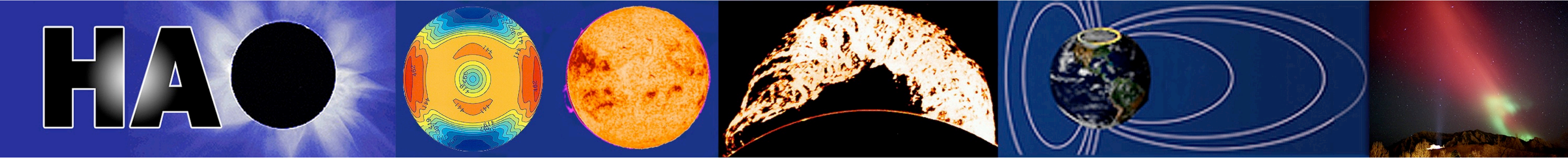

Solar/stellar oscillations
and
helioseismic inversions

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and University Corporation for Atmospheric Research

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NCAR

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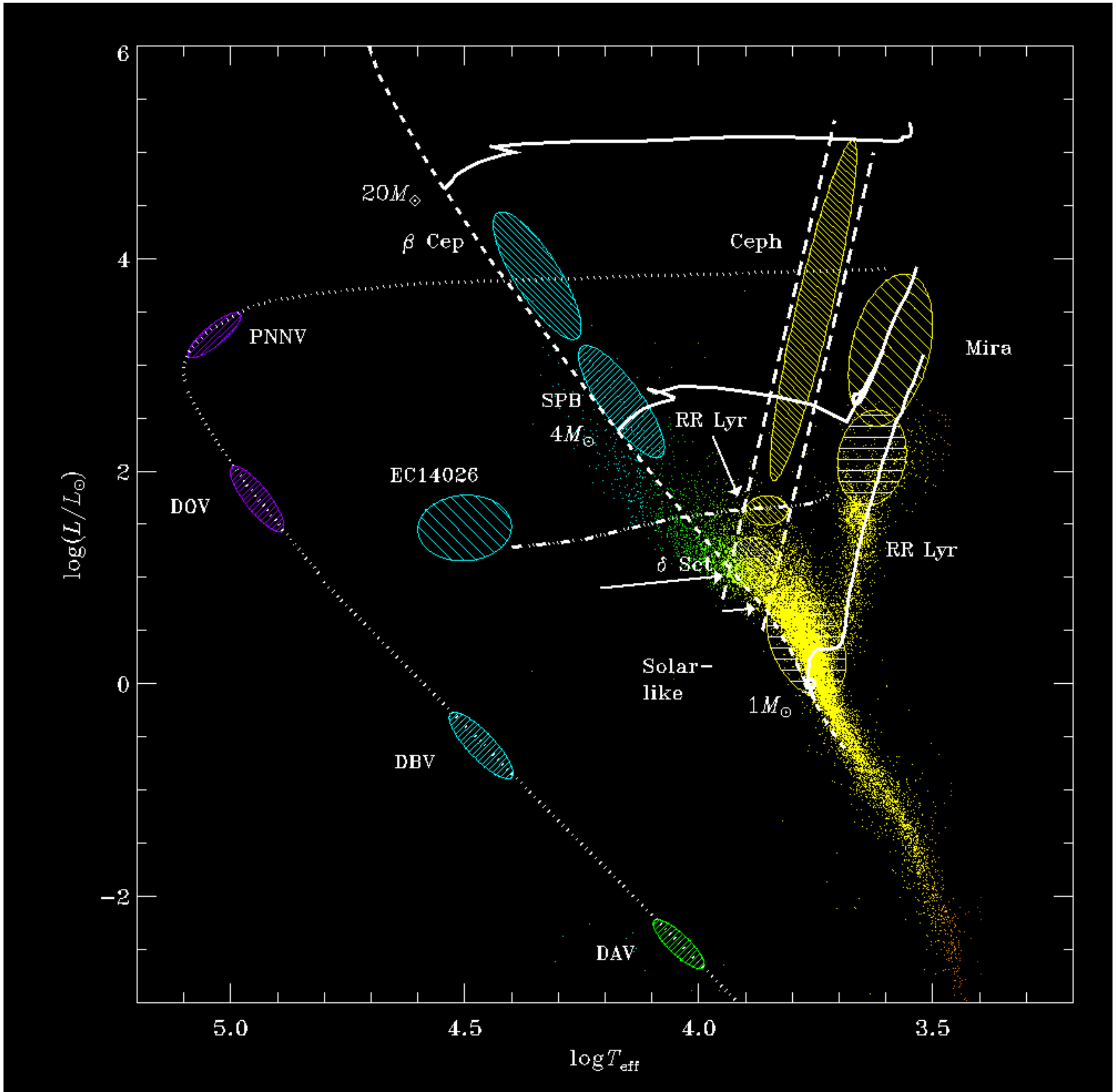
NSF

Properties of solar oscillations

- Small amplitude; hence consider linear perturbations around an equilibrium structure
- Periods much shorter than thermal timescale in most of the star; hence assume adiabatic oscillations.

- Modes are likely damped, predominantly by convective flux and turbulent pressure perturbations.
- **Modes are excited stochastically by convection**

Overview



$20M_\odot$
β Cep
Ceph
PNNV
Mira
SPB
$4M_\odot$
RR Lyr
EC14026
DOV
RR Lyr
δ Sct
Solar-like
$1M_\odot$
DBV
DAV
$\log(L/L_\odot)$
$\log T_{\rm eff}$
6
4
2
0
−2
5.0
4.5
4.0
3.5

Basic equations of (non-viscous) hydrodynamics

Continuity:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0$$

Motion:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{g}$$

Gravity:

$$\mathbf{g} = -\nabla \Phi \, , \qquad \nabla^2 \Phi = 4\pi G \rho$$

Energy:

$$\rho \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{(\gamma_3 - 1)} \left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) = \rho\,\epsilon - \mathrm{div}\,\mathbf{F}$$

Small perturbations around an equilibrium

No motion: $\mathbf{v}_0 = 0$

Hydrostatic equilibrium: $\nabla p_0 = \rho_0 \mathbf{g}_0$

Gravity: $\mathbf{g}_0 = -\dfrac{G m_0}{r^2} \mathbf{a}_r$

Energy: $\rho_0 \, \epsilon_0 = \mathrm{div}\, \mathbf{F}_0 = \dfrac{1}{r^2} \dfrac{\mathrm{d}}{\mathrm{d}r}(r^2 F_0) = \dfrac{1}{4\pi r^2} \dfrac{\mathrm{d}L_0}{\mathrm{d}r}$

Perturbations: $p(\mathbf{r}, t) = p_0(r) + p'(\mathbf{r}, t)$, etc.

Velocity: $\mathbf{v}' = \dfrac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t}$

Eulerian (p') and Lagrangian (δp) perturbations:

$$\delta p = p' + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0,$$

Basic linearized equations

Continuity equation

$$\rho' + \mathrm{div}(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0 \ .$$

Momentum equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \ ,$$

Poisson's equation

$$\nabla^2 \Phi' = 4\pi G \rho' \ , \qquad g' = -\nabla \Phi'$$

Adiabaticity

$$\delta p = \frac{\gamma_{1,0} p_0}{\rho_0} \delta\rho = c_0^2 \delta\rho \ ,$$

Basic linearized equations

Continuity equation

$$\rho' + \mathrm{div}(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0 \ .$$

Momentum equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \ ,$$

Poisson's equation

$$\nabla^2 \Phi' = 4\pi G \rho' \ , \qquad g' = -\nabla \Phi'$$

Adiabaticity

$$\delta p = \frac{\gamma_{1,0} p_0}{\rho_0} \delta\rho = c_0^2 \delta\rho \ ,$$

Separation of (θ, ϕ)

Separate displacement

$$\boldsymbol{\delta}\mathbf{r} = \xi_r \mathbf{a}_r + \boldsymbol{\xi}_{\mathrm{h}} \; .$$

Equations of motion

$$\rho_0 \frac{\partial^2 \xi_r}{\partial t^2} = -\frac{\partial p'}{\partial r} - \rho' g_0 - \rho_0 \frac{\partial \Phi'}{\partial r} \; .$$

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}_{\mathrm{h}}}{\partial t^2} = -\nabla_{\mathrm{h}} p' - \rho_0 \nabla_{\mathrm{h}} \Phi' \; .$$

Continuity equation

$$\rho' = -\frac{1}{r^2}\frac{\partial}{\partial r}(\rho_0 r^2 \xi_r) - \rho_0 \nabla_{\mathrm{h}} \cdot \boldsymbol{\xi}_{\mathrm{h}} \; .$$

Separation of (θ, ϕ)

Separation of scalar quantity

$$p'(r,\theta,\phi,t) = \sqrt{4\pi}\, p'(r) Y_l^m(\theta,\phi) \exp(-i\omega t) \ ,$$

Displacement vector

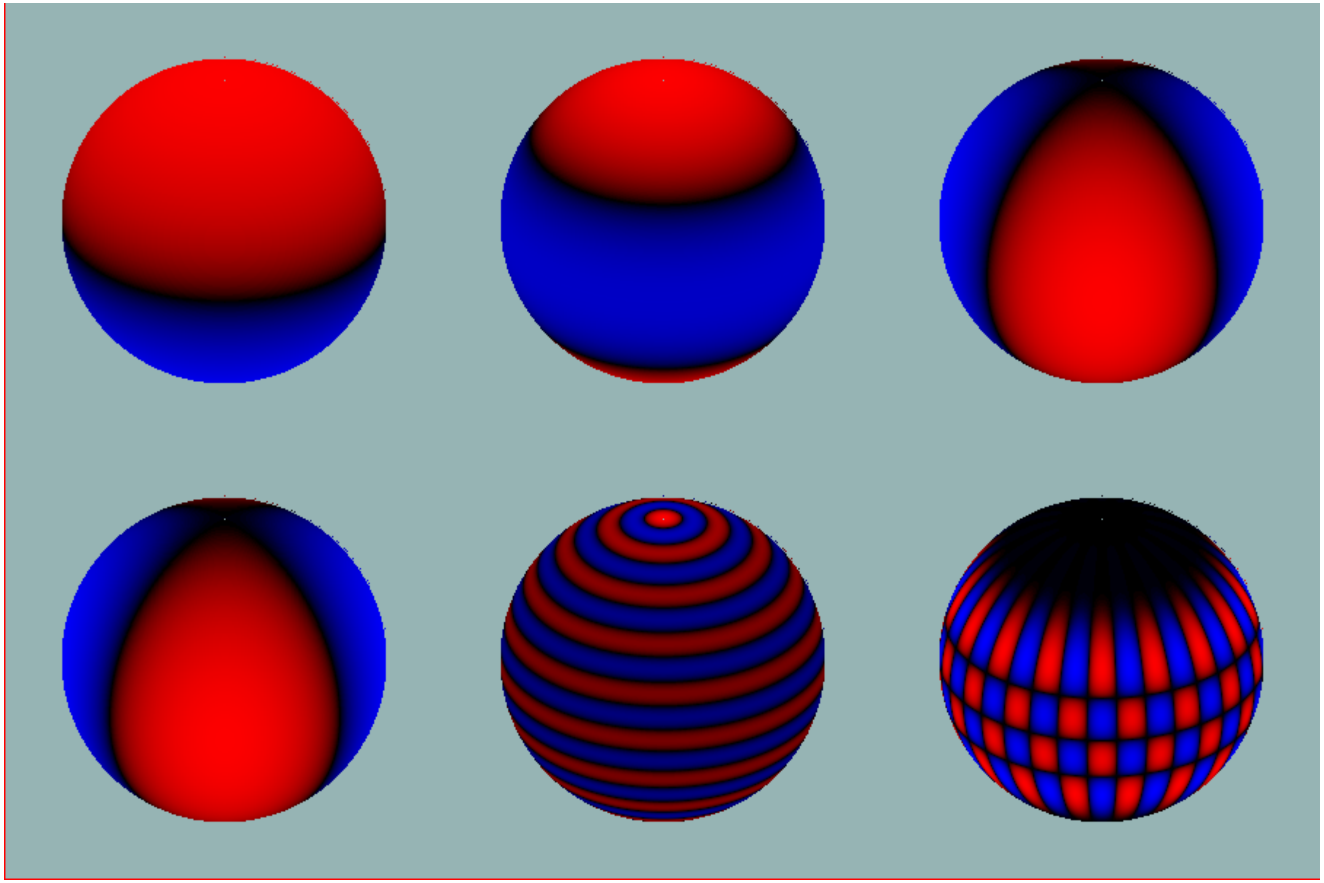
$$\boldsymbol{\delta}\mathbf{r} \ = \ \sqrt{4\pi}\Re\left\{\left[\xi_r(r)Y_l^m(\theta,\phi)\mathbf{a}_r + \xi_\mathrm{h}(r)\left(\frac{\partial Y_l^m}{\partial\theta}\mathbf{a}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\phi\right)\right]\exp(-i\omega t)\right\} \ .$$

$$\xi_\mathrm{h}(r) = \frac{1}{r\omega^2}\left(\frac{1}{\rho_0}p' + \Phi'\right) \ .$$

Effect of horizontal Laplacian for any perturbation ψ'

$$\nabla_\mathrm{h}^2\psi' = -\frac{l(l+1)}{r^2}\psi' \ , \qquad k_\mathrm{h}^2 = \frac{l(l+1)}{r^2} \ .$$

Spherical harmonics



- Behave like spherical harmonics: $P_l^m(\cos\theta)\cos(m\phi - \omega t)$
- $k_h = 2\pi / \lambda_h = [l(l+1)]^{1/2}/r$

