Lecture 7: FROM QED TO THE GENERAL RENORMALIZATION GROUP

Without a minimal understanding of quantum (or statistical) field theory and renormalization group, the theoretical basis of a notable part of the physics of the second half of the twentieth century remains incomprehensible.

Indeed, quantum field theory, in its various forms, describes completely the physics of fundamental interactions at the microscopic scale, the singular properties of continuous phase transitions (like liquid–vapour, ferromagnetism, superfluidity, binary mixtures...) near a transition point, the properties of dilute quantum gases beyond the model of Bose–Einstein condensation, the statistical properties of long polymer chains (as self-avoiding random walk), percolation...

In fact, quantum field theory (QFT) offers, up to now, the most powerful framework in which physical systems characterized by many strongly interacting, fluctuating degrees of freedom can be discussed.

However, at its birth, QFT has been confronted with a somewhat unexpected problem, the problem of infinities. The calculation of physical processes was yielding infinite results. An empirical recipe, renormalization, was eventually discovered, which allowed deriving finite predictions from divergent expressions. The procedure would hardly have been convincing if the corresponding predictions would not have been confirmed with increasing precision by experiments.

A new concept, Renormalization Group (RG), first abstracted from formal properties of QFT, but whose full meaning, in a more general form, was only completely appreciated in the general framework of continuous, macroscopic phase transitions (a process in which Wilson's contribution was essential), has led, later, to a satisfactory interpretation of the origin and role of renormalizable QFT and of the renormalization process.

As a consequence, in the modern interpretation, QFT's are only effective large distance, low energy field theories.

7.1 A brief history

1925– Heisenberg formulates the basis of Quantum Mechanics as a mechanics of matrices.

1926– Schrödinger publishes his famous equation, which bases Quantum Mechanics on the solution of a non-relativistic wave equation. Since the relativistic theory was already well-established at the birth of quantum mechanics, this may surprise. Indeed, for accidental reasons, the spectrum of the hydrogen atom is better reproduced by the non-relativistic Schrödinger equation than a relativistic spinless equation,the Klein–Gordon equation (1926).

1928– Dirac introduces his famous equation, a relativistic wave equation that incorporates the spin 1/2 of the electron, and leads to a spectrum of the hydrogen atom in much better agreement with experiment, and this opens the way for a relativistic quantum theory.

1929–1930, Heisenberg and Pauli establish the general principles of Quantum Field Theory.

1934– First correct calculation of a Quantum Electrodynamics (QED) correction by Weisskopf and confirmation of the existence of infinities, called UV divergences (since due, in this case, to very short wave length photons).

1937– Landau publishes his general theory of phase transitions.

1944– Exact solution of the two-dimensional Ising model by Onsager.

1947– Measurement of the Lambshift and surprising agreement with the QED prediction, after cancellation of infinities between physical observables.

1947–1949 Development of a general empirical strategy to eliminate divergences called Renormalization.

1954–1956 Discovery of a formal property of massless QED, called renormalization group, whose deeper meaning is not fully understood (Peterman–Stückelberg, Gellman–Low, Bogoliubov–Shirkov).

1967–1975 The Standard Model (Glashow, Weinberg, Salam), a renormalizable Quantum Field Theory based on the concept of non-Abelian gauge symmetry (Yang–Mills 1954) and spontaneous symmetry breaking (Higgs... 1964), is formulated, quantized (Faddeev–Popov, DeWitt) and shown to be consistent ('t Hooft–Veltman, Lee–Zinn-Justin). It describes, up to now, with remarkable precision fundamental microscopic physics, except neutrino masses and oscillations, but does not include gravitation.

1971–1972 Inspired by some premonitory ideas of Kadanoff, Wilson, Wegner... develop a more general Renormalization Group, based on the iterative integration over short-distance degrees of freedom, which includes the field theory RG in some limit, and which is able to explain universal non mean-field (or non quasi-Gaussian)-like properties of continuous phase transitions (like liquid–vapour, binary mixtures, superfluidity, ferromagnetism) or statistical properties of long polymeric chains.

1972–1975 Several groups, including Brézin, Le Guillou and Z.-J., develop efficient quantum field techniques to prove universality and calculate universal quantities.

1973–Politzer, Gross–Wilczek establish the asymptotic freedom of a class of non-Abelian gauge theories, which provides a RG explanation to the free particle behaviour of quarks at short distance inside nucleons.

1975–1976 Additional insight in the universal properties of phase transitions is provided by the study of non-linear σ model and the $(d-2)$ expansion (Polyakov, Brézin–Z-J).

1977–Nickel determines the perturbative expansion of the RG functions in the ϕ^4 field theory in three dimensions, within the Callan–Symanzik scheme as suggested by Parisi, up to six loops. The first precise estimates of critical exponents, based on Borel summation and conformal mapping applied to the perturbative expansion, are reported by Le Guillou and Z-J.

7.2 QED and the problem of infinities

At the beginning of the thirties, a quantum and relativistic theory had been proposed, which allows describing electromagnetic interactions between protons et electrons.

