar{z})\right] \tag{4.43}$$

with the euclidean action

$$\mathcal{S}(z,\bar{z}) = \int_{t'}^{t''} \mathrm{d}t \, \left[-\bar{z}(t)\dot{z}(t) + H\big(z(t), \bar{z}(t); t\big)\right] \tag{4.44}$$

and the boundary conditions $z(t'') = z''$, $\bar{z}(t') = \bar{z}'$.

Remarks

(i) Note that, in the boson interpretation, only Hamiltonians functions of the product $z\mathrm{d}/\mathrm{d}z$ conserve the boson number. This translates into Hamiltonians with kernels function only of $z\bar{z}$. The boson number conservation is thus associated with the $U(1)$ symmetry

$$z \mapsto \mathrm{e}^{i\theta}\, z\,, \quad \bar{z} \mapsto \bar{z}\, \mathrm{e}^{-i\theta}\ .$$

This property generalizes to several pairs of z and $\bar{z}$ variables.

(ii) The problems already encountered in section 4.6.3, induced by the ordering in products of the operators $z, \mathrm{d}/\mathrm{d}z$ that do not commute, clearly become more severe for more general quantum Hamiltonians.

4.7.1 Holomorphic formalism and position-momentum phase space

The classical Hamiltonian function of $z, \bar{z}$ and the usual Hamiltonian function of momentum p and position q, are related by a mapping of the form (4.1) (up to normalizations of p and q):

$$z = (q - ip)/\sqrt{2}\,,\ \ \bar{z} = (q + ip)/\sqrt{2}\,, \tag{4.45}$$

Holomorphic path integrals (4.43) and phase space integrals of the form

$$U = \int [\mathrm{d}p(t)\mathrm{d}q(t)]\,\mathrm{e}^{-\mathcal{S}(p,q)}, \tag{4.46}$$

which involves an integration over phase space trajectories and the classical euclidean action in the Hamiltonian formalism,

$$\mathcal{S}(p,q) = \int \mathrm{d}t\,\left[-ip(t)\dot{q}(t) + H\big(p(t), q(t)\big)\right],$$

are similarly related by a simple change of variables of the form (4.45), but have different boundary conditions and boundary terms.

4.7.2 Partition function

In the case of a time-independent Hamiltonian, from the statistical operator one derives the partition function by taking its trace. In terms of matrix elements, the relation reads:

$$\mathcal{Z}(\beta) = \int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi}\, \mathrm{e}^{-z\bar{z}} \langle z|\, U(\beta/2, -\beta/2)\, |\bar{z}\rangle\,.$$

One verifies that the trace operation leads to periodic boundary conditions in the path integral. One then obtains

$$\mathcal{Z}(\beta) = \int \left[\frac{\mathrm{d}\bar{z}(t)\mathrm{d}z(t)}{2i\pi}\right] \exp\left[-\mathcal{S}(z,\bar{z})\right] \tag{4.47}$$

with

$$\mathcal{S}(z,\bar{z}) = \int_{-\beta/2}^{\beta/2} \mathrm{d}t\, \left[-\bar{z}(t)\dot{z}(t) + H\big(z(t),\bar{z}(t)\big)\right]$$

and the boundary conditions $z(-\beta/2) = z(\beta/2)$, $\bar{z}(-\beta/2) = \bar{z}(\beta/2)$.

4.8 Several complex variables

The generalization of the holomorphic formalism to entire functions of N complex variables $z_i, \bar{z}_i$ is simple. The scalar product of two functions is defined by

$$(g, f) = \int \left(\prod_{i=1}^{N} \frac{\mathrm{d}z_i \, \mathrm{d}\bar{z}_i}{2i\pi} \, \mathrm{e}^{-z_i \bar{z}_i} \right) \overline{g(\mathbf{z})} f(\mathbf{z}). \qquad (4.48)$$

In terms of the Taylor series expansion

$$f(\mathbf{z}) = \sum_{k=0} \frac{1}{k!} \sum_{i_1, \ldots, i_k} f_{i_1 \ldots i_k} z_{i_1} \ldots z_{i_k} ,$$

where the coefficients $f_{i_1 \ldots i_k}$ are symmetric in the k indices, the norm of the function is given by

$$\|f\|^2 = (f, f) = \sum_{k=0} \frac{1}{k!} \sum_{i_1, \ldots, i_k} |f_{i_1 \ldots i_k}|^2 .$$

The functions with finite norm span a Hilbert space. Such spaces will be encountered again in the discussion of boson systems in lecture 4.9.

On these functions act hermitian conjugate creation and annihilation operators $z_i, \partial/\partial z_i$, with the commutation relations

$$[\partial/\partial z_i, z_j] = \delta_{ij} .$$

Associated kernels and partition function. The kernel of the identity is

$$\mathcal{I}(\mathbf{z}, \bar{\mathbf{z}}) = \exp \sum_i z_i \bar{z}_i . \tag{4.49}$$

It is then simple to derive kernels associated to operators written in normal form by acting on $\mathcal{I}$.

One infers a path integral representation for the partition function that generalizes expression (4.47),

$$\mathcal{Z}(\beta) = \int \left[\frac{\mathrm{d}^N z(t)\mathrm{d}^N \bar{z}(t)}{(2i\pi)^N}\right] \exp\left[-\mathcal{S}(\mathbf{z}, \bar{\mathbf{z}})\right] \tag{4.50}$$

with an action of the form

$$\mathcal{S}(\mathbf{z}, \bar{\mathbf{z}}) = \int_{-\beta/2}^{\beta/2} \mathrm{d}t \left[-\sum_i \bar{z}_i(t)\dot{z}_i(t) + H\big(\mathbf{z}(t), \bar{\mathbf{z}}(t)\big)\right]$$

and periodic boundary conditions

$$\mathbf{z}(-\beta/2) = \mathbf{z}(\beta/2), \quad \bar{\mathbf{z}}(-\beta/2) = \bar{\mathbf{z}}(\beta/2).$$

4.9 Boson states and Hamiltonian in second quantization

We have now set up a formalism that allows to discuss boson systems in the grand canonical formulation, obtained by summing over subspaces corresponding to any number of particles. We assume first that bosons can occupy only a finite number N of states that, for example, correspond to positions on a finite lattice.

