Hands on computer I
Ray tracing in stars

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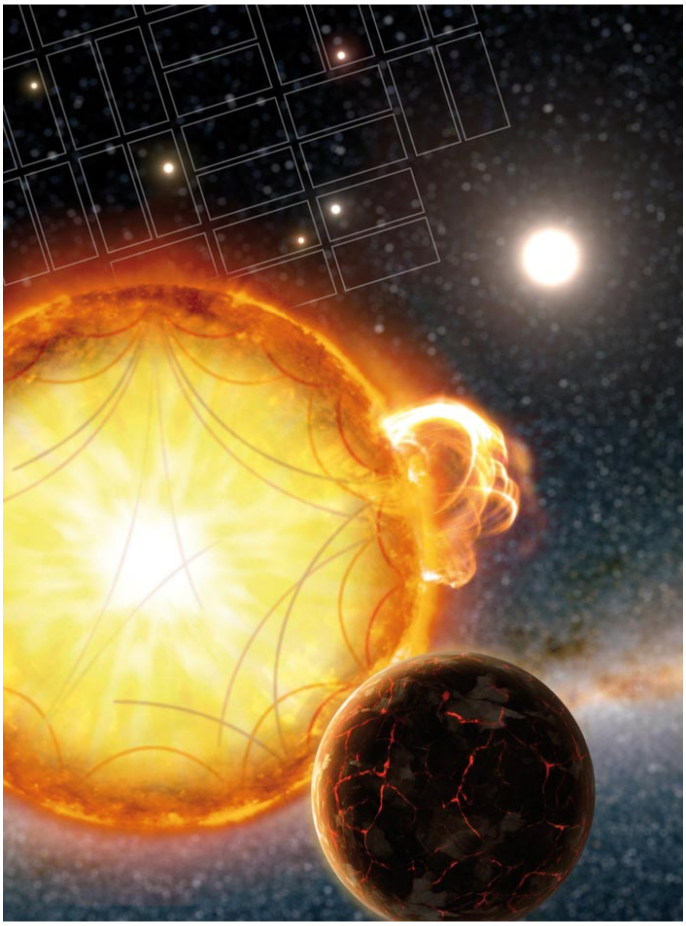

FIGURE 1 – Artist view of the internal structure of a solar-type star with an orbiting exoplanet (G Perez, IAC, SMM).

1 Preliminary lecture : acoustic and gravity modes (in a polytropic star)

In this session, we propose to use a simplified stellar polytropic model, in which $\overline{P} = K\overline{\rho}^{\Gamma}$. $\overline{P}$ and $\overline{\rho}$ are the density and pressure hydrostatic profiles, respectively, and Γ is the adiabatic index. We introduce the polytropic index $n = 1/(\Gamma - 1)$; from now on, we have $n = 3$. If time is sufficient, we will also consider Solar models.

If we linearize the stellar hydrodynamics equations and we combine them, we get the following equation

$$\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} + \omega_c^2\right)\Psi - \frac{\partial^2}{\partial t^2}\nabla^2\Psi - N^2\nabla_h^2\Psi = 0, \tag{1}$$

where $\Psi = \bar{\rho}^{1/2}c_s^2\left(\vec{\nabla}.\vec{\xi}\right)$ and $\vec{\xi}$ is the Lagrangian displacement of the stellar fluid defined by $\partial_t\vec{\xi} = \vec{v}$.

★ c_s is the sound speed in the star :

$$c_s^2 = \Gamma_1\frac{\bar{P}}{\bar{\rho}}. \tag{2}$$

★ N is the Brunt-Väisälä frequency, which describes the density (and entropy) stratification in the star :

$$N^2 = -\bar{g}\left(\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial r} - \frac{1}{\Gamma_1\bar{P}}\frac{\partial\bar{P}}{\partial r}\right). \tag{3}$$

★ ω_c is the acoustic cut-off frequency, which we will neglect here.

Equation (1) governs the behavior of stellar pulsations. There are two main families of waves : the acoustic and the gravity modes of oscillation. The acoustic waves have the compressibility as restoring force while the gravity waves result from the action of the buoyancy Archimedean force applied on stellar fluid elements. We will here study separately each type of waves.

1.1 Acoustic modes

In the case of acoustic waves, we assume that N can be neglected when compared to other terms in Eq. (1) ; it leads us to the d'Alembert-like equation :

$$\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\Psi - \nabla^2\Psi = 0. \tag{4}$$

To solve this equation, we introduce the asymptotic WKBJ [1] method, in which wave's related quantities are assumed to have characteristic space-scale of variation shorter than the ones of the background quantities. It allows us to look for solution for Ψ in the following form : $\Psi(\vec{x},t) = A(\vec{x},t)e^{i\Phi(\vec{x},t)}$, where A, the amplitude, is slowly varying compared to the phase Φ ($\vec{x}$ is the position vector). We define the wave vector $\vec{k} = \vec{\nabla}\Phi$ and the frequency $\omega = -\partial_t\Phi$.

Substituting Ψ in Eq. (4), we obtain the dispersion relation for acoustic modes (also called p modes, where "p" stands for pressure), where the frequency is expressed as a fonction of the wave vector :

$$\omega = c_s k\,; \tag{5}$$

k is the norm of the wave vector $\vec{k} = k_r\vec{e}_r + k_\theta\vec{e}_\theta + k_\varphi\vec{e}_\varphi$.

1.2 Gravity modes

For gravity modes (also called g modes, where "g" stands for gravity), we consider that terms scaling as $1/c_s$ are negligible. Applying again the WKBJ method, we derive their dispersion relation :

$$\omega = N\frac{k_h}{k}, \tag{6}$$

where k_h is the norm of the horizontal wave vector $\vec{k}_h$ defined as $\vec{k}_h = \vec{\nabla}_h\Phi = k_\theta\vec{e}_\theta + k_\varphi\vec{e}_\varphi$.

