Lecture 8: THE SCALAR FIELD: RENORMALIZATION AND RENORMALIZATION GROUP

We now proceed with the study of local quantum field theories (QFT). A local QFT is defined by the property that the action is a space integral of a local function of the field and its derivatives, as a consequence of an assumption that the initial microscopic model is characterized by a scale much smaller than the physical scale one investigates. In statistical physics this is an assumption of short-range interactions.

We immediately emphasize the notion of effective field theory, even though every quantum field theory is an effective field theory. The main reason is that initially it was thought by many physicists that in particle physics quantum field theory, in some form, would become a truly fundamental theory. For reasons we have explained this is no longer the case.

Note that often the denomination of effective field theory (EFT) is restricted to a context in which a field theory is a low energy approximation to another, more fundamental, known theory. An toy example is discussed in section 8.1.

In a naive interpretation of local quantum field theory, divergences are met in the perturbative expansion.

The problem of divergences is then dealt with renormalization theory. We will use its results without giving technical details.

However, we describe the modern point of view on the renormalization process, which is at strong variance with the more traditional point of view.

In the traditional presentation, one introduces a large momentum cut-off to render the perturbative expansions finite, one calculates physical observables as functions of the parameters of the Lagrangian and the cut-off, in particular, physical masses and coupling constants.

One then eliminates the parameters of the Lagrangian in favour of direct relations between physical observables and takes the infinite cut-off limit. When the quantum field theory is renormalizable, a form of short distance insensitivity, the infinite cut-off limit exists and defines a perturbative renormalized field theory independent of the specific cut-off procedure.

This programme, based on focusing on renormalizable theories, was met with considerable success, leading to the construction of the Standard Model of particle physics.

However, this process relies on tuning the initial parameters of the Lagrangian as functions of the cut-off. This tuning is hard to justify. Moreover, since the initial parameters diverge in the infinite cut-off limit, this led, at some point, led to the claim that the initial Lagrangian itself was unphysical.

Also the process provides only a technical and not a physical rationale for disregarding non-renormalizable interactions.

By contrast, we take the point of view of effective field theory. We assume that a true cut-off exists, provided by a, as yet unknown, more fundamental theory, which could be another quantum field theory or, eventually, a fundamental necessarily non-local theory.

This viewpoint is supported by the study of continuous phase transitions in statistical physics.

There, large momentum divergences are not present in the initial microscopic model due to the inherent short distance structure (like in lattice models), but this structure may be complicate to deal with.

One thus replace it by an *ad hoc* cut-off regularization. The results obtained by the method of regularization can then be physical only if they are largely independent of the specific regularization. The proof relies on perturbative renormalization group and, technically, renormalization theory.

Moreover, tuning should be avoided as much as possible: the parameters in the Lagrangian are fixed and expected to be of order unity (the naturalness assumption) at the cut-off scale.

In the more general context of local field theories, the issue of short distance insensitivity relies on a renormalization group (RG) analysis. RG methods allow evaluating the effective parameters at the physical scale and to show that physics at an energy or mass scale much lower than the cut-off is indeed independent, asymptotically, of the cut-off procedure.

In this framework, renormalization theory remains an important tool because it allows proving RG equations for renormalizable field theories.

Finally, the success of this programme initiated, in particular, by Wilson in the study of continuous phase transitions in macroscopic physics, gives confidence that it should also apply to particle physics.

Therefore, in this framework, the fine-tuning of the Higgs particle mass in the Standard Model remains a troubling issue, while in the corresponding example of macroscopic phase transitions, the experimentalist tunes the temperature to bring it close to the critical temperature.

We analyse more thoroughly the example of a theory involving a scalar field ϕ with a $\mathbb{Z}_2$ reflection symmetry $\phi \mapsto -\phi$ in dimension $d = 4$.

There, a problem will be discovered and discussed, the triviality of the renormalized ϕ^4 theory if one insists to take the truly infinite cut-off limit.

Finally, we briefly mention the properties of the ϕ^4 theory in dimension $d = 4 - \varepsilon$, a situation directly relevant for the study of macroscopic continuous phase transitions.

In the $\mathbb{Z}_2$ symmetric ϕ^4 theory considered in this lecture, as well as in the more general $O(N)$ symmetric $(\boldsymbol{\phi}^2)^2$, theory symmetry may be spontaneously broken. We postpone the discussion of spontaneously symmetry breaking of continuous symmetries to lecture 12 but we show how a combination of spontaneous symmetry breaking and renormalization group provided an early bound on the Higgs particle mass.

8.1 The notion of effective field theory

The basic idea behind the notion of effective field theory is the following: one starts from a microscopic model involving an infinite number of degrees of freedom whose dynamics is characterized by a microscopic scale and in which, as a result of the interaction between these degrees of freedom a length (called correlation length in statistical physics) much larger than the microscopic scale or, equivalently, a mass much smaller than the characteristic mass scale of the initial model are generated.

In this situation, the model exhibits a non-trivial large distance and, in a number of examples, universal physics.

Even if the initial model has a non-local dynamics, because the non-locality is confined to the microscopic scale, the large scale physics may be describable by a general local field theory, called effective field theory.

Though we will eventually concentrate on the four-dimensional example, it is convenient to first discuss arbitrary dimensions d.

A general even effective action for a scalar field can be written as (equation (6.14)),

$$\mathcal{S}(\phi) = \mathcal{S}_{\rm G}(\phi) + \mathcal{V}_{\rm I}(\phi), \tag{8.1}$$

where

$$\mathcal{S}_{\rm G}(\phi) = \tfrac{1}{2} \int {\rm d}^d x \left[\left(\nabla_x \phi(x)\right)^2 + u_0 \phi^2(x) \right] + \tfrac{1}{2} \sum_{k=1} u_{k+1} \int {\rm d}^d x \left[\nabla_x \phi(x) (-\nabla_x^2)^k \nabla_x \phi(x)\right]$$

and $\mathcal{V}_{\rm I}(\phi)$ is the space integral of a general, expandable, even function of the field and its derivatives:

$$\mathcal{V}_{\rm I}(\phi) = \int {\rm d}^d x \, V_{\rm I}[\phi(x), \partial_\mu \phi(x), \ldots]$$

with terms at least of degree 4 in ϕ.

The action has been split into the sum of the action of a free massless field, a ϕ^2 contribution, a number of terms quadratic in the field with a sufficient number of derivatives, in order to regularize all large-momentum (or short-distance) divergences and general even local interactions.

One calls such a field theory an effective field theory to emphasize that it is not a microscopic model, but only a model that reproduces correctly the asymptotic behaviour at large distance (this denomination has nowadays become almost a pleonasm in the sense that almost all quantum or statistical field theories that are encountered in physics have such an interpretation).

8.1.1 An illustrative example

Effective field theory is a notion that depends on the context. It can apply to the fundamental field theory of particle physics but it can also apply to an approximation to another quantum field theory when one mass is much larger than all other ones or in the high temperature regime.

We illustrate this notion with a simple example. We consider a model involving two scalar fields, ϕ and χ, and a renormalizable action of the form

$$\mathcal{S}(\phi,\chi)=\mathcal{S}_1(\phi)+\mathcal{S}_2(\chi,\phi)$$

where $\mathcal{S}_1(\phi)$ contains a ϕ^4 interaction and

$$\mathcal{S}_2(\chi,\phi)=\tfrac{1}{2}\int \mathrm{d}^4x\,\Big[\big(\nabla_x\chi(x)\big)^2+M^2\chi^2(x)+g\chi^2(x)\phi^2(x)+g'\chi^4(x)\Big].$$

An unspecified regularization with a large momentum cut-off Λ is assumed. For simplicity, we neglect the χ^4 interaction in what follows.

We assume that the physical mass M of the field χ is much larger than the physical mass of the field ϕ and the energies one is probing.

Since the particle associated to the field χ is not observed, one can integrate out the field χ. We set

$$\mathrm{e}^{-\mathcal{V}(\phi)}=\frac{1}{\mathcal{N}}\int[\mathrm{d}\chi]\,\mathrm{e}^{-\mathcal{S}_2(\chi,\phi)},\quad \mathcal{N}=\int[\mathrm{d}\chi]\,\mathrm{e}^{-\mathcal{S}_2(\chi,0)}.$$

The integral over χ is Gaussian and yields ($\ln\det = \operatorname{tr}\ln$)

$$\mathcal{V}(\phi) = \tfrac{1}{2}\operatorname{tr}\ln\left[-\nabla_x^2 + M^2 + g\phi^2(x)\right]\left[-\nabla_x^2 + M^2\right]^{-1},$$

leading to a non-local addition to the action $\mathcal{S}_1(\phi)$ for the ϕ field.

However, after expanding $\mathcal{V}(\phi)$ in powers of ϕ^2, for M large we can make a local expansion of each term.

The coefficient of $g\phi^2(x)$, in Fourier representation

$$\frac{1}{(2\pi)^4}\int\frac{d^4p}{p^2 + M^2},$$

is a divergent constant that renormalizes the ϕ mass term.