One-boson state. A one-boson state is defined by a vector, which we denote by ψ_i, which belongs to a complex vector space $\mathfrak{H}_1$ of finite dimension N.

n-particle state. A state of n identical bosons is described by a vector $\psi_{i_1 i_2 \dots i_n}$ where the indices i_k take N values. The statistical properties of bosons imply the invariance of the vector $\psi_{i_1 i_2 \dots i_n}$ under all permutations P of the indices $\{i_1, \dots, i_n\}$, $\psi_{i_1 i_2 \dots i_n} = \psi_{i_{P_1} i_{P_2} \dots i_{P_n}}$.

The vectors $\psi_{i_1 i_2 \dots i_n}$ are symmetric tensors with n indices and belong to a complex vector space $\mathfrak{H}_n$ of dimension $\binom{N+n-1}{n}$.

4.9.1 Hamiltonian

Independent particles. An one-body or one-particle Hamiltonian $\mathbf{H}^{(1)}$ is defined by its action on one-particle states: it is then represented by a hermitian $N \times N$ matrix $H^{(1)}_{ij}$, which thus can be diagonalized.

In terms of its eigenvalues ω_i, its action on one-particle states is given by

$$[\mathbf{H}^{(1)}\,\psi]_i = \omega_i \psi_i \,.$$

Its action on an n-particle state is additive:

$$[\mathbf{H}^{(1)}\,\psi]_{i_1 i_2 \ldots i_n} = \sum_{\ell} \omega_{i_\ell}\, \psi_{i_1 i_2 \ldots i_n} \,.$$

If the total Hamiltonian reduces to this simple form, the bosons do not interact: one then speaks of independent particles.

Two-body or pair interaction. A pair or two-body interaction $\mathbf{H}^{(2)}$ is defined by its action on two-particle states:

$$[\mathbf{H}^{(2)}\,\psi]_{i_1 i_2} = \sum_{j_1,j_2} H^{(2)}_{i_1 i_2, j_1 j_2}\psi_{j_1 j_2}\,,$$

where $H^{(2)}_{i_1 i_2, j_1 j_2}$ is a hermitian matrix that satisfies

$$H^{(2)}_{i_1 i_2, j_1 j_2} = H^{(2)}_{i_2 i_1, j_2 j_1} = (H^{(2)}_{j_1 j_2, i_1 i_2})^*$$

and, therefore, is an internal mapping in the vector space $\mathfrak{H}_2$ of symmetric tensors. When $\mathbf{H}^{(2)}$ acts only on symmetric tensors, the matrix can be symmetrized and then satisfies

$$H^{(2)}_{i_1 i_2, j_1 j_2} = H^{(2)}_{i_1 i_2, j_2 j_1} = H^{(2)}_{i_2 i_1, j_1 j_2}\,.$$

The action of $\mathbf{H}^{(2)}$ on an n-particle state, then, has the form

$$[\mathbf{H}^{(2)}\,\psi]_{i_1 i_2 \dots i_n} = \tfrac{1}{2}\sum_{\ell\neq m}\sum_{j,k} H^{(2)}_{i_\ell i_m, jk}\psi_{i_1 i_2 \dots i_{\ell-1} j i_{\ell+1}\dots i_{m-1} k\, i_{m+1}\dots i_n}\,.$$

A simple generalization of this construction allows defining k-particle (k-body) interactions but, in what follows, for simplicity, we restrict the discussion to at most two-particle interactions.

4.9.2 Second quantization formalism: state vectors

We now consider the set of all vectors corresponding to an arbitrary number of bosons, that is, which belong to the union $\mathcal{E} \equiv \oplus_n \mathfrak{H}_n$, $i = 0, 1, \ldots, \infty$ of all spaces. The space $\mathfrak{H}_0$, which has not been defined yet, corresponds to the zero-particle or empty state (also called the vacuum).

We then introduce a complex vector $\mathbf{z} \in \mathbb{C}^{\mathbf{N}}$, of components z_i, $i = 1, \ldots, N$, and the function of N complex variables

$$\Psi(\mathbf{z}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, i_2, \ldots, i_n} \psi_{i_1 i_2 \ldots i_n} z_{i_1} z_{i_2} \ldots z_{i_n} ,$$

which is a generating function of all vectors.

Because we deal with bosons, the coefficients $\psi_{i_1 i_2 \dots i_n}$ are symmetric in all indices and, thus, can be recovered by differentiating the function $\Psi(\mathbf{z})$, which thus parametrizes the elements of $\mathcal{E}$.

We choose the functions $\Psi(\mathbf{z})$ to be normalizable with respect to the scalar product (4.48),

$$\|\Psi\|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, i_2, \dots, i_n} |\psi_{i_1 i_2 \dots i_n}|^2 < \infty \,.$$

When $\|\Psi\| = 1$, the nth term in the sum gives the probability for the quantum system to be in an n-particle state, the factor $1/n!$ cancelling the over-counting of states implied by the unrestricted summation over all indices $\{i_k\}$.

This gives to $\mathcal{E}$ the structure of a Hilbert space, the Hilbert space of entire functions of N complex variables and this explains the name of second quantization given to the formalism.

One then notices that

$$\sum_j \omega_j z_j \frac{\partial \Psi(\mathbf{z})}{\partial z_j} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sum_j z_j \omega_j \sum_{i_1, i_2, \ldots, i_{n-1}} z_{i_1} z_{i_2} \ldots z_{i_{n-1}} \psi_{i_1 i_2 \ldots i_{n-1} j}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1, i_2, \ldots, i_n} z_{i_1} z_{i_2} \ldots z_{i_n} \sum_{\ell} \omega_{i_\ell} \psi_{i_1 i_2 \ldots i_n} .$$

The operator

$$\mathbf{H}^{(1)} \equiv \sum_i z_i \omega_i \frac{\partial}{\partial z_i} , \tag{4.51}$$

thus, represents the one-particle Hamiltonian acting on the vector $\Psi(\mathbf{z})$.

