Lecture 6: THE SCALAR FIELD: FORMAL PERTURBATION THEORY

In this lecture we introduce the generating functionals of connected correlation functions and vertex functions. Expressed in terms of field (functional) integrals, these functions can be calculated by perturbation theory, the successive perturbative contributions being Gaussian expectation values which can be evaluated with the help, for example, of Wick's theorem.

They have a graphical representation in the form of Feynman diagrams. We illustrate diagrammatically the relations between the first connected correlation functions and the corresponding vertex functions.

Perturbative calculations have an algebraic structure which is already apparent in the calculations of path integrals presented in section 3.10 .

The calculation of a field integral by the steepest descent method organizes the perturbative expansion as an expansion in the number of loops in the Feynman diagram representation.

We define quite generally dimensional continuation of Feynman diagrams and introduce the concept of dimensional regularization.

6.1 Correlation functions and generating functionals

Let $\phi(x)$ be a classical field endowed with a probability distribution, a normalized positive measure in the space of fields $[\mathrm{d}\phi]\,\mathrm{e}^{-\mathcal{S}(\phi)}/\mathcal{Z}$, where $\mathcal{S}(\phi)$ is the euclidean action and

$$\mathcal{Z} = \int [\mathrm{d}\phi]\,\mathrm{e}^{-\mathcal{S}(\phi)}$$

the associated partition function.

Correlation functions. The n-point correlation function is defined by

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}}\int [\mathrm{d}\phi]\phi(x_1)\phi(x_2)\ldots\phi(x_n)\,\mathrm{e}^{-\mathcal{S}(\phi)}, \qquad (6.1)$$

where the normalization ensures that $\langle 1\rangle = 1$.

6.1.1 Generating functional

Correlation functions can be inferred by functional differentiation from the generating functional (see also section 3.6)

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi) + \int \mathrm{d}^d x\, J(x)\phi(x)\right] \tag{6.2a}$$

$$= \mathcal{Z}(0)\left\langle \exp \int \mathrm{d}^d x\, \phi(x) J(x) \right\rangle, \quad \mathcal{Z}(0) \equiv \mathcal{Z}\,. \tag{6.2b}$$

Indeed,

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}(0)}\left(\prod_i \frac{\delta}{\delta J(y_i)}\right)\mathcal{Z}(J)\bigg|_{J=0}. \tag{6.3}$$

6.1.2 Connected correlation functions and cluster properties

The functional $\mathcal{W}(J) = \ln \mathcal{Z}(J)$ generates connected correlation functions (generalized cumulants):

$$\mathcal{W}(J) = \sum_{n=0} \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, W^{(n)}(x_1, \ldots, x_n) J(x_1) \ldots J(x_n).$$

It can be proved that in local field theories, connected correlation functions $W^{(n)}(x_1, \ldots, x_n)$ decays at least algebraically when the points $x_1, x_2, \ldots, x_n$ belong to two largely separated non-empty subsets $\{x_1 \ldots x_p\}$, $\{x_{p+1} \ldots x_n\}$:

$$W^{(n)}(x_1 \ldots x_p, x_{p+1} \ldots x_n) \to 0 \quad \text{when} \quad \min_{\substack{i=1\cdots p \\ j=p+1\cdots n}} |x_i - x_j| \to \infty \, .$$

This property, called cluster property, is a characteristic property of connected correlation functions.

As a consequence, in a perturbative expansion to connected correlation functions contribute only connected Feynman diagrams.

6.1.3 Generating functional of vertex functions

We also define the generating functional $\Gamma(\varphi)$ of vertex functions:

$$\Gamma(\varphi)=\sum_{n=0}\frac{1}{n!}\int \mathrm{d}^d x_1\ldots \mathrm{d}^d x_n\,\Gamma^{(n)}(x_1,\ldots,x_n)\varphi(x_1)\ldots\varphi(x_n).$$

where φ is a classical field (the local magnetization in magnetic systems) and $\Gamma(\varphi)$ is the Legendre transform of $\mathcal{W}(J)$:

$$\mathcal{W}(J)+\Gamma(\varphi)=\int \mathrm{d}^d x\, J(x)\varphi(x),\quad \varphi(x)=\frac{\delta\mathcal{W}(J)}{\delta J(x)}, \tag{6.4}$$

The role of the vertex functions $\Gamma^{(n)}$ will be discussed later. In statistical physics, $\Gamma(\varphi)$ is the thermodynamic potential, a function, for example, of the local magnetization for magnetic systems.

In a perturbative expansion only one-line or one-particle irreducible (1PI) Feynman diagrams contribute to vertex functions.

A general property. We now assume that $\mathcal{W}(J)$ depends on a parameter h. Then, after differentiating the first equation (6.4) with respect to h at J fixed, one finds

$$\left.\frac{\partial\Gamma}{\partial h}\right|_{\varphi\,\text{fixed}} + \int \mathrm{d}^d x\, \frac{\delta\Gamma}{\delta\varphi(x)}\frac{\partial\varphi(x)}{\partial h} + \left.\frac{\partial\mathcal{W}}{\partial h}\right|_{J\,\text{fixed}} = \int \mathrm{d}^d x\, J(x)\frac{\partial\varphi(x)}{\partial h}\,.$$

Then, using the second equation (6.4), one notes that the two terms proportional to $\partial\varphi/\partial h$ cancel and one infers

$$\left.\frac{\partial\Gamma}{\partial h}\right|_{\varphi\,\text{fixed}} + \left.\frac{\partial\mathcal{W}}{\partial h}\right|_{J\,\text{fixed}} = 0\,. \tag{6.5}$$

6.1.4 Fourier representation

We have assumed that the field theory is translation invariant. This hypothesis implies that for all $a \in \mathbb{R}^d$, the n-point connected correlation function satisfies

$$W^{(n)}(x_1 + a, \ldots, x_n + a) = W^{(n)}(x_1, \ldots, x_n).$$

As a consequence, the relation between an n-point function and its Fourier transform can be written as

$$\begin{aligned} &(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{n} p_i\right) \widetilde{W}^{(n)}(p_1, \ldots, p_n) \\ &\quad = \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, W^{(n)}(x_1, \ldots, x_n) \exp\left(i \sum_{j=1}^{n} x_j p_j\right). \end{aligned} \tag{6.6}$$

Cluster properties of connected correlation functions imply regularity properties of $\widetilde{W}^{(n)}(p_1, \ldots, p_n)$ in the variables p_i.

Vertex functions. In section 6.1.3, we have also defined vertex functions. The Fourier transform of the n-point vertex function is defined by

$$(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{n} p_i\right) \tilde{\Gamma}^{(n)}(p_1, \ldots, p_n)$$
$$= \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \Gamma^{(n)}(x_1, \ldots, x_n) \exp\left(i \sum_{j=1}^{n} x_j \cdot p_j\right). \qquad (6.7)$$

Vertex functions, due to the regularity properties of the $\tilde{\Gamma}^{(n)}$, play a central role in the perturbative expansion and in renormalization theory.

Note that in a translation invariant theory, as assumed above, the relations between connected and vertex functions are purely algebraic.

6.2 Gaussian or free field theory. Wick's theorem

In field theory, as in other stochastic processes, the simplest measure is the Gaussian measure, as several examples have already illustrated.

6.2.1 Gaussian or free field theory

In quantum field theory, unitarity (necessary for conservation of probabilities) is only satisfied for the simple Gaussian theory of the form

$$\mathcal{S}_{\mathrm{G}}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \left[\left(\nabla_x \phi(x) \right)^2 + m^2 \phi^2(x) \right].$$

The limit $m = 0$ corresponds to a massless theory (a critical theory in the terminology of phase transitions). For $m > 0$ the theory is massive (in the terminology of statistical physics the correlation length $\xi = \hbar / mc$ is finite).

However, in presence of local interactions, such a theory leads to large momentum or short distance divergences that have to be regularized. This can be achieved, for example, by adding unphysical higher derivative terms.

Therefore, we consider a more general Gaussian theory with a translation and rotation invariant euclidean action of the form

$$\mathcal{S}_{\mathrm{G}}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \left\{ \nabla_x \phi(x) \left(1 + \sum_{k=1} u_{k+1} (-\nabla_x^2)^k \right) \nabla_x \phi(x) + m^2 \phi^2(x) \right\}, \tag{6.8}$$

where the coefficients u_k are such that the quadratic form is positive.

The field integral,

$$\begin{aligned} \mathcal{Z}_{\mathrm{G}}(J) &= \int [\mathrm{d}\phi] \exp[-\mathcal{S}_{\mathrm{G}}(\phi)] \exp\left[\int \mathrm{d}^d x \, \phi(x) J(x) \right] \\ &= \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}_{\mathrm{G}}(\phi) + \int \mathrm{d}^d x \, \phi(x) J(x) \right], \end{aligned}$$

a functional of the external field $J(x)$, is proportional to the generating functional of correlation functions corresponding to the measure $\mathrm{e}^{-\mathcal{S}_{\mathrm{G}}(\phi)} / \mathcal{Z}_{\mathrm{G}}(0)$,

$$\left\langle \exp\left[\int \mathrm{d}^d x \, \phi(x) J(x) \right] \right\rangle_{\mathcal{H}_{\mathrm{G}}} = \mathcal{Z}_{\mathrm{G}}(J) / \mathcal{Z}_{\mathrm{G}}(0).$$

6.2.2 Calculation of the integral

To calculate the field integral, we solve the classical field equation derived from the action (6.8),

$$\mathcal{K}(\nabla_x)\phi(x) = J(x)$$

with

$$\mathcal{K}(\nabla_x) = -\nabla_x^2 + \sum_{k=1} u_{k+1}(-\nabla_x^2)^{k+1} + m^2.$$

In terms of Δ, the inverse of $\mathcal{K}$, the solution can be written as

$$\phi(x) = \int \mathrm{d}^d y \, \Delta(x-y) J(y). \tag{6.9}$$

One then change variables $\phi(x) \mapsto \phi'(x)$ with

$$\phi(x) = \phi'(x) + \int \mathrm{d}^d y \, \Delta(x-y) J(y). \tag{6.10}$$

This shift of $\phi(x)$ eliminates the term linear in ϕ in the exponential. The measure is invariant and the integrand becomes

$$\mathcal{Z}_{\mathrm{G}}(J) = \exp\left[\frac{1}{2}\int \mathrm{d}^d y\, J(x)\Delta(x-y)J(y)\right]\int [\mathrm{d}\phi']\exp[-\mathcal{S}_{\mathrm{G}}(\phi')].$$

The dependence in J is now explicit. The residual integral yields only a normalization. Its calculation may be difficult but it cancels in the calculation of correlation functions. Indeed, the measure must be normalized in such a way that $\langle 1\rangle_{\mathrm{G}} = 1$, where $\langle \bullet \rangle_{\mathrm{G}}$ means Gaussian expectation value (or free field in the context of quantum field theory). One concludes

$$\begin{aligned}\left\langle \exp\left[\int \mathrm{d}^d x\, \phi(x)J(x)\right]\right\rangle_{\mathrm{G}} &= \mathcal{Z}_{\mathrm{G}}(J)/\mathcal{Z}_{\mathrm{G}}(0) \\ &= \exp\left[\tfrac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x)\Delta(x-y)J(y)\right]. \quad (6.11)\end{aligned}$$

The kernel Δ, the inverse of $\mathcal{K}$, is the Gaussian two-point function, and is also called the propagator.

In a translation-invariant theory, it is convenient to introduce the Fourier representation:

$$\mathcal{K} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{-ip\cdot x}\, \tilde{\mathcal{K}}(p), \qquad \tilde{\mathcal{K}}(p) = p^2 + \sum_{k=1} u_{k+1}(p^2)^{k+1} + m^2, \tag{6.12a}$$

$$\Delta(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{ip\cdot x}\, \tilde{\Delta}(p), \qquad \tilde{\Delta}(p) = \int \mathrm{d}^d x\, \mathrm{e}^{ip\cdot x}\, \Delta(x), \tag{6.12b}$$

and thus

$$\tilde{\Delta}(p)\tilde{\mathcal{K}}(p) = 1\,.$$

In order for the field integral to exist, $\tilde{\mathcal{K}}(p)$ must be positive for $p \neq 0$. Since $\mathcal{K}(p)$ is rotation-invariant (the special orthogonal group $SO(d)$), the function $\Delta(x)$ is a function only of $|x|$, and $\tilde{\Delta}(p)$ only of $|p|$.