Separated equations

Separation of time as exp(- i ω t)

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \frac{l(l+1)}{\omega^2 r^2}\Phi' \, .$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}p' - \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \, ,$$

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_\mathrm{h}^2 c^2 \, , \quad c^2 = \frac{\gamma_1 p}{\rho} \, , \quad N^2 = g\left(\frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) \, .$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi' \, .$$

Boundary conditions

At centre

$$\xi_r \simeq l \xi_{\rm h} \, , \qquad \text{for } r \to 0 \, .$$

At surface

$$\Phi' = A \, r^{-l-1} \, , \qquad \frac{{\rm d}\Phi'}{{\rm d}r} + \frac{l+1}{r} \Phi' = 0 \qquad \text{at } r = R \, .$$

$$\delta p = p' + \xi_r \frac{{\rm d}p}{{\rm d}r} = 0 \qquad \text{at } r = R \, .$$

Equations and boundary conditions determine frequencies ω_{nl}

Frequency dependence on solar structure

Frequencies depend on dynamical quantities:

$$p(r)\ , \quad \rho(r)\ , \quad g(r)\ , \quad \gamma_1(r)$$

However, from hydrostatic equilibrium and Poisson's equation p and g can be determined from ρ
Hence adiabatic oscillations are fully characterized by

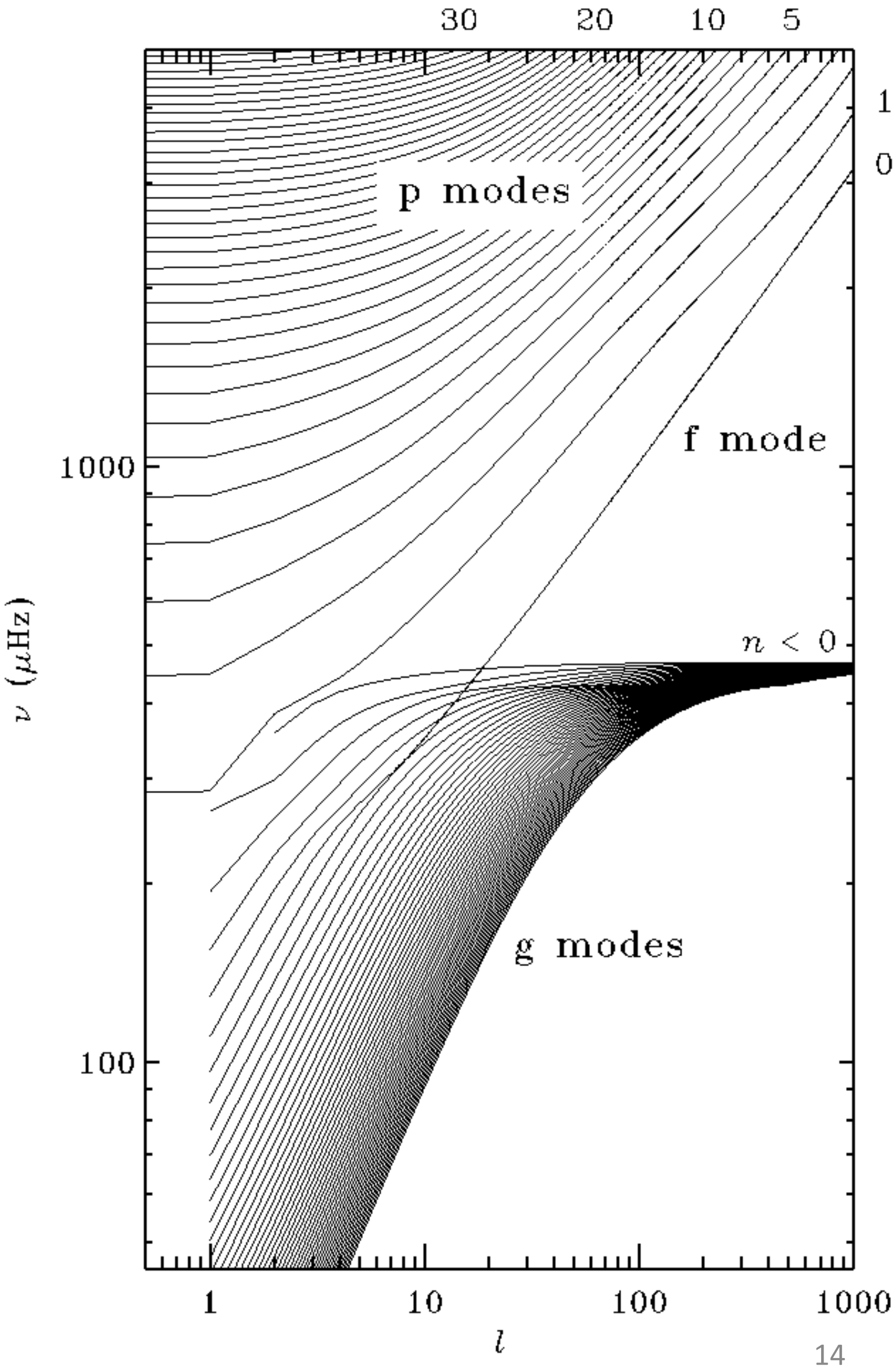
$$\rho(r)\ , \quad \gamma_1(r)$$

or, equivalently

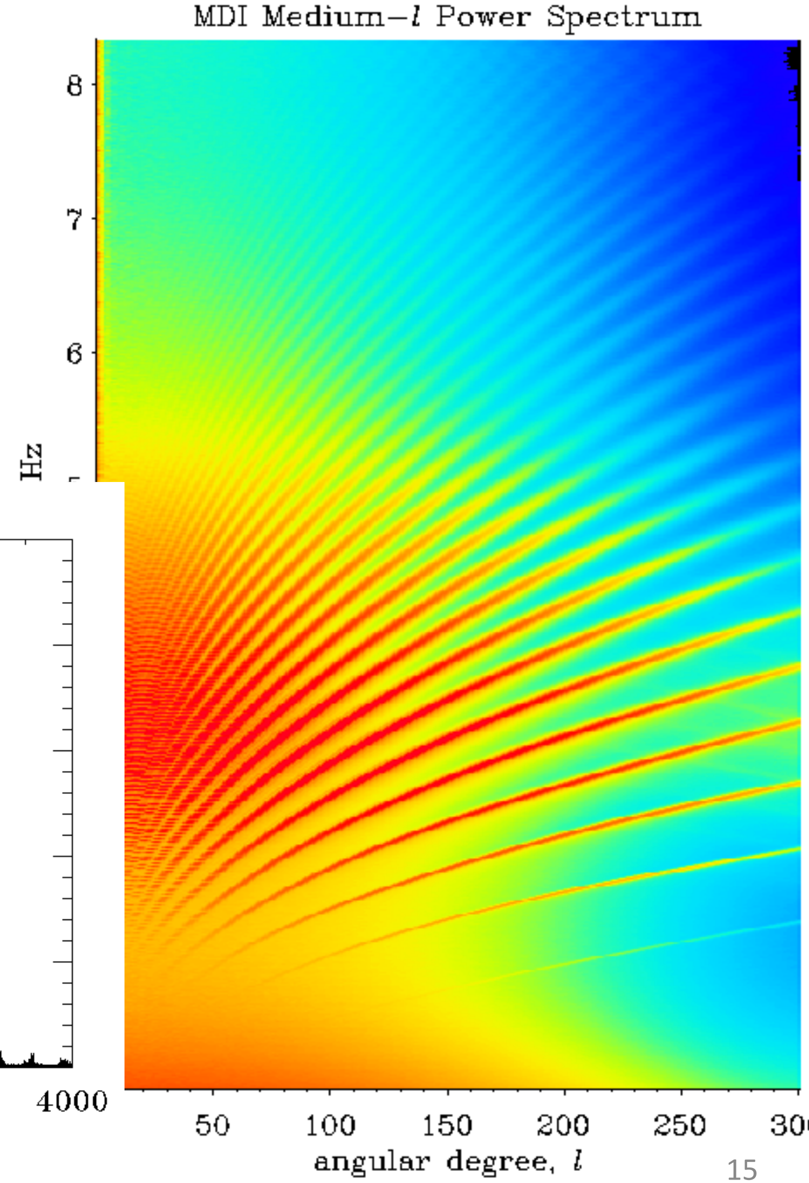
$$\rho(r)\ , \quad c^2(r)$$

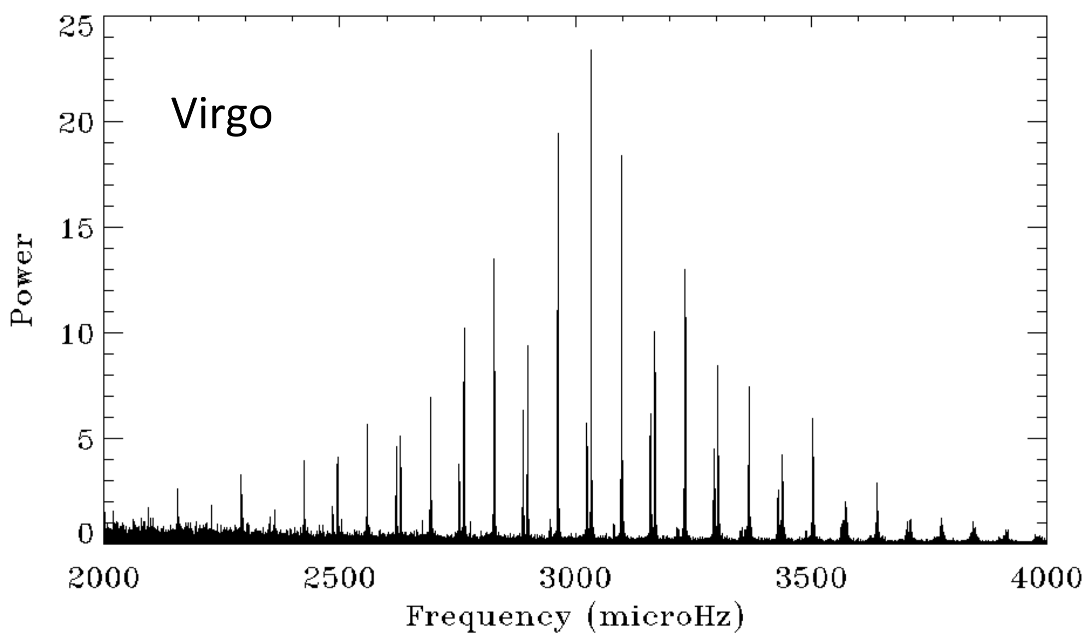
Frequencies of Model S

$\nu = \omega / 2\pi$



Data on solar oscillations





Separated equations

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\cancel{\left(\frac{2}{r} + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r} + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \cancel{\frac{l(l+1)}{\omega^2 r^2}\Phi'}\,.$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \cancel{\frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}p'} - \cancel{\rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r}}\,,$$

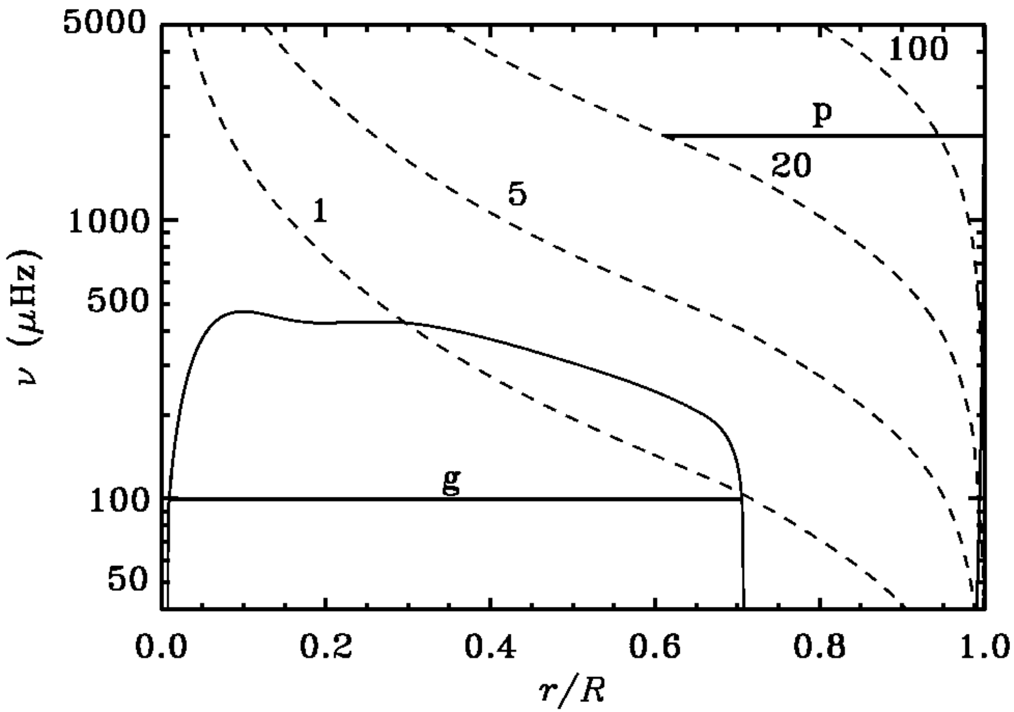
Cowling approximation

$$\cancel{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi'}\,.$$

High radial order

$$\frac{\mathrm{d}^2\xi_r}{\mathrm{d}r^2} \simeq -\frac{\omega^2}{c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)\left(\frac{N^2}{\omega^2} - 1\right)\xi_r$$