An analogous calculation shows that a pair interaction is represented by the operator

$$\mathbf{H}^{(2)} = \frac{1}{2} \sum_{i_1, i_2, j_1, j_2} z_{i_1} z_{i_2} H^{(2)}_{i_1 i_2, j_1 j_2} \frac{\partial^2}{\partial z_{j_1} \partial z_{j_2}} . \tag{4.52}$$

The total Hamiltonian

$$\mathbf{H} = \mathbf{H}^{(1)} + \mathbf{H}^{(2)} \tag{4.53}$$

has exactly the form of the Hamiltonians discussed in the framework of the holomorphic representation, generalized to N complex variables. In particular, it is hermitian with respect to the scalar product (4.48).

Occupation number and particle number. In this framework, one can introduce the occupation number operator of state i, whose action on $\Psi(\mathbf{z})$ is given by

$$\mathbf{n}_i = z_i \frac{\partial}{\partial z_i} \ \Rightarrow \ [\mathbf{n}_i, \mathbf{n}_j] = 0\,. \tag{4.54}$$

The sum

$$\mathbf{N} = \sum_i \mathbf{n}_i \tag{4.55}$$

is then the operator total number of particles. This operator is a special example of an one-body operator of type (4.51).

The Hamiltonian (4.53) conserves the number of particles because there is an equal number of factors z and $\partial/\partial z$ in each term. This property is equivalent to the commutation relation

$$[\mathbf{N}, \mathbf{H}] = 0 \,.$$

4.9.4 Associated kernels

The kernel associated with the identity is (expression (4.49))

$$\mathcal{I}(z, \bar{z}) = \exp \sum_i z_i \bar{z}_i \,. \tag{4.56}$$

The Hamiltonian (4.51) and the particle number operator are then represented, respectively, by

$$H^{(1)}(z, \bar{z}) = \mathcal{I}(z, \bar{z}) \sum_i \omega_i z_i \bar{z}_i \,, \quad N(z, \bar{z}) = \mathcal{I}(z, \bar{z}) \sum_i z_i \bar{z}_i \,. \tag{4.57}$$

Finally, the pair interaction (4.52) has the representation

$$H^{(2)}(z,\bar{z}) = \frac{1}{2}\mathcal{I}(z,\bar{z}) \sum_{i_1,i_2,j_1,j_2} z_{i_1} z_{i_2} H^{(2)}_{i_1 i_2, j_1 j_2} \bar{z}_{j_1} \bar{z}_{j_2} \,. \tag{4.58}$$

The conservation of the number of particles leads to a symmetry $U(1) \sim SO(2)$ of the Hamiltonian, corresponding to the transformation

$$z_i \mapsto \mathrm{e}^{i\theta}\, z_i \,, \quad \bar{z}_i \mapsto \mathrm{e}^{-i\theta}\, \bar{z}_i \,,$$

since only monomials with an equal number of factors z and $\bar{z}$ are present.

Of course, the same formalism allows also studying systems where the number of particles is not conserved. The symmetry is then absent and the chemical potential (*cf.* section 4.10) useless.

4.10 Quantum statistical physics: the partition function

From the viewpoint of quantum statistical physics, two different strategies are available to study the thermodynamic limit of a system of particles: one can either work with a fixed number n of particles and then take the limit $n \to \infty$, or consider the direct sum $\oplus \mathfrak{H}_n$, $n = 0, 1$, ..., ∞ of Hilbert spaces, and fix the average number of particles by varying the chemical potential (this assumes a weak coupling to a reservoir of particles, the analogue of a thermal bath for the temperature).

We study, here, the statistical properties of Bose systems in the latter framework, called the grand canonical formulation of statistical physics, using the formalism presented in section 4.9.

4.10.1 Chemical potential

In quantum mechanics, conservation laws correspond to operators that commute with the Hamiltonian. From the statistical viewpoint, conservation laws lead to a breaking of ergodicity. States corresponding to different values of the conserved quantity do not thermalize.

In such a case, in the second quantization formulation, it is necessary to add to the Hamiltonian a term proportional to the corresponding operator, to determine the expectation value of the conserved quantity. In the case of the conservation of the number of particles, one thus replaces the Hamiltonian $\mathbf{H}$ by the operator

$$\mathbf{H} - \mu \mathbf{N}\,,$$

which amounts simply here to modifying $\mathbf{H}^{(1)}$. The real parameter μ, which is coupled to the particle number operator $\mathbf{N}$, is called the chemical potential. It allows varying the average number of particles.

4.10.2 Partition function and equation of state

To calculate the partition function

$$\mathcal{Z}(\beta,\mu)=\operatorname{tr}\mathrm{e}^{-\beta(\mathbf{H}-\mu\mathbf{N})},$$

we now use of the holomorphic formalism described in sections 4.3-4.7. The partition function is given by a path integral of the form (4.50):

$$\mathcal{Z}(\beta,\mu)=\int[\mathrm{d}z_i(t)\mathrm{d}\bar{z}_i(t)]\exp\left[-\mathcal{S}(z,\bar{z})\right] \tag{4.59}$$

with an action

$$\mathcal{S}(z,\bar{z})=\int_{-\beta/2}^{\beta/2}\mathrm{d}t\left\{-\sum_i\bar{z}_i(t)\left[\dot{z}_i(t)+\mu z_i(t)\right]+H\big(z(t),\bar{z}(t)\big)\right\} \tag{4.60}$$

and periodic boundary conditions

$$z_i(-\beta/2)=z_i(\beta/2),\quad \bar{z}_i(-\beta/2)=\bar{z}_i(\beta/2).$$

The kernel associated to the Hamiltonian (4.53) can be written as

$$\langle z|\, H \,|\bar{z}\rangle = \mathcal{I}(z,\bar{z})H(z,\bar{z}),$$

where $\mathcal{I}$ is defined in (4.56) and $H(z,\bar{z})$ can be inferred from expressions (4.57, 4.58). One then finds

$$H(z,\bar{z}) = \sum_i \omega_i z_i \bar{z}_i + \tfrac{1}{2} \sum_{i_1, i_2, j_1, j_2} z_{i_1} z_{i_2} H^{(2)}_{i_1 i_2, j_1 j_2} \bar{z}_{j_1} \bar{z}_{j_2}\,.$$