The coefficient of the term of order g^2 in the Fourier representation is proportional to

$$g^2 \int \mathrm{d}^4x \, \mathrm{d}^4y \, \phi^2(x)\phi^2(y)\Delta^2(x-y),$$

where $\Delta(x)$ is the χ-field propagator. Then,

$$\Delta^2(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \; \mathrm{e}^{ikx} \, B(k) \tag{8.2}$$

with

$$B(k) = \frac{1}{(2\pi)^4} \int \frac{\mathrm{d}^4p}{(p^2+M^2)\big((p+k)^2+M^2\big)}.$$

We rewrite the integral by using an example of Feynman's parametrization, which takes the form of the identity

$$\frac{1}{\alpha\beta} = \int_0^1 \frac{\mathrm{d}s}{[\alpha s + \beta(1-s)]^2}\,.$$

We apply it to $\alpha = (p+k)^2 + M^2$, $\beta = p^2 + M^2$ and, after shifting $p + sk \mapsto p$, the integral can be rewritten as

$$B(k) = \frac{1}{(2\pi)^4} \int_0^1 \mathrm{d}s \int \frac{\mathrm{d}^4 p}{[p^2 + M^2 + s(1-s)k^2]^2} .$$

The integrand can be expanded in powers of $(p^2 + M^2)$,

$$\frac{1}{[p^2 + M^2 + k^2 s(1-s)]^2} = \frac{1}{(p^2 + M^2)^2} - \frac{2k^2 s(1-s)}{(p^2 + M^2)^3} + \frac{3k^4 s^2 (1-s)^2}{(p^2 + M^2)^4} + \cdots . \tag{8.3}$$

The first term gives a divergent constant, which after integration over k yields a local contribution renormalizing the coefficient of $\phi^4(x)$.

The second term is finite and proportional to k^2/M^2. It yields a local, non renormalizable interaction proportional to $\phi^2(x) \nabla_x^2 \phi^2(x)/M^2$.

More generally, the higher order terms yield finite local, non renormalizable interactions proportional to $\phi^2(x) (\nabla_x^2)^n \phi^2(x)/M^{2n}$.

The term of order ϕ^6 is finite and, at leading order, yields a contribution proportional to $\phi^6(x)/M^2$. The next term has two additional derivatives and a factor $1/M^4$. Quite generally, all contributions are local and their M dependence can be inferred from dimensional analysis: all contributions have mass dimension 4 and ϕ and ∇_x have mass dimension 1.

Therefore, the large M expansion generates a set of local interactions, two which do not vanish for M large but simply renormalize terms already present in the action $\mathcal{S}_1(\phi)$ and all others that correspond to non-renormalizable interactions and are suppressed by powers of M.

In an effective field theory, small non-renormalizable interactions are the observable remnants of a new massive particle or a new energy scale, as the history of the theory of weak interactions illustrates.

Successive perturbative contributions of these new interactions, which come endowed with a natural cut-off of order M, again renormalize the effective action and yield new contributions that vanish faster for M large.

8.2 Rescaling, Gaussian renormalization and power counting

Quite generally, one assumes that the physical mass is small with respect to the mass or energy scale of the microscopic theory and that the deviations from a free massless theory are in some sense small, justifying a local expansion of the initial microscopic model in the form of the action (8.1).

Initially, distances are measured in microscopic units and, generically, one expects the coefficients of the local expansion to be numbers of order 1. To describe only large distance physics, it is convenient to rescale distances to take as a reference scale the macroscopic scale relevant to large-distance physics, rather than the initial microscopic scale.

The ratio of scales Λ, which has a momentum dimension, becomes also the cut-off scale in the QFT terminology because the local expansion now breaks down at scale $1/\Lambda$ where non-localities that render the field theory finite appear and, thus, momentum integrals have to be cut at scale Λ.

8.2.1 Gaussian renormalization

We thus introduce a large parameter Λ that has a momentum dimension and rescale distances in the form

$$x = \Lambda x', \tag{8.4}$$

The initial microscopic scale is now characterized by the parameter $1/\Lambda$ (related, for example, to the spacing of an initial lattice model).

Instead of studying the large-distance or small momentum limit, one then studies the limit $\Lambda \to \infty$.

After this rescaling, the leading terms in the effective action are those with the smallest number of derivatives. For $\phi(x)$ small, the leading term then is $\phi^2(x)$, which implies generically a mass scale of order Λ. To generate a small mass, we must tune the coefficient of ϕ^2 in the action in such a way that the kinetic $(\nabla_x \phi(x))^2$ term, which alone leads to a massless free theory, becomes a leading term.

We thus renormalize the field, $\phi \mapsto \phi'$, to cancel its Λ dependence,

$$\int \mathrm{d}^d x \, \big(\nabla_x \phi(x)\big)^2 \mapsto \Lambda^{d-2} \int \mathrm{d}^d x \, \big(\nabla_x \phi(x)\big)^2 = \int \mathrm{d}^d x \, \big(\nabla_x \phi'(x)\big)^2$$

with

$$\phi'(x) = \Lambda^{(d-2)/2} \phi(x). \tag{8.5}$$

After this renormalization, the field has mass dimension $\frac{1}{2}(d-2)$.

In these new variables, in the action

$$\mathcal{S}(\phi) = \mathcal{S}_{\mathrm{G}}(\phi) + \mathcal{V}_{\mathrm{I}}(\phi), \tag{8.6}$$

the quadratic part becomes (now omitting primes)

$$\mathcal{S}_{\mathrm{G}}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \, (\nabla_x \phi(x))^2 + \tfrac{1}{2} \Lambda^2 u_0 \int \mathrm{d}^d x \, \phi^2(x) \\ + \tfrac{1}{2} \int \mathrm{d}^d x \sum_{k=1} u_{k+1} \Lambda^{-2k} \nabla_x \phi(x) (-\nabla^2)^k \nabla_x \phi(x) \,. \tag{8.7}$$

More generally, a monomial $V_{n,k}$ contributing to $V(\phi)$, which involves n powers of ϕ and $2k$ derivatives acting in an unspecified way on the fields ϕ, is transformed into

$$V_{n,k}(\phi) \mapsto \Lambda^{d-2k-n(d-2)/2} V_{n,k}(\phi). \tag{8.8}$$

One recognizes in the power of Λ the opposite of the dimension (6.36) of the vertex in the power counting analysis.

8.2.2 Summary and consequences

Quantities have now dimensions characterized by powers of the parameter Λ. Momenta acquire a dimension 1, position coordinates dimension -1, the field ϕ has dimension $\frac{1}{2}(d-2)$ and all local monomials in the action get a dimension in units of Λ. In particular, interaction vertices acquire the dimension $[V]$ assigned to them by the power counting analysis and are multiplied by $\Lambda^{-[V]}$.

This has several consequences on the interpretation of the terms in the action, which can also be rephrased in terms of a stability of the massless theory (the critical theory in the terminology of phase transitions, also a Gaussian fixed point) from the point of view of renormalization group, with respect to local perturbations.

In particular, we observe that the contributions in the action which are the most innocuous from the viewpoint of power counting now grow when Λ increases, while the 'dangerous ones', corresponding to non-renormalizable interactions, are the most suppressed.

The marginal situation corresponds to strictly renormalizable interactions, which are dimensionless, and whose RG flow with Λ, even close to the free field theory, cannot be predicted by this leading order analysis but requires studying renormalization group equations.

8.2.3 Mass term and fine tuning problem

The coefficient of ϕ^2 (in a free theory the mass term) is multiplied by Λ^2. This indicates that, generically, no small scale is generated and a scalar particle, if it exists, has a mass of the order of Λ. A physical mass $m \ll \Lambda$ requires a fine-tuning of the coefficient u_0 of ϕ^2 of the order of $(m/\Lambda)^2$.

We denote by u_{0c} the value of u_0 (negative for a ϕ^4 field theory) for which the theory is massless (or critical because the correlation length diverges) in the full theory,

$$u_0 = u_{0c} \;\Leftrightarrow\; \tilde{\Gamma}^{(2)}(p=0) = 0\,.$$

If we set

$$\Lambda^2 u_0 = \Lambda^2 u_{0c} + r\,, \tag{8.9}$$

where r characterizes the deviation from the massless theory, we note that since $u_0 - u_{0c} \mapsto r/\Lambda^2$, the factor $1/\Lambda^2$ cancels the Gaussian renormalization.

8.2.4 Interactions

Super-renormalizable interactions. In $d = 3$ dimensions, the ϕ^4 interaction is super-renormalizable and the coefficient of ϕ^4 is proportional to Λ. If only the ϕ^2 coefficient can be tuned, the interaction diverges with the cut-off. This is the theory one has to investigate to describe critical phenomena in continuous phase transitions if only the temperature can be adjusted.

Renormalizable interactions. This is the situation of the ϕ^4 interaction in dimension 4. In this situation, the ϕ^4 coefficient has no cut-off dependence in this leading order analysis. The same applies to the ϕ^3 interaction in six dimensions and to the ϕ^6 interaction in three dimensions.

A more detailed study based on renormalization group equations shows that, beyond leading order, the effective interaction strength then has a slow logarithmic behaviour when the ratio between the cut-off and the physical momentum scale increases. A one-loop calculation determines whether the effective coupling increases or decreases.