Connected correlation functions. The generating functional of connected correlation functions $\mathcal{W}_{\mathrm{G}} = \ln \mathcal{Z}_{\mathrm{G}}$ reduces to a simple quadratic form:

$$\mathcal{W}_{\mathrm{G}}(J) - \mathcal{W}_{\mathrm{G}}(0) = \tfrac{1}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x)\Delta(x-y)J(y).$$

As we have already pointed out, in the case of Gaussian integrals, connected functions (generalized cumulants) with more than two points vanish.

Vertex functions. Finally, we calculate the generating functional $\Gamma_{\mathrm{G}}(\varphi)$ of vertex functions by a Legendre transformation. First,

$$\varphi(x) = \frac{\delta \mathcal{W}_{\mathrm{G}}}{\delta J(x)} = \int \mathrm{d}^d y\, \Delta(x-y) J(y) \ \Rightarrow\ J(x) = \mathcal{K}(\nabla_x)\varphi(x).$$

Thus,

$$\begin{aligned}\Gamma_{\mathrm{G}}(\varphi) &= \int \mathrm{d}^d x\, \varphi(x) J(x) - \mathcal{W}(J) \\ &= \tfrac{1}{2} \int \mathrm{d}^d x\, \varphi(x) \mathcal{K}(\nabla_x) \varphi(x) = \mathcal{S}_{\mathrm{G}}(\varphi),\end{aligned}$$

is directly related to the action.

Wick's theorem. Expression (6.11) combined with the arguments of section 3.8.3, leads to an immediate generalization of equation (3.43), which expresses Wick's theorem in a scalar field theory:

$$\begin{aligned} \left\langle \prod_{1}^{2s} \phi(x_i) \right\rangle_{\mathrm{G}} &= \left[\prod_{i=1}^{2s} \frac{\delta}{\delta J(x_i)} \exp \left[\mathcal{W}_{\mathrm{G}}(J) - \mathcal{W}_{\mathrm{G}}(0) \right] \right] \Bigg|_{J \equiv 0} \\ &= \sum_{\substack{\text{all pairings} \\ \text{of } \{1,2,\ldots,2s\}}} \Delta(x_{i_1} - x_{i_2}) \ldots \Delta(x_{i_{2s-1}} - x_{i_{2s}}). \end{aligned} \tag{6.13}$$

6.3 Perturbative expansion

We now consider a more general (euclidean) action of the form

$$\mathcal{S}(\phi) = \mathcal{S}_{\mathrm{G}}(\phi) + \mathcal{V}_{\mathrm{I}}(\phi), \tag{6.14}$$

where $\mathcal{S}_{\mathrm{G}}(\phi)$ is the quadratic form (6.8) and $\mathcal{V}_{\mathrm{I}}(\phi)$ is a local polynomial in the field, which, in the context of quantum field theory, is called an interaction. In a local field theory, that is, in the class that we have introduced in the preceding lectures, $\mathcal{V}_{\mathrm{I}}(\phi)$ is the space integral of a function of the field and its derivatives:

$$\mathcal{V}_{\mathrm{I}}(\phi) = \int \mathrm{d}^d x \, V_{\mathrm{I}}[\phi(x), \partial_\mu \phi(x), \ldots]. \tag{6.15}$$

Although most results presented in this section will be illustrated only by actions of ϕ^3 and ϕ^4 types, these results apply to more general theories.

6.3.1 Perturbative expansion

The perturbative expansion of correlation functions is obtained by expanding expression (6.14) in powers of $\mathcal{V}_{\mathrm{I}}$, keeping only the quadratic term $\mathcal{S}_{\mathrm{G}}(\phi)$ in the exponential. The interaction (6.15) is a sum of monomials called interaction vertices. The expansion then reduces to the calculation of Gaussian expectation values of products of fields of the form

$$\left\langle \phi(x_1)\cdots\phi(x_n) \int \mathrm{d}^d y_1\, \phi^{p_1}(y_1) \int \mathrm{d}^d y_2\, \phi^{p_2}(y_2) \cdots \int \mathrm{d}^d y_k\, \phi^{p_k}(y_k) \right\rangle_{\mathrm{G}} \tag{6.16}$$

(to simplify, we have omitted possible derivatives) and, thus, to Wick's theorem. Wick's theorem involves the Gaussian two-point function or propagator Δ (equation (6.9)). Each contribution takes the form of a product of propagators integrated over all points corresponding to interaction vertices and has a graphical representation in terms of Feynman diagrams.

The perturbative expansion has a formal global representation. Using systematically the property,

$$\frac{\delta}{\delta J(x)} \exp\left[\int \mathrm{d}^d y\, J(y)\phi(y)\right] = \phi(x) \exp\left[\int \mathrm{d}^d y\, J(y)\phi(y)\right],$$

one can express $\mathcal{Z}(J)$ in terms of the Gaussian functional $\mathcal{Z}_{\mathrm{G}}(J)$ in the form

$$\begin{aligned}\mathcal{Z}(J) &= \exp\left[-\mathcal{V}_{\mathrm{I}}\left(\frac{\delta}{\delta J}\right)\right] \mathcal{Z}_{\mathrm{G}}(J) = \exp\left[-\mathcal{V}_{\mathrm{I}}\left(\frac{\delta}{\delta J}\right)\right] \exp\left[\mathcal{W}_{\mathrm{G}}(J)\right] \\ &= \mathcal{Z}_{\mathrm{G}}(0) \exp\left[-\mathcal{V}_{\mathrm{I}}\left(\frac{\delta}{\delta J}\right)\right] \exp\left[\tfrac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x)\Delta(x-y)J(y)\right]. \quad (6.17)\end{aligned}$$

Combining identities (6.3) and (6.17), one can calculate correlation functions of the field ϕ as formal series in powers of the interaction $\mathcal{V}_{\mathrm{I}}$, to use quantum field theory terminology. To each monomial contributing to $\mathcal{V}_{\mathrm{I}}$ corresponds a differential operator: a product of derivatives $\delta/\delta J$ that generates a product of propagators Δ.

Discussion. As we have already stressed, the kernel $\mathcal{K}$ cannot be reduced to the Gaussian form $-\nabla_x^2 + m^2$ because the perturbative expansion then contains short-distance divergences. This is a manifestation of a coupling between the different physical scales.

In the Fourier representation, short-distance divergences become large-momentum or ultraviolet (UV) divergences (in quantum field theory the arguments of the Fourier transform are momenta or energies).

These divergences are unphysical in the theory of phase transitions since the fixed-point two-point function is only the asymptotic form at large distance, and the lattice, or more generally the microscopic structure, modify the theory at short distance.

In particular physics, the existence of such a necessary short distance modification can only be guessed and its precise form is unknown.

Here, we work in the continuum and we add to the action irrelevant (subleading at large distance) terms with higher order derivatives, to generate a propagator $\Delta(x-y)$ sufficiently regular for $|x-y| \to 0$. This modification is called a regularization. In the Fourier representation, finiteness of the perturbative expansion requires that $\tilde{\Delta}(p)$, the Fourier transform of $\Delta(x-y)$ decays fast enough for $p \to \infty$.

These regularization terms are different from those of the true theory. In quantum field theory they even lead to unphysical properties. However, renormalization theory and renormalization group arguments imply that the precise form of the regularization does not affect large distance properties.

A fictitious lattice can also be used because, since momenta vary in a bounded domain, a Brillouin zone, these divergences are absent.

Here, we assume that the large momentum decay of the propagator, in the Fourier representation, is sufficiently fast to render the perturbative expansion finite to all orders.

6.4 Feynman diagrams: loops

We now define Feynman diagrams. To each monomial contributing to the interaction, one associates a vertex, a point from which originates a number of lines equal to the degree of the monomial. A propagator is represented by a line that joins the points that correspond to its arguments. These points are either vertices, or points corresponding to external arguments of a correlation function.

In what follows we call an internal line a line that joins two vertices. By contrast, an external line joins a vertex to a point of a correlation function.

In a local action, a vertex corresponds to a space integral of a product of fields and their derivatives (representation (6.15)). Each vertex, in a diagram, thus corresponds to an argument on which one integrates.

After Fourier transformation, to each line is attached a momentum, the argument of the propagator in the Fourier representation. This assumes an orientation of the lines: changing the orientation changes the sign of the momentum attached to the line.

In the Fourier representation, due to translation invariance, at each vertex the sum of the entering momenta vanishes: this is analogous to Kirchoff's laws for current intensities in an electric circuit.

Finally, one integrates over all remaining free, independent momenta.

A topological relation. For any connected diagram, the number of loops L, the number I of internal lines, or propagators, relating vertices and the number of vertices n are related by

$$L = I - n + 1 . \tag{6.18}$$

A method for proving the relation is the following:

(i) Cutting one internal line, one also suppresses one loop and $L - I$ is unchanged.

(ii) A diagram without loops is a tree. Suppressing one vertex on the boundary of a tree, one transforms an internal line into an external line and $I - n$ is unchanged. Eventually, every diagram is reduced to a vertex which corresponds to $L = I = 0$, $n = 1$.

In the Fourier representation, L is also the number of independent momenta over which to integrate: it is equal to the number of propagators minus the number of vertices, due to momentum conservation at each vertex, plus 1 because the conservation of the total momentum entering in a diagram is then automatic.

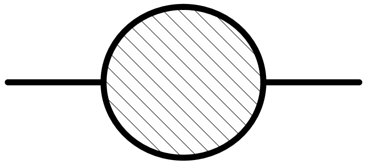
Remark. Local interactions may also contain derivatives of the fields. Then, the evaluation of expression (6.16) involves also derivatives of the propagator. The representation in terms of Feynman diagrams, as they have been defined so far, is no longer faithful since the presence of derivatives is not indicated. One can construct a more faithful representation by splitting vertices and by placing arrows on lines.

6.4.1 Connected and one-line irreducible diagrams

In the Feynman diagram representation, the perturbative expansion of $\mathcal{Z}(J)$ contains non-connected contributions in the sense of graphs. By contrast, $\mathcal{W}(J) = \mathcal{Z}(J)$, is the sum of connected contributions.

Finally, the functional $\Gamma(\varphi)$ has the simplest perturbative properties: indeed, one proves that its expansion contains only one-line irreducible diagrams, that is, diagrams that cannot be decomposed into several connected components by cutting only one line.

Renormalization theory deals directly with one-line irreducible diagrams also called 1PI for one-particle irreducible.

$W^{(2)} =$  ,

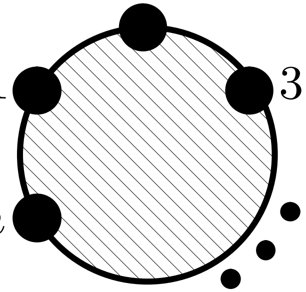
$\Gamma^{(n)} =$ 

Fig. 6.1 – Representations of the connected two-point correlation function and the n-point vertex function.

For illustration, we give a graphical representation of the first relations between connected and vertex functions.

In figure 6.1, we define the graphical representation of $W^{(2)}$ and $\Gamma^{(n)}$. In the representation of $\Gamma^{(n)}$, we have emphasized the property that no propagator is attached to the points of the boundary of the graph, in contrast with the diagrams contributing to connected functions.

The relation between two-point functions is

$$\int \mathrm{d}^d z\, W^{(2)}(x-z)\Gamma^{(2)}(z-y) = \delta^{(d)}(x-y).$$

It is convenient to set

$$\Gamma^{(2)}(x-y) = \mathcal{K}(x-y) + \Sigma(x-y),$$

where we have separated the Gaussian contribution $\mathcal{K}$ from the sum Σ of contributions generated by the interactions $\mathcal{V}_\mathrm{I}$, also called the mass operator. In terms of Σ, the perturbative expansion of $W^{(2)}$ can be organized as a geometrical series:

$$W^{(2)}(x-y) = \Delta(x-y) - \int \mathrm{d}^d z_1\, \mathrm{d}^d z_2\, \Delta(x-z_1)\Sigma(z_1-z_2)\Delta(z_2-y) + \cdots\ .$$

Fig. 6.2 displays one term of the sum expressed in terms of Σ, that is, of 1-irreducible components.

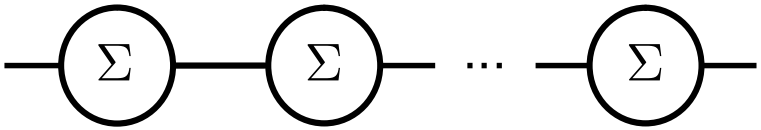


Fig. 6.2 – Contribution to the connected two-point function $W^{(2)}$.