Equation of state. The equation of state is the relation between average number of particles, temperature and chemical potential. It can be derived from the partition function by differentiating with respect to the chemical potential:

$$\langle \mathbf{N} \rangle = \frac{1}{\beta} \frac{\partial \ln \mathcal{Z}}{\partial \mu}\,.$$

From the path integral representation, using time translation invariance, one derives

$$\langle \mathbf{N} \rangle = \frac{1}{\beta} \sum_i \int_{-\beta/2}^{\beta/2} \mathrm{d}t \, \langle z_i(t) \bar{z}_i(t) \rangle = \sum_i \langle z_i(0) \bar{z}_i(0) \rangle . \tag{4.61}$$

Operator ordering. In any explicit calculation of the path integral, one is confronted with ambiguities, which are related to the problem of operator ordering. The products that appear in expressions (4.51, 4.52) are naturally written in normal order. Therefore, if one insists using the symmetric convention $\mathrm{sgn}(0) = 0$, for reasons that have been explained in section 4.7, one must modify $\mathbf{H}^{(1)}$ by a term generated by $\mathbf{H}^{(2)}$ and add a constant term to cancel the ground state energy.

4.11 S-matrix and holomorphic formalism

The holomorphic formalism has been discussed in section 4.3, and we adapt it here to real time evolution. The holomorphic formalism in real time is useful when the asymptotic states are eigenstates of the harmonic oscillator, a situation that one encounters naturally in quantum many-body theory and relativistic quantum field theory, as we start discussing in section 5.5.

4.11.1 Path integral representation

The representation of the evolution operator is formally obtained from the statistical operator by the continuation $t \mapsto it$. Then,

$$U(z'', \bar{z}'; t'', t') = \int \left[\frac{\mathrm{d}\bar{z}(t)\mathrm{d}z(t)}{2i\pi}\right] \exp\left[i\mathcal{A}(z, \bar{z})\right], \tag{4.62}$$

$$\mathcal{A}(z, \bar{z}) = -i\bar{z}(t')z(t') - \int_{t'}^{t''} \mathrm{d}t \left[i\bar{z}(t)\dot{z}(t) + h\big(z(t), \bar{z}(t)\big)\right],$$

with the boundary conditions $z(t'') = z''$, $\bar{z}(t') = \bar{z}'$.

From the evolution operator one can derive the corresponding S-matrix. Defining the S-matrix by expression (3.48) (for $\hbar = 1$),

$$S = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \mathrm{e}^{iH_0 t''}\, U(t'', t')\, \mathrm{e}^{-iH_0 t'},$$

where H_0 is the Hamiltonian of the harmonic oscillator,

$$H_0 = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega^2 \hat{q}^2,$$

$(\omega > 0)$ one finds

$$S(z,\bar{z}) = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int \frac{\mathrm{d}z''\mathrm{d}\bar{z}''}{2i\pi} \frac{\mathrm{d}z'\mathrm{d}\bar{z}'}{2i\pi}\, \mathrm{e}^{-z''\bar{z}''}\, \mathrm{e}^{-z'\bar{z}'}\, \mathrm{e}^{i\omega t''/2} \exp\left(z\bar{z}''\, \mathrm{e}^{i\omega t''}\right)$$
$$\times\, U(z'', \bar{z}'; t'', t')\, \mathrm{e}^{-i\omega t'/2} \exp\left(z'\bar{z}\, \mathrm{e}^{-i\omega t'}\right).$$

Using equation (4.27), or integrating directly, one obtains

$$S(z,\bar{z}) = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \mathrm{e}^{i\omega t''/2}\, U(z\, \mathrm{e}^{i\omega t''}, \bar{z}\, \mathrm{e}^{-i\omega t'}; t'', t')\, \mathrm{e}^{-i\omega t'/2}\,. \tag{4.63}$$

The coefficients of the expansion of $S(z,\bar{z})$ in powers of z and $\bar{z}$ yield the matrix elements S_{mn} of the transition between the corresponding eigenstates of the harmonic oscillator,

$$S(z,\bar{z}) = \sum_{m,n} S_{mn} \frac{z^m}{\sqrt{m!}} \frac{\bar{z}^n}{\sqrt{n!}} \,,$$

which are also states with different boson number. As in the position representation (section 3.11.2), the configurations in the path integral which contribute to the S-matrix are, for large time, asymptotic to the solutions of the classical equation of motion. For the harmonic oscillator H_0 this means

$$z(t'') \underset{t''\to+\infty}{\sim} z\, \mathrm{e}^{i\omega t''}, \qquad \bar{z}(t') \underset{t'\to-\infty}{\sim} \bar{z}\, \mathrm{e}^{-i\omega t'} \,.$$

4.11.2 Time-dependent force

In the case of a finite number of degrees of freedom a simple application is the evaluation of transition rates between eigenstates of the harmonic oscillator induced by a time-dependent perturbation that vanishes for large positive and negative times. As an example we apply the result (4.63) to the harmonic oscillator perturbed by a linear, time-dependent coupling to a creation and annihilation operators of the form

$$H(z, \bar{z}) = \omega z \bar{z} - b(t)\bar{z} - b^*(t)z\,,$$

where $b(t)$ vanishes for $|t| \to \infty$. The perturbed Hamiltonian is still hermitian and thus the S-matrix unitary.