Non-renormalizable interactions. Non-renormalizable interactions, like the ϕ^6 interaction in six dimensions, which are obstacles in renormalization theory, appear in the effective field theory framework quite innocuous because they are suppressed by powers of the cut-off. They lead to very weak interactions. Moreover, when iterated in perturbation theory, though they lead to increasing divergences, these divergences are cancelled by the cut-off factors. Actually one can show, as indicated in the preceding example, that the contributions that do not vanish for infinite cut-off simply renormalize the parameters of the renormalizable part of the action and the other ones give small additional perturbative contributions, which vanish for infinite cut-off.

These results can be proved by studying the renormalization of local polynomials of the fields, also called composite operators.

Warning. This analysis is based on perturbation theory. Its validity beyond perturbation theory relies on the assumption that the perturbative effects do not modify qualitatively power counting. This issue will be discussed when we describe renormalization group (RG) properties.

8.3 Renormalization group and Gaussian fixed point

The preceding analysis can be reformulated in the RG terminology. The scale transformation, analogous to the scale transformation (8.5),

$$\phi(x) \mapsto \lambda^{(2-d)/2}\phi(x/\lambda) \tag{8.10}$$

where $\lambda > 0$ is a scale factor, can be considered as an RG transformation. The action of the free massless scalar field is invariant under the transformation and is thus a fixed point, the Gaussian fixed point.

The scaling behaviour of all other local terms in the action, dictated by power counting, determines the local stability of the Gaussian fixed point for $\lambda \to \infty$, that is, at large distance.

8.3.1 Quadratic terms

In d dimensions, the mass dimension of the monomial

$$\int \mathrm{d}^d x \, \phi(x)(\nabla_x)^{2k}\phi(x)$$

is $-d+(d-2)+2k = 2k-2$ and, thus, the monomial is multiplied by a factor λ^{2-2k}. The term $\phi^2(x)$ $(k=0)$ corresponds to a direction of instability, as we have already seen in expression (8.7). It is called a relevant operator.

The term with $k=1$ only renormalizes the field and has no effect. It is called redundant.

All terms with $k>1$ correspond to directions of stability and are called irrelevant. This analysis indicates the origin of the insensitivity of the large distance physics to the explicit form of the regularization.

8.3.2 Interactions

We specialize here to theories that have a $\phi \mapsto -\phi$ reflection symmetry.

Quartic terms. The interactions of the smallest degree in ϕ are thus quartic. Power counting is given by equation (8.8) and yields a factor λ^{4-d-2k}.

To $k = 0$ corresponds the ϕ^4 interaction and a power λ^{4-d}. The scaling properties of the ϕ^4 thus depend on space dimensions. For $d > 4$, it is irrelevant. For $d < 4$, it is relevant (a situation encountered in the theory of macroscopic phase transitions).

For $d = 4$, the situation of interest for particle physics it is marginal. Whether the Gaussian fixed point is then marginally stable or marginally unstable, requires an analysis that takes into account the ϕ^4 interaction beyond leading order, even for weak coupling.

For $k = 1$, one finds λ^{2-d} and the interactions are irrelevant for $d > 2$ and marginal for $d = 2$. More derivatives lead to irrelevant operators.

Higher degree in ϕ. The ϕ^6 interaction is irrelevant for $d > 3$, marginal at $d = 3$ (a situation met in phase transitions at tricritical points where the ϕ^4 interaction is suppressed) and relevant for $d = 2$.

All interactions with two derivatives are irrelevant except for $d = 2$ where they are marginal. Higher powers of ϕ lead to interactions that are irrelevant in all dimensions except for $d = 2$ where they are relevant.

8.3.3 Example: the ϕ^4 interaction in $d > 4$

We have stated that interactions that are multiplied by negative powers of the cut-off, at higher orders generate corrections non-vanishing with the cut-off that renormalize the relevant or marginal terms in the action together with additional corrections that again vanish with the cut-off. To illustrate this properties we consider the ϕ^4 interaction in $d > 4$.

The action reads

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\bigl(\nabla_x \phi(x)\bigr)^2 + \tfrac{1}{2}(\Lambda^2 u_{0c}(g) + r)\phi^2(x) + \tfrac{1}{4!} g \Lambda^{4-d} \phi^4(x)\right]$$
$$+ \text{regularization}\,, \tag{8.11}$$

where u_{0c} is defined by the condition that the theory is massless for $r = 0$.

The two-point vertex function. In the momentum representation,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + r + \tfrac{1}{2} g \Lambda^{4-d} \left[\Omega_d(r) - \Omega_d(0)\right] + O(g^2),$$

where $\Omega_d(r)$ is given by the diagram of figure 8.1,

$$\Omega_d(r) = \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d p}{p^2 + r}\,.$$

(The cut-off regularization is indicated symbolically by an upper-bound on the momentum integral.)

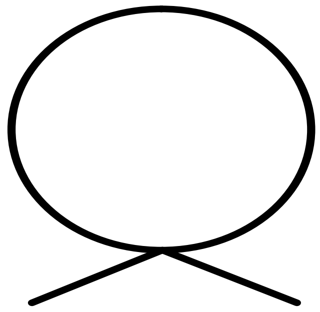

Fig. 8.1 – One-loop ϕ^4 contribution to the two-point function.

Then,

$$\Omega_d(r) - \Omega_d(0) = -\frac{r}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d p}{p^2(p^2+r)}.$$

For $d > 4$, the integral is dominated by large momenta and the leading contribution is proportional to Λ^{d-4}. For example, for $4 < d < 6$,

$$\Omega_d(r) - \Omega_d(0) = C(d) r \Lambda^{d-4} + K(d) r^{d/2-1} + \cdots \ ,$$

where $C(d)$ is regularization dependent and $K(d)$ universal. Then,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + r + \tfrac{1}{2} g r \left[C(d) + K(d)(r/\Lambda^2)^{d/2-2} \right] + \cdots \ .$$

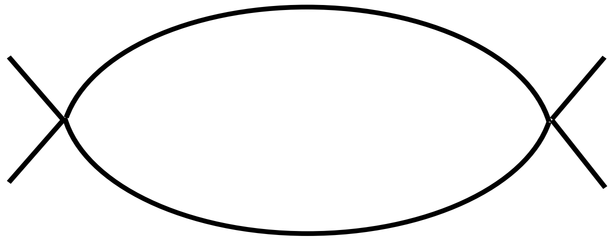

Fig. 8.2 – One-loop contribution to the four-point function.

The first term is simply a renormalization of the parameter r while the second term is a new singular contribution but which is suppressed by a factor Λ^{4-d}.

Similarly, one verifies that the leading contribution of order g^2 (figure 8.2) to the four-point function is proportional to

$$g^2\Lambda^{8-2d}\int^{\Lambda}\frac{\mathrm{d}^d p}{(p^2+r)\big((p+k)^2+r\big)}=g^2\Lambda^{4-d}\int^{1}\frac{\mathrm{d}p}{p^4}+O\left(g^2\Lambda^{8-2d}\right).$$

The first term yield a finite, regularization-dependent renormalization of the quartic interaction, while the second one vanishes faster with Λ.

For $d > 4$, we verify that, after tuning of the coefficient of $\phi^2(x)$, the free theory is stable and that the effect of the quartic interaction is to modify (renormalize) the parameters of the initial action and to generate new corrections that are suppressed at least by a factor Λ^{4-d}.

By contrast, for $d < 4$, the two momentum integrals converge but the diagrams are multiplied by powers of the divergent factors $g\Lambda^{4-d}$. This is a reflection of the instability of the Gaussian fixed point in presence of the relevant ϕ^4 interaction. We will indicate later that large distance physics is then governed by another non-Gaussian IR fixed point.

In the critical dimension four, the cut-off dependence is no longer generated by the coupling constant but, instead, is generated by the divergences of Feynman diagrams and takes the form of powers of $\ln \Lambda$, the degree increasing with the order in g. This marginal situation is examined separately in the next section.

8.4 Renormalization theorem and renormalization group

We do not describe here how the results of renormalization theory can be derived but only borrow them.

8.4.1 Renormalization theorem

One considers a quantum field theory renormalizable by power counting. One assumes that the action includes all local monomials allowed by power counting and possible symmetries.

Then, one can prove that, in presence of a suitable regularization, the parameters of the Lagrangian (this includes a field renormalization) can be adjusted order by order in a loop expansion in such a way that all correlation functions (called renormalized) have an infinite cut-off limit. Moreover, the limit is independent of the regularization scheme.