1. Wentzel-Kramers-Brillouin-Jeffreys.

1.3 Ray tracing

The ray tracing method allows us to visualize the energy transport along the wave propagation. It is based on Hamilton's equations verified by $W(\vec{x}, \vec{k}, t) = \omega$.
In our simplified polytropic stellar model (and in all 1-D stellar models), the values of c_s and N are only function of the radius r. It allows us to obtain a quantization for k_h

$$k_h = \frac{\sqrt{l(l+1)}}{r}, \tag{7}$$

where we have introduced the integer orbital number l of spherical harmonics. The vertical wave vector k_r can then be deduced from the dispersion relation where ω is a fixed parameter. The system of equations that must be solved is :

$$\begin{cases} \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial W}{\partial k_r} \\ \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{r}\frac{\partial W}{\partial k_h}. \end{cases} \tag{8}$$

★ Step 1 : Take the file containing the sound velocity (c_s) and the buoyancy frequency (N) profiles which has been sent to you by mail.

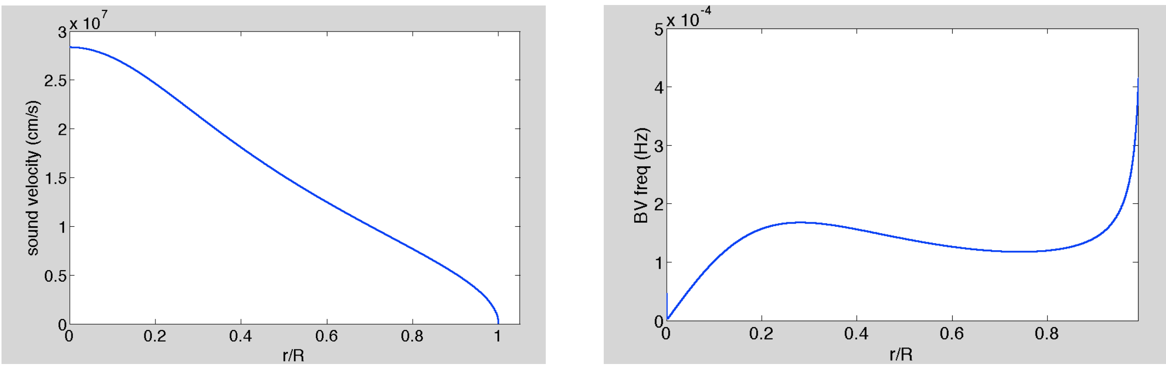

FIGURE 2 – Sound speed (left panel) and buoyancy frequency (right panel) radial profiles (the radius has been normalized by the total radius of the polytropic star).

★ Step 2 : Express $\frac{\partial W}{\partial k_r}$ and $\frac{\partial W}{\partial k_h}$ for p and g modes and solve the system (8) using the numerical method of your choice. Figure 3 gives examples of polar trajectories for each type of modes.

★ For p modes, what do you note when modifying l keeping ω constant ? and for g modes ?

★ What does happen if you consider a g mode with a frequency ω greater than Max (N) ? Why ?

★ Stellar oscillation modes are used to probe the interior of stars as seismic waves are used to probe the Earth's interior. For you, based on ray tracing, what are the best waves to probe stellar cores ?

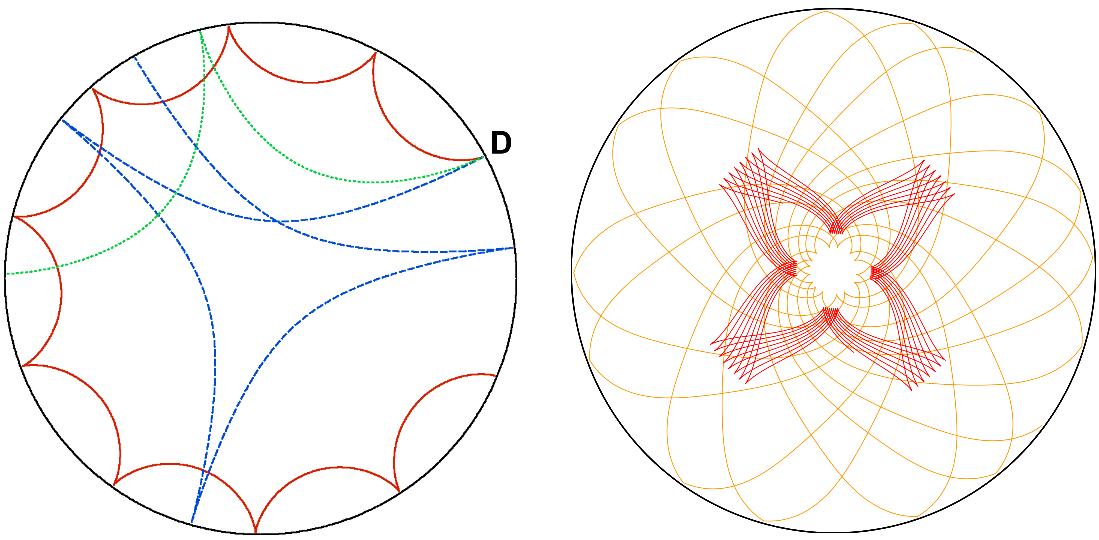

Figure 3 – Examples of rays obtained for p modes (left panel) and g modes (right panel). On the left panel, D is the initial "launching" point common to the three rays. Varying (ω,l) will allow you to explore the modes' behavior.