After a straightforward calculation, one finds

$$S(z,\bar z) = \exp\Big[z\bar z + i\int_{-\infty}^{+\infty} \mathrm{d}t\, \big(z\,\mathrm{e}^{i\omega t}\, b^*(t) + \bar z\,\mathrm{e}^{-i\omega t}\, b(t)\big) \\ - \int_{-\infty}^{+\infty} \mathrm{d}t_1\, \mathrm{d}t_2\, b^*(t_1)\theta(t_2-t_1)\,\mathrm{e}^{-i\omega(t_2-t_1)}\, b(t_2)\Big]. \qquad (4.64)$$

Using the formalism of section 4.3, one verifies the unitarity of the S-matrix. Moreover, it is convenient to express the result in terms of the Fourier components of $b(t)$. Setting

$$b(t) = \int_{-\infty}^{+\infty} \mathrm{d}\nu\ \mathrm{e}^{i\nu t}\, \tilde b(\nu),$$

one obtains ($\varepsilon \to 0_+$)

$$S(z,\bar z) = \exp\left[z\bar z + 2i\pi\big(z\tilde b^*(\omega) + \bar z\tilde b(\omega)\big) - \int_{-\infty}^{+\infty} \mathrm{d}\nu \frac{2i\pi}{\nu-\omega+i\varepsilon}|b(\nu)|^2\right]. \qquad (4.65)$$

Coupling to position only. If the function $b(t)$ is real the perturbation is coupled to the position $q(t)$ only. From (4.45) we infer that the coefficient of $b(t)$ is $\sqrt{2\omega}q(t)$. Then the expression (4.64) can be symmetrized in time and becomes

$$S(z,\bar{z}) = \exp\left[z\bar{z} + i\int_{-\infty}^{+\infty} \mathrm{d}t\left(z\,\mathrm{e}^{i\omega t} + \bar{z}\,\mathrm{e}^{-i\omega t}\right)b(t)\right.$$
$$\left. -\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t_1\,\mathrm{d}t_2\,b(t_1)\,\mathrm{e}^{-i\omega|t_2-t_1|}\,b(t_2)\right]. \tag{4.66}$$

In terms of Fourier components, one finds

$$S(z,\bar{z}) = \exp\left[z\bar{z} + 2i\pi\left(z\tilde{b}(-\omega) + \bar{z}\tilde{b}(\omega)\right) - \pi\int_{-\infty}^{+\infty} \mathrm{d}\nu\frac{2i\omega}{\nu^2 - \omega^2 + i\varepsilon}|b(\nu)|^2\right]. \tag{4.67}$$

4.11.3 Calculation with the path integral in the position basis

It is interesting to compare the expression with a direct calculation of the real path integral

$$\mathcal{Z}(b) = \int [\mathrm{d}q] \exp i \int_{-\infty}^{+\infty} \mathrm{d}t \left(\tfrac{1}{2}\dot{q}^2(t) - \tfrac{1}{2}\omega^2 q^2(t) + \sqrt{2\omega}\, b(t) q(t) \right).$$

As such the path integral is not defined because the classical equation of motion has non-trivial solutions. We thus define the real time path integral as the analytic continuation of the euclidean path integral.

The $i\varepsilon$ rule. We perform a rotation in the time complex plane $t \mapsto t\,\mathrm{e}^{i\theta}$ where θ varies between 0 (the euclidean theory) and $\pi/2$, (the real time theory). In the Fourier variable ν the corresponding rotation is $\nu \mapsto \nu\,\mathrm{e}^{-i\theta}$. Following the rotation, we find that this amounts to giving to ω^2 an infinitesimal negative imaginary part $\omega^2 \mapsto \omega^2 - i\varepsilon$ with $\varepsilon \to 0_+$.

The $i\varepsilon$ term then ensures the convergence of the Gaussian path integral and one finds

$$\mathcal{Z}(b) = \mathcal{Z}(0) \exp\left[-\tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t_1\, \mathrm{d}t_2\, b(t_1)\, \mathrm{e}^{-i\omega|t_2-t_1|}\, b(t_2)\right].$$

We recognize the contribution quadratic in b in the expression (4.66). To recover the complete expression (4.66), one has to slightly modify the path integral.

Defining $q_0(t) = (2\omega)^{-1/2}\left(z\,\mathrm{e}^{i\omega t} + \bar{z}\,\mathrm{e}^{-i\omega t}\right)$, where $q_0(t)$ is also the most general solution of the equation of motion of the classical harmonic oscillator, one finds

$$\mathcal{Z}(b) = \int [\mathrm{d}q] \exp i \int_{-\infty}^{+\infty} \mathrm{d}t \Big(\tfrac{1}{2}\dot{q}^2(t) - \tfrac{1}{2}\omega^2 q^2(t) + \sqrt{2\omega}\, b(t)\big(q(t) + q_0(t)\big)\Big) \tag{4.68}$$

and then $S(z, \bar{z}) = \mathrm{e}^{z\bar{z}}\, \mathcal{Z}(b)/\mathcal{Z}(0)$.

One can then shift $q_0(t)+q(t)\mapsto q(t)$. Taking into account the equation of motion, one finds that the action in expression (4.68) can be written as

$$\mathcal{A}(q)=\int_{-\infty}^{+\infty}\mathrm{d}t\left(\tfrac{1}{2}\dot{q}^2(t)-\tfrac{1}{2}\omega^2q^2(t)+\sqrt{2\omega}\,b(t)q(t)-\tfrac{1}{2}\dot{q}_0^2(t)+\tfrac{1}{2}\omega^2q_0^2(t)\right),$$

where the contribution of q_0 does not vanish, though q_0 satisfies the equation motion derived from the harmonic action, because it does not vanish asymptotically.

Actually, the contribution of q_0 ensures the convergence of the time integral because the function $q(t)$ now satisfies scattering boundary conditions,

$$q(t)\underset{|t|\to\infty}{\sim}q_0(t)=(2\omega)^{-1/2}\left(z\,\mathrm{e}^{i\omega t}+\bar{z}\,\mathrm{e}^{-i\omega t}\right),$$

in agreement with general arguments.