In the example of the initial action (8.11) in $d = 4$,

$$\mathcal{S}(\phi) = \int \mathrm{d}^4 x \left[\tfrac{1}{2}\bigl(\nabla_x \phi(x)\bigr)^2 + \tfrac{1}{2}(\Lambda^2 u_{oc}(g) + r)\phi^2(x) + \tfrac{1}{4!} g\phi^4(x)\right] + \text{ reg.}\,, \tag{8.12}$$

the renormalized action takes the form

$$\begin{aligned}\mathcal{S}_{\mathrm{r}}(\phi) = \int \mathrm{d}^4 x \Bigl[\tfrac{1}{2} Z(\Lambda)\bigl(\nabla_x \phi(x)\bigr)^2 + \tfrac{1}{2}\bigl(\Lambda^2 u_{oc}(g) + r(\Lambda)\bigr) Z(\Lambda)\phi^2(x) \\ + \tfrac{1}{4!} Z^2(\Lambda) g(\Lambda)\phi^4(x)\Bigr] + \text{ regularization}\,.\end{aligned}$$

It involves three, cut-off dependent, renormalization constants multiplying the three independent monomials in the action. They have to be adjusted order by order in a loop expansion to generate finite correlation or vertex functions (denoted below by $\Gamma_{\mathrm{r}}^{(n)}$) that are finite in the infinite cut-off limit.

The algebraic part of the proof is based on a recursion on the number of loops and the properties of the generating functional $\Gamma(\varphi)$ of vertex (one-line or one-particle irreducible) functions.

The renormalized action is determined in the following way: one assumes that all divergences have been cancelled up to loop order $(L-1)$ by adding the proper counter-terms to the action. One then evaluates the remaining divergences of vertex functions at loop order L. We denote by $\Gamma_L^{\text{div}}(\varphi)$ the generating functional of the L loop divergent contributions ($\Gamma_L^{\text{div}}(\varphi)$ is defined up a finite part).

One proves that it involves, in general, all local monomials consistent with power counting and symmetries. One adds to the $(L-1)$ loop renormalized action $-\Gamma_L^{\text{div}}(\phi)$. The action remains local. Using the relation between $\Gamma(\varphi)$ and the action at leading order, one verifies that vertex functions are then finite at L loop order and, moreover, that $\Gamma_{L+1}^{\text{div}}(\varphi)$ is again local.

8.4.2 One-loop calculations: the massless theory

At one-loop order, the two-point function (Fig. 8.1) is given by

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \tfrac{1}{2}\Omega_4(0)g + O(g^2)$$

with

$$\Omega_4(0) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p\, \tilde{\Delta}(p)$$

and

$$\tilde{\Delta}(p) = \frac{1}{p^2 + O(p^4/\Lambda^2)}\,.$$

The one-loop correction is a constant, which implies $Z = 1 + O(g^2)$. The counter-term is determined by imposing the massless condition $\tilde{\Gamma}^{(2)}_{\mathrm{r}}(0) = 0$ and yields a contribution to $u_{0c}(g)$,

$$\Lambda^2 u_{0c}(g) = -\tfrac{1}{2}\Omega_4(0)g + O(g^2).$$

The four-point function at one-loop is given by (see Fig. 8.2)

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) = g - \tfrac{1}{2}g^2 \left[B(p_1 + p_2) + B(p_1 + p_3) + B(p_1 + p_4)\right]$$

with

$$B(p) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 q \, \tilde{\Delta}(q) \tilde{\Delta}(p - q).$$

For $\Lambda \to \infty$, at leading order the momentum p can be neglected and

$$B(p) \sim \frac{1}{(2\pi)^4} \int^{\Lambda} \frac{\mathrm{d}^4 q}{q^4} \sim \frac{1}{8\pi^2} \ln \Lambda \, .$$

One infers

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) \underset{\Lambda \to \infty}{=} g - \frac{3g^2}{16\pi^2} \ln \Lambda + O(g^2 \times 1) + O(g^3). \tag{8.13}$$

This divergent contribution determines the coefficient of ϕ^4 at one-loop order, up to a finite part, as

$$g(\Lambda) = g + \frac{3}{16\pi^2} g^2 \ln \Lambda \, .$$

8.4.3 Renormalization: the massless or critical theory

We consider the action (8.12). The renormalization theorem can be reformulated most conveniently in terms of vertex functions. We first restrict the discussion to the massless (critical) theory $r = 0$.

We introduce a momentum scale $\mu \ll \Lambda$, called the renormalization scale and a parameter g_{r} characterizing the effective interaction at scale μ, called the renormalized interaction.

One can then find two functions $Z(\Lambda/\mu, g_{\mathrm{r}})$ and $Z_g(\Lambda/\mu, g_{\mathrm{r}})$, which satisfy

$$g = Z_g(\Lambda/\mu, g_{\mathrm{r}})g_{\mathrm{r}} = g_{\mathrm{r}} + O(g_{\mathrm{r}}^2), \quad Z(\Lambda/\mu, g_{\mathrm{r}}) = 1 + O(g_{\mathrm{r}}), \tag{8.14}$$

calculable order by order in an expansion in powers of g_{r}, such that, order by order, all renormalized vertex functions

$$\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) = Z^{n/2}(g_{\mathrm{r}}, \Lambda/\mu)\tilde{\Gamma}^{(n)}(p_i; g, \Lambda), \tag{8.15}$$

have finite limits $\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu)$ when $\Lambda \to \infty$ at p_i, μ, g_{r} fixed.

Remarks

(i) There is some arbitrariness in the choice of the renormalization constants Z and Z_g since they can be multiplied by arbitrary finite functions of $g_{\rm r}$. The constants can be completely determined, for example, by renormalization conditions. One can, for example, impose the three conditions (the first one being the criticality or massless condition),

$$\tilde{\Gamma}^{(2)}_{\rm r}(p=0,\mu,g_{\rm r})=0\,,\quad \frac{{\rm d}}{{\rm d}p^2}\tilde{\Gamma}^{(2)}_{\rm r}(p=\mu,\mu,g_{\rm r})=1\,,$$
$$\tilde{\Gamma}^{(4)}_{\rm r}(p_i=\mu\theta_i,\mu,g_{\rm r})=g_{\rm r}\ \text{with}\ \sum_i\theta_i=0\,,\ \theta_i\cdot\theta_j=\tfrac{4}{3}\delta_{ij}-\tfrac{1}{3}\,. \quad (8.16)$$

One verifies that $\tilde{\Gamma}^{(n)}_{\rm r}$ has dimension $4-n$ (equation (8.21)). Thus, $\tilde{\Gamma}^{(2)}_{\rm r}$ has dimension 2 and $\tilde{\Gamma}^{(4)}_{\rm r}$ is dimensionless in such a way that $g_{\rm r}$ is dimensionless.

Then, one proves, order by order in an expansion in powers of $g_{\rm r}$, that the functions $\tilde{\Gamma}^{(n)}_{\rm r}$ are unique, that is, independent of the specific regularization scheme.

(*ii*) In the traditional terminology of quantum field theory, the parameters or correlation functions of the initial theory are called bare parameters or bare correlation functions.

(*iii*) The renormalization momentum scale μ is chosen to correspond to the momentum or energy scale at which phenomena are measured and the parameter g_{r} plays the role of the effective quartic interaction $g(\Lambda/\mu)$, where Λ/μ is the ratio between the momentum scale of the initial microscopic model and the physical momentum scale.

(*iv*) Beyond perturbation theory, the interpretation of renormalized correlation functions is subtle. In formal renormalization theory, the parameters of the renormalized theory are fixed, and the initial parameters, which are the coefficients of all operators relevant or marginal with respect to the Gaussian fixed point, are adjustable parameters and thus vary when the scale factor Λ/μ varies. This corresponds to a generalized fine-tuning, generalizing the required fine-tuning of the coefficient of ϕ^2.

However, the possibility of such a systematic fine-tuning and its physical relevance have to be investigated. The study of the RG equations will provide some information about this question.

Finally, of course, the renormalization theorem has implications for the large-distance behaviour of statistical models only if the property of renormalizability remains true at fixed dimension $d < 4$, without fine-tuning of the quartic interaction.

8.5 Renormalization group equations for the massless theory

In the framework of perturbation theory, that is, in an expansion in powers of the parameter g, the large distance behaviour of correlation functions differs from the tree level behaviour only by powers of logarithms. These logarithms are organized by the RG equations.

8.5.1 RG equations

From equation (8.15) and the existence of a limit $\Lambda \to \infty$, a new equation obtained by differentiation of the equation with respect to Λ at μ, g_{r} fixed, follows:

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu \text{ fixed}} Z^{n/2}(g, \Lambda/\mu)\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = o(\Lambda^{-2+\upsilon}), \quad 0 < \upsilon \ll 1\,, \tag{8.17}$$

where the renormalization factor Z has been expressed in terms of g instead of g_{r}.

In agreement with the perturbative philosophy, one first neglects all contributions that, order by order, decay as powers of Λ. Thus, we define asymptotic functions $\tilde{\Gamma}^{(n)}_{\text{as.}}(p_i; g, \Lambda)$ and $Z_{\text{as.}}(g, \Lambda/\mu)$ as sums of the perturbative contributions to the functions $\tilde{\Gamma}^{(n)}(p_i; g, \Lambda)$ and $Z(g, \Lambda/\mu)$, respectively, that do not go to zero when $\Lambda \to \infty$.

One can show that such correlation functions can also be obtained by adding to the action all possible irrelevant terms and adjusting order by order their amplitudes as functions of g in order to eliminate systematically the contributions that go to zero.