This theory is a field theory and not a theory of individual particles to unify the description of photons and charged particles.

One expected answer: a solution to the puzzle of the infinite Coulomb contribution to the electron mass.

Unfortunately, divergences survived, though less severe, for deep reasons: They were unavoidable consequences of the point-like character of the electron and conservation of probabilities (and it is very difficult to construct a relativistic theory of non point-like particles).

Eventually, an empirical procedure, called renormalization, was discovered that led to finite results, based on elimination of initial parameters in favour of direct relations between physical observables.

The method allowed calculations with increasing precision for physical processes governed by QED. The concept of renormalizable QFT became so fruitful that it could later be applied to all fundamental interactions but gravitation: the Standard Model of weak, electromagnetic and strong interactions has successfully survived, except for the neutrino sector, to more than 40 years of confrontation with experiment and again been confirmed by the discovery of the Higgs particle.

However, the deep logic behind the renormalization procedure has itself remained for a long time a mystery for some theorists. A set of convergent ideas, arising both from microscopic physics and the theory of macroscopic phase transitions, which can be collected under the general name of renormalization group, has finally led to a new and consistent picture.

As a result of the essential coupling of physics at very different scales, renormalizable field theories have a limited range of validity.

One speaks of effective field theories, low energy or large distance approximations to more fundamental theories, often known in statistical physics but as yet unknown in particle physics.

This is the evolution in ideas that we want to briefly explain here.

7.3 QED: a local quantum field theory

QED, which describes, in a quantum relativistic framework, interactions between charged particles is not a theory of individualized particles, like in non-relativistic quantum mechanics, but a Quantum Field Theory. It is a quantum extension of a classical relativistic field theory, where the dynamic variables are fields, the electric and magnetic fields.

Such a theory differs drastically from a theory of particles in the sense that fields have an infinite number of coupled, fluctuating degrees of freedom, the values of fields at each point in space. The non-conservation of particles in high-energy scattering is a manifestation of this property.

The field theory that describe microscopic physics is local, a generalization of the notion of point-like particles: it lacks a short-distance structure. The infinite number of fluctuating degrees of freedom combined with locality are the basic reasons why QFT's have somewhat unusual properties.

7.4 First calculations: the problem of infinities

Shortly after the work of Dirac, Heisenberg and Pauli, the first, but wrong, calculations of the order $\alpha = e^2/4\pi\hbar c \approx 1/137$ (the fine structure constant) correction to the electron propagation in the photon field were published (Oppenheimer, Waller 1930). (The electric charge e being defined in terms of the Coulomb potential written as e^2/R.)

One motivation: Cure the disease of the 'classical relativistic model' of the point-like electron. In a model where the electron is represented by a charged sphere of radius R, the contribution to its mass coming from the Coulomb self-energy diverges as e^2/R when $R \to 0$.
The first correct result was published by Weisskopf (1934) in an erratum after correction of a a last mistake in a preceding article, which had been pointed out by Furry.

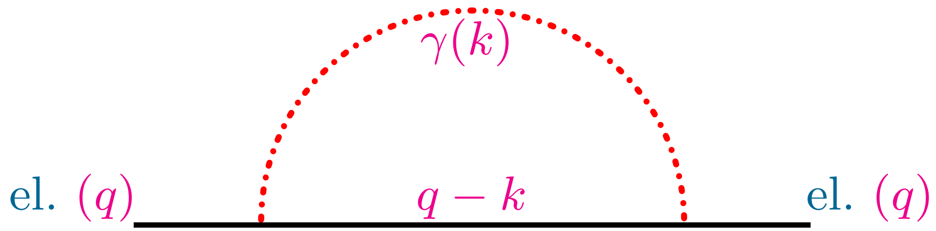


Fig. 7.1 – Electron propagation: dotted line for the photon and full line for the electron.

The contribution to the mass was still infinite, the linear classical divergence being only replaced by a softer logarithmic UV divergence:

$$\delta m_{\text{el. QED}} = -3\frac{\alpha}{2\pi} m_{\text{el.}} \ln(m_{\text{el.}}/\Lambda_\gamma) \quad \text{with} \quad \Lambda_\gamma \approx \hbar/cR\,.$$

In terms of Feynman diagrams (a representation imagined only much later), the relevant physical process consists in the emission and re-absorption of a virtual photon of energy-momentum k by an electron of energy-momentum q, (Fig. 7.1).

It became slowly clear that the problem was very deep; these divergences seemed unavoidable consequences of locality (point-like particles with contact interactions) and unitarity (conservation of probabilities). Indeed,

(i) one must sum over the contribution of virtual photons with arbitrarily high energies because there is no short-distance structure.

(ii) Due to conservation of probabilities, all processes contribute additively.

These divergences seemed to indicate that QED was an incomplete theory, but it was hard to figure out how to modify it without giving away some fundamental physical principle.