The graphical representations of the correlation functions $W^{(3)}$ and $W^{(4)}$, for example, in terms of the corresponding vertex functions and of $W^{(2)}$ are given in Figs. 6.3 and 6.4, respectively.

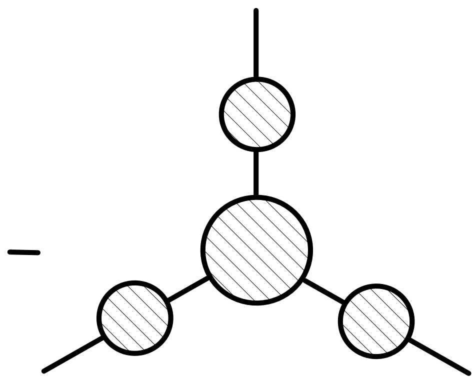


Fig. 6.3 – The connected three-point function $W^{(3)}$.

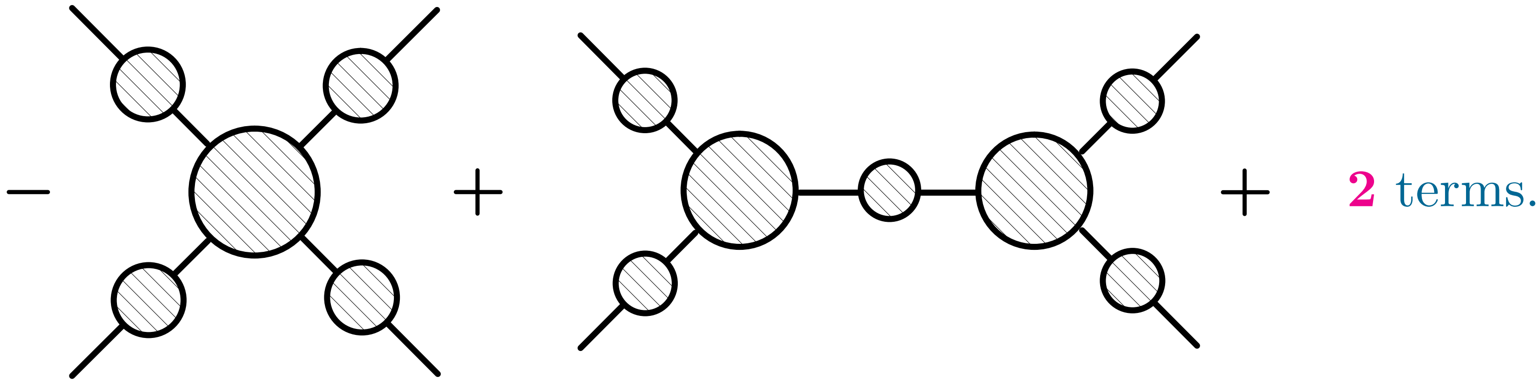


Fig. 6.4 – The connected four-point function $W^{(4)}$.

6.5 Example: the ϕ^4 interaction

We now consider the example, specially useful both for particle physics and the theory of macroscopic phase transitions, of the quartic interaction

$$\mathcal{V}_{\mathrm{I}}(\phi) \equiv \frac{g}{4!} \int \mathrm{d}^d x \, \phi^4(x). \tag{6.19}$$

The first non-trivial order in g of the RG functions can be derived from the expansion of the two- and four-point functions up to order g^2.

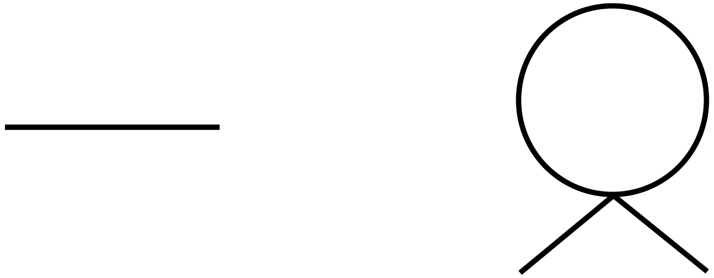


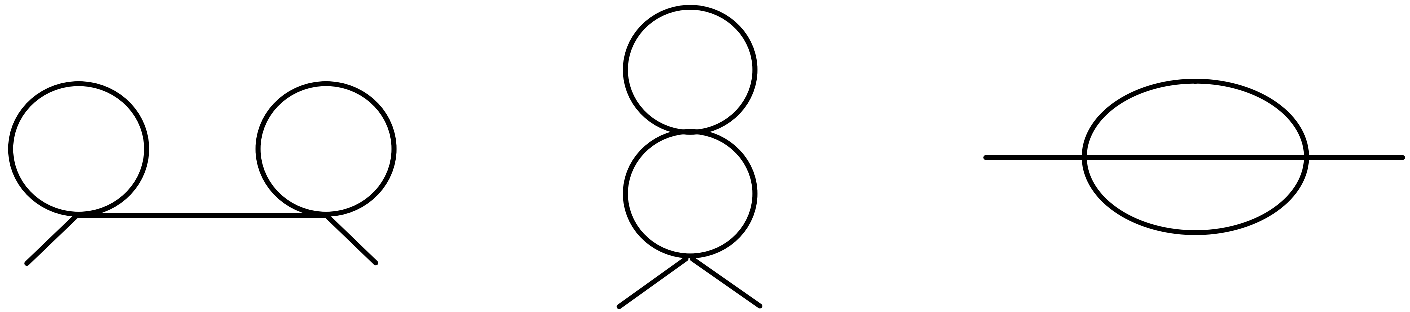
Fig. 6.5 – The two-point function: contributions of order 1 and g.

6.5.1 *Two-point function*

The two-point function at order g^2 is given by

$$\langle\phi(x_1)\phi(x_2)\rangle = \text{(a)} - \tfrac{1}{2}g\,\text{(b)} + \tfrac{1}{4}g^2\,\text{(c)} + \tfrac{1}{4}g^2\,\text{(d)} + \tfrac{1}{6}g^2\,\text{(e)} + \; O\left(g^3\right).$$

In the expansion, (a) is the propagator and (b) the Feynman diagram that appears at order g, and both are displayed in figure 6.5. The diagrams (c), (d), (e) of order g^2 are displayed in figure 6.6.



(c) (d) (e)

Fig. 6.6 – The two-point function: contributions of order g^2.

The three additional contributions that factorize into

$$\langle\phi(x_1)\phi(x_2)\rangle_{\mathrm{G}}\left\langle\phi^4(y)\right\rangle_{\mathrm{G}}, \quad \left\langle\phi(x_1)\phi(x_2)\phi^4(y_1)\right\rangle_{\mathrm{G}}\left\langle\phi^4(y_2)\right\rangle_{\mathrm{G}} \text{ and}$$
$$\langle\phi(x_1)\phi(x_2)\rangle_{\mathrm{G}}\left\langle\phi^4(y_1)\phi^4(y_2)\right\rangle_{\mathrm{G}},$$

cancel after division by the partition function $\mathcal{Z}$.

In quantum field theory, the connected diagrams contributing to $\ln\mathcal{Z}$ are also called vacuum diagrams, because in the quantum context they contribute to the ground state (the vacuum) energy.

More explicitly, one thus finds (here $W^{(2)} = Z^{(2)}$)

$$W^{(2)}(x_1 - x_2) = \Delta(x_1 - x_2) - \tfrac{1}{2}g \int \mathrm{d}^d y \, \Delta(x_1 - y)\Delta(0)\Delta(y - x_2)$$
$$+ g^2 \int \mathrm{d}^d y_1 \, \mathrm{d}^d y_2 \left[\tfrac{1}{4}\Delta^2(0)\Delta(x_1 - y_1)\Delta(y_1 - y_2)\Delta(y_2 - x_2)\right.$$
$$+ \tfrac{1}{4}\Delta(0)\Delta^2(y_1 - y_2)\Delta(x_1 - y_2)\Delta(x_2 - y_2)$$
$$\left.+\tfrac{1}{6}\Delta(x_1 - y_1)\Delta^3(y_1 - y_2)\Delta(y_2 - x_2)\right] + O(g^3).$$

Only the diagram (c) is 1-reducible and it cancels in the Legendre transformation. Also, the external propagators are removed. One then finds

$$\Gamma^{(2)}(x_1 - x_2) = \mathcal{S}^{(2)}(x_1 - x_2) + \tfrac{1}{2}g\delta^{(d)}(x_1 - x_2)\Delta(0)$$
$$- \tfrac{1}{4}g^2\Delta(0) \int \mathrm{d}^d y \, \Delta^2(y)\delta^{(d)}(x_1 - x_2) - \tfrac{1}{6}g^2\Delta^3(x_1 - x_2) + O(g^3),$$

which is a simpler expression.

Fourier transformation. As we have already pointed out, in a translation-invariant theory the relations between correlation and vertex functions take simpler forms in the Fourier representation. We thus introduce the functions defined in (6.6), (6.7) and the representations (6.12) of the propagator and of its inverse. With this notation, the vertex two-point function becomes

$$\tilde{\Gamma}^{(2)}(p) = \tilde{\mathcal{K}}(p) + \frac{g}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\,\tilde{\Delta}(q) - \frac{g^2}{4}\int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\tilde{\Delta}(q_1)\int \frac{\mathrm{d}^d q_2}{(2\pi)^d}\tilde{\Delta}^2(q_2)$$
$$- \frac{g^2}{6}\int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\tilde{\Delta}(q_1)\tilde{\Delta}(q_2)\tilde{\Delta}(p-q_1-q_2) + O(g^3). \quad (6.20)$$

The connected two-point function is then obtained by expanding the relation

$$\widetilde{W}^{(2)}(p)\tilde{\Gamma}^{(2)}(p) = 1\,.$$

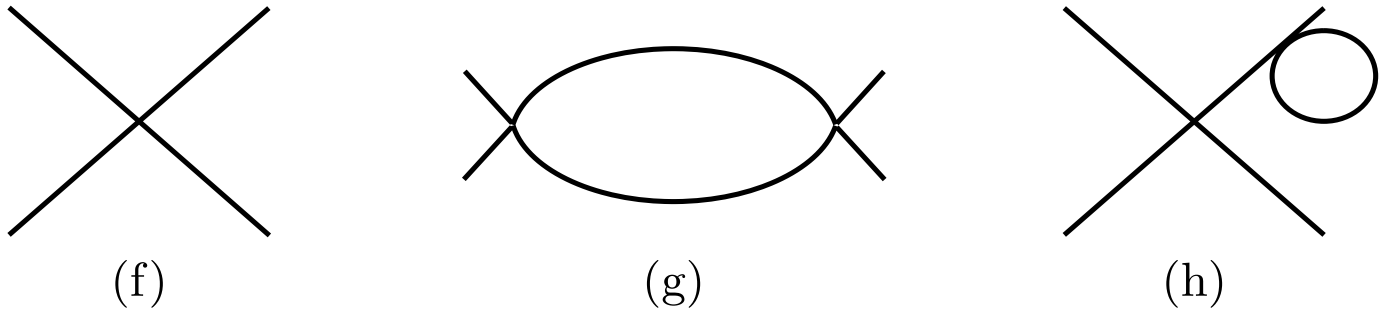


Fig. 6.7 – Feynman diagrams, connected four-point function: contributions of order g and g^2.

6.5.2 Four-point function

At order g^2, the four-point function is given by

$$\begin{aligned}
&\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\
&\quad = [\,(\mathrm{a})_{12}\ (\mathrm{a})_{34}\ + 2\,\text{terms}\,] - \tfrac{1}{2}g\,[\,(\mathrm{a})_{12}\ (\mathrm{b})_{34} + 5\,\text{terms}\,] - g\,(\mathrm{f}) \\
&\qquad + g^2\left\{(\mathrm{a})_{12}\left[\tfrac{1}{4}\left((\mathrm{c})_{34} + (\mathrm{d})_{34}\right) + \tfrac{1}{6}(\mathrm{e})_{34}\right] + 5\ \text{terms}\right\} \\
&\qquad + \tfrac{1}{4}g^2\,[\,(\mathrm{b})_{12}\ (\mathrm{b})_{34}\ + 2\,\text{terms}\,] + \tfrac{1}{2}g^2\,[(\mathrm{g}) + 3\ \text{terms}\,] \\
&\qquad + \tfrac{1}{2}g^2\,[(\mathrm{h}) + 2\ \text{terms}\,] + O\left(g^3\right).
\end{aligned}$$

The diagrams (f), (g), (h) are displayed in Fig. 6.7. The notation $(\mathrm{a})_{12}$, for example, means diagram (a), contributing to the two-point function, with arguments x_1 and x_2. Finally, the terms that must be added in order to restore the permutation symmetry of the four-point function are obtained by exchanging the external arguments.