The asymptotic functions then satisfy equation (8.17) with a right-hand side that vanishes exactly. Using the chain rule, one derives from equation (8.17)

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g,\Lambda/\mu)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g,\Lambda/\mu)\right]\tilde{\Gamma}^{(n)}_{\text{as.}}(p_i;g,\Lambda)=0\,, \tag{8.18}$$

where the functions β and η are defined by

$$\beta(g,\Lambda/\mu)=\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_{\rm r},\mu}g\,, \tag{8.19a}$$

$$\eta(g,\Lambda/\mu)=-\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_{\rm r},\mu}\ln Z_{\text{as.}}(g,\Lambda/\mu). \tag{8.19b}$$

Since the two functions are dimensionless, they can depend on g, Λ, μ only through the dimensionless combinations g and Λ/μ. Moreover, the functions β and η can be calculated directly, by using equation (8.18), in terms of the initial vertex functions, which do not depend on μ. Thus, the functions β and η cannot depend on Λ/μ.

Equation (8.18) then takes the simpler form (Zinn-Justin 1973)

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\eta(g)\right)\tilde{\Gamma}^{(n)}_{\text{as.}}(p_i;g,\Lambda)=0\,. \tag{8.20}$$

Formulated in terms of the cut-off Λ, the fundamental idea of the RG becomes: it is possible to modify the cut-off Λ and in a correlated way the normalization of the field ϕ and the coefficients of all interactions in a way that leaves all correlation functions invariant.

Equation (8.20) is satisfied by the functions $\tilde{\Gamma}^{(n)}$ asymptotically in the limit $|p_i| \ll \Lambda$. At all orders in the expansion in g the neglected terms are of the form $(\ln \Lambda)^L/\Lambda^2$, where the degree L increases with the order.

Notation. In what follows, the vertex functions are implicitly the functions $\tilde{\Gamma}^{(n)}_{\text{as.}}$ without corrections, which satisfy the RG equations exactly and we omit the subscript 'as.'.

8.5.2 Dimension of vertex functions

We have stated that it is convenient to discuss renormalization and renormalization group equations in terms of vertex functions. To solve the RG equations, we need the mass dimensions of the Fourier components of vertex functions.

In generic space dimension d, the dimension $[\tilde{\phi}]$ of the Fourier components $\tilde{\phi}$ of the field can be inferred from the relation

$$\phi(x) = \int \mathrm{d}^d p\, \mathrm{e}^{ipx}\, \tilde{\phi}(p) \quad \Rightarrow \quad [\tilde{\phi}] = -(d+2)/2\,.$$

The vertex functions $\tilde{\Gamma}^{(n)}$ are obtained by expanding the generating functional in powers of the field Fourier components $\tilde{\varphi}(p) = \langle \tilde{\phi}(p) \rangle$:

$$\Gamma(\varphi) = \sum_n \frac{1}{n!} \int \mathrm{d}^d p_1 \ldots \mathrm{d}^d p_n \, \delta^{(d)}\Big(\sum_i p_i\Big) \tilde{\varphi}(p_1) \ldots \tilde{\varphi}(p_n) \tilde{\Gamma}^{(n)}(p_1, \ldots, p_n).$$

One infers the dimension

$$[\tilde{\Gamma}^{(n)}] = -nd + n(d+2)/2 + d = d - n(d-2)/2 \,. \tag{8.21}$$

Equation (8.20) can then be written in a different form. Indeed, the dimensional relation (8.21) implies

$$\Lambda \frac{\partial}{\partial \Lambda} + \sum_i p_i \frac{\partial}{\partial p_i} = d - \frac{n}{2}(d-2)$$

and, thus,

$$\left(d - \sum_i p_i \frac{\partial}{\partial p_i} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} [d - 2 + \eta(g)] \right) \tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = 0 \,.$$

8.6 Renormalized correlation functions and RG

The renormalized vertex functions (8.15) and the initial vertex functions are by definition proportional and, thus, have the same small momentum behaviour.

The renormalized functions also satisfy RG equations that are derived from equation (8.15) (in which the initial correlation functions and the renormalized functions play almost a symmetric role) by differentiating with respect to μ at Λ and g fixed.

Moreover, the calculation of renormalized functions, within the framework of dimensional regularization and minimal subtraction scheme, is easier.

However, studying only renormalized functions does not allow determining the behaviour and the range of the parameters of the renormalized theory as functions of the initial parameters for large cut-off or the nature of all corrections to the asymptotic behaviour.

8.6.1 RG equations for renormalized vertex functions

One rewrites equation (8.15) as

$$Z^{-n/2}(g_{\mathrm{r}},\Lambda/\mu)\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i;g_{\mathrm{r}},\mu,\Lambda)=\tilde{\Gamma}^{(n)}(p_i;g,\Lambda), \tag{8.22}$$

and expresses that the right hand side does not depend on the renormalization scale μ:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{g,\Lambda\ \text{fixed}}\tilde{\Gamma}^{(n)}(p_i;g,\Lambda)=0\,.$$

The roles of g,Λ and g_{r},μ are thus formally interchanged and the algebraic transformations of section 8.5, in the large Λ limit, lead to an equation of the form

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_{\mathrm{r}}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}}-\frac{n}{2}\eta_{\mathrm{r}}(g_{\mathrm{r}})\right)\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i;g_{\mathrm{r}},\mu)=0\,. \tag{8.23}$$

After the limit $\Lambda\to\infty$ has been taken, the equation is exact. However, this assumes that the limit exists beyond perturbation theory, a non-trivial issue.

Note that the functions $\beta_{\rm r}$ and $\eta_{\rm r}$ differ from the functions β and η defined in (8.19) but coïncide at leading order for $g \to 0$.

In the renormalized scheme, functions can be calculated using dimensional regularization (see lecture 9) and the renormalized theory can also be precisely defined by minimal subtraction (or modified minimal subtraction) by subtracting from perturbative contributions the simple or multiple poles in $1/(d-4)$, without introducing renormalization conditions.

8.7 Solution of RG equations

The solution of equation (8.20), combined with perturbative calculations, allows determining the large distance behaviour of the effective interaction strength and of correlation functions.

Equation (8.20) can be solved, for example, by the method of characteristics. One introduces a dilatation parameter λ and one looks for two functions $g(\lambda)$ and $Z(\lambda)$ such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda/\lambda\big)\right] = 0\,. \tag{8.24}$$

By differentiating explicitly with respect to λ, one finds that equation (8.24) is consistent with equation (8.20) if

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = -\beta\big(g(\lambda)\big), \qquad g(1) = g\,; \tag{8.25a}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = -\eta\big(g(\lambda)\big), \qquad Z(1) = 1\,. \tag{8.25b}$$

Equations (8.24) and (8.25) then imply

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda/\lambda\big). \tag{8.26}$$

Remark. The equation becomes analogous to equation (8.15) if one chooses $\lambda = \Lambda/\mu$ and if one identifies $g_{\rm r}$ with $g(\Lambda/\mu)$, the effective interaction at scale μ, and $Z(\Lambda/\mu)$ with the renormalization of the field. One may then wonder why it was necessary to introduce the partial differential equations.

The main reason is the following: it allows showing that the coefficients of the RG equations do not depend on the ratio μ/Λ, in contrast with the renormalization constants. This implies properties of renormalization constants that are contained in equation (8.15) only implicitly. This also proves that the flow equations (8.25) are independent of λ, a property that is important to allow for fixed points.

Other forms. First, it is convenient to multiply Λ by a factor λ in equation (8.26):

$$\tilde{\Gamma}^{(n)}(p_i; g, \lambda\Lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\big). \tag{8.27}$$

The dimensional considerations of section 8.5.2, in particular the relation (8.21), imply

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda\lambda) = \lambda^{4-n}\tilde{\Gamma}^{(n)}(p_i/\lambda; g, \Lambda).$$

Then, equation (8.27) can be rewritten as

$$\tilde{\Gamma}^{(n)}(p_i/\lambda; g, \Lambda) = \lambda^{n-4}Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}\big(p_i; g(\lambda), \Lambda\big). \tag{8.28}$$

The appearance of the scale factor $Z^{1/2}(\lambda)/\lambda$ reflects the meaning of $Z^{1/2}(\lambda)$, which is the ratio between the complete scaling and the Gaussian scaling transformation (8.10). It also corresponds to the initial renormalization of the field ϕ performed in section 8.2, which has the form of the field RG transformation adapted to the Gaussian fixed point.

Equations (8.25) and (8.28) realize asymptotically (because terms sub-leading by powers of Λ have been neglected) the general ideas of the RG.

The parameter $g(\lambda)$ characterizes the effective action $\mathcal{S}_\lambda$ at scale λ and, therefore, equation (8.25a) governs the flow of interactions.

8.7.1 Integrated RG flow equations

Integrating equations (8.25), one finds

$$\int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\beta(g')} = -\ln\lambda\,, \tag{8.29a}$$

$$\int_1^{\lambda} \frac{\mathrm{d}s}{s}\eta\bigl(g(s)\bigr) = -\ln Z(\lambda). \tag{8.29b}$$

Equation (8.20) is the RG equation in differential form. Equations (8.28) and (8.29) are the integrated RG equations. In what follows, we assume explicitly that the RG functions, $\beta(g)$ and $\eta(g)$, are regular functions of g, for $g > 0$.