Mode trapping



$$\frac{\mathrm{d}^2\xi_r}{\mathrm{d}r^2} \simeq -\frac{\omega^2}{c^2}\left(\frac{S_l^2}{\omega^2}-1\right)\left(\frac{N^2}{\omega^2}-1\right)\xi_r$$

Eigenfunction oscillates as function of *r* when

$\omega^2 > S_l^2, N^2$ **p modes**

$\omega^2 < S_l^2, N^2$ **g modes**

Asymptotics of frequencies

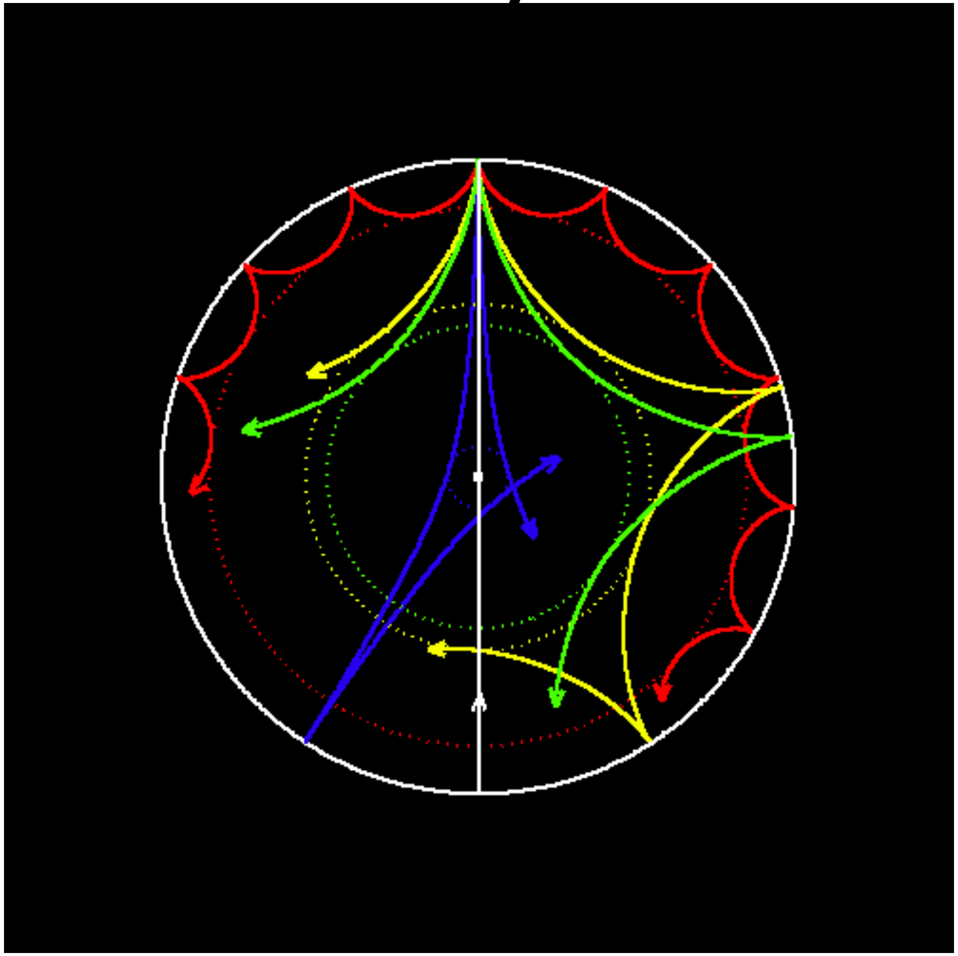
Acoustic-wave dispersion relation

$$\omega^2 = c^2|\mathbf{k}|^2 = c^2(k_r^2 + k_\mathrm{h}^2) = c^2\left(k_r^2 + \frac{l(l+1)}{r^2}\right)$$

Hence

$$k_r = \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right]^{1/2}$$

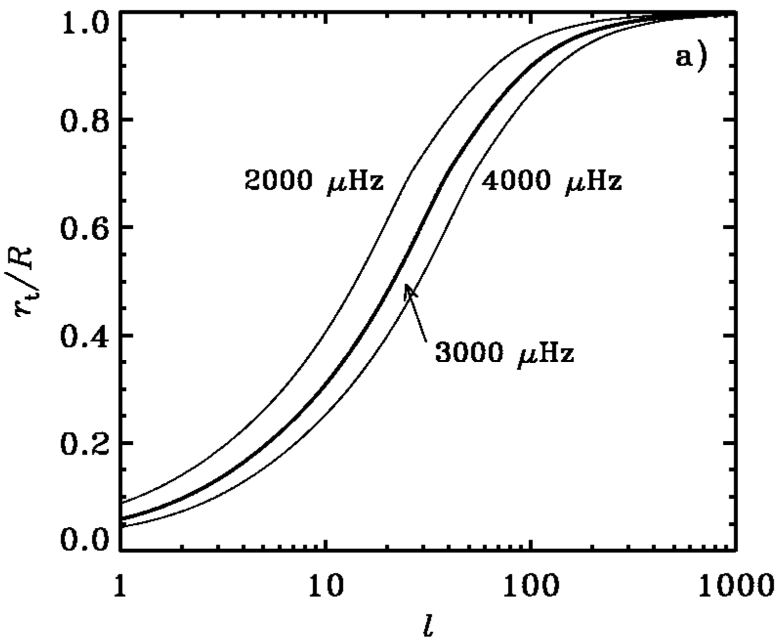
Rays

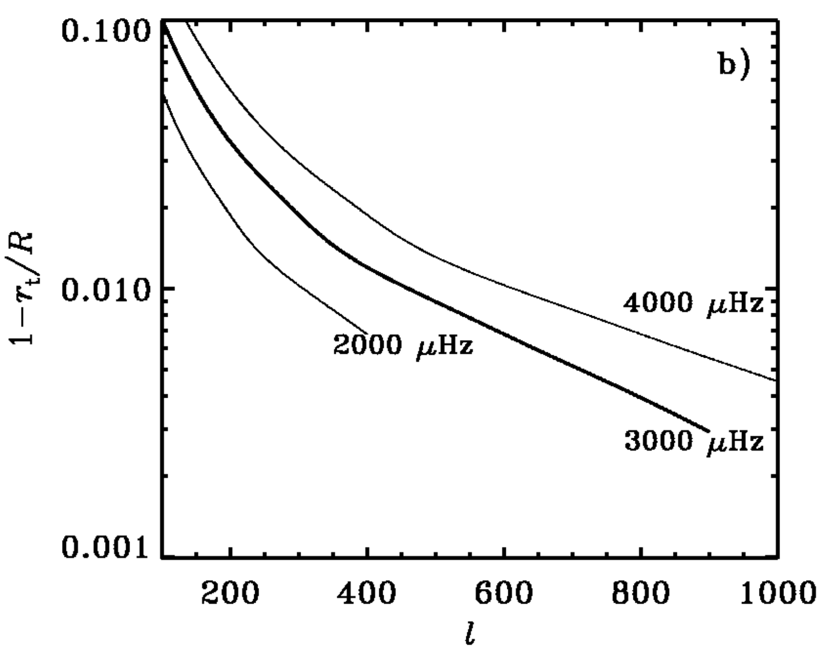


$$k_r = \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right]^{1/2}$$

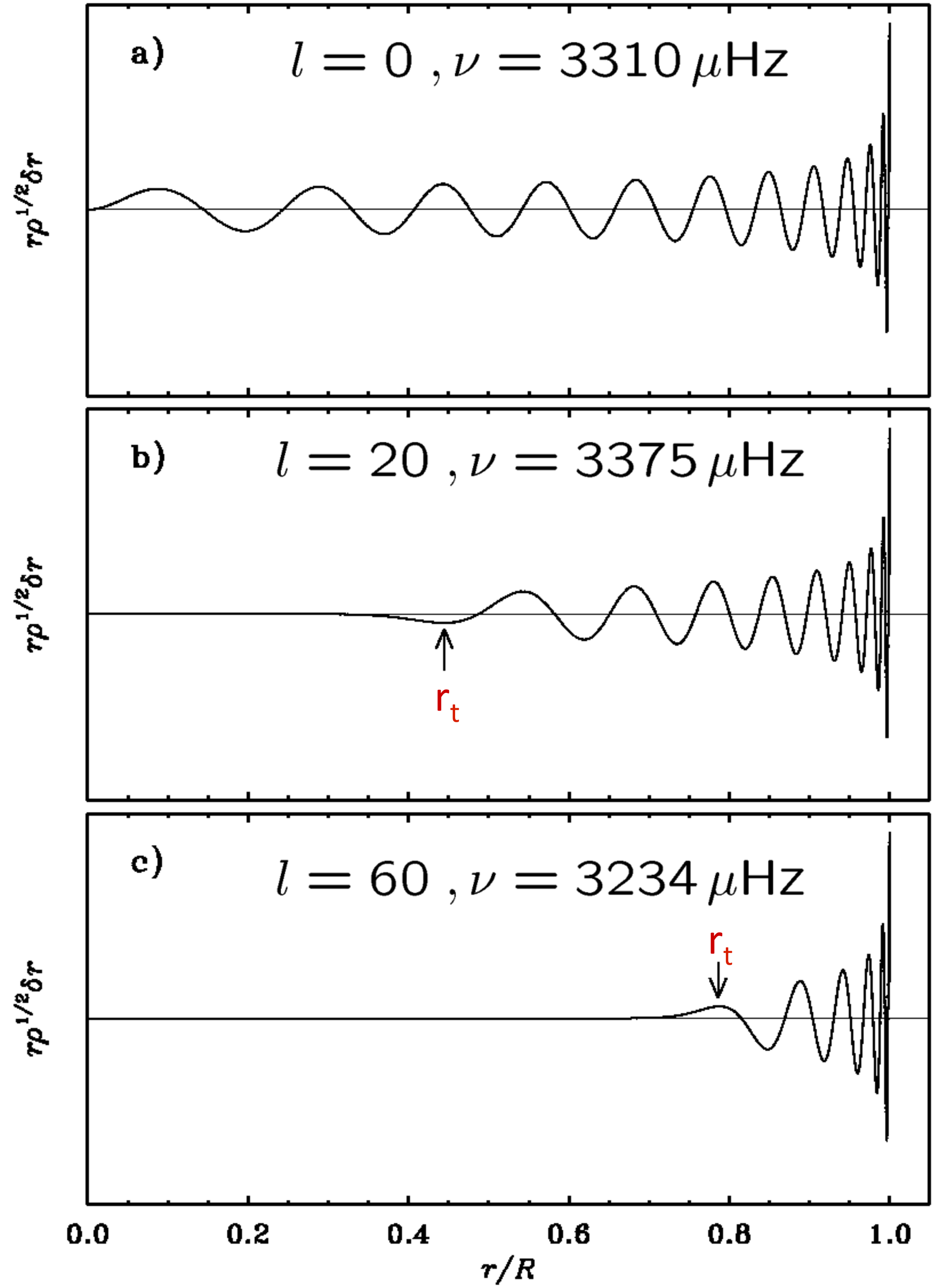
Turning point: $\dfrac{c(r_\mathrm{t})}{r_\mathrm{t}} = \dfrac{\omega}{\sqrt{l(l+1)}}$