Exercises

Exercise 4.1

Space and momentum translation operators. In this exercise, one uses for the momentum and position operators $\hat{p}, \hat{q}$ the holomorphic representation

$$\hat{q} = \frac{1}{\sqrt{2}} \left(z + \frac{\mathrm{d}}{\mathrm{d}z} \right), \ \hat{p} = \frac{i}{\sqrt{2}} \left(z - \frac{\mathrm{d}}{\mathrm{d}z} \right).$$

1. Find the eigenvectors of the position operator $\hat{q}$ in the space of entire functions. Calculate the scalar product of two eigenvectors (it may be useful to return to real integration).

Solution. The eigenvector $f_q(z)$, corresponding to the real eigenvalue q, is solution of

$$\frac{1}{\sqrt{2}} \left(z + \frac{\mathrm{d}}{\mathrm{d}z} \right) f_q(z) = q f_q(z).$$

The solution can be written as

$$f_q(z) = \mathrm{e}^{-(z-q\sqrt{2})^2/2}\,.$$

Then,

$$(f_{q'}, f_q) = \sqrt{\pi}\,\mathrm{e}^{-q^2}\,\delta(q-q').$$

2. One considers the operator $T(\alpha)$ as defined by its action on holomorphic vectors:

$$[T(\alpha)f](z) \equiv f(z,\alpha) = \mathrm{e}^{-\alpha z/\sqrt{2}-\alpha^2/4}\,f(z+\alpha/\sqrt{2}),$$

with α real. Verify the Abelian multiplication law

$$T(\alpha)T(\beta) = T(\alpha+\beta),$$

by acting on an arbitrary vector f.

Verify explicitly that the operator $T(\alpha)$ preserves the scalar product (and, thus, that it is unitary).

Show by acting with $T(\alpha)$ on the eigenvectors f_q of $\hat{q}$:

$$[T(\alpha)f_q](z) = f_{q+\alpha}(z).$$

Show that $f(z,\alpha)$ satisfies the partial derivative equation

$$\frac{\partial f(z,\alpha)}{\partial \alpha} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} - z\right) f(z,\alpha). \tag{4.69}$$

Infer that $T(\alpha)$ is the translation operator: $T(\alpha) = \mathrm{e}^{i\alpha\hat{p}}$.

3. Determine the kernel associated with $T(\alpha)$ and directly verify unitarity.

Solution. One acts with $T(\alpha)$ on the identity kernel $\mathcal{I}$:

$$\begin{aligned} T(\alpha, z, \bar{z}) \equiv [T(\alpha)\mathcal{I}](z,\bar{z}) &= \mathrm{e}^{-\alpha z/\sqrt{2}-\alpha^2/4}\,\mathcal{I}(z+\alpha/\sqrt{2},\bar{z}) \\ &= \mathrm{e}^{z\bar{z}+\alpha(\bar{z}-z)/\sqrt{2}-\alpha^2/4}\,. \end{aligned}$$

The proof of unitarity then relies on the verification

$$\int \frac{\mathrm{d}v\mathrm{d}\bar{v}}{2i\pi}\, \mathrm{e}^{-v\bar{v}}\, \overline{T}(\alpha, z, \bar{v}) T(\alpha, v, \bar{z}) = \mathrm{e}^{z\bar{z}}\,.$$

4. One now defines the operator

$$[V(\beta)f](z) = \mathrm{e}^{-i\beta z/\sqrt{2}-\beta^2/4}\, f(z - i\beta/\sqrt{2}),$$

with β real. Same questions as for $T(\alpha)$, equation (4.69) being replaced by

$$\frac{\partial[V(\beta)f]}{\partial\beta} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial z} + z\right)[V(\beta)f].$$

Infer that $V(\beta)$ is the momentum translation operator $V(\alpha) = \mathrm{e}^{-i\alpha\hat{q}}$.

5. Acting on holomorphic vectors, derive the commutation relation

$$V(\beta)T(\alpha) = \mathrm{e}^{i\alpha\beta}\, T(\alpha)V(\beta).$$

Exercise 4.2

Boson systems in the holomorphic representation.

One considers the Hilbert space of analytic functions $f(z)$ endowed with the scalar product (4.10).

The unperturbed Hamiltonian, in the holomorphic representation,

$$H_0 = z\frac{\mathrm{d}}{\mathrm{d}z}\,,$$

describes a system of bosons that can occupy only one energy state. One then adds an interaction between the bosons and a medium that can absorb and emit boson pairs with equal probability. This corresponds to adding a potential

$$V = \frac{\alpha}{1+\alpha^2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 + z^2\right]$$

to H_0, where α is chosen real and the parametrization is convenient for what follows. With these assumptions, the Hamiltonian $H = H_0 + V$ remains hermitian.

1. Introducing the operator

$$B = \frac{\mathrm{d}}{\mathrm{d}z} + \alpha z \,,$$

express H in terms of $B^\dagger B$.

Solution.

$$B^\dagger = \alpha \frac{\mathrm{d}}{\mathrm{d}z} + z \,, \; H = \frac{1}{1+\alpha^2} \left(B^\dagger B - \alpha^2 \right) \,.$$

2. Determine the holomorphic eigenvectors $f_\pm(z)$ with $f_\pm(0) = 1$, such that

$$B f_+ = 0 \,, \; B^\dagger f_- = 0 \,,$$

and calculate their norm. Under which conditions are the norms finite?

Solution. $f_+(z) = \mathrm{e}^{-\alpha z^2/2}$ and thus

$$(f_+, f_+) = \frac{1}{\sqrt{1-\alpha^2}}.$$

The norm is finite if $|\alpha| < 1$.

Moreover,

$$[B^\dagger f_-](z) = \alpha f'(z) + z f(z)$$

and thus $f_-(z) = \mathrm{e}^{-z^2/2\alpha}$ and

$$(f_-, f_-) = \frac{|\alpha|}{\sqrt{\alpha^2 - 1}}.$$

One finds the condition $\alpha^2 > 1$. Both vectors cannot be normalized simultaneously and for $\alpha^2 = 1$, neither is normalizable.

3. Calculate the commutator $[B, B^\dagger]$ and relate B and $B^\dagger$ to creation and annihilation operators of a harmonic oscillator (distinguishing the two cases

$|\alpha| < 1$ and > 1). Infer the spectrum of H. Show that for $|\alpha| \neq 1$, it is a spectrum of independent particles, which one can call quasi-bosons.