In equation (8.27), one notes that it is equivalent to increase Λ or λ. To study the limit $\Lambda \to \infty$, one must thus determine the behaviour of the amplitude $g(\lambda)$ of the effective interaction for $\lambda \to \infty$. Equation $(8.29a)$ shows that $g(\lambda)$ increases if the β-function is negative, or decreases in the opposite case. Fixed points correspond to zeros g^* of the β-function which, therefore, play an essential role in the study of the large distance behaviour. Those for which the function β has a negative slope are repulsive fixed points in the IR: $g(\lambda)$ moves away from such zeros, except if initially $g(1) = g^*$. On the contrary, those for which the slope is positive are attractive fixed points from the viewpoint of the large-distance behaviour.

8.8 RG functions at one-loop order and triviality of ϕ^4_4

The RG functions β and η can be calculated perturbatively by expressing that the two- and four-point functions satisfy the RG equations (8.20). For the two- and four point functions, they respectively read

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\eta(g)\right)\tilde{\Gamma}^{(2)}_{\text{as.}}(p;g,\Lambda)=0\,,$$
$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-2\eta(g)\right)\tilde{\Gamma}^{(4)}_{\text{as.}}(p_i;g,\Lambda)=0\,.$$

8.8.1 One-loop calculations

The one-loop correction to the two-point function (Fig. 8.1) is a constant and induces only a modification of the critical parameter $u_{0\mathrm{c}}$ in such a way that

$$\tilde{\Gamma}^{(2)}(p)=p^2+O(g^2).$$

The four-point function at one-loop given by (see Fig. 8.2)

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) = g - \tfrac{1}{2} g^2 \left[B(p_1 + p_2) + B(p_1 + p_3) + B(p_1 + p_4) \right] + O(g^3)$$

for $\Lambda \to \infty$ is given by equation (8.13):

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) \underset{\Lambda \to \infty}{=} g - \frac{3g^2}{16\pi^2} \ln \Lambda + O(g^2 \times 1) + O(g^3).$$

Inserting the two expansions in the respective RG equations, one first concludes $\eta(g) = O(g^2)$, then

$$-\frac{3g^2}{16\pi^2} + \beta(g)\big(1 + O(g)\big) + O(g^3) = 0$$

and, thus,

$$\beta(g) = \frac{3g^2}{16\pi^2} + O(g^3).$$

8.8.2 The problem of triviality

Assuming that initially the constant g is small enough, we can integrate the flow equation (8.29a) and find

$$g(\lambda) = \frac{g}{1 + (3g/16\pi^2)g \ln \lambda}$$

and thus, for $\lambda \to +\infty$,

$$g(\lambda) \sim \frac{16\pi^2}{3 \ln \lambda}\,.$$

Therefore, because the leading term of the β-function is positive, the Gaussian fixed point is marginally stable. More generally, if the β-function remains positive for $g > 0$, $g(\lambda)$ is a decreasing function of λ and $g(\lambda)$ converges toward zero for $\lambda \to \infty$, indicating a screening effect.

Although such a result has not been established rigorously yet, there is much numerical evidence that this is the right scenario.

If this situation is realized, for any initial value of the interaction parameter g, the effective interaction when $\Lambda/\mu \to \infty$ vanishes and the infinite cut-off limit is a free field theory. This is the so-called triviality problem. As a consequence, a renormalized quantum field theory with a ϕ^4 interaction, which would be a field theory consistent for all scales, does not exist.

However, from the viewpoint of effective field theories, this is an acceptable scenario. Indeed, the cut-off scale, insofar it represents the scale of some new microscopic physics, is not infinite and the field theory is not required to remain consistent up to the cut-off scale.

The main prediction of this analysis is that for a cut-off large but finite, one expects the effective interaction to be small, but since the decrease is only logarithmic, it will remain much larger than the interactions due to non-renormalizable (irrelevant) interactions.

For example, under the extreme assumption that Λ is Planck's scale and $\mu = 100$ GeV, one finds

$$\frac{g(\Lambda/\mu)}{8\pi^2} \sim \frac{2}{3\ln(\Lambda/\mu)} \approx 0.017\,,$$

where, to give a better appreciation of the magnitude of the parameter g, we have divided the coupling constant by the loop factor $8\pi^2 = S_4/(2\pi)^4$, S_4 being the area of the sphere in four dimensions.

From the point of view of effective field theories, the existence of a consistent renormalized quantum field theory on all scales may be an interesting mathematical problem but with little physics relevance.

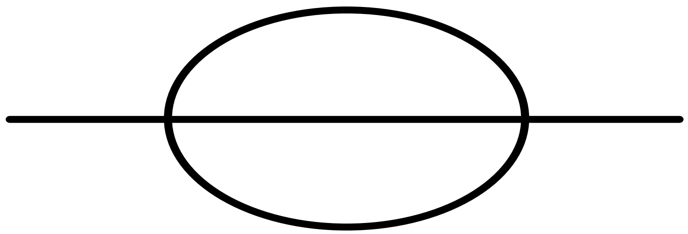

Fig. 8.3 – Two-loop contribution to the two-point function

8.8.3 The function $\eta(g)$ at two loops

A somewhat longer calculation of the two-loop diagram displayed in Fig. 8.3 yields the leading contribution of order g^2 to $\eta(g)$. The diagram contributing to $\tilde{\Gamma}^{(2)}(p)$ is given by (Some elements of the calculation can be found in section 9.2.2.)

$$\tilde{\Gamma}_2^{(2)}(p) = -\frac{1}{6(2\pi)^8} \int \mathrm{d}^4 q_1 \, \mathrm{d}^4 q_2 \, \tilde{\Delta}(q_1)\tilde{\Delta}(q_2)\tilde{\Delta}(p - q_1 - q_2).$$

The Feynman diagram can be calculated more easily by using the propagator in position variables.

It then takes the form

$$\tilde{\Gamma}_2^{(2)}(p) = -\tfrac{1}{6}\int \mathrm{e}^{ipx}\,\Delta^3(x)\mathrm{d}^4x\,.$$

As a function of position variables, the propagator with a cut-off can be parametrized as

$$\Delta(x) = \frac{\vartheta(\Lambda x)}{4\pi^2 x^2}\,, \tag{8.30}$$

where the large-distance behaviour is determined by the massless free propagator and the cut-off ensures regularity at short distance:

$$\lim_{x\to\infty}\vartheta(x) = 1\,, \quad \vartheta(x) \underset{x\to 0}{\propto} x^2.$$

The condition $\tilde{\Gamma}^{(2)}(0) = 0$ determines u_{0c} at order g^2. The vertex function then takes the form

$$\tilde{\Gamma}^{(2)}(p) = p^2 - \frac{g^2}{6}Kp^2\ln(\Lambda/p) + O(g^2\times 1 + O(g^3)).$$

The coefficient K is given by

$$K = \frac{\partial}{\partial p^2} \Lambda \frac{\partial}{\partial \Lambda} \int \mathrm{e}^{ipx}\, \Delta^3(x) \mathrm{d}^4 x \Big|_{p=0} .$$

Notice the identity

$$\sum_{\mu=1}^{4} \left(\frac{\partial}{\partial p_\mu} \right)^2 \Phi(p^2) = 8\Phi'(p^2) + 4p^2 \Phi''(p^2).$$

Thus,

$$K = -\frac{1}{8(4\pi^2)^3} \int \frac{\mathrm{d}^4 x}{x^4} \Lambda \frac{\partial}{\partial \Lambda} \vartheta^3(\Lambda x) = -\frac{1}{(4\pi)^4} \int_0^\infty \mathrm{d}x \, \frac{\partial}{\partial x} \vartheta^3(\Lambda x).$$

The integrand is an explicit derivative. Only $x = \infty$ contributes and the result thus is independent of the cut-off function ϑ. One finds

$$K = -\frac{1}{(4\pi)^4} .$$

One infers

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \frac{1}{24}\frac{g^2}{(8\pi^2)^2}p^2 \ln(\Lambda/p) + O(g^2 \times 1, g^3).$$

The RG equation (8.20) then implies

$$\frac{g^2}{6(4\pi)^4}p^2 - \eta(g)p^2 = 0 \quad \Rightarrow \quad \eta(g) = \frac{1}{6(4\pi)^4}g^2 + O(g^3). \tag{8.31}$$

Solving equation (8.29b), rewritten in the form

$$\ln Z(\lambda) = \int_g^{g(\lambda)} \mathrm{d}g' \frac{\eta(g')}{\beta(g')},$$

we see that $\ln Z(\lambda)$ has the limit for $\lambda \to \infty$, which due to the signs of β and η, is consistent with the general result $Z < 1$:

$$\lim_{\lambda\to\infty} Z(\lambda) = \exp\left[-\int_0^g \mathrm{d}g' \frac{\eta(g')}{\beta(g')}\right] < 1\,. \tag{8.32}$$

8.9 The massive ϕ^4 theory in the symmetric phase

We now study the massive phase $r > 0$. We thus modify the action:

$$\mathcal{S}(\phi) \mapsto \mathcal{S}(\phi) + \frac{r}{2} \int \mathrm{d}^4x \, \phi^2(x),$$

where r, the coefficient of ϕ^2, characterizes in statistical physics the deviation from the critical temperature: $r \propto T - T_\mathrm{c}$.