Dirac (1942) proposed to abandon unitarity, but physical consequences seemed hardly acceptable. A non-local relativistic extension (which would correspond to give an inner structure to all particles) was hard to imagine in a relativistic context, though Heisenberg (1938) proposed the introduction of a fundamental length. In fact, only in the eighties were plausible candidates proposed in the form of string theories.

Even more drastic: Wheeler (1937) et Heisenberg (1943) proposed to completely abandon QFT in favour of a theory of physical observables (scattering data): the so-called S-matrix theory, an idea that became very popular in the 1960's in the theory of Strong Interactions (responsible for nuclear forces).

7.4.1 Infinities and charged scalar bosons

More pragmatic physicists in the meantime explored the nature and form of divergences in quantum corrections, calculating other physical quantities. An intriguing remark (Weisskopf 1939): while in the case of charged fermions logarithmic divergences are numerically acceptable if some reasonable momentum cut-off can be found (the range of nuclear forces, about 100 MeV, seemed a plausible candidate), charged scalar bosons lead to large quadratic divergences, which are totally unacceptable because they would spoil the classical results. Thus, can scalar bosons be fundamental particles?

The problem remains very much relevant because the Standard Model of interactions at the microscopic scale contains a scalar particle, called Higgs boson, and, indeed, recently a boson with a 125 GeV mass has been discovered at the Large Hadron Collider (CERN) with apparently the right properties.

It becomes even more acute if one assumes that the Standard Model is valid up to a possible grand unification ($\sim 10^{15}$ Gev) or gravitation (Planck's mass) (10^{19} GeV) scales. It leads to the fine tuning problem and has been one motivation for introducing supersymmetry (a symmetry relating bosons and fermions). Supersymmetric particles are thus intensively searched at LHC, with little success up to now.

7.5 Renormalization procedure

An empirical observation. It was eventually noticed that in some combinations of physical observables, divergences cancelled (see, for instance, Weisskopf 1936) but the meaning of this observation remained obscure.

An essential experimental input. In 1947 Lamb et Retheford measured precisely the splitting between the levels $2s_{1/2}2p_{1/2}$ of the hydrogen atom, Rabi's group in Columbia measured the anomalous magnetic moment of the electron.

The first QED results. Remarkably enough, it was possible to organize the calculation of the Lambshift in such a way that all infinities cancel (first approximate calculation by Bethe) and the result agreed beautifully with experiment. Shortly after, Schwinger obtained the leading contribution to the anomalous magnetic moment of the electron.

Soon, the idea of divergence subtraction was generalized to the concept of renormalization (building up on work of Kramers) and in 1949 Dyson, following work by Feynman, Schwinger and Tomonaga, gave the first proof that, after renormalization, divergences cancel to all orders of perturbation theory. The principles of renormalization theory were thus established.

The general idea. To render the theory finite, one first introduces a momentum cut-off $c\Lambda$ which modifies in a somewhat arbitrary and unphysical way the theory at a very short distance of order $\hbar/c\Lambda$.

One then calculates physical observables in terms of the initial bare parameters of the Lagrangian, the bare mass m_0, the bare charge e_0 of the electron (mass and charge for vanishing interaction), or equivalently the bare fine structure constant $\alpha_0 = e_0^2/4\pi\hbar c$, as series in α_0.

In particular, one determines the physical mass m and physical charge e as renormalized by the interaction.

The results take the form (β_2, γ_1 and C_1 are numerical constants):

$$e^2/4\pi\hbar c \equiv \alpha = \alpha_0 - \beta_2\alpha_0^2 \ln(\Lambda/m_0) + \cdots,$$
$$m = m_0 - \gamma_1\, m_0\alpha_0 \ln(\Lambda C_1/m_0) + \cdots.$$

One inverts the relations,

$$\alpha_0 = \alpha + \beta_2\alpha^2 \ln(\Lambda/m) + \cdots,$$
$$m_0 = m + \gamma_1\, m\alpha \ln(\Lambda C_1/m) + \cdots$$

expressing then all other physical observables in terms of the renormalized parameters m and α as expansions in α.

Most surprisingly, all physical observables expressed in terms of renormalized fields and renormalized parameters then have an infinite cut-off Λ limit.

This *a priori* somewhat strange renormalization procedure, had led to QED predictions that agree, with unprecedented precision, with experiment.

Moreover, the success of renormalization theory has led to the very important concept of renormalizable QFT. Since the renormalization procedure works only for a limited number of theories, this has strongly constrained the structure of possible QFT's.

7.6 The nature of divergences and the meaning of renormalization

Renormalized QED was obviously the right theory because predictions agreed with experiment, but why? Several answers were proposed, for example:

(i) Divergences were a disease of the perturbative expansion in α_0 and a proper mathematical handling of the theory with non-perturbative input would free it from infinities. In the same spirit, Axiomatic QFT tried to establish rigorous non-perturbative results from general principles.