Location of turning point

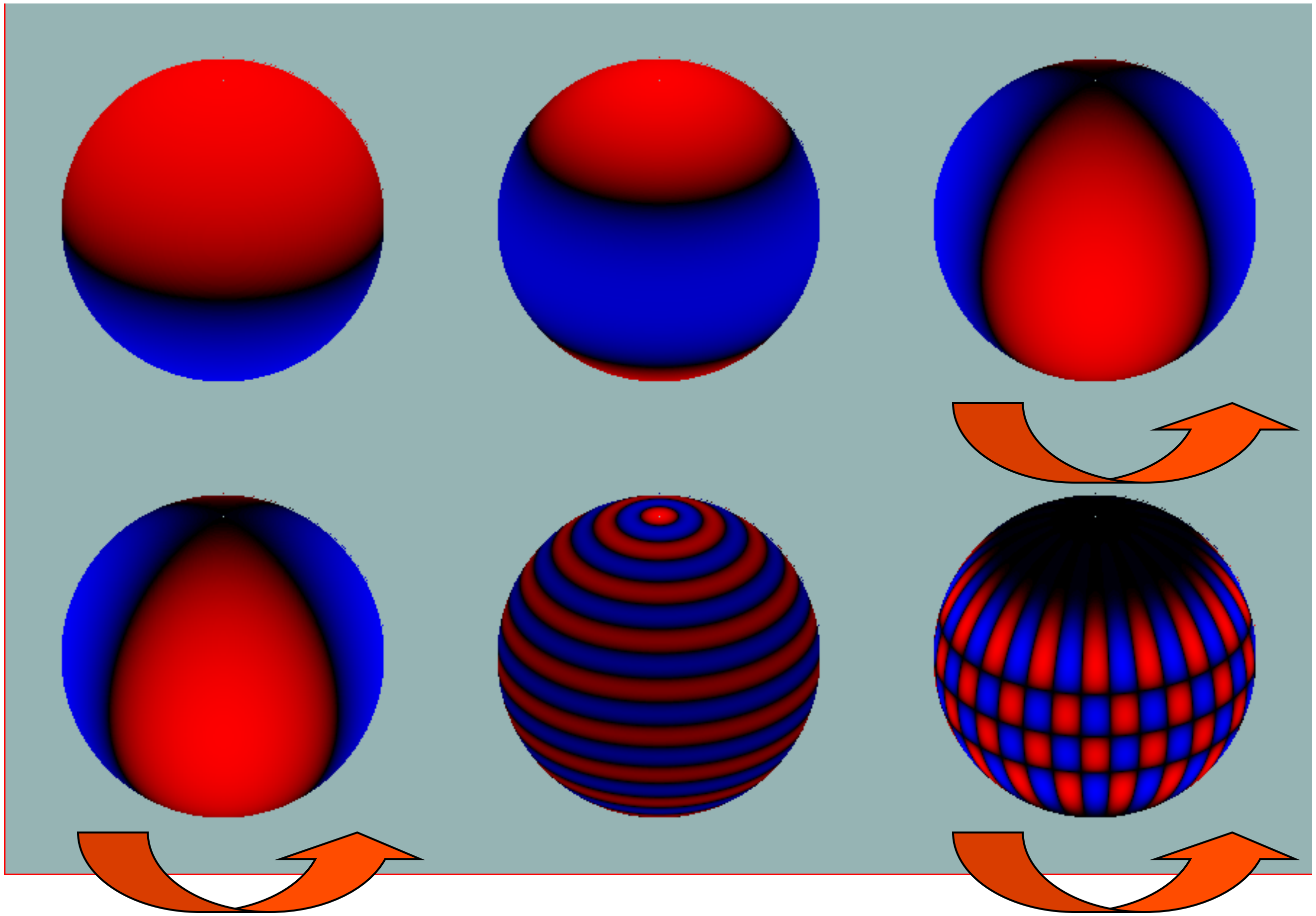




Effect on eigen-functions



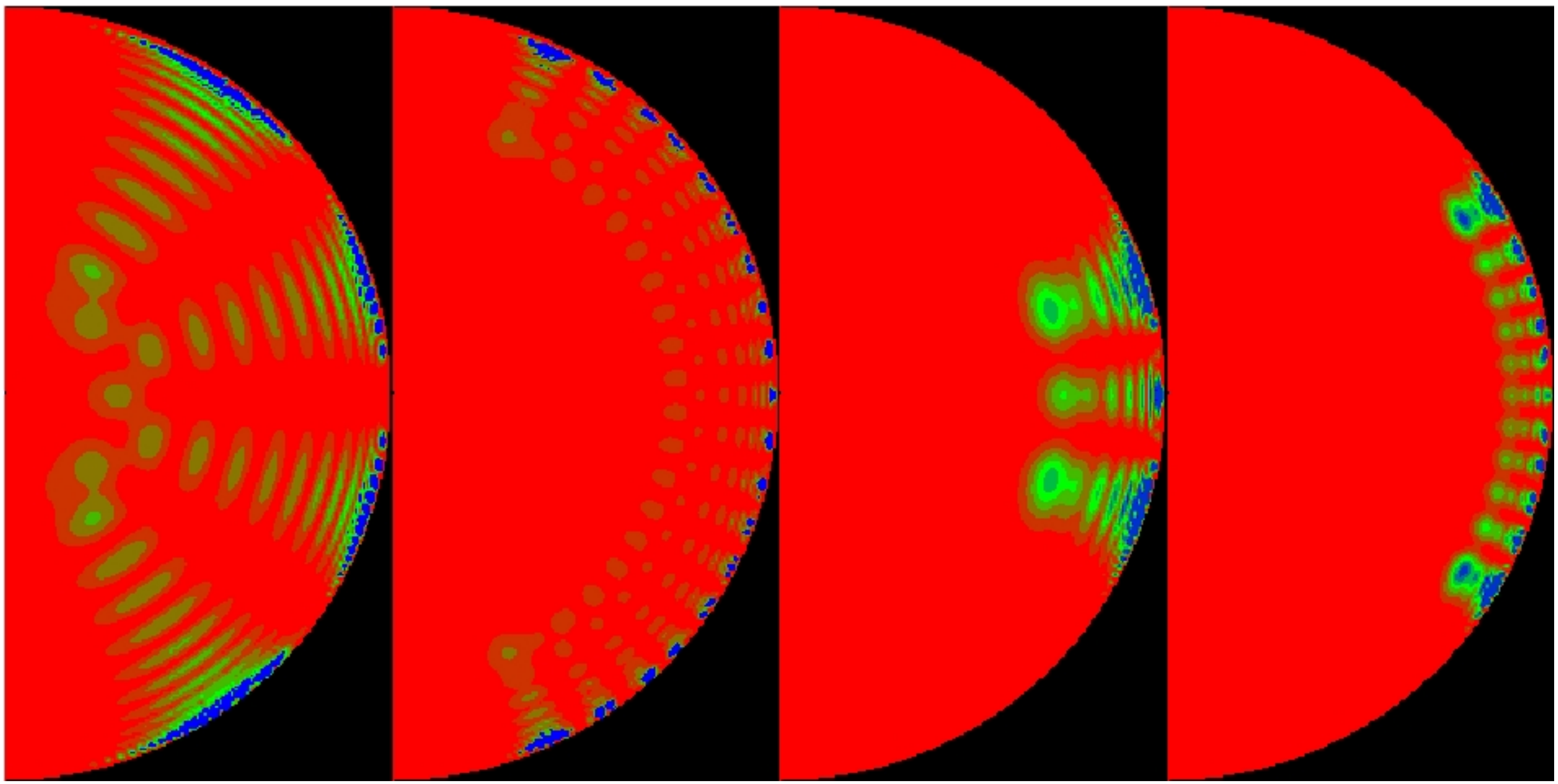
Rotational splitting



$$\omega_{nlm} = \omega_{nl0} + m\langle\Omega\rangle$$

Kernels for rotational splitting

$$\omega_{nlm} = \omega_{nl0} + m \int_0^R K_{nlm}(r)\Omega(r) r \mathrm{d}r$$



Helioseismic inversion

$$\text{Model} \underset{\text{Inversion}}{\overset{\text{Forward modeling}}{\rightleftarrows}} \text{Observables}$$

Aim of inversion: to make inferences about (usually) localized properties of the solar interior

Linear inversion methods

Many of the inversion methods used in helioseismology are linear: the solution is a linear function of the data.

In the rest of this lecture I shall introduce some linear techniques and a framework for understanding and comparing them.

Prototypical example: 1-D rotation law $\Omega(r)$

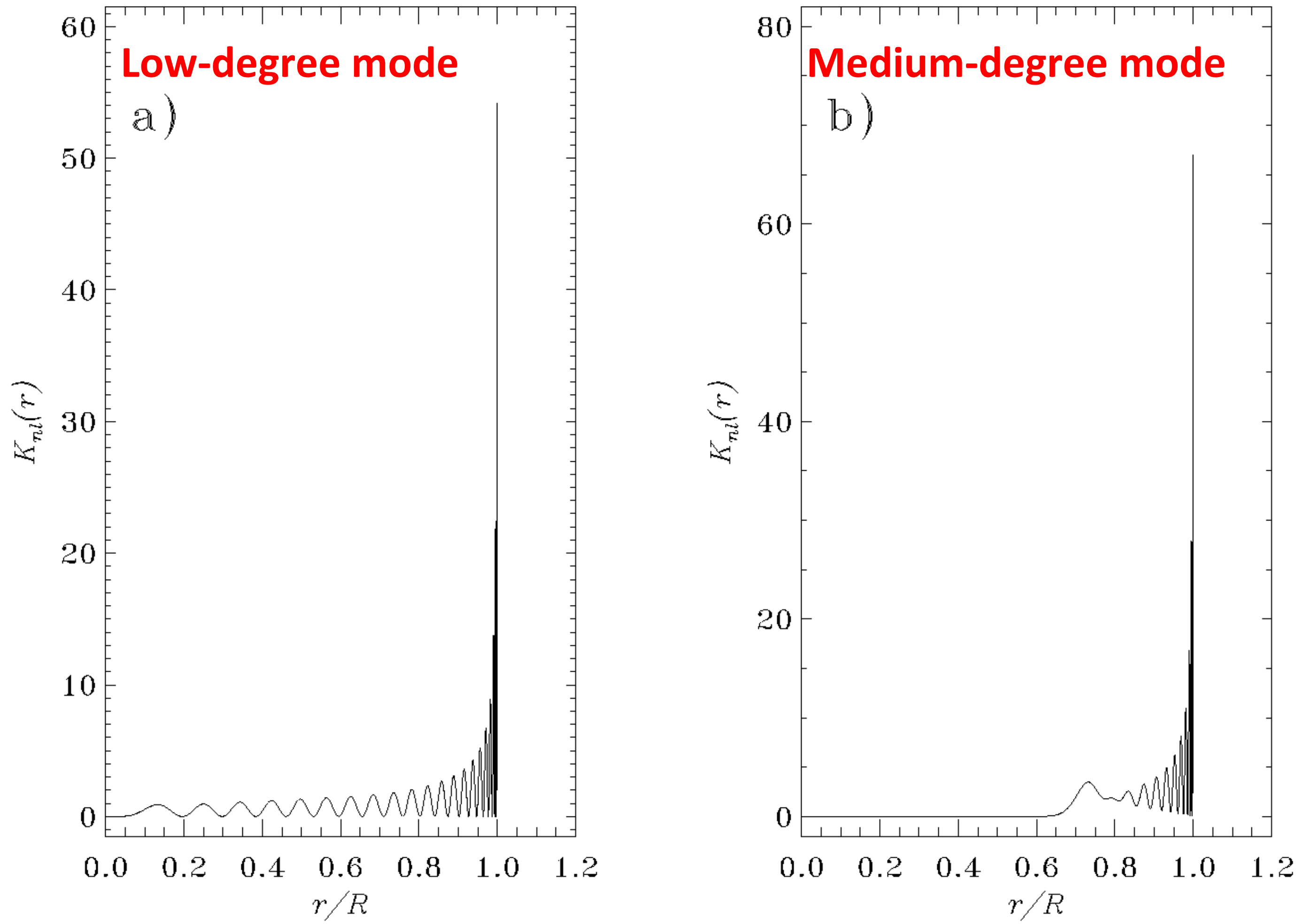
As already discussed, rotation raises the degeneracy of global mode frequencies and introduces a dependence on azimuthal order m.

The dependence is particularly simple if we consider a rotation profile $\Omega(r)$ depending only on the radial coordinate:

$$\omega_{nlm} = \omega_{nl0} + m \int K_{nl}(r)\Omega(r)\, dr$$

The kernels $K_{nl}(r)$ are different for different modes.

Kernels $K_{nl}(r)$ for 1-D rotation



Let $d_{nl} = (\omega_{nlm} - \omega_{nl0})/m$ be our data. Then

$$d_{nl} \;=\; \int K_{nl}(r)\Omega(r)\,dr \;+\; \epsilon_{nl}$$

where ε_{nl} are noise in the data, each with with standard deviation (s.d.) σ_{nl}.

For simplicity, we shall use single subscript "i" in place of "nl".

Least-squares fitting

Idea of least-squares (LS) fitting:

Approximate the unknown function $\Omega(r)$ in terms of a chosen set of basis functions $\phi_k(r)$: $\Omega(r) \approx \bar{\Omega}(r) = \Sigma x_k \, \phi_k(r)$.

Choose coefficients x_k to minimize

$$\sum_i \left(\frac{d_i - \int K_i \bar{\Omega} \, dr}{\sigma_i} \right)^2$$

This can be written as a matrix equation:
minimize $\quad | A\underline{x} - \underline{b} |^2$.

The solution of this is

$$\underline{x} = (A^T A)^{-1} A^T \underline{b} \ .$$

Unfortunately, unless we choose a highly restrictive representation for $\overline{\Omega}$, the matrix A is usually ill-conditioned in helioseismic inversions and so the LS solution $\underline{x}$ and hence $\overline{\Omega}$ also are dominated by data noise and thus useless.

Regularized Least-Squares (RLS) fitting

We can get better-behaved solutions out of LS by adding a "regularization term" to the minimization: e.g. to minimize

$$\sum \left(\frac{d_i - \int K_i \bar{\Omega}\, dr}{\sigma_i} \right)^2 + \lambda^2 \int \bar{\Omega}^2 dr$$

trade-off parameter

or

$$\sum \left(\frac{d_i - \int K_i \bar{\Omega}\, dr}{\sigma_i} \right)^2 + \lambda^2 \int (d^2\bar{\Omega}/dr^2)^2 dr$$

This can again be written as a matrix equation: minimize $|\, A\underline{x} - \underline{b}\, |^2 + \lambda^2 |\, L\underline{x}\, |^2$. The solution is $\underline{x} = (A^T A + \lambda^2 L^T L)^{-1} A^T \underline{b}$.

Optimally Localized Averages (OLA) method

$$d_i = \int K_i(r)\Omega(r)\,dr + \epsilon_i \quad i = 1, ..., M$$

Idea: for each radial location r_0, try to find a linear combination of the kernels that is localized there.

$$\mathcal{K}(r, r_0) = \sum_{i=1}^{M} c_i(r_0) K_i(r)$$

If successful, then the same linear combination of the data is a localized average of the rotation rate near $r=r_0$:

$$\begin{aligned} \bar{\Omega}(r_0) \equiv \sum c_i d_i &= \int (\sum c_i K_i)\Omega\,dr + \sum c_i \epsilon_i \\ &= \int \mathcal{K}\Omega\,dr + \sum c_i \epsilon_i \end{aligned}$$

How can the coefficients c_i be found?