8.9.1 RG equations

The renormalization theorem generalizes to correlation functions of $\phi(x)$ and $\phi^2(x)$, and leads to the appearance of a new renormalization factor $Z_2(\Lambda/\mu, g_\mathrm{r})$ associated with the parameter r. By the same arguments as in the massless situation, one derives a more general RG equation of the form

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \eta_2(g) r \frac{\partial}{\partial r}\right] \tilde{\Gamma}^{(n)}(p_i; r, g, \Lambda) = 0 \,, \qquad (8.33)$$

where a new function $\eta_2(g)$ appears.

The additional term is proportional to r since it must vanish for $r = 0$. The dimensionless function η_2 may still have a regular dependence in the ratio r/Λ^2, but we have neglected such a possible dependence for the same reason that we have already neglected all other contributions of the same order.

To determine $\eta_2(g)$, one can calculate the two-point function and apply the RG equations. At order g, one finds

$$\tilde{\Gamma}^{(2)}(p) = p^2 + r + \frac{g}{32\pi^4} \int^{\Lambda} \mathrm{d}^4 q \left(\frac{1}{q^2 + r} - \frac{1}{q^2} \right) + \cdots \, .$$

Using

$$\frac{1}{32\pi^2} \int^{\Lambda} \mathrm{d}^4 q \left(\frac{1}{q^2 + r} - \frac{1}{q^2} \right) \sim -\frac{r}{16\pi^2} \ln(\Lambda/\sqrt{r}), \tag{8.34}$$

and applying the RG equations, one finds

$$-\frac{gr}{16\pi^2} - \eta_2(g) r = 0 \quad \Rightarrow \quad \eta_2(g) = -\frac{g}{16\pi^2} + O(g^2)\,. \tag{8.35}$$

8.9.2 Renormalized RG equations

Again a renormalized deviation r_{r} from the massless theory can be introduced and the analogue of equation (8.33) is obtained:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_{\mathrm{r}}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}}-\frac{n}{2}\eta_{\mathrm{r}}(g_{\mathrm{r}})-\eta_{2,\mathrm{r}}(g_{\mathrm{r}})r_{\mathrm{r}}\frac{\partial}{\partial r_{\mathrm{r}}}\right]\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i;r_{\mathrm{r}},g_{\mathrm{r}},\mu)=0\,. \tag{8.36}$$

The solutions of these RG equations then follows immediately from the corresponding solutions of equations (8.33).

8.9.3 Solution of RG equations

To study the behaviour of correlation functions of the massive theory in the unbroken phase, we integrate equation (8.33) by the method of characteristics, as we have done for previous RG equations. In addition to the functions $g(\lambda)$ and $Z(\lambda)$ of equations (8.29), one must now introduce a function $r(\lambda)$.

It is determined by imposing that equation (8.33) is consistent with equation

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}\big(p_i; r(\lambda), g(\lambda), \Lambda/\lambda\big) \right] = 0 \,. \tag{8.37}$$

The consistency condition is equivalent to the set of equations

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = -\beta\big(g(\lambda)\big), \qquad g(1) = g \,, \tag{8.38}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln r(\lambda) = \eta_2\big(g(\lambda)\big), \qquad r(1) = r \,, \tag{8.39}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = -\eta\big(g(\lambda)\big), \qquad Z(1) = 1 \,. \tag{8.40}$$

Dimensional analysis (see section 8.2.1), which reflects the action of the Gaussian renormalization group, implies

$$\tilde{\Gamma}^{(n)}\big(p_i; r(\lambda), g(\lambda), \Lambda/\lambda\big) = (\Lambda/\lambda)^{4-n} \tilde{\Gamma}^{(n)}\big(\lambda p_i/\Lambda; \lambda^2 r(\lambda)/\Lambda^2, g(\lambda), 1\big).$$

We are only interested in the region in parameter space for which the physical mass is much smaller than the cut-off, which implies $|r| \ll \Lambda^2$ and this is the source of the IR singular behaviour that is present in the perturbative expansion. We now assume that the equation

$$r(\lambda) = \Lambda^2/\lambda^2, \tag{8.41}$$

has a solution in λ. Then, after a dilatation of parameter λ, the physical mass is of order unity. This also corresponds, by relating the dilatation parameter λ and r, to choosing the initial value (for $\lambda = 1$) of the coefficient of the relevant term ϕ^2 in such a way that its value after dilatation is of order unity.

Combining equations (8.37)–(8.41), one finds

$$\tilde{\Gamma}^{(n)}(p_i; r, g, \Lambda) = Z^{-n/2}(\lambda) m^{4-n} \tilde{\Gamma}^{(n)}\big(p_i/m; 1, g(\lambda), 1\big), \tag{8.42}$$

where we have introduced the notation

$$m = \Lambda/\lambda\,. \tag{8.43}$$

The constant $Z(\lambda)$ goes to a limit (equation (8.32). The solution of equation (8.39) using equation (8.25a) can be written as

$$\ln r(\lambda) = \ln r + \exp\left[\int_1^\lambda \frac{\mathrm{d}\sigma}{\sigma}\eta_2\big(g(\sigma)\big)\right] = \ln r + \int_{g(\lambda)}^g \mathrm{d}g'\,\frac{\eta_2(g')}{\beta(g')}.$$

From the expansions

$$\beta(g) = \frac{3g^2}{16\pi^2} + O(g^3), \quad \eta_2(g) = -\frac{g}{16\pi^2} + O(g^2),$$

since the integral is dominated by the lower-bound and the behaviour of the integrand near zero, one infers

$$\ln\big(r(\lambda)/r\big) = \int_{g(\lambda)}^g \mathrm{d}g'\,\frac{\eta_2(g')}{\beta(g')} \sim \tfrac{1}{3}\ln g(\lambda) \;\Rightarrow\; \frac{r(\lambda)}{r} \propto \big(g(\lambda)\big)^{1/3} \propto (\ln\lambda)^{-1/3}$$

and thus, using equations (8.41) and (8.43),

$$m = \frac{\Lambda}{\lambda} \propto \frac{\sqrt{r}}{(\ln(\Lambda/\sqrt{r}))^{1/6}}\,.$$

8.9.4 Callan–Symanzik equations

Callan–Symanzik (CS) equations are useful, inhomogeneous, variants of the renormalized RG equations (8.36). They apply only to a massive (non-critical) theory but can thus be derived for dimensions $d \le 4$.

In $d = 4 - \varepsilon$ dimensions, the renormalized vertex functions are defined by the conditions

$$\tilde{\Gamma}_{\mathrm{r}}^{(2)}(p, m, g_r) = m^2 + p^2 + O(p^4), \quad \tilde{\Gamma}_{\mathrm{r}}^{(4)}(0,0,0,0) = m^{\varepsilon} g_{\mathrm{r}}\,,$$

where $m \ll \Lambda$ is proportional to the inverse of the correlation length and g_{r} is dimensionless.

One needs also the vertex (1PI) function $\Gamma^{(1,n)}$ associated to the correlation function (see also section 8.9.1)

$$\left\langle \tfrac{1}{2}\phi^2(y)\phi(x_1)\dots\phi(x_n)\right\rangle .$$

Its renormalization is determined by the condition

$$\tilde{\Gamma}_{\mathrm{r}}^{(1,2)}(0;0,0;g_{\mathrm{r}},m)=1\,.$$

Dimensional analysis then implies

$$\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i;g_{\mathrm{r}},m)=m^{d-n(d-2)/2}F_{+}^{(n)}(p_i/m),$$

a form consistent for $d=4$ with equation (8.42), since the missing factor Z factor is provided by the field renormalization (8.22).

One then proves the RG equations in the CS form

$$\begin{aligned}\left(m\frac{\partial}{\partial m}+\beta_{\mathrm{r}}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}}-\frac{n}{2}\eta_{\mathrm{r}}(g_{\mathrm{r}})\right)&\tilde{\Gamma}_{\mathrm{r}}^{(n)}(p_i;g_{\mathrm{r}},m)\\ &=\left(2-\eta_r(g_{\mathrm{r}})\right)m^2\tilde{\Gamma}_{\mathrm{r}}^{(1,n)}(0;p_i;g_{\mathrm{r}},m)\,,\end{aligned}\tag{8.44}$$

where the right hand side refers to the vertex function involving the expectation value of n fields ϕ and $\frac{1}{2}\int \mathrm{d}^d x\,\phi^2(x)$.

The advantage of the CS equations is that they can be derived even at fixed dimension $d < 4$ because the theory is massive while the massless theory suffers from perturbative IR divergences.

However, they are predictive only if the right hand side becomes negligible for large momenta $|p_i| \gg m$ (but still $|p_i| \ll \Lambda$), a property that can only be explicitly verified in four dimensions or within the framework of the ε-expansion.