(ii) More drastic, the problem was fundamental: QFT was only defined by perturbation theory but the procedure that generated the perturbative expansion had to be modified in order to generate automatically finite renormalized quantities. The initial bare theory, based on a Lagrangian with divergent coefficients, was physically meaningless. It provided a simple book-keeping device to generate the perturbative expansion.

This line of thought led, in particular, to the BPHZ formalism (Bogoliubov, Parasiuk, Hepp, Zimmerman), and eventually to the Epstein–Glaser work, where renormalized Feynman diagrams where generated directly and taken as fundamental building blocks of the theory.

While this approach clarified the properties of renormalized perturbation theory, it had also the effect of disguising the problem of infinities as if it had never existed in the first place.

(iii) The cut-off had a real physical meaning, being generated by additional interactions beyond QED (like Strong Interactions), but then the meaning of renormalizability, which reflected some form of short-distance insensitivity, had still to be understood.

This last viewpoint is the closest to modern thinking, except that the cut-off is no longer linked to Strong Interactions but to a as yet unknown much higher energy scale.

7.7 QFT and renormalization group

An intriguing consequence of renormalization in massless QED (Peterman–Stückelberg (1953), Gell-Mann–Low (1954), Bogoliubov–Shirkov (1955)).

In a QED with massless electrons, the renormalized charge cannot be defined in terms of the interaction between non-existing static electrons since they propagate at the speed of light.

One must introduce some arbitrary mass or energy or momentum-scale μ to define the renormalized charge e in terms of the strength of the e.m. interaction at scale μ: it is the effective charge at scale μ.

But then the same physics can be parametrized by the effective charge e' at another scale μ'. The set of transformations of physical quantities associated with this change of scale and required to keep physics constant was called Renormalization Group (RG).

Moreover, one could show that in an infinitesimal scale change, the variation (or the flow) of the effective charge satisfies a differential equation of the form

$$\mu \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu} = \beta\bigl(\alpha(\mu)\bigr), \quad \beta(\alpha) = \beta_2 \alpha^2 + O(\alpha^3). \tag{1}$$

In fact, even in a massive theory such a definition can be used. It also leads to a parametrization in terms of the effective charge at some mass scale μ.

Interpretation: At large distance, the strength of the electromagnetic interaction remains constant at the value measured through the Coulomb force. However, at distances much smaller that the wave length $\hbar/mc$ associated with the electron (one explore in some way the 'interior' of the particle), one observes screening effects. What has to be noticed is that these short-distance screening effects are related to renormalization.

Gell-Mann and Low's initial hope, which was to use this flow equation to determine the bare charge as the large momentum limit of the effective charge, failed because due to the sign $\beta_2 > 0$, the effective charge increases at large momentum (a phenomenon verified experimentally at the Z boson mass at CERN) until perturbation theory becomes useless.

A related issue: Landau's ghost. A leading log summation of high energy contributions to the electron propagator exhibits an unphysical (a ghost) pole (Landau and Pomeranchuk (1955)) at a mass $M \propto m\, \mathrm{e}^{1/\beta_2\alpha} \approx 10^{30}$GeV. For Landau, this was a sign of QED inconsistency, but Bogoliubov and Shirkov noticed that this amounted to solving RG flow equation for α small and using it for α large.

Still, it seems that Landau's intuition was right but inconsistency at such high energy is physically irrelevant.

7.8 The triumph of renormalizable QFT: The Standard Model

The principle of looking for renormalizable QFT's led, at the beginning of the 70s, to the construction of the Standard Model describing all interactions (but gravity) at the microscopic scale, based on non-Abelian gauge theories. In the perplexing sector of Strong Interactions, the negative sign of the RG β-function allowed explaining the weakness of interactions between quarks at short distance as seen in deep inelastic experiments (Gross–Wilczek, Politzer 1973) in a way consistent with quark confinement and led to Quantum Chromodynamics (QCD).