OLA Classic (Multiplicative OLA – MOLA)

Choose the coefficients c_i so as to minimize

$$\int_0^R J\mathcal{K}^2 dr \; + \; \tan\theta \sum \sigma_i^2 c_i^2$$

E.g. $J=12(r-r_0)^2$. This penalizes K for being large except at $r=r_0$. Parameter θ trades off between localizing K and keeping the error term small

Subtractive OLA (SOLA)

Choose the coefficients c_i so as to minimize

$$\int_0^R (\mathcal{K} - \mathcal{T})^2 dr \; + \; \tan\theta \sum \sigma_i^2 c_i^2$$

E.g. $T=A \exp(-(r-r_0)^2/\delta^2)$. This penalizes K for deviating from the target function T. Trade-off parameters: θ and δ.

Error propagation

Assume the errors in the individual data d_i are independent (i.e. uncorrelated) and the standard deviation of each d_i is σ_i, say.

If the solution is $\Omega(r_0) = \Sigma\, c_i\, d_i$ then the standard deviation $\sigma[\Omega(r_0)]$ in the solution is is given by

$$\sigma[\Omega(r_0)] = (\Sigma\, c_i^2\, \sigma_i^2)^{1/2}$$

Error correlation

Consider the solution at two points r1 and r2:

$$\Omega(r_1) = \Sigma\, c_{1i}\, d_i \;\; , \Omega(r_2) = \Sigma\, c_{2i}\, d_i$$

These are constructed from the same (noisy) data and so in general their errors are correlated, i.e.

$$\mathrm{cov}[\Omega(r_1)\, ,\, \Omega(r_2)\,] = E[(\Omega(r_1) - E[\Omega(r_1)])\; (\Omega(r_2) - E[\Omega(r_2)])] \neq 0$$

If the data errors are independent:

$$\mathrm{cov}[\Omega(r_1)\, ,\, \Omega(r_2)\,] = \Sigma\, c_{1i}\, c_{2i}\, \sigma_i^{\,2}$$

A common framework for discussing *any* linear inversion

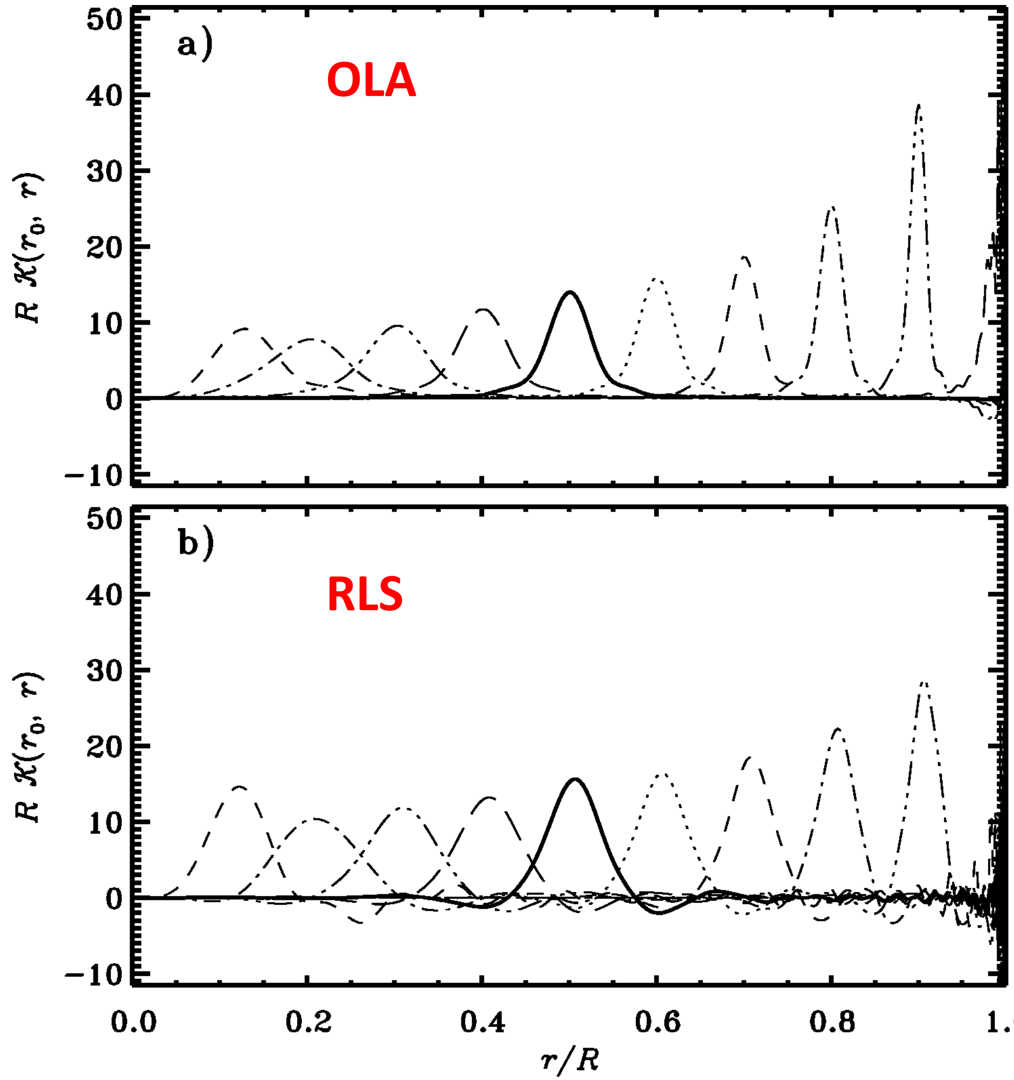
LS, RLS, MOLA, SOLA techniques above are all examples of linear methods: the solution is a linear combination of the data.

For any linear method, we can find inversion coefficients $c_i(r_0)$, look at averaging kernels $\Sigma c_i(r_0)K_i(r)$ and calculate error propagation, using the same expressions as in OLA.

Examples of averaging kernels

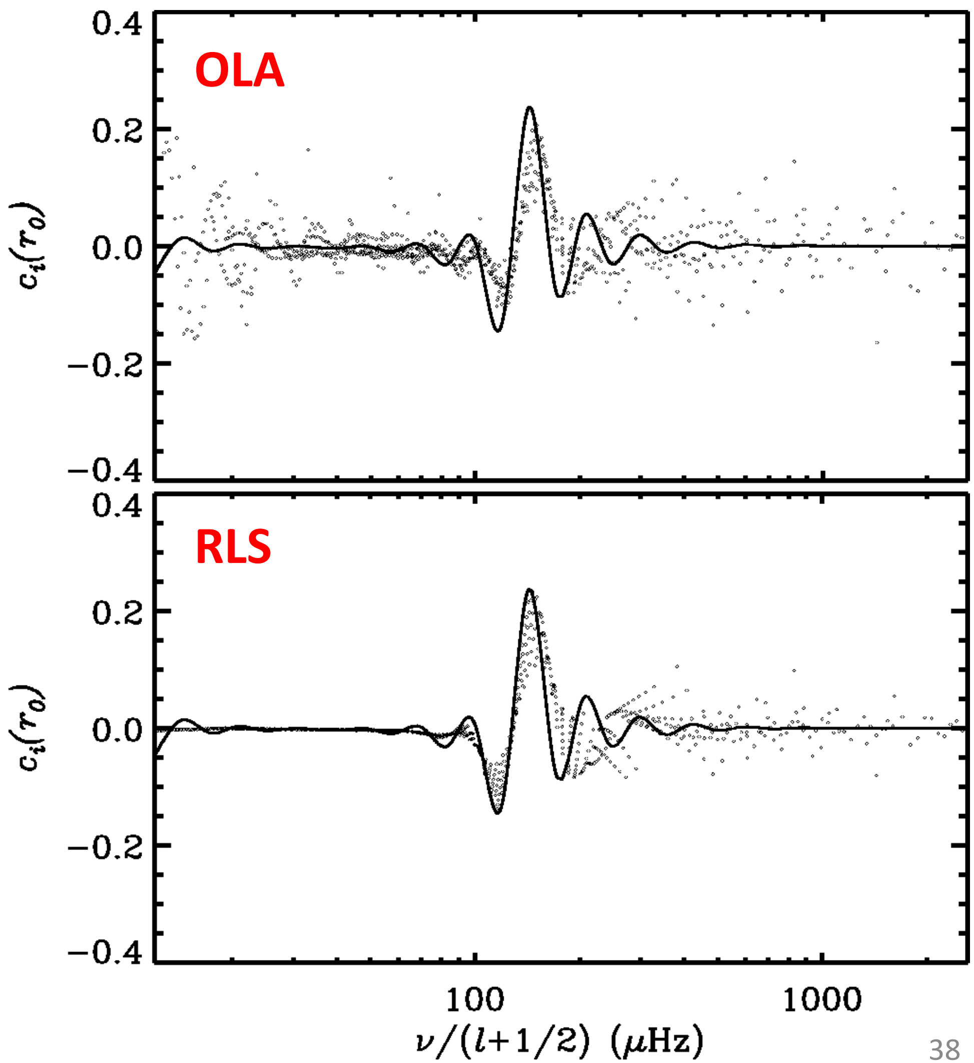
Averaging kernels for Ω(r) constructed with 834 p-modes with $1 \leq l \leq 200$

Note that the RLS kernels have negative sidelobes and near-surface structure.