Diagrams such as $\mathrm{a}_{12}\mathrm{a}_{34}$, which are expressed in terms of two-point function contributions, are not connected and factorize into a product of functions that depend on disjoint subsets of variables.

Again, as in the case of the two-point function, we have omitted non-connected diagrams in which one factor has no external arguments. These diagrams are cancelled by the perturbative contributions of the partition function $\mathcal{Z}$ in expression (6.3).

The connected four-point function, in a more explicit notation, reduces to

$$
\begin{aligned}
&W^{(4)}(x_1, x_2, x_3, x_4) \\
&\quad = -g \int \mathrm{d}^d y \, \Delta(x_1 - y)\Delta(y - x_2)\Delta(x_3 - y)\Delta(x_4 - y) + \tfrac{1}{2} g^2 \\
&\qquad \times \int \mathrm{d}^d y_1 \mathrm{d}^d y_2 \, \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(x_3 - y_2)\Delta(x_4 - y_2)\Delta^2(y_1 - y_2) \\
&\qquad\quad + \ 2 \text{ terms} \\
&\qquad + \tfrac{1}{2} g^2 \int \mathrm{d}^d y_1 \mathrm{d}^d y_2 \, \Delta(y_1 - y_1)\Delta(y_1 - y_2)\Delta(x_1 - y_1)\Delta(x_2 - y_2) \\
&\qquad \times \Delta(x_3 - y_2)\Delta(x_4 - y_2) + \ 3 \text{ terms} + O(g^3).
\end{aligned}
$$

The Legendre transformation is simple also for the four-point function in this theory: here it suffices to remove the contributions of the two-point functions on the external lines, an operation called an amputation, and to change the sign.

One finds

$$\Gamma^{(4)}(x_1,x_2,x_3,x_4) = g\delta^{(d)}(x_1-x_2)\delta^{(d)}(x_1-x_3)\delta^{(d)}(x_1-x_4)$$
$$-\tfrac{1}{2}g^2\delta^{(d)}(x_1-x_2)\delta^{(d)}(x_3-x_4)\Delta^2(x_1-x_3) + \text{ 2 terms} + O(g^3).$$

The respective Fourier transforms are then given by

$$\begin{aligned}\widetilde{W}^{(4)}(p_1,p_2,p_3,p_4) = {} & \tilde{\Delta}(p_1)\tilde{\Delta}(p_2)\tilde{\Delta}(p_3)\tilde{\Delta}(p_4) \\ & \times\left[-g + \frac{g^2}{2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\,\tilde{\Delta}(p_1+p_2-q)\tilde{\Delta}(q) + \text{ 2 terms}\right. \\ & \left. + \frac{g^2}{2}\tilde{\Delta}(p_1)\int\frac{\mathrm{d}^d q}{(2\pi)^d}\,\tilde{\Delta}(q) + \text{ 3 terms}\right] + O(g^3)\end{aligned}$$

and

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) = g - \frac{g^2}{2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\,\tilde{\Delta}(p_1+p_2-q)\tilde{\Delta}(q) + \text{ 2 terms} + O(g^3).$$

6.6 Loop expansion

The perturbative expansion can be organized in the form of a loop expansion. For this purpose, it is convenient to reinstate a factor $\hbar$ and consider the partition function

$$\mathcal{Z}(J) = \int [\mathrm{d}\phi] \exp\left[-\frac{1}{\hbar}\left(\mathcal{S}(\phi) - \int \mathrm{d}^d x\, J(x)\phi(x)\right)\right].$$

In the formal limit $\hbar \to 0$, the integral can be calculated by the steepest descent method applied to field integrals.

The saddle point ϕ_c is given by the minimum of the functional

$$\mathcal{S}(\phi, J) = \mathcal{S}(\phi) - \int \mathrm{d}^d x\, J(x)\phi(x),$$

and thus is solution of the equation

$$\left.\frac{\delta \mathcal{S}(\phi, J)}{\delta \phi(x)}\right|_{\phi=\phi_\mathrm{c}} = \left.\frac{\delta \mathcal{S}(\phi)}{\delta \phi(x)}\right|_{\phi=\phi_\mathrm{c}} - J(x) = 0\,.$$

After the change of variables $\phi \mapsto \chi$,

$$\phi(x) = \phi_{\mathrm{c}}(J;x) + \chi(x),$$

one expands $\mathcal{S}(\phi, J)$ in powers of χ:

$$\mathcal{S}(\phi, J) = \mathcal{S}(\phi_{\mathrm{c}}, J) + \frac{1}{2!}\int \mathrm{d}^d x_1 \mathrm{d}^d x_2\, \chi(x_1) \left.\frac{\delta^2 \mathcal{S}(\phi)}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi=\phi_{\mathrm{c}}} \chi(x_2) \\ + O(\chi^3).$$

One keeps the term quadratic in χ in the exponential and expands the terms of higher degree in χ. This reduces the calculation of each term to a Gaussian expectation value but with a propagator and vertices that have now a non-trivial dependence in the classical field ϕ_c.

It is convenient to normalize connected correlation functions as

$$\mathcal{W}(J) = \hbar \ln \mathcal{Z}(J). \tag{6.21}$$

6.6.1 Leading order: tree diagrams

Approximating the field integral by its value at the saddle point, one finds the leading contribution to the connected functional

$$\mathcal{W}_0(J) = -\mathcal{S}(J, \phi_c) = -\mathcal{S}(\phi_c) + \int \mathrm{d}^d x\, J(x)\phi_c(x)\,. \tag{6.22}$$

One then verifies that the Legendre transform Γ_0 of $\mathcal{W}_0(J)$ is given by

$$\Gamma_0(\varphi) = \mathcal{S}(\varphi).$$

Indeed, first

$$\varphi(x) = \frac{\delta \mathcal{W}}{\delta J(x)} = \phi_c(x) - \int \mathrm{d}^d y\, \frac{\delta \mathcal{S}(J, \phi_c)}{\delta \phi_c(y)} \frac{\delta \phi_c(y)}{\delta J(x)} = \phi_c(x). \tag{6.23}$$

Then, from equation (6.22),

$$\Gamma_0(\varphi) = \int \mathrm{d}^d x\, J(x)\varphi(x) - \mathcal{W}_0(J) = \mathcal{S}(\varphi).$$

Loop expansion and perturbative expansion. The direct perturbative expansion is obtained by expanding the solution $\phi_{\mathrm{c}}(J)$ in powers of J. The diagrams generated in this way are tree diagrams (without loops). In the example of an action of the form (6.14),

$$\mathcal{S}(\phi)=\mathcal{S}_{\mathrm{G}}(\phi)+\mathcal{V}_{\mathrm{I}}(\phi),$$

the expansion takes the form

$$\begin{aligned}\phi_{\mathrm{c}}(x)&=\int \mathrm{d}^d y\,\Delta(x-y)J(y)-\int \mathrm{d}^d y\,\Delta(x-y)\frac{\delta\mathcal{V}_{\mathrm{I}}(\phi_{\mathrm{c}})}{\delta\phi(y)}\\&=\int \mathrm{d}^d y\,\Delta(x-y)J(y)-\int \mathrm{d}^d y\,\Delta(x-y)\frac{\delta\mathcal{V}_{\mathrm{I}}}{\delta\phi_{\mathrm{c}}(y)}(\Delta J)+\cdots\,,\end{aligned}$$

where the argument ΔJ of $\mathcal{V}_{\mathrm{I}}$ represents the substitution

$$\phi_{\mathrm{c}}(x)\mapsto\Delta J\equiv\int \mathrm{d}^d y\,\Delta(x-y)J(y).$$

If, for example,

$$\mathcal{V}_{\mathrm{I}}(\phi) = \frac{g}{4!}\int \mathrm{d}^d x\, \phi^4(x),$$

the expansion of ϕ_c in powers of J takes the diagrammatic form shown in figure 6.8.

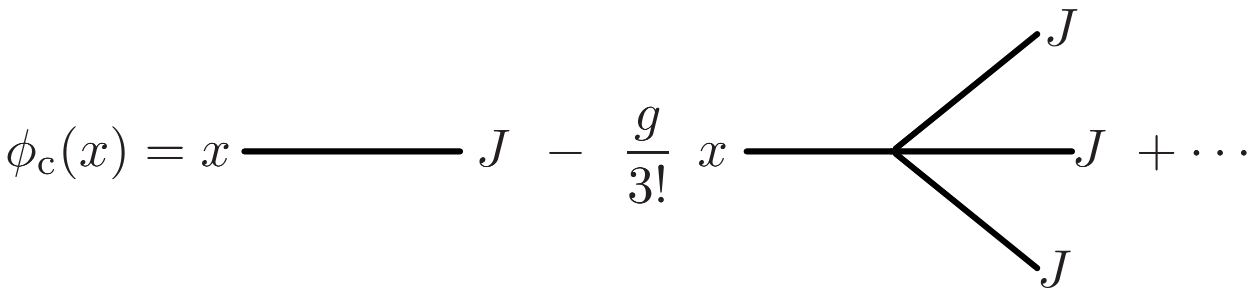


Fig. 6.8 – Expansion of ϕ_c in powers of J.

6.6.2 Next order: one-loop diagrams

Keeping only the term quadratic in χ in the expansion of $\mathcal{S}(\phi)$, one calculates the Gaussian integral and one finds

$$\mathcal{Z}(J) \propto \mathcal{Z}_0(J) \left[\det \frac{\delta^2 \mathcal{S}}{\delta\phi_{\mathrm{c}}(x_1)\delta\phi_{\mathrm{c}}(x_2)}\right]^{-1/2},$$

where the normalization, which is independent of J, depends on the continuum limit of a specific lattice regularization. Setting

$$\mathcal{W}(J) = \mathcal{W}_0(J) + \hbar\mathcal{W}_1(J) + \cdots,$$

one finds the next contribution to the connected functional

$$\mathcal{W}_1(J) = -\tfrac{1}{2}\operatorname{tr}\ln \frac{\delta^2 \mathcal{S}}{\delta\phi_{\mathrm{c}}(x_1)\delta\phi_{\mathrm{c}}(x_2)},$$

where the general identity $\ln\det = \operatorname{tr}\ln$ has been used. The functional $\mathcal{W}_1(J)$, expanded in powers of J, generates all one-loop connected diagrams. Fig. 6.9 exhibits a typical contribution to $\mathcal{W}_1(J)$.

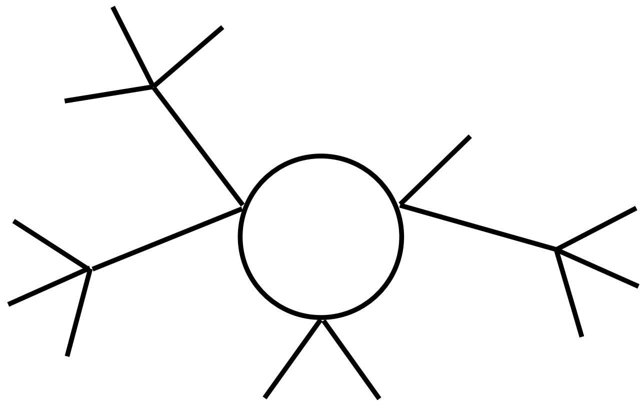


Fig. 6.9 – Example of a connected one-loop contribution.

Vertex functions. As a consequence of the property (6.5) applied to $\hbar$,

$$\left.\frac{\partial\Gamma}{\partial\hbar}\right|_{\varphi\,\text{fixed}} + \left.\frac{\partial\mathcal{W}}{\partial\hbar}\right|_{J\,\text{fixed}} = 0\,,$$

the leading corrections to $\mathcal{W}$ and Γ are opposite. Since $\varphi = \phi_c$ at leading order (equation (6.23)), the one-loop contribution $\Gamma_1(\varphi)$ to the functional $\Gamma(\varphi)$ is

$$\Gamma_1(\varphi) = \tfrac{1}{2}\,\mathrm{tr}\ln\frac{\delta^2\mathcal{S}}{\delta\varphi(x_1)\delta\varphi(x_2)}. \tag{6.24}$$

Loop expansion. Quite generally, the successive terms generated by the steepest descent method correspond to a power series in $\hbar$ and to Feynman diagrams with an increasing number of loops. The expansion is thus called a loop expansion.