Solution. $[B, B^\dagger] = 1 - \alpha^2$.

For $|\alpha| < 1$, one can set $B = A\sqrt{1-\alpha^2}$, where A is the annihilation operator with the standard normalization $[A, A^\dagger] = 1$ and, thus,

$$H = \frac{(1-\alpha^2)A^\dagger A - \alpha^2}{1+\alpha^2} \ \Rightarrow \ E_N = \frac{(1-\alpha^2)N - \alpha^2}{1+\alpha^2}\,, \ N \geq 0\,.$$

For $|\alpha| > 1$, one can set $B^\dagger = A\sqrt{\alpha^2 - 1}$ and the spectrum of H is then given by

$$E_N = \frac{(\alpha^2-1)N - 1}{1+\alpha^2}\,, \quad N \geq 0\,.$$

These results are consistent with the normalizability conditions of $f_\pm$. One verifies that, like the Hamiltonian, the spectrum is invariant by $\alpha \mapsto 1/\alpha$. The change $z \mapsto iz$ shows also directly that $H(\alpha)$ has the same spectrum as $H(-\alpha)$, which is consistent with the explicit result.

For $|\alpha| \neq 1$, one thus finds a spectrum of independent particles, of quasi-bosons in the sense that these boson states consist in a superposition of states with 1, 3, ..., initial bosons.

4. Determine the kernel $\langle z| H |\bar{z}\rangle$ corresponding to the Hamiltonian H. Infer, then, the path integral representation of the kernel $\langle z| U(\beta, 0) |\bar{z}\rangle$ associated with the statistical operator $U(\beta, 0) = \mathrm{e}^{-\beta H}$.

Solution.

$$\langle z| H |\bar{z}\rangle = \mathrm{e}^{\bar{z}z} H(z, \bar{z}), \; H(z, \bar{z}) = z\bar{z} + \frac{\alpha}{1+\alpha^2}\left(z^2 + \bar{z}^2\right).$$

One infers the path integral representation

$$\langle z| U(\beta, 0) |\bar{z}\rangle = \int \left[\frac{\mathrm{d}\bar{z}(t)\mathrm{d}z(t)}{2i\pi}\right] \mathrm{e}^{\bar{z}(0)z(0)} \exp\left[-\mathcal{S}(z, \bar{z})\right]$$

with the euclidean action

$$\mathcal{S}(z, \bar{z}) = \int_0^{\beta} \mathrm{d}t \left[-\bar{z}(t)\dot{z}(t) + \bar{z}(t)z(t) + \frac{\alpha}{1+\alpha^2}\left(z^2(t) + \bar{z}^2(t)\right)\right] \qquad (4.70)$$

and the boundary conditions $z(\beta) = z$, $\bar{z}(0) = \bar{z}$.

5. From now on, one restricts the discussion to $0 < \alpha < 1$ and sets

$$\beta(1-\alpha^2)/(1+\alpha^2) = \lambda \,, \ \ \alpha = \mathrm{e}^{-\mu} \,.$$

Infer from the calculation of the path integral, the dependence of $\langle z|U(\beta,0)|\bar{z}\rangle$ on $z, \bar{z}$.

One may use linear combinations of the classical equations of motion to cast the action into the form

$$\mathcal{S}(z_c, \bar{z}_c) = -\tfrac{1}{2}\left[z_c(\beta)\bar{z}_c(\beta) - z_c(0)\bar{z}_c(0)\right],$$

where $z_c(t), \bar{z}_c(t)$ are solutions. Then, to solve the equations, one may introduce the two linear combinations $z(t) \pm \bar{z}(t)$.

Solution. The integral is Gaussian and can thus be evaluated. One first solves the equations of motion to eliminate the boundary conditions:

$$-\dot{z}(t) + z(t) + \frac{2\alpha}{1+\alpha^2}\bar{z}(t) = 0\,, \tag{4.71}$$

$$\dot{\bar{z}}(t) + \bar{z}(t) + \frac{2\alpha}{1+\alpha^2} z(t) = 0\,. \tag{4.72}$$

Adding $\bar{z}(t)$ times equation (4.71) to $z(t)$ times equation (4.72), one finds

$$z(t)\bar{z}(t) + \frac{\alpha}{1+\alpha^2}\left(z^2(t) + \bar{z}^2(t)\right) = \frac{1}{2}\left(\bar{z}(t)\dot{z}(t) - \dot{\bar{z}}(t)z(t)\right).$$

Substituting this identity into the action (4.70), one obtains a total derivative, which leads to the first result:

$$\mathcal{S}(z,\bar{z}) = -\tfrac{1}{2}\left[z(\beta)\bar{z}(\beta) - z(0)\bar{z}(0)\right].$$

Since $z(\beta)$ and $\bar{z}(0)$ are fixed by the boundary conditions, it suffices to calculate $z(0)$ and $\bar{z}(\beta)$. The solutions of the equations can be written as

$$\begin{aligned} z(t) &= a\,\mathrm{e}^{\omega t} - \alpha b\,\mathrm{e}^{-\omega t} \\ \bar{z}(t) &= -\alpha a\,\mathrm{e}^{\omega t} + b\,\mathrm{e}^{-\omega t}, \end{aligned}$$

with $\omega = (1-\alpha^2)/(1+\alpha^2)$ and

$$a = \frac{z\,\mathrm{e}^{\mu} + \bar{z}\,\mathrm{e}^{-\lambda}}{2\sinh(\lambda+\mu)}, \quad b = \frac{z + \bar{z}\,\mathrm{e}^{\lambda+\mu}}{2\sinh(\lambda+\mu)}.$$

Thus,

$$z(0) = \frac{z\sinh\mu - \bar{z}\sinh\lambda}{\sinh(\lambda+\mu)}, \quad \bar{z}(\beta) = \frac{\bar{z}\sinh\mu - z\sinh\lambda}{\sinh(\lambda+\mu)},$$

and, finally,

$$\mathcal{S}(z,\bar{z}) - \bar{z}(0)z(0) = \frac{(z^2+\bar{z}^2)\sinh\lambda - 2z\bar{z}\sinh\mu}{2\sinh(\lambda+\mu)}.$$

6. Infer the normalization from the calculation of $\operatorname{tr} U(\beta,0)$ by comparing the result, for example, with the expression obtained directly by using the spectrum determined previously.