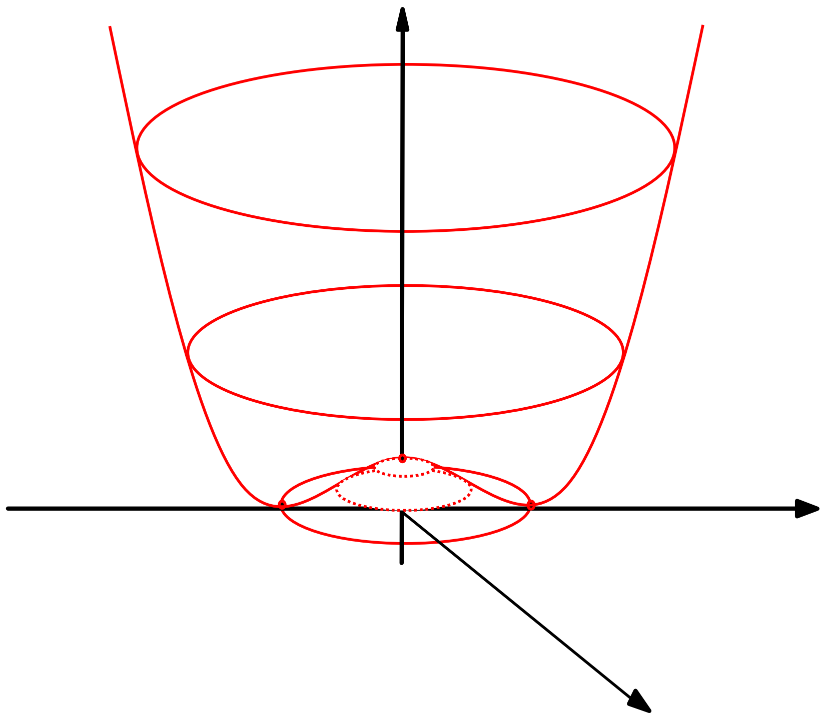


Fig. 7.2 – $O(2)$-symmetric potential with spontaneous symmetry breaking.

The short range of weak interactions could be explained by intermediate vector bosons getting masses due a combination of spontaneous symmetry breaking and gauge invariance.

One outstanding problem remained: to include gravitation into the framework of renormalizable theories.

The failure of reaching this goal has led to a search for non-field theoretical extensions of the theory in the form, in particular, of string theories.

7.9 Critical phenomena: other infinities

Second order or continuous macroscopic phase transitions, with short-range interactions, are characterized by a collective behaviour leading to correlations on large scales (compared to the microscopic defining scale) near the critical temperature. The scale of these dynamically generated correlations is characterized by the correlation length, which diverges at the critical temperature.

At the scale of the correlation length, non-trivial macroscopic physics is observed. The usual idea of scale decoupling leads to expect that macroscopic physics could be described by a small number of well-chosen effective parameters, without explicit reference to the initial microscopic interactions.

This idea leads to Mean Field Theory and, more generally, to Landau's theory of critical phenomena (1937), theories that can be called perturbed or quasi-Gaussian with reference to the central limit theorem of probabilities.

Among the simplest universal predictions of such a theory, one finds the universality of the singular behaviour of thermodynamic quantities near the critical temperature T_c.

For example, in magnetic systems the spontaneous magnetization vanishes

$$M \propto \sqrt{T_c - T},$$

the correlation length diverges as

$$\xi \propto 1/\sqrt{T - T_c}$$

..., and these properties are independent of dimension of space, of symmetries, and the form of the microscopic dynamics.

However, it came slowly apparent that these predictions disagreed with more precise experiments and lattice model calculations. They also disagreed with the exact solution of the 2D Ising model (Onsager 1944).

Moreover, an attempt to calculate corrections to the Gaussian theory leads to infinities for all space dimensions $d \le 4$ when the correlation length diverges.

In fact, numerical investigations seemed to indicate that some universality survived but in a more limited form, the critical behaviour depending on the dimension of space as well as some general qualitative properties of models but not on the detailed form of the interaction.

7.10 Scale decoupling in physics

A basic paradigm in physics:

The decoupling of physical phenomena corresponding to too different scales of distances.

Example: the period τ of the pendulum: $\tau \propto \sqrt{\ell/g}$.

Implicit hypothesis: the sizes that are very different from the pendulum length ℓ, like the size of the atoms or the radius of the earth play no role in the period of the pendulum.

In the same way, orbits of planets can be determined to a very good approximation by replacing planets and the sun by point-like objects, and by forgetting all other stars in the galaxy.

Provided one is able to discover the relevant degrees of freedom and parameters, one can devise models adapted to the scale of the phenomena.

This empirical property is essential for the predictivity of models. If physics would be sensitive to all scales, prediction would be impossible, since it would require a complete knowledge of all physical laws and all parameters of nature. Nevertheless, in the twentieth century, in two *a priori* very different domains of physics, this commonly accepted paradigm has been challenged.

To explain the compatibility between a non-decoupling of scales and, nevertheless, the relative insensitivity of large distance physics to the microscopic structure, a new tool had to be invented:

the Renormalization Group.

This rather abstract concept has allowed not only understanding such a property, but has also inspired number of precise calculation methods.

It has provided a new interpretation to renormalizable Quantum Field Theories as effective large distance field theories.

KADANOFF–WILSON'S RG IDEA:

We consider a lattice model with classical spins ϕ, short range interactions and a $\mathbb{Z}_2$ reflection symmetry $\phi \mapsto -\phi$. The partition function has the form (a is the lattice spacing)

$$\mathcal{Z} = \sum_{\{\phi(\mathbf{n}a)\},\ \mathbf{n}\in\mathbb{Z}^d} \exp\left[-\mathcal{H}(\phi)\right], \quad \mathcal{H}(\phi) = \mathcal{H}(-\phi)$$

where the interactions in the spin Hamiltonian $\mathcal{H}(\phi)$ are short range.

Kadanoff and Wilson suggested to sum iteratively over short-distance degrees of freedom. For example, one sums over the initial spins with the constraint that their average values on a cell are fixed:

$$\phi(\mathbf{n}a) \mapsto \phi'(\mathbf{n}a) = \frac{\sqrt{Z}}{2d} \sum_{a\mathbf{n}'\ \text{neighbours of}\ 2a\mathbf{n}} \phi(a\mathbf{n}'),$$

where Z is a renormalization factor.