Since we have normalized $\mathcal{W}(J)$ as (equation (6.21))

$$\mathcal{W} = \hbar \ln \mathcal{Z},$$

the tree contributions to $\mathcal{W}$ and Γ are of order $\hbar^0$. The introduction of the parameter $\hbar$ has the effect of replacing the propagator Δ by $\hbar\Delta$ (it is the inverse of the coefficient of $\phi\phi$) and to divide all vertices by $\hbar$.

For a 1-irreducible diagram, calling I the number of propagators and n the total number of vertices, one thus finds a factor $\hbar^{I-n+1}$, the last factor $\hbar$ coming from the normalization (6.21) of $\mathcal{W}$. Using the topological relation (6.18), one recognizes the factor $\hbar^L$ where L is the number of loops.

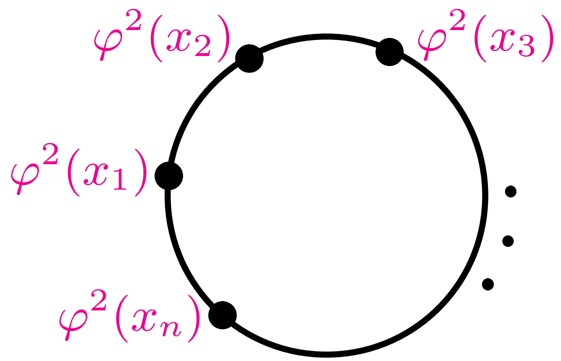


Fig. 6.10 – Contribution to the 1-irreducible functional at one-loop.

6.6.3 The ϕ^4 field theory

Again, we illustrate the result with the ϕ^4 field theory (equation (6.19)). The first term is the expansion is the action. Then, in symbolic notation,

$$\Gamma_1(\varphi) - \Gamma_1(0) = \frac{1}{2} \operatorname{tr} \ln \left(\mathbf{1} + \frac{1}{2} g \Delta \varphi^2 \right),$$

where $\Delta\varphi^2$ is the operator associated with the kernel

$$[\Delta\varphi^2](x, y) = \Delta(x - y)\varphi^2(y).$$

Expanding in powers of φ^2,

$$\Gamma_1(\varphi) - \Gamma_1(0) = \sum_{n=1} (-1)^{n+1} \frac{g^n}{n\, 2^{n+1}} \operatorname{tr}(\Delta\varphi^2)^n,$$

one generates the one-loop, 1-irreducible diagrams of Fig. 6.10:

$$\operatorname{tr}(\Delta\varphi^2)^n = \int \prod_{i=1}^{n} \mathrm{d}^d x_i\, \varphi^2(x_i)\Delta(x_i - x_{i-1}) \quad \text{with} \quad x_n = x_0\,.$$

We introduce the Fourier representations (6.12b),

$$\Delta(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{ip\cdot x}\, \tilde{\Delta}(p)$$

and

$$\varphi(x) = \int \mathrm{d}^d p\, \mathrm{e}^{ipx}\, \tilde{\varphi}(p).$$

We can rewrite the terms in the expansion as

$$\mathrm{tr}(\Delta\varphi^2)^n = \int \left(\prod_{i=1}^{2n} \mathrm{d}^d p_i\, \tilde{\varphi}(p_i)\right) D_n(p_1, \ldots, p_{2n})$$

with

$$\begin{aligned} D_n(p_1, \ldots, p_{2n}) &= \int \prod_{j=1}^{n} \left(\mathrm{d}^d x_j\, \frac{\mathrm{d}^d q_j}{(2\pi)^d}\ \mathrm{e}^{i(x_j(p_{2j}+p_{2j-1})}\, \mathrm{e}^{iq_j(x_{j-1}-x_j)}\, \tilde{\Delta}(q_j)\right) \\ &= (2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{2n} p_i\right) \int \frac{\mathrm{d}^d q}{(2\pi)^d} \prod_{j=1}^{n} \tilde{\Delta}(q_j) \end{aligned}$$

and

$$q_1 = q\,, \quad q_j - q_{j+1} = p_{2j-1} + p_{2j}\,, \quad q_{n+1} = q_1\,.$$

6.7 Divergences in perturbation theory

We have already anticipated the existence of divergences in perturbation theory by introducing immediately a regularized form of the action. We now investigate the problem of divergences in naive perturbation theory when the action is not regularized. We examine the problem first in the example of the ϕ^3 at one-loop order.

6.7.1 The ϕ^3 field theory

The euclidean action for a local quantum field theory in d dimensions involving a scalar field ϕ with a ϕ^3 self-interaction can be written as

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\bigl(\nabla_x \phi(x)\bigr)^2 + \tfrac{1}{2} m^2 \phi^2(x) + \tfrac{1}{3!} g\, \phi^3(x)\right], \tag{6.25}$$

where g and m are constants. This theory is somewhat unphysical because the ϕ^3 potential is not bounded from below.

However, it has a well-defined perturbative expansion where this non-perturbative pathology does not show up. Note that the expansion parameter is g^2, which plays the role of $\hbar$.

Moreover, for g imaginary, it makes sense beyond perturbation theory and describes in classical statistical physics universal properties of the Yang–Lee edge singularity of the Ising model.

Tree approximation. In the tree approximation the 1PI functional $\Gamma(\varphi)$ reduces to the action $\mathcal{S}(\varphi)$. The inverse or 1PI two-point function is thus

$$\Gamma^{(2)}_{\text{tree}}(x,y) = \left(-\nabla_x^2 + m^2\right)\delta(x-y),$$

and after Fourier transformation,

$$\widetilde{\Gamma}^{(2)}_{\text{tree}}(p) = p^2 + m^2 .$$

More generally, the Fourier components of the 1PI n-point functions are

$$\widetilde{\Gamma}^{(3)}_{\text{tree}}(p_1, p_2, -p_1 - p_2) = g\,, \quad \widetilde{\Gamma}^{(n)}_{\text{tree}}(p_1, \ldots, p_n) = 0 \quad \text{for } n > 3\,.$$

6.7.2 Perturbation theory at one-loop order

The 1PI functional $\Gamma(\varphi)$ has been calculated at one-loop order in section 6.6.2. Specializing equation (6.24), we obtain the one-loop order contribution

$$\Gamma_1(\varphi) - \Gamma_1(0) = \tfrac{1}{2}\operatorname{tr}\ln\left[1 + g\left(-\nabla_x^2 + m^2\right)^{-1}\varphi\right].$$

The expansion of $\Gamma_1(\varphi)$ in powers of φ generates all one-loop contributions to the 1PI functions $\Gamma^{(n)}$. After Fourier transformation, for the first three functions (Fig. 6.11) one finds

$$\widetilde{\Gamma}^{(1)}_{1\,\text{loop}} = \tfrac{1}{2} g \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + m^2}\,,$$

$$\widetilde{\Gamma}^{(2)}_{1\,\text{loop}} = -\tfrac{1}{2} g^2 \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)\big((p+q)^2 + m^2\big)}\,,$$

$$\widetilde{\Gamma}^{(3)}_{1\,\text{loop}} = g^3 \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)\big((p_1+q)^2 + m^2\big)\big((p_1+p_2+q)^2 + m^2\big)}$$

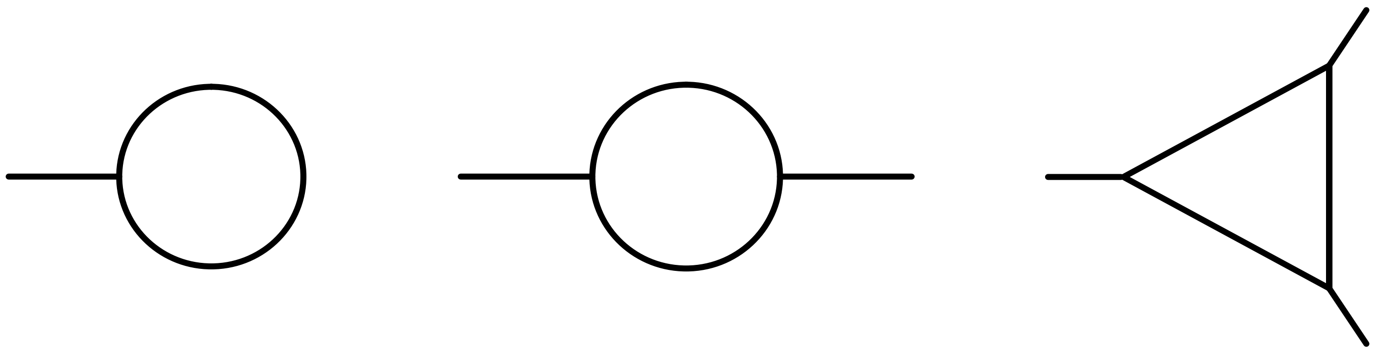


Fig. 6.11 – First one-loop diagrams in a ϕ^3 field theory.

More generally, $\widetilde{\Gamma}^{(n)}_{1\,\mathrm{loop}}$ is the sum of $\frac{1}{2}(n-1)!$ terms obtained by symmetrizing over $\{p_1, p_2, \ldots, p_n\}$ with $\sum_i p_i = 0$ the integral

$$(-1)^{n+1} g^n \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2+m^2} \frac{1}{(q+p_1)^2+m^2} \cdots \times \frac{1}{(q+p_1+\cdots+p_{n-1})^2+m^2}, \tag{6.26}$$

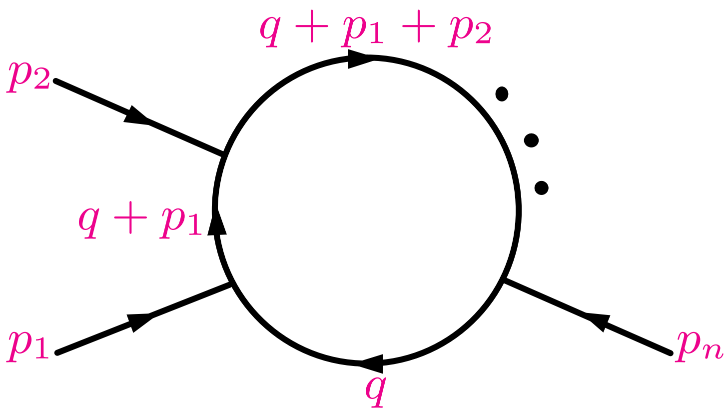


Fig. 6.12 – One-loop 1PI diagrams.

$$\frac{(-1)^{n+1}g^n}{(2\pi)^d}\int\frac{\mathrm{d}^d q}{(q^2+m^2)\big((q+p_1)^2+m^2\big)\cdots\big((q+p_1+\cdots+p_{n-1})^2+m^2\big)}, \tag{6.27}$$

which is represented by the Feynman diagram of Fig. 6.12.

The integrand in expression (6.27) behaves, for large momentum q, like $1/q^{2n}$, and the integral thus diverges for $2n \le d$.

Except for $d = 1$ (quantum mechanics) divergences appear. For $d = 2$ the one-point function, which has no momentum dependence, diverges like $\int d^2q/q^2$.

With increasing dimension d more correlation functions diverge. For $d = 6$, the one, two and three-point functions diverge. If the momentum integrals are cut at a large momentum $|\mathbf{q}| = O(\Lambda)$, then the one- and two-point functions diverge like powers of Λ, while $\widetilde{\Gamma}^{(3)}$ diverges logarithmically.

To calculate the contributions which diverge with Λ explicitly, in massive theory one can expand the integrand in a Taylor series in the external momenta (the massless theory has to be defined as a limit). It is simple to verify, using dimensional analysis, that the coefficients of the terms of global degree k in the momenta are given by integrals which diverge only for $d \geq k + 2n$.

Therefore, the divergent part of a one-loop contribution to the n-point function is a polynomial of degree $(d - 2n)$.

The first important observation is that since the divergences are polynomials in the external momenta, the divergent part $\Gamma_1^{\text{div.}}(\varphi)$ of the 1PI functional $\Gamma(\varphi)$ is local, that is, takes the form of the space integral of a function of the field and its derivatives, like the action itself.

To define more precisely a divergent part, we introduce a specific regularization.