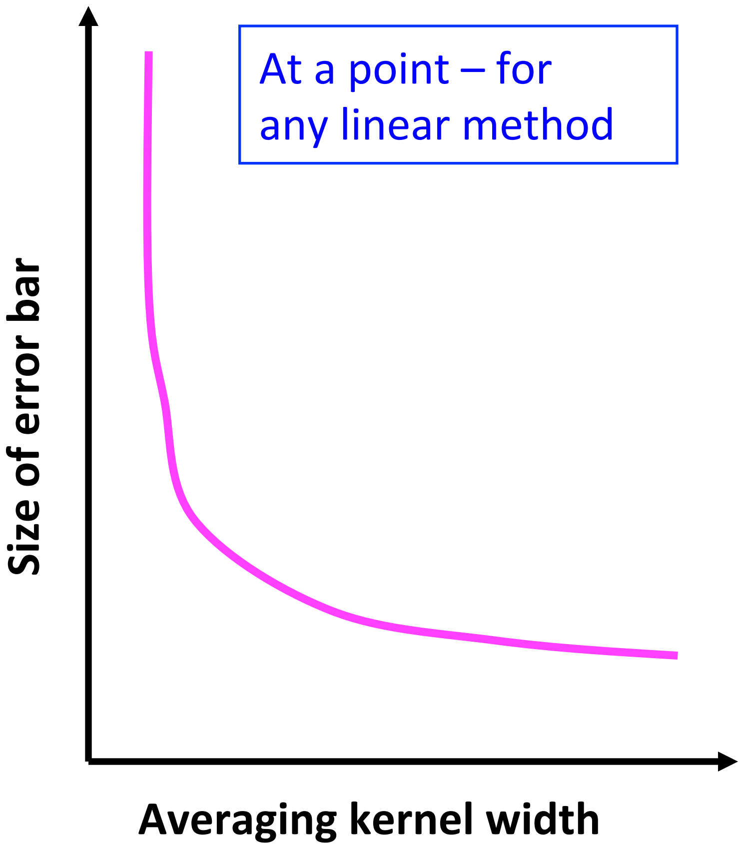


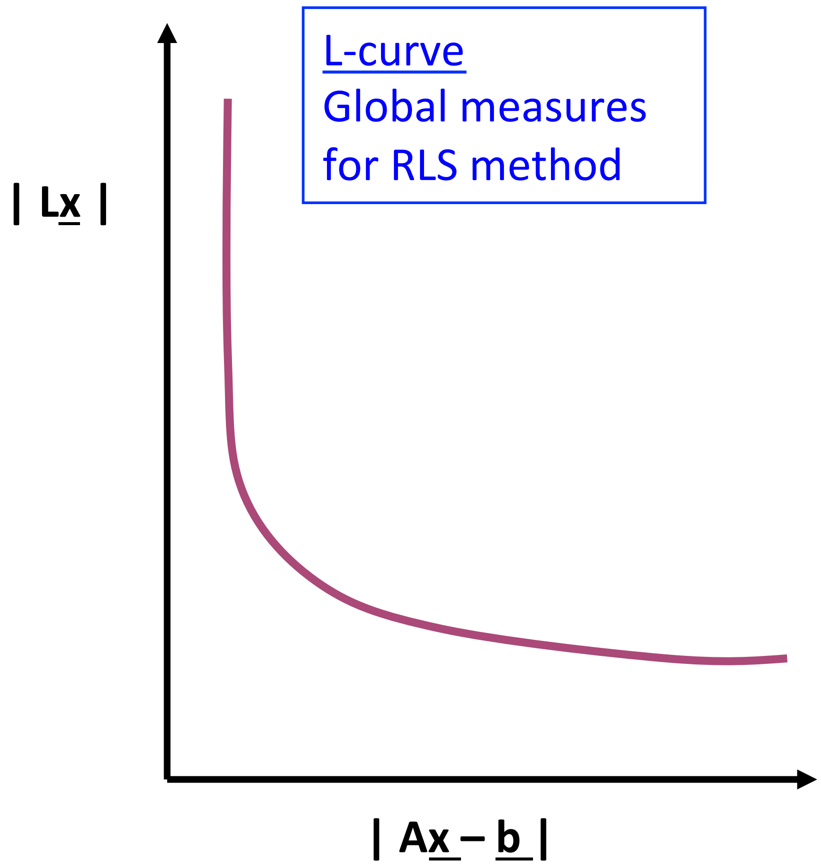
Inversion coefficients

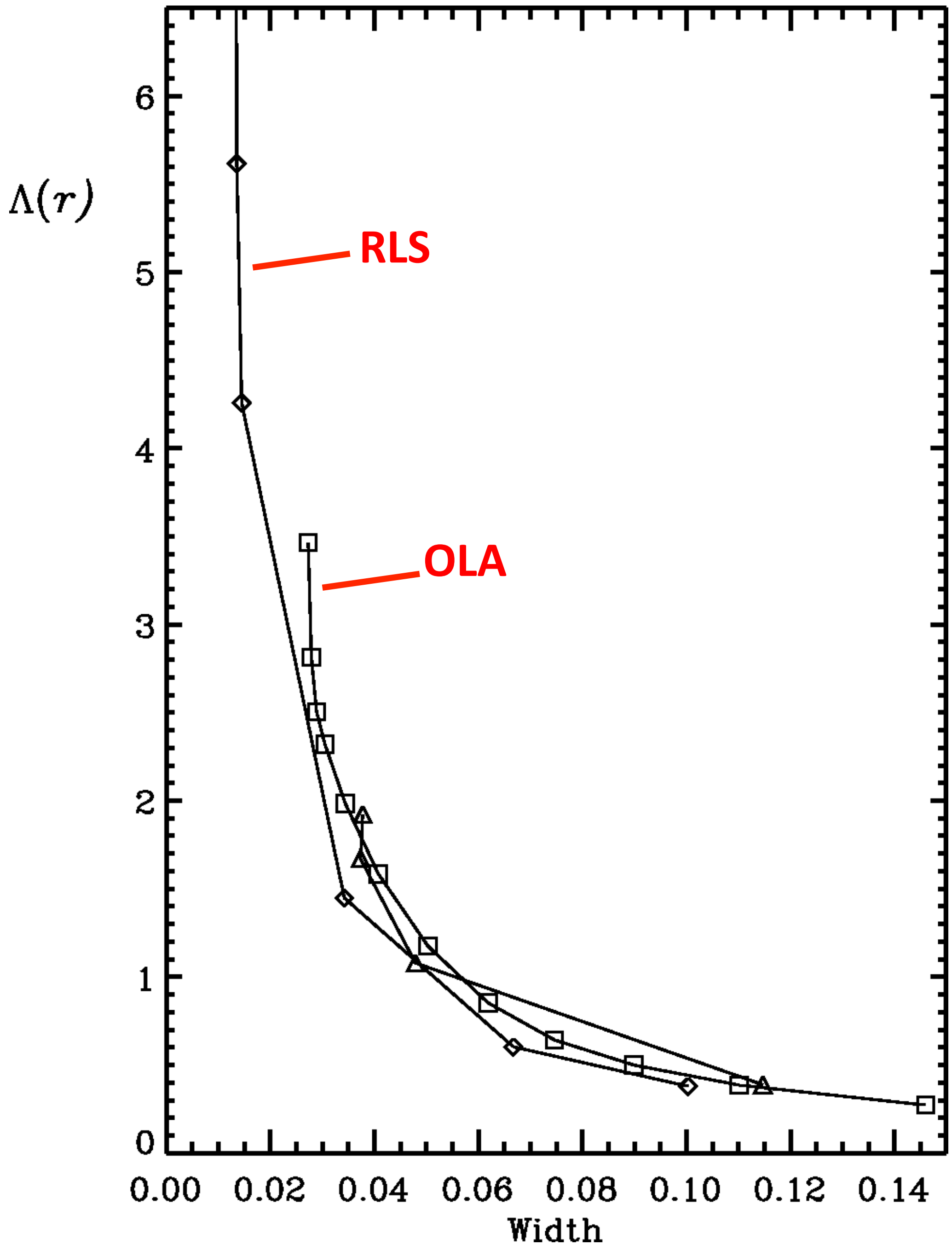
Inversion coefficients for solution at r_0=0.5R, for OLA and RLS inversions and (continuous curve) for a linear asymptotic inversion method.



Trade-off curves







$\Lambda(r)$
RLS
OLA
6
5
4
3
2
1
0
0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14
Width

Linearized inversion for solar structure

The dependence of the frequencies on solar structure is inherently nonlinear.

But we can use linear inversion techniques if we assume that the Sun's structure and frequencies are small perturbations to those of a known reference model.

No time to discuss it in detail now, but frequencies can be written in terms of a variational principle

$$\omega^2 = \frac{\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle}{\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle}$$

such that

$$\text{if } \mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F} \,, \quad \mathcal{F}_0(\boldsymbol{\xi}_0) = \omega_0^2 \boldsymbol{\xi}_0$$

$$\text{then } \mathcal{F}(\boldsymbol{\xi}) = \omega^2 \boldsymbol{\xi} \,, \qquad \omega^2 = \omega_0^2 + \delta\omega^2 \,, \quad \delta\omega^2 \simeq \frac{\langle \boldsymbol{\xi}_0, \delta\mathcal{F}(\boldsymbol{\xi}_0) \rangle}{\langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_0 \rangle}$$

This can describe differences between the Sun and a reference model.
In which case, δF depends on differences in quantities p, ρ and Γ_1.

Primary seismic variables

The model quantities that appear in the equations governing the adiabatic oscillations are p, ρ, Γ_1 (and combinations such as c).
These are therefore the

primary seismic variables.

Except for additional physics (e.g. rotation) they are the ONLY quantities that can be inferred from the frequencies unless we introduce additional assumptions.

NB In this lecture I use both γ_1 and Γ_1 for the first adiabatic exponent 

Can write previous equation $\omega^2 = \langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi})\rangle / \langle \boldsymbol{\xi}, \boldsymbol{\xi}\rangle$ as $\omega^2 = K/I$ where (in Cowling approximation)

$$I = \langle \rho |\boldsymbol{\xi}|^2 \rangle \ , \qquad K = \langle \rho c^2 |\nabla \cdot \boldsymbol{\xi}|^2 + 2\xi_r (\nabla \cdot \boldsymbol{\xi}) \frac{dp}{dr} + |\xi_r|^2 \frac{d \ln \rho}{dr} \frac{dp}{dr} \rangle$$

Then

$$\frac{\delta\omega}{\omega} = \left(\delta K - \omega^2 \delta I\right) / 2I\omega^2$$

where

$$\delta I = \langle \delta\rho \, |\boldsymbol{\xi}|^2 \rangle \ ,$$

$$K = \langle \delta(\rho c^2) \, |\nabla \cdot \boldsymbol{\xi}|^2 + 2\xi_r (\nabla \cdot \boldsymbol{\xi}) \frac{d\delta p}{dr} + |\xi_r|^2 \frac{d\delta \ln \rho}{dr} \frac{d\delta p}{dr} \rangle$$

For full details see Gough & Thompson (1991), in Solar Interior and Atmosphere, eds Cox, Livingston & Matthews, p. 519-561 (Univ. of Arizona Press)

Invoking hydrostatic equilibrium

There appear to be three independent unknown functions: $\delta p/p$, $\delta\rho/\rho$, and $\delta\Gamma_1/\Gamma_1$. But the oscillations are presumed to take place about an equilibrium background in hydrostatic equilibrium:

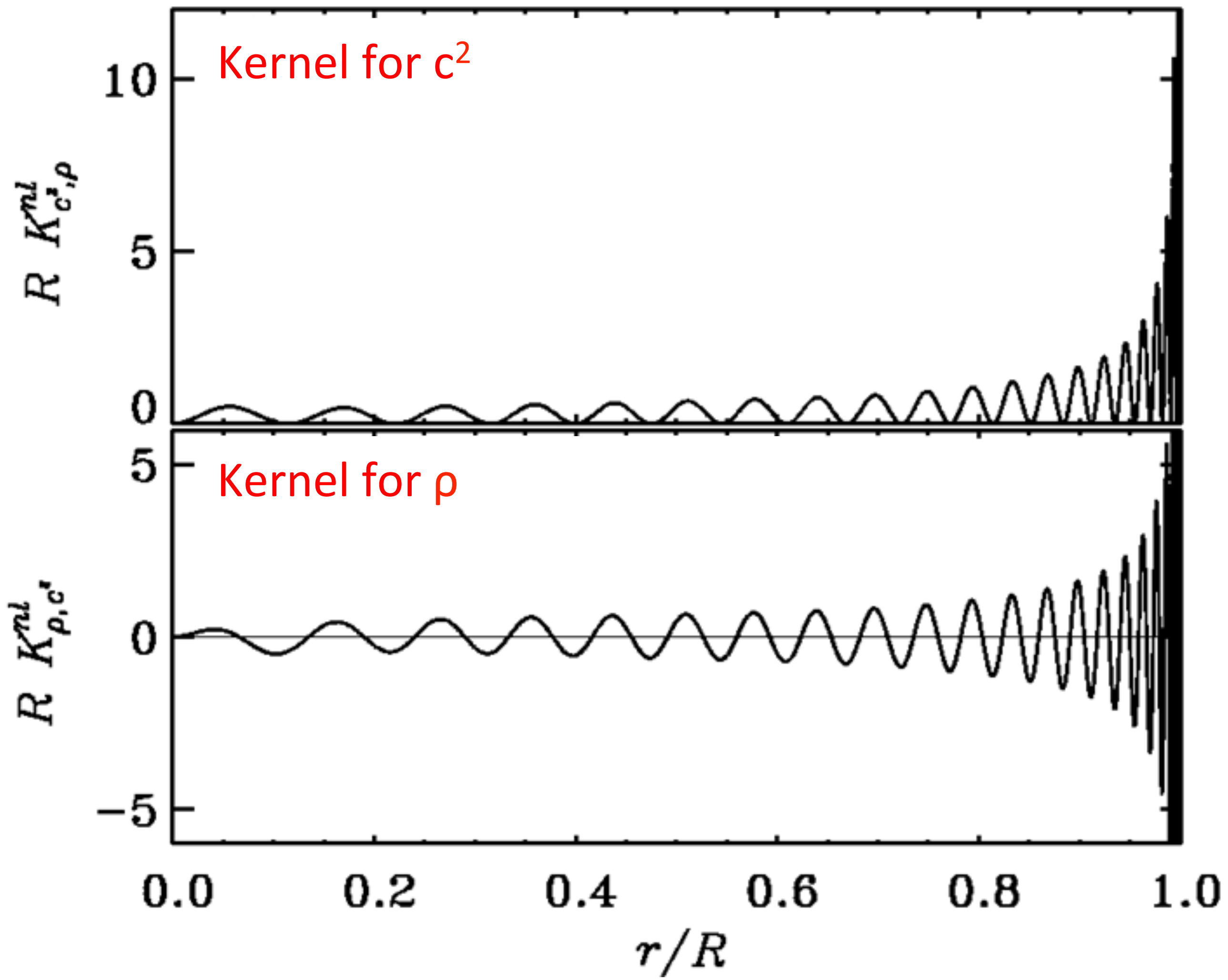
$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}$$

Perturbing this gives

$$\frac{d\delta p}{dr} = -\frac{Gm\delta\rho}{r^2} - \frac{G\delta m\,\rho}{r^2}$$

Likewise, using the mass equation, δm can be written in terms of $\delta\rho$. Hence $\delta p/p$ can finally be expressed in terms of $\delta\rho/\rho$, and the number of unknown functions reduced from 3 to 2.

Kernels for sound speed and density



Mass conservation - an additional constraint

Density perturbations cannot be chosen arbitrarily, as the mass of the Sun is known, i.e.

$$\int_0^R 4\pi r^2 \rho \, dr = M$$

Hence

$$\int_0^R 4\pi r^2 \delta\rho \, dr = 0$$

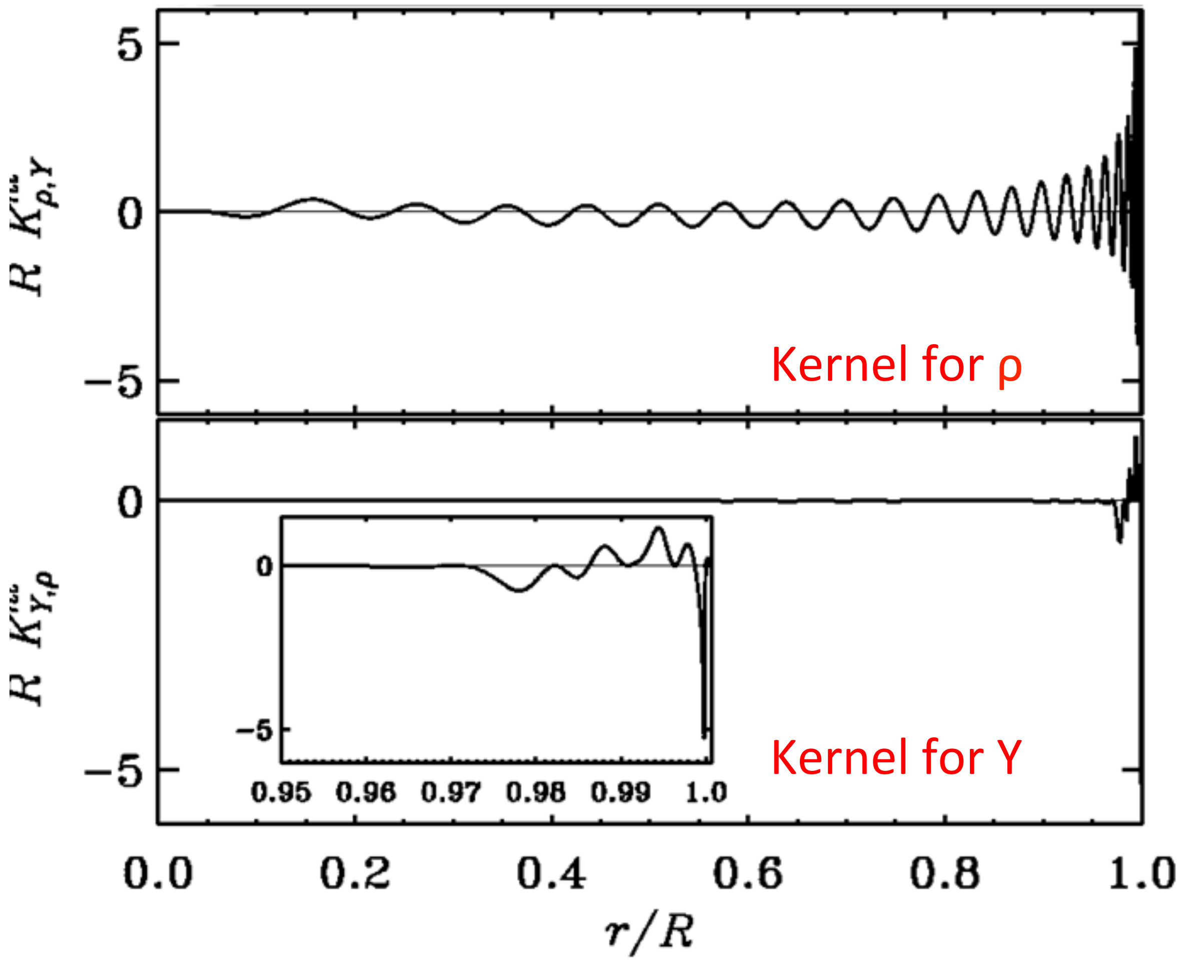
Introducing additional assumptions

E.g. assume the equation of state $\Gamma_1 = \Gamma_1(p,\rho,Y)$ known, where Y is the helium abundance. So one can express $\delta\Gamma$ in terms of other perturbations