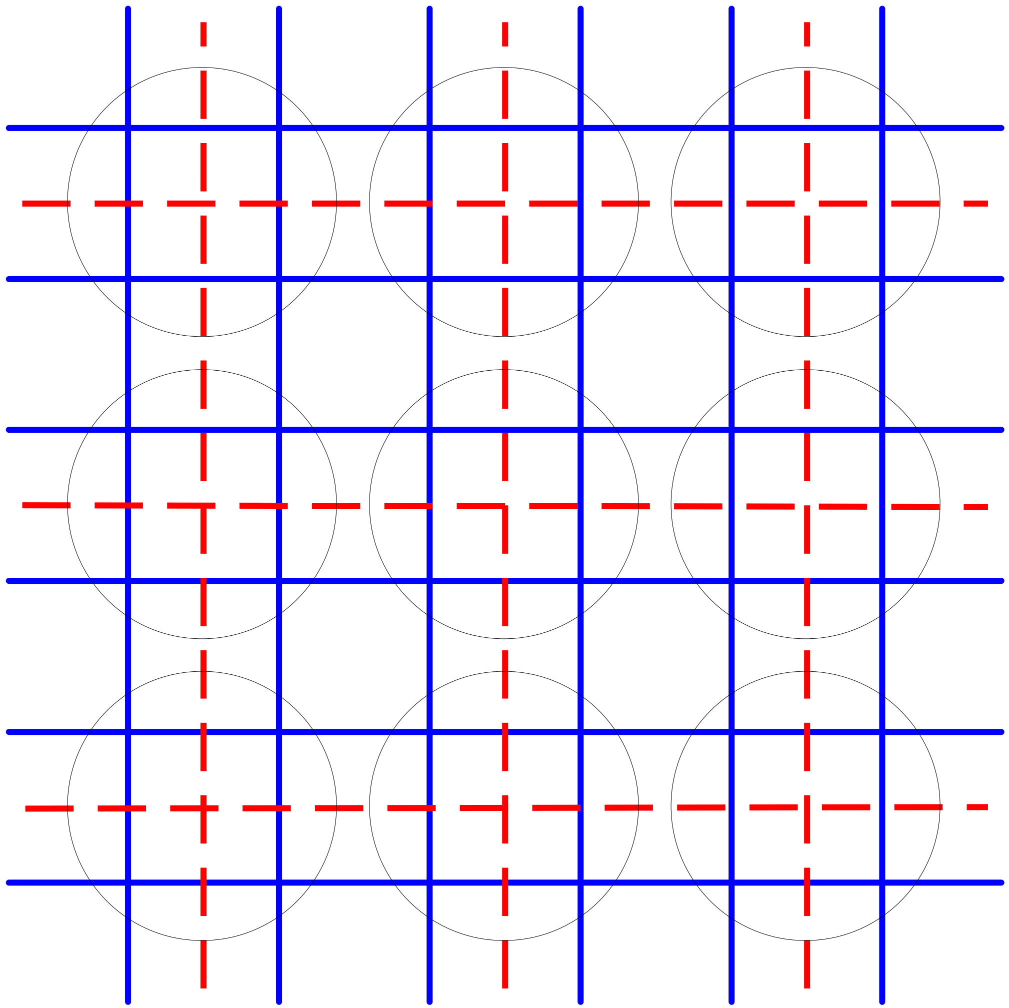


Fig. 7.3 – Initial (blue) lattice with lattice size a and (red) lattice with lattice size $2a$.

This transformation leads to a new Hamiltonian, function of average spins on a lattice with double lattice size. Iterating, one constructs a renormalization group:

$$\mathcal{H}(\phi; 2^n a) = \mathcal{T}\left[\mathcal{H}(\phi; 2^{n-1} a)\right].$$

Fixed points: One looks for fixed points, solution of (this requires adjusting the value of the renormalization Z)

$$\mathcal{H}^*(\phi) = \mathcal{T}\left[\mathcal{H}^*(\phi)\right].$$

If attractive fixed points can be found, then

$$\mathcal{H}(\sqrt{Z}\phi; 2^n a) \underset{n\to\infty}{\to} \mathcal{H}^*(\phi).$$

The existence of attractive fixed points of a RG allows understanding universality, within universality classes, when scales do not decouple.

Fixed points and QFT. Even if initially the spin variables take only discrete values and space has a lattice structure, after a large number of iterations the effective spins can be replaced by continuous variables and the lattice by continuum space: a fixed point Hamiltonian, if it exists, thus corresponds to a local statistical field theory (Wilson 1971):

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{H}(\phi)\right].$$

The Gaussian fixed point. One verifies that the Gaussian field theory (free massless scalar theory),

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{H}_{\mathrm{G}}(\phi)\right], \quad \mathcal{H}_{\mathrm{G}}(\phi) = \tfrac{1}{2}\int \mathrm{d}^d x \left(\nabla_x \phi(x)\right)^2,$$

is a fixed point (related to the central limit theorem of probabilities).