For $d = 6$, cutting the momentum integral according to Schwinger's regularization (a specific cut-off scheme), one finds

$$\begin{aligned}
\widetilde{\Gamma}^{(1)}_{1\,\text{loop}} &= \frac{g}{2^7\pi^3}\left[\tfrac{1}{2}\Lambda^4 - \tfrac{1}{2}m^2\Lambda^2 + m^4\ln(\Lambda/m) + O(1)\right],\\
\widetilde{\Gamma}^{(2)}_{1\,\text{loop}} &= -\frac{g^2}{2^7\pi^3}\left[\Lambda^2 - \left(2m^2 + p^2/3\right)\ln(\Lambda/m) + O(1)\right],\\
\widetilde{\Gamma}^{(3)}_{1\,\text{loop}} &= \frac{g^3}{2^6\pi^3}\ln(\Lambda/m) + O(1).
\end{aligned} \tag{6.28}$$

The three divergent one-loop diagrams are displayed in Fig. 6.11.

We note that the dimension 6 is special in the following sense: the 1PI correlation functions that diverge are all those which are already non-trivial in the tree approximation (a term linear in ϕ can be added to the initial action (6.25) by translating ϕ by a constant). Moreover, the divergent terms and the tree approximation have the same momentum dependence.

By contrast, for $d \geq 8$ the four-point function, which vanishes in the tree approximation, is also divergent.

6.7.3 Empirical removal of divergences at one-loop order

For $d \leq 6$ dimensions, the divergent parts of the one-loop correlation functions have the structure of the initial action. For example in $d = 6$, $\Gamma_1^{\text{div.}}(\varphi)$, the divergent part at one-loop of $\Gamma(\varphi)$, has the structure

$$\Gamma_1^{\text{div.}}(\varphi) = \int \mathrm{d}^6 x \left[\tfrac{1}{2} g^2 a_0(\Lambda) \bigl(\nabla_x \varphi(x)\bigr)^2 + g a_1(\Lambda)\varphi(x) + \tfrac{1}{2} g^2 a_2(\Lambda)\varphi^2(x) \right.$$
$$\left. + \tfrac{1}{3!} g^3 a_3(\Lambda)\varphi^3(x)\right] . \tag{6.29}$$

The functions $a_i(\Lambda)$ follow from equations (6.28) and are, therefore, defined only up to additive finite parts. In the minimal subtraction scheme, one defines them as the sum of the divergent terms in the asymptotic expansion in a dimensionless parameter. Choosing here Λ/m, one finds

$$\begin{aligned}
2^7\pi^3 a_0(\Lambda) &= \tfrac{1}{3}\ln(\Lambda/m),\\
2^7\pi^3 a_1(\Lambda) &= \tfrac{1}{2}\Lambda^4 - \tfrac{1}{2}m^2\Lambda^2 + m^4\ln(\Lambda/m),\\
2^7\pi^3 a_2(\Lambda) &= -\Lambda^2 + 2m^2\ln(\Lambda/m),\\
2^7\pi^3 a_3(\Lambda) &= 2\ln(\Lambda/m).
\end{aligned} \tag{6.30}$$

Adding the local counter-term $-\Gamma_1^{\text{div.}}(\phi)$ to the initial action $\mathcal{S}(\phi)$, $\mathcal{S}_1(\phi) = \mathcal{S}(\phi) - \Gamma_1^{\text{div.}}(\phi)$, one obtains

$$\begin{aligned}
\mathcal{S}_1(\phi) = \mathcal{S}_1(\phi) = \mathcal{S}(\phi) - \Gamma_1^{\text{div.}}(\phi) = \int \mathrm{d}^6x \Big[& \tfrac{1}{2}\left(1 - g^2 a_0(\Lambda)\right)(\nabla_x\phi)^2 \\
& - g a_1(\Lambda)\phi + \tfrac{1}{2}\left(m^2 - g^2 a_2(\Lambda)\right)\phi^2 + \tfrac{1}{3!}\left(g - g^3 a_3(\Lambda)\right)\phi^3\Big].
\end{aligned}$$

The new action $\mathcal{S}_1$ can be reduced to the form (6.25) by a multiplicative renormalization of the field and a constant shift to cancel the linear term. It thus differs from the initial action by a field redefinition and by its parametrization, but involves the same monomials of the field.

The new 1PI functional $\Gamma(\varphi)$ at one-loop order is then

$$\begin{aligned}\Gamma(\varphi) &= \mathcal{S}_1(\varphi) + \Gamma_1(\varphi) + O\,(\text{two loops})\\ &= \mathcal{S}(\varphi) - \Gamma_1^{\text{div.}}(\varphi) + \Gamma_1(\varphi) + O\,(\text{two loops})\,,\end{aligned}$$

where $\Gamma_1(\varphi)$ is the sum of one-loop diagrams calculated only with $\mathcal{S}(\phi)$. Therefore, $\Gamma(\varphi)$ now has a limit at one-loop order when the cut-off becomes infinite.

Finally, note that a change in the definition of the divergent part changes $\Gamma_1^{\text{div.}}(\varphi)$ by a finite local polynomial, and the conclusions are the same.

Generalizing this method to all loop orders, one defines a renormalized action whose coefficients are functions of the renormalized parameters and the cut-off.

The renormalization idea. To solve, at least formally, the problem of divergences in perturbation theory, one tries to generalize the previous method to all orders in the loop expansion and to more general field theories.

One introduces a large momentum cut-off in the theory, or equivalently modify the field theory at short distance, as we have done here, to characterize the divergences of Feynman diagrams (the physical reality of such a cut-off is from this point of view irrelevant).

One then investigates the possibility of choosing the initial parameters of the theory as functions of the cut-off in such a way that correlation functions have a finite large cut-off limit. When such a limit exists, one can show that it is independent of the cut-off procedure (under some general conditions).

The local field theories, for which this procedure works, are called renormalizable (or super-renormalizable if some parameters in the interaction are cut-off independent).

Renormalizable theories are to some extent short distance insensitive in the sense that even if a large mass or a microscopic scale in space provide a true physical cut-off, their long distance or low momentum properties can be described, without detailed knowledge of the short distance structure, in terms of a small number of effective parameters.

However, and this is a deep issue, the necessity of an underlying microscopic scale is implied by the existence of divergences in the field theory before regularization.

We already emphasize these ideas here because they motivate the technical analysis that follows. Renormalization group will eventually provide us with the necessary tool to understand the renormalization procedure.

Note, that in the ϕ^3 theory, for $d \geq 8$, the renormalization method fails already at one-loop order. Indeed to generate in the tree approximation a term proportional to the divergence of the four-point function, for example, a ϕ^4 term is required in the action.

We show below that such an interaction induces, in turn, worse divergences which cannot be reproduced. Power counting will show that for $d > 6$ no polynomial interaction leads to a renormalizable theory.

6.8 Divergences: general analysis and power counting

Divergences in perturbation theory and field integrals. Contributions to perturbation theory are Gaussian expectation values,which are expressed in terms of the Gaussian two-point function. The perturbative divergences are thus related to the singularity of the two-point function $\Delta(x)$ at short distance. In the Fourier representation

$$\Delta(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \; \mathrm{e}^{ipx} \, \tilde{\Delta}(p),$$

they are related to the behaviour of $\tilde{\Delta}(p)$ for large momentum.

We now assume that $\tilde{\Delta}(p)$ behaves like

$$\tilde{\Delta}(\lambda p) \propto \lambda^{-\sigma} \quad \forall\, p \neq 0 \quad \text{for } \lambda \to \infty\,,$$

From the viewpoint of field integrals, these singularities are related to the regularity of fields contributing to the field integrals. Indeed, calculating the Gaussian expectation value, we find

$$\tfrac{1}{2}\left\langle \left(\phi(x) - \phi(y)\right)^2 \right\rangle = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \left(1 - \mathrm{e}^{ip(x-y)}\right) \tilde{\Delta}(p) \underset{|x-y|\to 0}{\propto} |x-y|^{\sigma-d},$$

for $d < \sigma \leq d+2$ for boson fields and $d < \sigma \leq d+1$ for fermion fields, while the integral diverges for $\sigma \leq d$.

Thus, typical fields are continuous only for $\sigma > d$ and, thus, $[\phi] < 0$, and this condition ensures that the perturbative expansion is finite in the absence of derivative couplings. With derivative couplings, typical fields have to be differentiable and this implies at least $\sigma \geq d+2$. In all cases except free field theory, a regularization is thus required for $d \geq 2$.

6.8.1 Power counting: dimension of fields

We now introduce the notion of canonical, or engineering, dimension of fields and interaction vertices.

We explicitly assume that the propagator of every field ϕ is $O(d)$ covariant in d dimensions, or at least has a uniform large momentum behaviour of the form

$$C_1\lambda^{-\sigma} < \left|\tilde{\Delta}(\lambda p)\right| < C_2\lambda^{-\sigma} \quad \forall\, p \neq 0 \quad \text{for } \lambda \to \infty\,,$$

in which C_1 is a strictly positive constant. Other cases require a special analysis. The canonical, or engineering dimension $[\phi]$ of a field $\phi(x)$ is then defined in terms of the large momentum behaviour of the ϕ propagator by

$$[\phi] = \tfrac{1}{2}(d - \sigma). \tag{6.31}$$

Fields contributing to the field integral are continuous only for $[\phi] < 0$.

6.8.2 Field dimensions: unregularized field theory

We now determine the field dimensions in the initial, unregularized field theory.

In the scalar field theory (6.25),

$$\Delta(\lambda p) \propto 1/\lambda^2 , \quad \text{for } \lambda \to \infty \quad \Rightarrow \quad [\phi] = \tfrac{1}{2}(d-2). \tag{6.32}$$

For the fermions that will be considered in section 11.9, the propagator of the field Fourier components reads

$$\widetilde{W}^{(2)}_{\alpha\beta}(p) = \left\langle \bar{\psi}_\alpha(-p)\psi_\beta(p)\right\rangle = (m + i\not{p})^{-1}_{\beta\alpha}$$

and, thus,

$$\Delta(\lambda p) \propto 1/\lambda , \quad \text{for } \lambda \to \infty \quad \Rightarrow \quad [\psi] = [\bar{\psi}] = \tfrac{1}{2}(d-1). \tag{6.33}$$

In the case of scalar and spinor fermion fields, the definition (6.31) coincides with the natural mass dimension of the field as deduced from the quadratic part of the action by dimensional analysis.

Indeed, let us assign a dimension $+1$ to momenta p and masses m,

$$[p] = [m] = 1\,,$$

correspondingly a dimension -1 to length and position variables,

$$[x] = -1 \quad \Rightarrow \quad [\partial/\partial x] = +1\,.$$

If one expresses the condition that the action is dimensionless, one finds

$$\left[\int \mathrm{d}^d x \left(\nabla_x \phi(x)\right)^2\right] = 0 \quad \Rightarrow \quad -d + 2 + 2[\phi] = 0\,,$$
$$\left[\int \mathrm{d}^d x\, \bar{\psi}(x) \partial\!\!\!/ \psi(x)\right] = 0 \quad \Rightarrow \quad -d + 1 + 2[\psi] = 0\,.$$

This property no longer holds for higher spin fields, in general.

For example, the free action for a massive vector field A_μ is

$$\mathcal{S}(A) = \int \mathrm{d}^d x \left(\tfrac{1}{4} \sum_{\mu,\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \tfrac{1}{2} m^2 A_\mu A_\mu \right).$$

The A-field propagator in the Fourier representation then reads

$$\tilde{\Delta}_{\mu\nu}(p) = \frac{\delta_{\mu\nu} + p_\mu p_\nu / m^2}{p^2 + m^2}\,.$$

It is such that

$$\tilde{\Delta}_{\mu\nu}(\lambda p) \sim \lambda^0 \frac{p_\mu p_\nu}{m^2 p^2} \;\Rightarrow\; [A_\mu] = d/2\,, \tag{6.34}$$

while dimensional analysis indicates that A_μ has the dimension of a scalar field. This property is directly related to the presence of negative powers of m^2 in the propagator: the quadratic form in the action is not invertible for $m = 0$.

A similar phenomenon occurs with higher spin fields. A spin s massive field propagator has the form

$$\tilde{\Delta}(p) = \frac{P_{2s}(p/m)}{p^2 + m^2}\,,$$

in which $P_{2s}(p)$ is a polynomial of degree $2s$ in p, which is a projector on 'mass-shell', that is, for $p^2 = -m^2$. The dimension of the corresponding field ϕ_s, which generalizes equations (6.32, 6.33, 6.34), is

$$[\phi_s] = \tfrac{1}{2}(d - 2 + 2s). \tag{6.35}$$

6.8.3 Power counting: dimension of vertices

We consider here only theories invariant under space translations. The interaction term in the action is thus the space integral of a linear combination of monomials which are products of fields and their derivatives.