$$\frac{\delta \Gamma_1}{\Gamma_1} = \left(\frac{\partial \ln \Gamma_1}{\partial \ln p}\right)_{\rho,Y} \frac{\delta p}{p} + \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho}\right)_{p,Y} \frac{\delta \rho}{\rho} + \left(\frac{\partial \ln \Gamma_1}{\partial Y}\right)_{p,\rho} \delta Y$$

and hence (after some work) derive kernels for e.g. u and Y.

Kernels for density and helium abundance



Formulation of structure inversion

For each observed mean-multiplet frequency we have a datum (or constraint) of the form e.g.

$$\frac{\delta\omega_{nl}}{\omega_{nl}} = \int K_{c,\rho}^{nl}(r)\frac{\delta_r c^2(r)}{c^2(r)}\mathrm{d}r + \int K_{\rho,c}^{nl}(r)\frac{\delta_r \rho(r)}{\rho(r)}\mathrm{d}r$$

$$+\frac{G_{\mathsf{surf}}(\omega_{nl})}{Q_{nl}} + \epsilon_{nl}\ .$$

slowly varying function of frequency

"mode mass"

- $K^{nl}_{c,\rho}(r)$, $K^{nl}_{\rho,c}(r)$ are known functions
- $G_{surf}(\omega_{nl})$ is a term from near-surface errors in the mode

  ε_{nl} are errors in the observations.

Also have in this case the mass-conservation constraint, which can be written in the same form as the data constraints.

Problem: use these constraints to make inferences about e.g. δc^2 and $\delta\rho$ between Sun and model.

RLS inversion for structure

Perhaps the most obvious approach.

Adjust the unknown functions $\delta c^2/c^2$, $\delta\rho/\rho$ and G_{surf} to get the best fit to the data.

As for rotation, need to regularize, so include terms in the minimization to penalize solutions $\delta c^2/c^2$, $\delta\rho/\rho$ that have e.g. large second derivative. Typically choose $G_{surf}(\omega)$ to be a low-order polynomial.

OLA inversion for structure

Try to choose inversion coefficients $c_i(r_0)$ so that e.g.

$$\mathcal{K}_{c,\rho}(r_0, r) = \sum_{nl} c_{nl}(r_0) K_{c,\rho}^{nl}(r)$$

is localized near $r=r_0$. If successful, then

$$\sum_{nl} c_{nl}(r_0)\frac{\delta\omega_{nl}}{\omega_{nl}} = \int \sum_{nl} c_{nl}(r_0) K_{c,\rho}^{nl}(r)\frac{\delta_r c^2(r)}{c^2(r)}\mathrm{d}r + \int \sum_{nl} c_{nl}(r_0) K_{\rho,c}^{nl}(r)\frac{\delta_r \rho(r)}{\rho(r)}\mathrm{d}r$$

$$+\sum_{nl} c_{nl}(r_0)\frac{G_{\mathrm{surf}}(\omega_{nl})}{Q_{nl}} + \sum_{nl} c_{nl}(r_0)\epsilon_{nl} \; .$$

is a localized estimate of the relative difference in sound-speed squared between Sun and model near $r=r_0$.

OLA inversion for structure

Choose inversion coefficients c_i to minimize

$$\int_0^R \left[\mathcal{K}_{c^2,\rho}(r_0,r) - \mathcal{T}(r_0,r)\right]^2 \mathrm{d}r + \beta \int_0^R \mathcal{C}_{\rho,c^2}(r_0,r)^2 \mathrm{d}r$$

$$+\mu \sum_{ij} E_{ij} c_i(r_0) c_j(r_0) \; ,$$

where

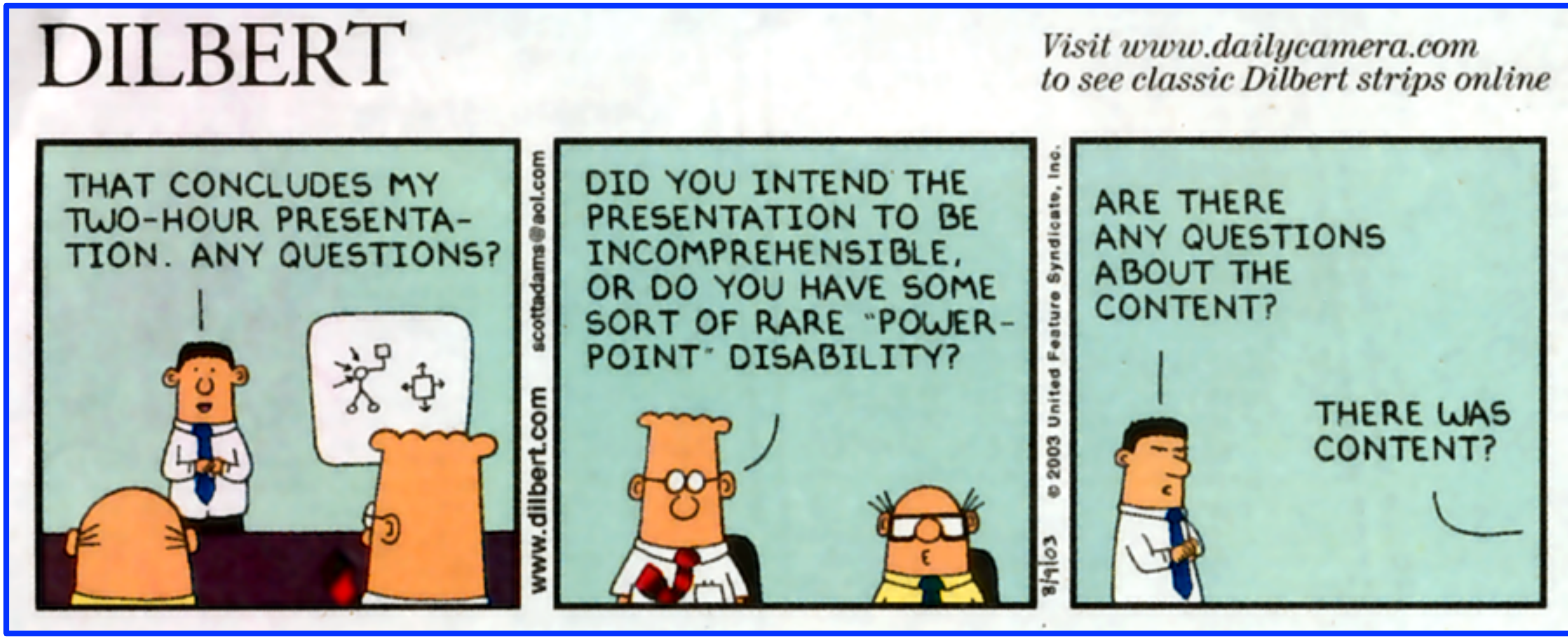
$$\mathcal{K}_{c^2,\rho}(r_0,r) = \sum_i c_i(r_0) K^i_{c^2,\rho}(r)$$ averaging kernel

$$\mathcal{C}_{\rho,c^2}(r_0,r) = \sum_i c_i(r_0) K^i_{\rho,c^2}(r)$$ cross-talk kernel

subject to the constraints

$$\int_0^R \mathcal{K}_{c^2,\rho}(r_0,r) \mathrm{d}r = 1$$ unimodular constraint

$$\sum_i c_i(r_0) Q_i^{-1} \psi_\lambda(\omega_i) = 0 \; , \lambda = 0, \ldots, \Lambda \; .$$ surface constraints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8/9/03

Thank you!

PHYSICS OF THE SOLAR INTERIOR

Basic equations:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} \ ,$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \ ,$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \nabla \frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}r} \ ,$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \left[\rho\epsilon - \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{\rho}\right) + \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right] \ .$$

$$\frac{\partial X}{\partial t} = \mathcal{R}_{\mathrm{H}} + \frac{1}{r^2\rho}\frac{\partial}{\partial r}\left[r^2\rho\left(D_{\mathrm{H}}\frac{\partial X}{\partial r} + V_{\mathrm{H}}X\right)\right] \ ;$$

Composition characterized by abundances *X*, *Y*, *Z* of H, He and the rest

Energy equation, adiabatic approximation

$$\frac{1}{(\gamma_3 - 1)} \left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) = \rho\,\epsilon - \mathrm{div}\,\mathbf{F}$$

Radiative flux: $\mathbf{F} = -\frac{a\tilde{c}}{3\kappa\rho} \nabla T^4$

(What about convection???)

$$\frac{|\mathrm{d}p/\mathrm{d}t|}{|\mathrm{div}\,\mathbf{F}|} \simeq \frac{p/\Pi}{|\mathbf{F}|/\ell} \simeq \frac{p/\Pi}{(a\tilde{c}/3\kappa\rho)(T^4/\ell^2)} = \frac{\tau_{\mathrm{rad}}}{\Pi} \gg 1$$

where $\tau_{\mathrm{rad}} = \frac{3\kappa\rho p \ell^2}{a\tilde{c}T^4}$

Adiabatic approximation: $\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \simeq 0$

Sound waves in a homogeneous medium

$\rho_0 \simeq \text{constant} \qquad \mathbf{g}_0 \simeq 0 \qquad \mathbf{g}' \simeq 0 \qquad \nabla p_0 \simeq 0$

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = -\nabla p' \, ,$$

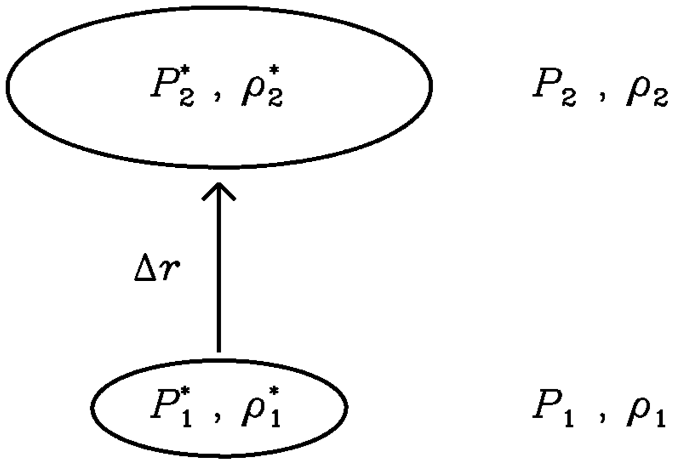
$$\rho' = \rho_0 \text{div}\, \boldsymbol{\delta}\mathbf{r} \qquad p' = \frac{\gamma_{1,0} p_0}{\rho_0} \rho' \equiv c_0^2 \rho'$$

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p' = c_0^2 \nabla^2 \rho'$$

Wave equation: Solution $\rho' \propto \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$

if $\omega^2 = c_0^2 |\mathbf{k}|^2$

Internal gravity waves



Buoyancy force:

$$\rho \frac{\mathrm{d}^2 \Delta r}{\mathrm{d}t^2} = f_{\mathrm{buoy}} = -g(\rho_2^* - \rho_2)$$

$$= -g\rho \left(\frac{1}{\gamma_1} \frac{\mathrm{d} \ln p}{\mathrm{d}r} - \frac{\mathrm{d} \ln \rho}{\mathrm{d}r} \right) \Delta r$$

$$\equiv -\rho N^2 \Delta r$$

Oscillation with frequency $\omega = N$ if

$N^2 > 0$ or $\dfrac{1}{\gamma_1} \dfrac{\mathrm{d} \ln p}{\mathrm{d}r} > \dfrac{\mathrm{d} \ln \rho}{\mathrm{d}r}$ *i.e.*, for convective stability.