The Gaussian fixed point is related to Landau or mean field theory. It becomes unstable when the dimension d of space is lower than 4.

Near dimension 4, the instability is induced by the perturbation

$$\int \mathrm{d}^d x\, \phi^4(x).$$

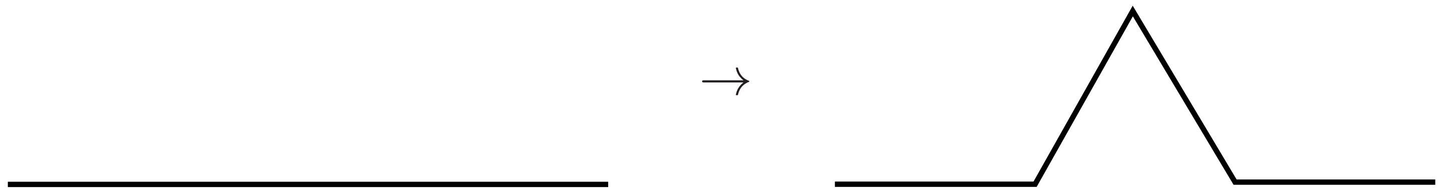
If another fixed point exists in a neighbourhood of the Gaussian fixed point, universal properties can be determined from the local statistical field theory (quantum field theory in imaginary time):

$$\mathcal{H}(\phi) = \mathcal{H}_{\mathrm{G}}(\phi) + \int \mathrm{d}^d x \, \left[\tfrac{1}{2} r \phi^2(x) + \tfrac{1}{4!} g \phi^4(x)\right],$$

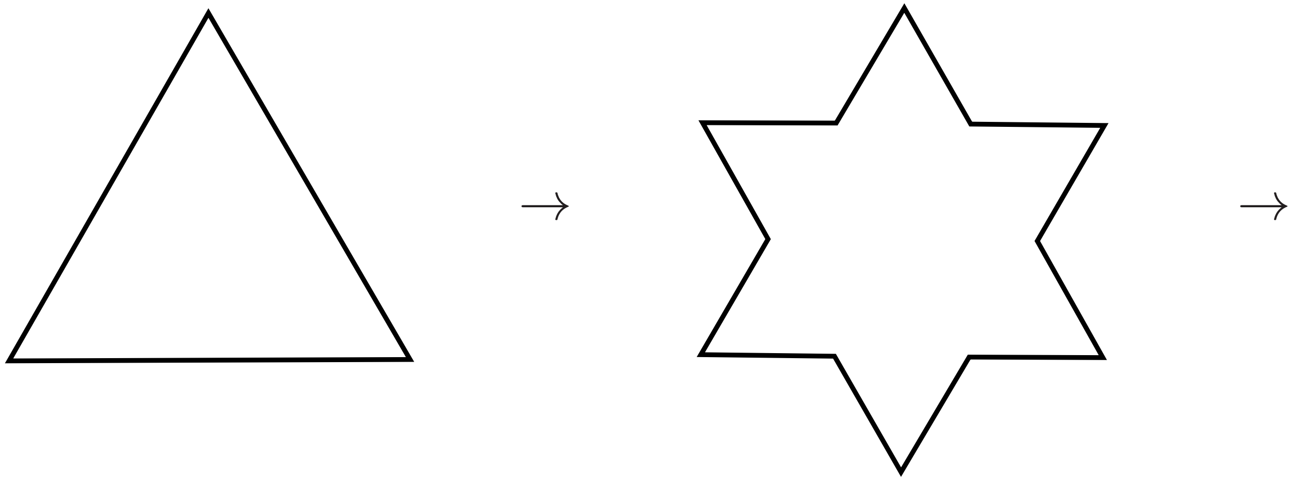
where the parameter r plays the role of temperature near T_c and allows exploring the neighbourhood of T_c (the critical domain). This statistical field theory is renormalizable at and near dimension 4 and, thus, admits a field theory renormalization group.

Fixed points, geometric analogies: straight line and fractals

A Gaussian fixed point analogue: the straight line.



A RG transformation analogue.



A geometric fractal: a non-trivial fixed point analogue.

In the continuum, the RG or flow equation for the Hamiltonian takes the general form of the functional equations (Wegner–Wilson)

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}\mathcal{H}(\phi,\Lambda) = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{D}_\Lambda(k)\left[\frac{\delta^2\mathcal{H}}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(-k)} - \frac{\delta\mathcal{H}}{\delta\tilde{\phi}(k)}\frac{\delta\mathcal{H}}{\delta\tilde{\phi}(-k)}\right] + \int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{L}_\Lambda(k)\frac{\delta\mathcal{H}}{\delta\tilde{\phi}(k)}\tilde{\phi}(k).$$

The renormalization group of QFT appears, in the general RENORMALIZATION GROUP framework, as an asymptotic renormalization group in the neighbourhood of the Gaussian fixed point (the free massless scalar field theory).