Solution. The Gaussian integration yields a normalization $\mathcal{N}$ and

$$\langle z|\, U(\beta,0)\,|\bar{z}\rangle = \mathcal{N}\,\mathrm{e}^{\bar{z}(0)z(0)-\mathcal{S}(z,\bar{z})}\,.$$

Calculating the trace, one obtains the partition function

$$\mathcal{Z}(\beta) = \mathcal{N}\int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi}\,\mathrm{e}^{-z\bar{z}}\,\mathrm{e}^{\bar{z}(0)z(0)-\mathcal{S}(z,\bar{z})} = \frac{\mathcal{N}}{2\sinh(\lambda/2)}\sqrt{\frac{\sinh(\lambda+\mu)}{\sinh\mu}}\,.$$

The direct calculation based on the already determined spectrum yields

$$\mathcal{Z}(\beta) = \frac{\mathrm{e}^{\beta/2}}{2\sinh(\lambda/2)} \;\Rightarrow\; \mathcal{N} = \mathrm{e}^{\beta/2}\sqrt{\frac{\sinh\mu}{\sinh(\lambda+\mu)}}\,.$$

Exercise 4.3

One considers the partition function (4.47) with

$$\mathcal{S}(z,\bar{z}) = \int_{-\beta/2}^{\beta/2} \mathrm{d}t\, \left\{ \bar{z}(t)\left[-\dot{z}(t) + \omega z(t)\right] + \tfrac{1}{4}\lambda \left[\bar{z}(t)z(t)\right]^2 \right\},$$

where ω, λ are two positive constants. Calculate the ground state energy E_0, up to order λ^2 with the convention $\theta(0) = \frac{1}{2}$, that is $\operatorname{sgn}(0) = 0$. Determine then the whole spectrum to order λ. More generally, determine the exact spectrum.

Solution. Help can be found in the calculations presented in section 4.2. First, using Wick's theorem, one obtains

$$\operatorname{tr} \mathrm{e}^{-\beta H} = \frac{1}{2\sinh(\omega\beta/2)} \left[1 - \tfrac{1}{2}\lambda \int \mathrm{d}t \left\langle \bar{z}(t)z(t)\right\rangle^2 + O\left(\lambda^2\right)\right],$$

where $\langle \bar{z}(t)z(t)\rangle$ is given by the propagator (4.35) at time 0:

$$\langle \bar{z}(t)z(t)\rangle \equiv \Delta(0) = \tfrac{1}{2}\left(\operatorname{sgn}(0) + 1/\tanh(\omega\beta/2)\right).$$

Expanding for β large, one obtains the spectrum to order λ:

$$E_k = (k + \tfrac{1}{2})\omega + \tfrac{1}{4}\lambda(k^2 + k + \tfrac{1}{2}).$$

This is also the exact spectrum. In the limit $\beta \to \infty$, the Gaussian two-point function (4.37) implies time ordering. Therefore, all connected diagrams, except the order λ that is ambiguous, vanish and this proves the property for the ground state energy. More generally, one verifies, before integrating over time, that all perturbative contributions are time-independent. Time integration then yields a factor β^n at order λ^n. The expansion (?.?) then implies that contributions of order higher than one no longer modify the spectrum.

Exercise 4.4

Calculate the two-point function $\langle \bar{z}(u) z(v) \rangle$ corresponding to the weight $\mathrm{e}^{-\mathcal{S}} / \mathcal{Z}$ to order λ^2, for $\beta \to \infty$. Infer the energy $\Omega = E_1 - E_0$, where E_1 corresponds to the first excited state (the one-particle state).

Solution. Expanding in powers of λ^2 and using Wick's theorem, one finds (again the causal property of the propagator much simplifies the calculation)

$$\begin{aligned}\langle \bar{z}(u) z(v) \rangle &= \theta(u-v)\,\mathrm{e}^{-\omega(u-v)}\left(1 - \tfrac{1}{2}\lambda(u-v) + \tfrac{1}{8}\lambda^2(u-v)^2\right) + O(\lambda^3) \\ &= \theta(u-v)\,\mathrm{e}^{-(\omega+\lambda/2)(u-v)} + O(\lambda^3).\end{aligned}$$

Thus, $\Omega = \omega + \frac{1}{2}\lambda + O(\lambda^3)$, in agreement with the exact spectrum.

Exercise 4.5

Explain these results in terms of the corresponding Hamiltonian expressed in terms of creation and annihilation operators.

Solution. The path integral defines the Hamiltonian up to the problem of operator ordering. Therefore, it has the form

$$H = \omega(a^\dagger a + \tfrac{1}{2}) + \tfrac{1}{4}\lambda\big((a^\dagger a)^2 + \alpha a^\dagger a + \beta\big),$$

where α, β are two numerical constants. The spectrum is then

$$E_k = \omega(k + \tfrac{1}{2}) + \lambda\big(k^2 + \alpha k + \beta\big).$$

Comparing with the result of the calculation, one concludes

$$\alpha = 1\,, \quad \beta = \tfrac{1}{2}\,.$$

Finally, since the eigenvectors of the Hamiltonian are those of the harmonic oscillator, the two-point function is given by

$$\begin{aligned}\langle \bar{z}(u) z(v) \rangle &= \theta(u-v)\, \langle 0|\, \mathrm{e}^{-(H-E_0)(u-v)}\, |0\rangle \\ &= \sum_{k=0} \langle 0|\, a\, |k\rangle\, \mathrm{e}^{-(E_k-E_0)(u-v)}\, \langle k|\, a^\dagger\, |0\rangle \\ &= \theta(u-v)\, \mathrm{e}^{-(E_1-E_0)(u-v)}\,.\end{aligned}$$