We write a monomial $V(\phi)$ symbolically as

$$V(\phi) \propto \int \mathrm{d}^d x \, (\nabla_x)^k \, \phi_1^{n_1}(x)\phi_2^{n_2}(x)\ldots\phi_s^{n_s}(x),$$

where the k derivatives act in an unspecified way on the fields ϕ_i.

We call these elementary interaction terms vertices because they are represented by vertices in Feynman diagrams.

We define the dimension $[V]$ of the corresponding vertex by

$$[V] = -d + k + \sum_{i=1}^{s} n_i \, [\phi_i] \, , \tag{6.36}$$

where n_i the power of the fields ϕ_i present in the vertex V.

In terms of the Fourier components $\tilde{\phi}_i(p)$ of the fields $\phi_i(x)$, and taking into account translation invariance, the vertex $V(\phi)$ can be written (in analogous symbolic notation) as

$$V(\phi) \propto \int \prod_{n=1}^{N} \mathrm{d}^d p_n \; \delta^{(d)}(p_1 + p_2 + \cdots + p_N)\, p^k \tilde{\phi}_1(p_1) \ldots \tilde{\phi}_s(p_{n_1+\cdots+n_s}),$$

where $N = n_1 + \cdots + n_s$ and p^k represents a linear combination of products of k momenta.

6.8.4 Superficial degree of divergence: power counting

We consider only 1PI diagrams. Each vertex yields a δ-function of momentum conservation. The number of independent integration momenta in a Feynman diagram, taking into account momentum conservation at vertices, thus equals the number of loops. This follows directly from one of the definitions of the number of loops L in a diagram given in section 6.4.

Finally, a vertex multiplies the numerator of a Feynman diagram by the product of k momenta.

Therefore, if all integration momenta in a diagram γ are scaled by a factor λ, for $\lambda \to \infty$ the diagram is scaled by a factor $\lambda^{\delta(\gamma)}$ with

$$\delta(\gamma) = dL - \sum_i I_i \sigma_i + \sum_\alpha v_\alpha k_\alpha, \tag{6.37}$$

in which v_α is the number of vertices of type α with k_α derivatives, and I_i the number of internal lines corresponding to propagators Δ_i joining the different vertices.

The number $\delta(\gamma)$ is called the superficial degree of divergence of the diagram γ. For a one-loop diagram regularized with a momentum cut-off, it characterizes the divergence of the diagram as a power of the cut-off.

More generally, if $\delta(\gamma)$ is positive a regularized diagram diverges at least like $\Lambda^{\delta(\gamma)}$. If $\delta(\gamma) = 0$, it diverges at least like a power of $\ln \Lambda$. If $\delta(\gamma)$ is negative, the diagram is superficially convergent, which means that divergences can come only from subdiagrams.

An example: the ϕ^3 field theory for $d = 6$. In the example of the ϕ^3 field theory in $d = 6$ dimensions, since $\sigma = 2$ and $k = 0$, expression (6.37) yields

$$\delta(\gamma) = 6 - 2I\,.$$

To $I = 1, 2, 3$, respectively, correspond the values 4, 2, 0 in agreement with equations (6.28). For $I > 3$ the diagrams are superficially convergent.

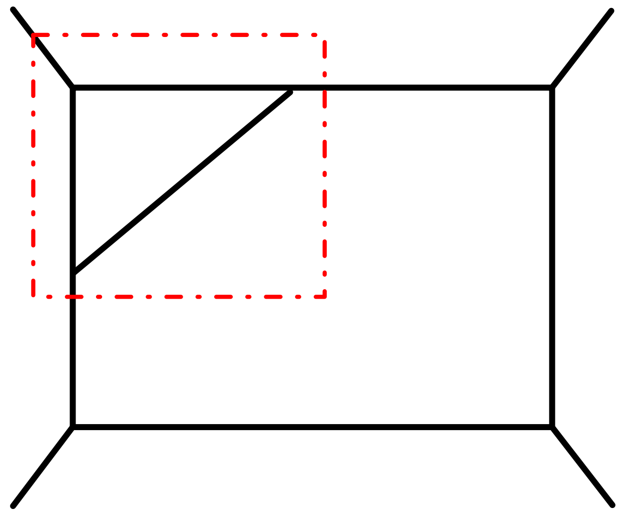


Fig. 6.13 – Superficially convergent diagram with divergent (red) subdiagram.

Fig. 6.13 exhibits a superficially convergent diagram with a divergent subdiagram in the ϕ^3 theory: the superficial degree of divergence is -2, the diagram is superficially convergent, but the subdiagram inside the dotted box is divergent.

6.8.5 Divergences: other expressions

Various topological relations on graphs allow to write $\delta(\gamma)$ in different forms.

Combining equation (6.37) with the relation (6.18) rewritten in the form

$$L = \sum_i I_i - \sum_\alpha v_\alpha + 1\,, \tag{6.38}$$

we eliminate L and obtain

$$\delta(\gamma) = d + 2\sum_i I_i[\phi_i] + \sum_\alpha v_\alpha(k_\alpha - d), \tag{6.39}$$

where $[\phi_i]$ the dimension of ϕ_i (equation (6.31)),

$$[\phi_i] = \tfrac{1}{2}(d - \sigma_i).$$

We verify that when the relevant fields in the field integral are continuous and thus $[\phi] < 0$ and the vertices have no derivatives, the only divergent term corresponds to all $v_\alpha = 0$ and, thus, to the normalization of the field integral in the free theory.

Another topological relation. We consider a diagram γ contributing to a 1PI correlation function with E_i (for external line) fields ϕ_i.

Then if we call n_i^α the number of fields ϕ_i at a vertex α belonging to the diagram, we have the relation

$$E_i + 2I_i = \sum_\alpha n_i^\alpha v_\alpha \,. \tag{6.40}$$

The derivation of the relation is simple: each internal line connects two vertices while an external line is only attached to one vertex.

Fig. 6.14 gives an example.

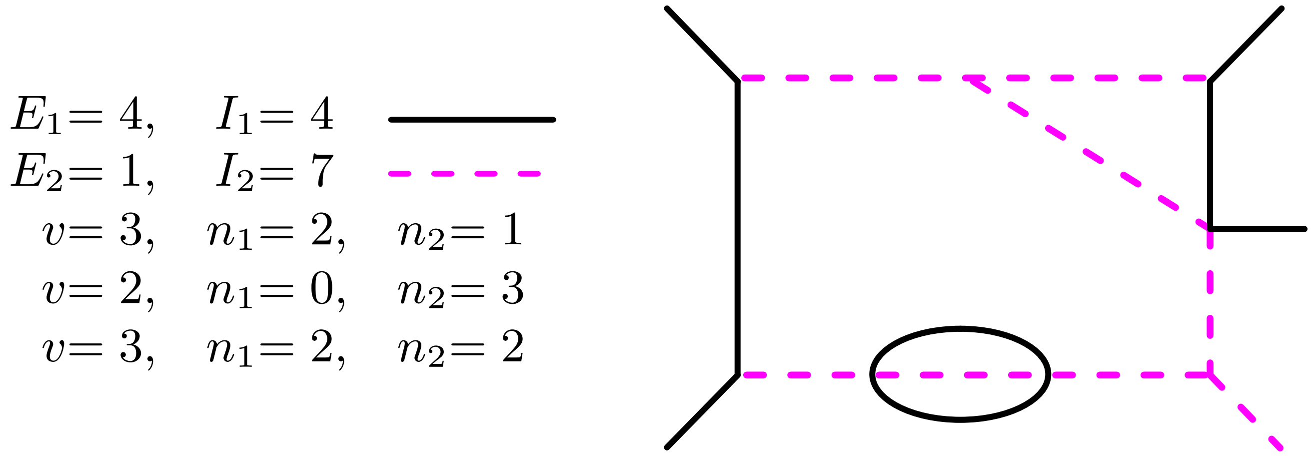


Fig. 6.14

Application. Combining equation (6.40) with the relation (6.39) to eliminate I_i in $\delta(\gamma)$, we obtain

$$\delta(\gamma) = d - \sum_i E_i[\phi_i] + \sum_\alpha v_\alpha [V_\alpha], \tag{6.41}$$

where $[\phi_i]$ is the dimension of ϕ_i and $[V_\alpha]$ the dimension of the vertex α,

$$[V_\alpha] = -d + k_\alpha + \sum_i n_i^\alpha [\phi_i].$$

6.9 Classification of renormalizable field theories

Equation (6.41) directly leads to a classification of renormalizable quantum field theories.

The program outlined in section 6.7.1 can only be realized if the superficial degree of divergence is bounded. When this condition is fulfilled we call the theory renormalizable by power counting.

Non-renormalizable theories. If at least one vertex V has a positive dimension, $[V] > 0$, then the degree of divergence of diagrams contributing to any 1PI correlation function can be rendered arbitrarily large by increasing the number v of vertices of this type.

A field theory with such a vertex is not renormalizable because, in order to cancel divergences, one has to add an infinite number of new interactions to the action and the final theory thus depends on an infinite number of parameters.

In the context of effective field theories (a notion we discuss later), such a theory may still be useful because it may retain some non-trivial information.

The ϕ^3 theory in $d > 6$ dimensions or the ϕ^4 theory in $d > 4$ dimensions provide some examples.

Super-renormalizable theories. When only a finite number of Feynman diagrams are superficially divergent the corresponding field theory is called super-renormalizable. This happens when all vertices have strictly negative dimensions.

Example. In the ϕ^4 field theory in $d = 3$ dimensions,

$$\delta(\gamma) = 3 - \tfrac{1}{2}E - v\,.$$

The superficially divergent diagrams are listed in figure 6.15.

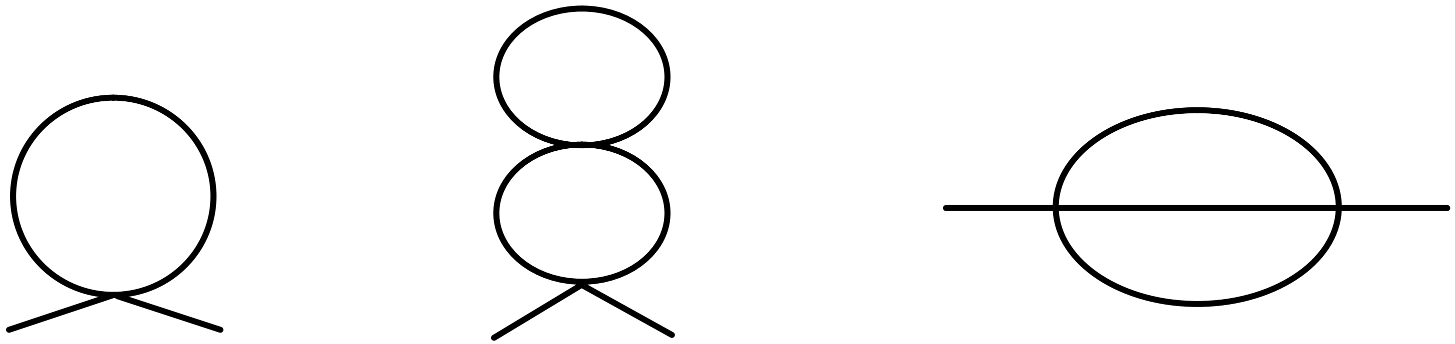


Fig. 6.15 – Superficially divergent diagrams in $\phi^4_{d=3}$.

Renormalizable theories. These theories are characterized by the property that at least one vertex has dimension zero, and no vertex has a positive dimension. Then, an infinite number of diagrams have a positive superficial degree of divergence; however, the maximal degree of divergence at E_i fixed does not increase with the number of loops, and is independent of the number of insertions of the vertices of dimension zero.

In addition, if all fields $[\phi_i]$ have a strictly positive dimension, then only a finite number of correlation functions are superficially divergent.

If at least one field has dimension zero, the situation is more complicate: the degree of divergence is bounded; however, an infinite number of correlation functions are superficially divergent. Generically this leads to field theories depending on an infinite number of parameters although, in contrast to the case of non-renormalizable theories, only a subclass of all possible interactions is generated by renormalization.

In addition, in some cases symmetries relate all these parameters so that only a finite number are really independent.