With this assumption,

$$\left(m \frac{\partial}{\partial m} + \beta_{\rm r}(g_{\rm r}) \frac{\partial}{\partial g_{\rm r}} - \frac{n}{2} \eta_{\rm r}(g_{\rm r}) \right) \tilde{\Gamma}^{(n)}_{\rm r}(p_i; g_{\rm r}, m) \underset{|p_i| \gg m}{\longrightarrow} 0 \,. \tag{8.45}$$

This equation can then be solved in the same way as the homogeneous RG equation (8.33).

8.10 An early RG upper-bound on the Higgs particle mass

In the Standard Model, the Higgs field through its expectation value and its various couplings gives masses to all particles. The observed masses determine the corresponding couplings. Before the LHC discovery, only the Higgs mass and thus the Higgs self-coupling were unknown parameters. However, it was likely that the renormalized $(\phi^2)^2$ self-coupling g would be small enough so that perturbation theory remained at least semi-quantitatively applicable. Otherwise, the successes of the Standard Model would have been difficult to understand.

In the perturbative regime, the Higgs mass increases with g. To obtain an upper-bound on the Higgs mass, one has to examine what happens when g increases. For g large enough, the Higgs mass is mostly determined by the Higgs self-coupling. In the pure $(\phi^2)^2$ field theory and in the perturbative regime, simple RG arguments are applicable.

8.10.1 The $O(4)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory in the broken phase

We consider here a field theory that is more directly relevant to particle physics, where $\boldsymbol{\phi}$ is a four-component field and the action has an $O(4)$ symmetry, which is locally equivalent to the $SU(2)\times SU(2)$ chiral symmetry, and a $(\boldsymbol{\phi}^2)^2$ interaction. The unregularized action reads

$$\mathcal{S}(\boldsymbol{\phi}) = \int \mathrm{d}^4x \left[\tfrac{1}{2}\left(\nabla_x \boldsymbol{\phi}(x)\right)^2 + \tfrac{1}{2}(u_{0c}+r)\boldsymbol{\phi}^2(x) + \tfrac{1}{4!}g_0\left(\boldsymbol{\phi}^2(x)\right)^2\right].$$

The field expectation value $\mathbf{v} = \langle \boldsymbol{\phi} \rangle$ is given by the minimum of the action for constant fields. At leading order,

$$\mathcal{S}(\mathbf{v})/\mathrm{volume} = \tfrac{1}{2}r\mathbf{v}^2 + \tfrac{1}{4!}g_0(\mathbf{v}^2)^2.$$

Differentiating with respect to $\mathbf{v}$, one obtains the equation

$$\mathbf{v}\left(r + \tfrac{1}{6}g_0\mathbf{v}^2\right) = 0\,.$$

For $r<0$, the minimum of the potential is not reached for the $O(4)$ symmetric extremum at $\boldsymbol{\phi}=0$ but for the sphere

$$|\mathbf{v}|=\sqrt{-6r/g_0}\,.$$

The choice of a specific point on the sphere, that is, the direction of the vector $\mathbf{v}$, breaks the $O(4)$ symmetry. The region $r<0$ corresponds to a phase with spontaneous broken $O(4)$ symmetry.

Setting then

$$\boldsymbol{\phi}=\mathbf{v}+\boldsymbol{\chi}$$

and expanding the action up to second order in $\boldsymbol{\chi}$, one finds

$$\mathcal{S}(\boldsymbol{\chi})=\int \mathrm{d}^4x\,\left[\tfrac{1}{2}\left(\nabla_x\boldsymbol{\chi}(x)\right)^2+\tfrac{1}{6}g_0\left(\mathbf{v}\cdot\boldsymbol{\chi}(x)\right)^2+O(|\chi|^3\right].$$

One verifies that the perturbative spectrum consists in one massive particle, the component of $\boldsymbol{\chi}$ along $\mathbf{v}$ with mass $|\mathbf{v}|\sqrt{g_0/3}$, which in this context we identify with the Higgs particle, and 3 massless particles, the Nambu–Goldstone modes (see lecture 12 for details).

8.10.2 Beyond leading order: a bound on the mass of the Higgs particle

Any quantum field theory requires eventually a cut-off to cure the unavoidable problem of infinities. The cut-off Λ then corresponds to the onset of some new physics beyond the Standard Model.

The renormalized coupling constant g is the effective coupling constant $g_0(\mu/\Lambda)$ at the renormalization scale μ, $\mu \ll \Lambda$. Then,

$$g \sim g_0(\mu/\Lambda), \quad \int_{g_0}^{g_0(\mu/\Lambda)} \frac{\mathrm{d}g'}{\beta(g')} = \ln(\mu/\Lambda).$$

For g_0 small, the perturbative expansion of the β-function is

$$\beta(g_0) = \beta_2 g_0^2 + \beta_3 g_0^3 + O\left(g_0^4\right), \quad 8\pi^2\beta_2 = 2\,, \beta_3/\beta_2^2 = -13/24\,.$$

For Λ/μ large, the effective coupling g at scale μ is small because β_2 is positive if the β-function remains positive for all values of g_0.

Then,

$$\ln(\Lambda/\mu) = \frac{1}{\beta_2 g} + \frac{\beta_3}{\beta_2^2} \ln g + K(g_0) + O(g),$$

where $K(g_0) = O(1)$ can only be determined by non-perturbative methods.

For g small, perturbation theory relates the Higgs field expectation value, which is known from the Z mass ($|\mathbf{v}| \equiv \langle H \rangle \sim 250\,\mathrm{GeV}$), and the Higgs mass. At leading order, one finds

$$m_H^2 = \tfrac{1}{3} g \mathbf{v}^2 + O\left(g^2\right).$$

To minimize higher order corrections, one chooses for g the renormalized coupling constant at scale $\langle \phi \rangle$. One can then eliminate g and finds

$$\ln\left(\frac{\Lambda}{|\mathbf{v}|}\right) = \frac{1}{3\beta_2} \frac{\mathbf{v}^2}{m_H^2} + \frac{2\beta_3}{\beta_2^2} \ln\left(\frac{m_H}{|\mathbf{v}|}\right) + \tilde{K}(g_0) + O(g).$$

If one can neglect in the right hand side all terms but the two first ones, one obtains a relation between the two ratios $\Lambda / \langle H \rangle$ and $m_H / \langle H \rangle$.

Moreover, if the Higgs is really associated to a physical particle, its mass must be smaller than the cut-off. Taking for the two coefficients of the β-function the values for $O(4)$, $8\pi^2\beta_2 = 2$, $\beta_3/\beta_2^2 = -13/24$, one obtains the upper-bound

$$m_H < 2.6 \langle\phi\rangle \;\Rightarrow m_H < 640 \text{ GeV} .$$

The value could be compared with computer simulation values, which vary in the range 670–700 GeV. Moreover, the corresponding value of g is such that perturbation theory at leading order should still be semi-quantitatively correct.

Conversely, from the value physical coupling constant at scale μ, one can infer from the equation an upper-bound on the cut-off or scale of new physics. Clearly, this bound is very sensitive to small corrections since the equation determines $\ln(\Lambda/\mu)$. Moreover, for smaller values of m_H the coupling to the quark top and vector bosons have to be taken into account.

The conclusion at the time when the possible design of a new hadron collider was discussed was that by exploring physics in the TeV range one would either find the Higgs particle or discover some new physics, or both.

Later, the study of radiative corrections, even though they vary only like $\ln m_H$, benefited from the precision of LEP measurements. Assuming a Standard Model Higgs particle, they actually indicated that the Higgs mass should lie between 114 and 200 GeV.

8.10.3 Summary

In this lecture, we have shown that in the framework of effective field theory (EFT), using scaling arguments, it is natural to focus on renormalizable theories, but 'small' non-renormalizable interactions may eventually show up.

We have argued that the momentum cut-off, required to give a meaning to perturbation theory, should be considered as an *ad hoc* substitute to an unknown or too complicate short distance structure.

However, it is then necessary to prove that the physical observables at a momentum scale much smaller than the cut-off are insensitive to the cut-off procedure and this is the purpose of renormalization theory.

In the traditional formulation, renormalization theory is based on fixing parameters at a physical scale and fine-tuning the parameters of the Lagrangian as functions of the cut-off.

By contrast, in the framework of EFT one considers the parameters in the Lagrangian as fixed since fine tuning has little physical justification. In a theory with a scalar field, this leads to the important (and unsolved in the Standard Model) problem of the fine tuning of the scalar mass term.

One has then to investigate, at fixed initial parameters, the behaviour of the effective couplings or masses at physical scale, when typical momenta or energies become much smaller than the cut-off.

This can be achieved by using renormalization group equations implied by renormalization theory. We have used them to study the behaviour of the effective ϕ^4 coupling constant at physical scale when the cut-off is sent to infinity. We have then uncovered the problem of triviality, consequence of the positivity of corresponding RG β-function: in the infinite cut-off limit the effective coupling vanishes.

However, this is not lethal in the framework of EFT because one is not required to sent the cut-off to infinity. Indeed, one does not insist that the EFT should be consistent on all scales and the cut-off has just to be assumed to be large enough in such a way that an effective large distance theory with proper physical properties can be defined.

As an early application, we have recalled how renormalization arguments have, long before discovery, provided bounds on the mass of the Higgs particle, indicating that a collider exploring the TeV range was bound either to find the Higgs particle or some new physics.