Indeed, in the neighbourhood of the Gaussian fixed point, all interactions corresponding to stable directions in Hamiltonian space at the Gaussian fixed point (irrelevant interactions) can be neglected .

The general discrete transformation $\mathcal{H} \mapsto \mathcal{T}(\mathcal{H})$ then reduces to $g(\ell) \mapsto g(2\ell)$. In continuum space, in an infinitesimal dilatation $\ell \mapsto \ell(1+\delta\ell/\ell)$,

$$\ell \frac{\partial g}{\partial \ell} = -\beta[g(\ell)].$$

The fixed point equation reduces to

$$\beta(g^*) = 0\,.$$

To determine universal properties in the critical domain (near the transition temperature), one must first determine the zeros g^* of the β-function, and then calculate all other physical quantities for $g = g^*$.

This program has led to precise determinations of various physical quantities like critical exponents (Le Guillou and ZJ) and equation of state (Guida and ZJ), in very good agreement with values extracted from lattice models and experiments.

Critical exponents from $O(N)$ symmetric $(\phi^2)^2_3$ field theory
(Le Guillou and Z.-J. (1980) updated by Guida and Z.-J. (1998))

N	0	1	2	3
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
θ	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

This success has confirmed that, somewhat unexpectedly, large distance physics near a continuous phase transition in systems with short-range interactions, can be described by local renormalizable quantum field theories.

7.11 Effective quantum field theories

The condition that microscopic physics should be describable by renormalizable QFT's has been one of the basic principles that have led to the Standard Model of microscopic interactions.

From the success of the program, one might have concluded that renormalizability was a new law of nature. The implication would have been that all interactions including gravity should be describable by renormalizable QFT.

The failure to exhibit a renormalizable version of quantum gravity has shed some doubt on such a viewpoint.

Indeed, if the Standard Model and its thinkable extensions are only low energy approximations, it becomes difficult to understand why they should obey such an abstract principle.

By contrast, the theory of critical phenomena shows that a dynamical generation of a large scale may generate a non-trivial large distance physics, which can be described by a renormalizable QFT.

This provides a simpler and more natural explanation for the appearance of renormalizable QFT's in physics.

One can then speculate that fundamental interactions are described at some more microscopic scale (like the Planck length) by a finite theory that has no longer the nature of a local quantum field theory.

Although such a theory should involve only some short microscopic scale, for reasons that can only be a matter of speculation, it generates strong correlations between a large number of degrees of freedom and a large distance physics with very light particles.

This line of thought has changed considerably our viewpoint on the renormalization process. In the traditional presentation, one introduces a large momentum cut-off to render the perturbative expansions finite, one calculates physical observables as functions of the parameters of the Lagrangian and the cut-off, in particular, physical masses and coupling constants, one eliminates the parameters of the Lagrangian in favour of direct relations between physical observables and takes the infinite cut-off limit.

When the quantum field theory is renormalizable, the infinite limit exists and defines a cut-off independent renormalized field theory.

However, this process relies on tuning all initial parameters of the Lagrangian as functions of the cut-off, which then in the infinite cut-off limit diverge. This tuning is so difficult to justify that, at some point, it led to the claim that the initial Lagrangian itself was unphysical.

Moreover, through this programme was met with considerable success, it did not provide any rationale for eliminating non-renormalizable interactions.

By contrast, we take the point of view of effective field theory; we assume that a true cut-off exists, which is provided by a more fundamental theory, which could be another quantum field theory or, eventually, the fundamental necessarily non-local theory. Of course, since this theory is unknown, the cut-off regularization is an *ad hoc* procedure and it is necessary to show that physical results are largely independent of the specific regularization.

Moreover, one wants to avoid tuning as much as possible and one assumes the parameters of the Lagrangian are fixed and expected to be of order unity (the naturalness assumption). In particular, the fine tuning of the Higgs bare mass is an important physical issue.

In this framework, renormalization group (RG) plays an essential role. It allows to evaluate the effective parameters at the physical scale and to show that physics at an energy of mass scale much lower than the cut-off is indeed independent, to a good approximation, of the cut-off procedure.

The success of this programme, in particular, initiated by Wilson, in the study of continuous phase transitions in macroscopic physics gives confidence that it should also apply to particle physics.

Moreover, additional, weak non-renormalizable interactions should be expected (the irrelevant interactions of renormalization group), suppressed by powers of the short-distance scale. Non-renormalizable quantum gravity could be an example of such an (RG) irrelevant interaction.

To summarize, renormalizable quantum field theories are only effective large distance theories in which all non-renormalizable interactions have been neglected.

They come endowed with a natural cut-off reflection of the existence of a more fundamental theory, but are somewhat short-distance insensitive, due to specific Renormalization Group properties.

They are not necessarily consistent on all scales, (like QED), since they have only a limited energy range of validity.