6.9.1 Classification of renormalizable theories

In what follows we restrict ourselves to the most frequent situation, $[\phi_i] > 0$ for all fields. Other cases require a special analysis. We consider simple theories involving fields of spin s with dimensions $[\phi_s]$ given by equation (6.35),

$$[\phi_s] = \tfrac{1}{2}(d - 2 + 2s).$$

The condition $[\phi_s] > 0$ is satisfied for $d \geq 2$ except for the $s = 0$, $d = 2$, case which must be examined separately.

All vertices should satisfy

$$-d + k + \tfrac{1}{2} \sum_s n_s(d - 2 + 2s) \leq 0\,. \tag{6.42}$$

This condition bounds k, the number of derivatives, s the spin, n_s the number of fields of spin s at the vertex and the dimension d.

Scalar field theories. For $k = s = 0$, the condition (6.42) implies for all vertices

$$n \le 2d/(d-2)\,.$$

The corresponding renormalizable interactions are ϕ^3 in $d = 6$ dimensions, ϕ^4 in $d = 4$ dimensions, ϕ^6 in $d = 3$ dimensions (ϕ^5 is either non- or super-renormalizable). Finally, any polynomial in ϕ is super-renormalizable in $d = 2$ dimensions.

With two derivatives, $k = 2$, the only solution is $d = 2$, but then $[\phi] = 0$.

Theories with spin $1/2$ *fermions.* We now consider vertices with one spinor fermion pair $\bar\psi\psi$ and n scalar fields. The condition (6.42) then becomes

$$n \le 2/(d-2)\,.$$

Renormalizable interactions are $\psi\bar\psi\phi$ (of Yukawa-type) in $d=4$ dimensions, $\bar\psi\psi\phi^2$ in $d=3$ dimensions.

In two dimensions, $P(\phi)\bar\psi\psi$, in which $P(\phi)$ is a polynomial in ϕ, is super-renormalizable. Finally, the vertex $(\bar\psi\psi)^2$ is renormalizable in two dimensions.

The vertices $P(\phi)(\bar\psi\psi)^2$ and $P(\phi)\bar\psi\partial\!\!\!/\psi$ also have dimension zero in two dimensions but again the dimension of ϕ vanishes.

Higher spins. For spin one vector fields, general $O(d)$ invariance leaves only dimension $d=2$ as a possibility. The only candidate with only fermions is the vertex

$$\bar{\psi}A_\mu\gamma_\mu\psi \equiv \bar{\psi}\not{A}\psi\,,$$

which is renormalizable in two dimensions. In addition the vertices $\phi\partial_\mu\phi A_\mu$ and $\phi^2 A_\mu^2$, which appear in gauge theories, are dimensionless. However, they again lead to a non-trivial renormalization problem because scalar fields are dimensionless ($[\phi]=0$).

No higher spin field leads to renormalizable theories by power counting. This includes Einstein–Hilbert action.

However, note that spin one vector fields associated with gauge symmetries do not enter into this classification because their propagator has in some gauges the behaviour of a scalar field propagator (for a discussion of gauge theories see coming lectures).

Finally, no physically acceptable, from the point of view of particle physics, and renormalizable theory by power counting exists above dimension 4. It is not known whether this property has a physics relation with the empirical fact that space–time has just four dimensions, or is a mere coincidence.

6.10 S-matrix and correlation functions

This section uses concepts and results that have been presented in section 5.10.2 but contains also material that has, naturally, its place in this lecture.

6.10.1 Connected correlation functions

By comparing the general form of the S-matrix with correlation functions, we have seen in section 5.10.2 that connected S-matrix elements generated by a functional $\mathcal{T}$ are related to the analytic continuation to real time ($t \mapsto it, k_0 \mapsto ip_0$) of euclidean correlation functions in the mass-shell limit.

More precisely, we introduce

$$\tilde{\Delta}(k) = \frac{i}{k^2 - m^2 + i\varepsilon},$$

which is the propagator with a pole at the physical mass.

Then in terms of $\mathcal{W}(J)$ the generating functional of connected correlation functions,

$$\mathcal{T}(\phi_0) = i\mathcal{W}(\Delta^{-1}J),$$

where J is inferred from equation (5.58),

$$\frac{i}{k^2 - m^2 + i\varepsilon}\tilde{J}(k) \mapsto -i\delta(k^2 - m^2)\left[\varphi(\hat{k})\theta(-k_0) + \bar{\varphi}(-\hat{k})\theta(k_0)\right],$$

and ϕ_0 is given by equation (5.53),

$$\tilde{\phi}_0(k_0, \hat{k}) = \delta(k_0^2 - \hat{k}^2 - m^2)\left[\varphi(\hat{k})\theta(-k_0) + \bar{\varphi}(-\hat{k})\theta(k_0)\right].$$

Connected correlation functions can then be expressed in terms of amputated functions by

$$\tilde{W}^{(n)}(p_1, \ldots, p_n) = \left[\prod_{i=1}^{n} \hat{W}^{(2)}(p_i)\right] \tilde{W}^{(n)}_{\text{amp.}}(p_1, \ldots, p_n).$$

The two-point function $\tilde{W}^{(2)}(p)$ has a pole located at $p^2 = -m^2$ where m is the physical mass. Near the pole

$$\tilde{W}^{(2)}(p) \underset{p^2 \to -m^2}{\sim} \frac{Z}{p^2 + m^2}\,, \tag{6.43}$$

where Z is the field renormalization constant (section 5.11). We conclude that the coefficient $\mathcal{T}_{\mathrm{r}}^{(n)}$ of $\mathcal{T}(\phi_0/\sqrt{Z})$ in the expansion in powers of ϕ_0 is given by

$$\mathcal{T}_{\mathrm{r}}^{(n)}(p_1, \ldots, p_n) = iZ^{n/2}\, \tilde{W}_{\mathrm{amp.}}^{(n)}(p_1, \ldots, p_n)\Big|_{p_i^2 = -m^2}\,. \tag{6.44}$$

The factor Z in the equation corresponds to a finite renormalization of the field such that the residue of the two-point function (equation (6.43)) on the physical pole $p^2 = -m^2$ of the renormalized field is 1. It ensures that the matrix elements $\mathcal{T}_{\mathrm{r}}^{(n)}$ satisfy the unitarity relations with the proper normalization.

The generalization to several particles is straightforward.

6.10.2 S-matrix and 1PI generating functional

We now also exhibit a direct relation with the analytic continuation of the generating functional of proper vertices $\Gamma(\varphi)$. We start from the euclidean form

$$\mathrm{e}^{\mathcal{W}(J)} = \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{A}(\phi)+J\cdot\phi}$$

with the symbolic notation

$$J \cdot \phi \equiv \int \mathrm{d}t\, \mathrm{d}^{d-1}x\, J(t,x)\phi(t,x).$$

We then substitute $\Gamma(\varphi)$, the Legendre transform of $\mathcal{W}(J)$:

$$\begin{aligned} \mathcal{W}(J) + \Gamma(\varphi) &= \int \mathrm{d}t\, \mathrm{d}^{d-1}x\, J(t,x)\varphi(t,x), \quad \varphi(t,x) = \frac{\delta \mathcal{W}}{\delta J(t,x)}\,; \\ \mathrm{e}^{-\Gamma(\varphi)+J\cdot\varphi} &= \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{A}(\phi)+J\cdot\phi}\,. \end{aligned} \tag{6.45}$$

Using

$$J(t,x) = \frac{\delta\Gamma}{\delta\varphi(t,x)},$$

we can write equation (6.45) as

$$\mathrm{e}^{-\Gamma(\varphi)} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{A}(\phi) + \int \mathrm{d}t\,\mathrm{d}^{d-1}x \big(\phi(t,x) - \varphi(t,x)\big) \frac{\delta\Gamma}{\delta\varphi(t,x)}\right],$$

or equivalently translating $\phi(t,x)$,

$$\mathrm{e}^{-\Gamma(\varphi)} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{A}(\phi+\varphi) + \int \mathrm{d}t\,\mathrm{d}^{d-1}x\,\phi(t,x) \frac{\delta\Gamma}{\delta\varphi(t,x)}\right]. \tag{6.46}$$

We now take the limit of a vanishing source J and φ is then a solution of

$$\frac{\delta\Gamma}{\delta\varphi(t,x)} = 0\,.$$

Clearly, this equation has propagating type solutions only after continuation to real time. One then verifies that $\Gamma(\varphi)$ coincides with $i\mathcal{T}(\varphi)$, when the solution $\varphi(x)$ of the equation is expanded around ϕ_0 defined in equation (5.53).

6.10.3 Change of field variables

In our previous discussion we have derived S-matrix elements from field correlation functions. We now show that S-matrix elements are to some extent invariant under local field transformations. Field correlation functions thus contain more information than the scattering matrix. This leads to problems in the point of view where only the scattering data are physical.

On the other hand, such a property is important in theories like gauge theories, where all gauges are equivalent, or models defined on Riemannian manifolds, where the fields $\phi_i(x)$ correspond only to a particular choice of coordinates on the manifold.

We have seen that S-matrix elements are calculated from connected correlation functions by taking the residues of the poles of the external propagator (see sections 5.10.2, 6.10). We consider, for simplicity, the case of only one species of field ϕ, which we have defined in such a way that it has a vanishing expectation value. Then the connected elements of the scattering matrix $\mathcal{S}^{(n)}$ can be expressed in terms of the amputated correlation functions (equation (6.44)):

$$\mathcal{S}^{(n)}(p_1, \ldots, p_n) = Z^{n/2}\, W^{(n)}_{\text{amp.}}(p_1, \ldots, p_n)\,\Big|_{p^2=-m^2}\ ,$$

where Z is the field renormalization constant.

S-matrix and field representation. We now compare the S-matrix obtained from the ϕ-field correlation functions to the S-matrix derived from the correlation functions of a different field $\phi'(x)$ related to $\phi(x)$ by

$$\phi'(x) = C_1\phi(x) + \sum_2^{\infty} \frac{C_k}{k!}\phi^k(x), \qquad C_1 \neq 0\,. \tag{6.47}$$

We assume, when necessary, that the theory has been regularized in such a way that the new correlation functions exist.

Using relation (6.47) we can express the ϕ' correlation functions in terms of the ϕ correlation functions.

The expansion of the ϕ' propagator shows immediately that the ϕ and ϕ' two-point functions have poles at the same location (see figure 6.16).

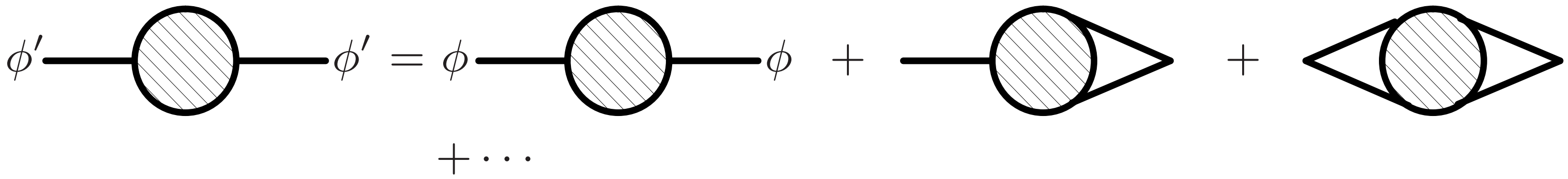


Fig. 6.16

The contributions to the n-point functions which have poles on the external lines then have the form shown in figure 6.17.

In the mass shell limit $(p_i^2 = -m^2)$, the ϕ' and ϕ correlation functions become proportional. The S-matrix elements are identical. Here, again, all fields related by transformation (6.43) are equivalent.

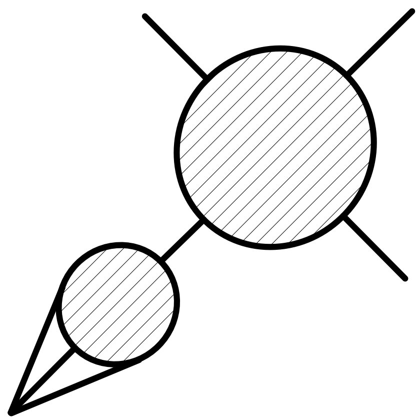


Fig. 6.17

In such situations not all parameters of the theory are physical. For example, the field amplitude renormalization is obviously unphysical. The same physical theory may have renormalizable and non-renormalizable realizations.

Using the background field method one can avoid the calculation of unphysical quantities (but to prove renormalizability the study of correlation functions cannot be avoided).