In reality increased inertia owing to horizontal motion

If perturbation $\propto \exp[i\mathbf{k} \cdot \mathbf{r}]$ $\mathbf{k} = k_r \mathbf{a}_r + \mathbf{k}_h$

$$\omega^2 = \left(1 + \frac{k_r^2}{|\mathbf{k}_\mathrm{h}|^2} \right)^{-2} N^2 = \left(1 + \frac{\lambda_\mathrm{h}^2}{\lambda_r^2} \right)^{-2} N^2$$

Characteristic frequencies

Acoustic frequency

$$S_l^2 = \frac{l(l+1)c^2}{r^2}$$

Buoyancy frequency:

$$N^2 = g\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln p}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) \simeq \frac{g^2\rho}{p}(\nabla_{\mathrm{ad}} - \nabla + \nabla_\mu)$$

$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln p}, \qquad \nabla_{\mathrm{ad}} = \left(\frac{\partial\ln T}{\partial\ln p}\right)_{\mathrm{ad}}, \qquad \nabla_\mu = \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln p}.$$

Asymptotics of frequencies

Acoustic-wave dispersion relation

$$\omega^2 = c^2|\mathbf{k}|^2 = c^2(k_r^2 + k_\mathrm{h}^2) = c^2\left(k_r^2 + \frac{l(l+1)}{r^2}\right)$$

Hence

$$k_r = \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right]^{1/2}$$

Standing-wave condition, with surface-induced phase shift α

$$\int_{r_\mathrm{t}}^{R} k_r \mathrm{d}r = (n+\alpha)\pi\ , \qquad \frac{c(r_\mathrm{t})}{r_\mathrm{t}} = \frac{\omega}{\sqrt{l(l+1)}}$$

Hence Duvall law

$$\int_{r_\mathrm{t}}^{R} \left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{1/2} \frac{\mathrm{d}r}{c} = \frac{[n+\alpha(\omega)]\pi}{\omega}\ , \qquad L = l+1/2$$

Functional analysis

$$\omega^2 \boldsymbol{\delta}\mathbf{r} = \mathcal{F}(\boldsymbol{\delta}\mathbf{r})$$

$$\mathcal{F}(\boldsymbol{\delta}\mathbf{r}) = \frac{1}{\rho_0}\nabla p' - \mathbf{g}' - \frac{\rho'}{\rho_0}\mathbf{g}_0 \ ,$$

since $\rho' = -\mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r})$ $\qquad \delta p = c_0^2 \delta\rho$ $\qquad \mathbf{g}' = -\nabla\Phi'$

$$\Phi' = -G \int_V \frac{\rho'(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}\,\mathrm{d}V$$

Domain $\mathcal{D}$: $\boldsymbol{\delta}\mathbf{r}$ such that $\delta p(R) = 0$

Inner product: $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \int_V \rho_0 \boldsymbol{\xi}^* \boldsymbol{\eta}\,\mathrm{d}V$ $\qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{D}$

Symmetry: $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\eta}) \rangle = \langle \mathcal{F}(\boldsymbol{\xi}), \boldsymbol{\eta} \rangle$

Consequences

$$\text{If } \mathcal{F}(\boldsymbol{\xi}_0) = \omega_0^2 \boldsymbol{\xi}_0$$

$$\text{Then } \omega_0^2 = \Sigma(\boldsymbol{\xi}_0) \equiv \frac{\langle \boldsymbol{\xi}_0, \mathcal{F}(\boldsymbol{\xi}_0) \rangle}{\langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_0 \rangle} \qquad \text{is real}$$

$$\text{If } \mathcal{F}(\boldsymbol{\xi}_1) = \omega_1^2 \boldsymbol{\xi}_1; \qquad \mathcal{F}(\boldsymbol{\xi}_2) = \omega_2^2 \boldsymbol{\xi}_2; \qquad \omega_1^2 \neq \omega_2^2 \ ,$$

$$\text{Then} \qquad \langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \rangle = 0 \ .$$

$$\text{If } \mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F} \ , \quad \mathcal{F}_0(\boldsymbol{\xi}_0) = \omega_0^2 \boldsymbol{\xi}_0$$

$$\text{Then } \mathcal{F}(\boldsymbol{\xi}) = \omega^2 \boldsymbol{\xi} \ , \qquad \omega^2 = \omega_0^2 + \delta\omega^2 \ , \quad \delta\omega^2 \simeq \frac{\langle \boldsymbol{\xi}_0, \delta\mathcal{F}(\boldsymbol{\xi}_0) \rangle}{\langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_0 \rangle}$$

Linearized numerical differences

Linearizing around a reference model, $\delta\,\omega_{nl} = \omega_{nl}^{(obs)} - \omega_{nl}^{(mod)}$

$$\frac{\delta\omega}{\omega} \simeq \frac{\langle \boldsymbol{\delta}\mathbf{r}_0, \delta\mathcal{F}(\boldsymbol{\delta}\mathbf{r}_0)\rangle}{2\omega^2\langle \boldsymbol{\delta}\mathbf{r}_0, \boldsymbol{\delta}\mathbf{r}_0\rangle} \qquad \delta\mathcal{F} \text{ determined by } \frac{\delta_r c^2}{c^2}, \quad \frac{\delta_r \rho}{\rho}$$

$$\frac{\delta\omega_{nl}}{\omega_{nl}} = \int K_{c,\rho}^{nl}(r)\frac{\delta_r c^2(r)}{c^2(r)}\mathrm{d}r + \int K_{\rho,c}^{nl}(r)\frac{\delta_r \rho(r)}{\rho(r)}\mathrm{d}r$$

$$+\frac{G_{\mathsf{surf}}(\omega_{nl})}{Q_{nl}} + \epsilon_{nl} \,.$$

- $K^{nl}_{c,\rho}(r)$, $K^{nl}_{\rho,c}(r)$ are known functions
- $G_{surf}(\omega_{nl})$ is a term from near-surface errors in the mode

  ε_{nl} are errors in the observations.

Linearized structure inversion

Choose coefficients $c_{nl}(r_0)$ to obtain solution at $r = r_0$

$$\sum_{nl} c_{nl}(r_0) \frac{\delta\omega_{nl}}{\omega_{nl}} = \int \sum_{nl} c_{nl}(r_0) K_{c,\rho}^{nl}(r) \frac{\delta_r c^2(r)}{c^2(r)} \mathrm{d}r + \int \sum_{nl} c_{nl}(r_0) K_{\rho,c}^{nl}(r) \frac{\delta_r \rho(r)}{\rho(r)} \mathrm{d}r$$

$$+ \sum_{nl} c_{nl}(r_0) \frac{F_{\mathsf{surf}}(\omega_{nl})}{Q_{nl}} + \sum_{nl} c_{nl}(r_0) \epsilon_{nl} \ .$$

$$\mathcal{K}_{c,\rho}(r_0, r) = \sum_{nl} c_{nl}(r_0) K_{c,\rho}^{nl}(r)$$

is localized near $r = r_0$, and other terms are small.

Linearized structure inversion

$$\int_0^R \left[\mathcal{K}_{c^2,\rho}(r_0,r) - \mathcal{T}(r_0,r)\right]^2 \mathrm{d}r + \beta \int_0^R \mathcal{C}_{\rho,c^2}(r_0,r)^2 \mathrm{d}r$$

$$+\mu \sum_{ij} E_{ij} c_i(r_0) c_j(r_0) \ ,$$

$$\int_0^R \mathcal{K}_{c^2,\rho}(r_0,r)\mathrm{d}r = 1$$

$$\sum_i c_i(r_0) Q_i^{-1} \psi_\lambda(\omega_i) = 0 \ , \lambda = 0, \ldots, \Lambda \ .$$

$$\mathcal{K}_{c^2,\rho}(r_0,r) = \sum_i c_i(r_0) K^i_{c^2,\rho}(r)$$

$$\mathcal{C}_{\rho,c^2}(r_0,r) = \sum_i c_i(r_0) K^i_{\rho,c^2}(r)$$

Inverse problem for EOS

From equation of state $\gamma_1 = \gamma_1(p, \rho, Y, Z)$

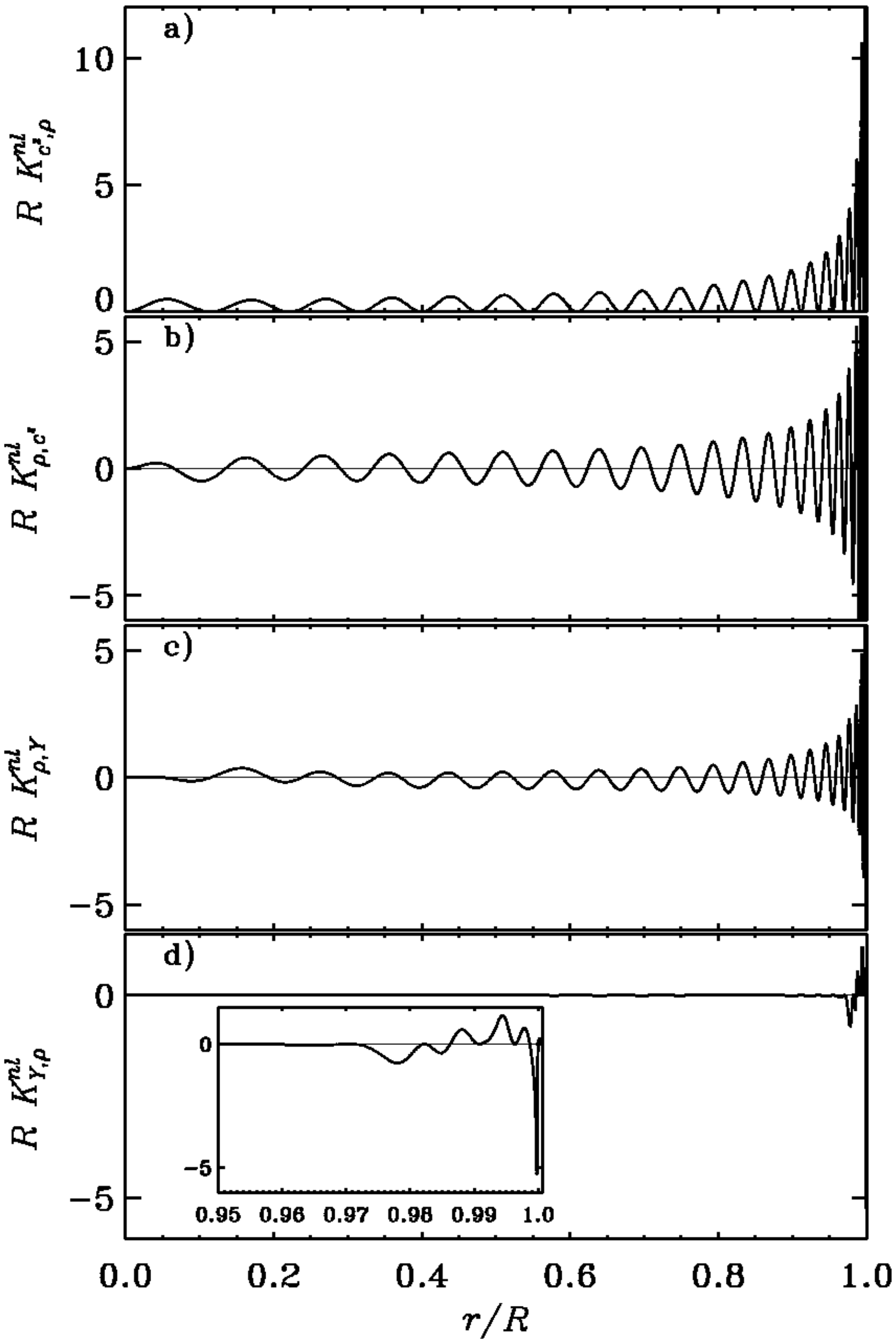
Hence

$$\frac{\delta_r \gamma_1}{\gamma_1} = \left(\frac{\partial \ln \gamma_1}{\partial \ln p}\right)_{\rho,Y} \frac{\delta_r p}{p} + \left(\frac{\partial \ln \gamma_1}{\partial \ln \rho}\right)_{p,Y} \frac{\delta_r \rho}{\rho} + \left(\frac{\partial \ln \gamma_1}{\partial Y}\right)_{p,\rho} \delta_r Y + \left(\frac{\delta \gamma_1}{\gamma_1}\right)_{\text{int}}$$

Using $c^2 = \gamma_1 u \,, \quad u = p/\rho$

$$\frac{\delta \omega_{nl}}{\omega_{nl}} = \int K_{u,Y}^{nl}(r) \frac{\delta_r u(r)}{u(r)} \mathrm{d}r + \int K_{Y,u}^{nl}(r) \delta_r Y(r) \mathrm{d}r + \int K_{c,\rho}^{nl}(r) \left(\frac{\delta \gamma_1}{\gamma_1}\right)_{\text{int}} \mathrm{d}r + \frac{G_{\text{surf}}(\omega_{nl})}{Q_{nl}} + \epsilon_{nl} \,.$$

Kernels


a)
10
5
0
$R\ K^{nl}_{c^2,\rho}$
b)
5
0
−5
$R\ K^{nl}_{\rho,c^2}$
c)
5
0
−5
$R\ K^{nl}_{\rho,Y}$
d)
0
−5
$R\ K^{nl}_{Y,\rho}$
0
−5
0.95 0.96 0.97 0.98 0.99 1.0
0.0 0.2 0.4 0.6 0.8 1.0
r/R

SVD analysis to understand what's happening in (R)LS

Can make singular value decomposition (SVD) of matrix A :

$A = U \Sigma V^T$

where U and V are orthogonal matrices (i.e. $U^TU=I$ and $V^TV=I$) with column vectors $\underline{u}^{(i)}$ and $\underline{v}^{(i)}$ say, and $\Sigma = \mathrm{diag}(s_1,s_2,\ldots,s_R)$ is a diagonal matrix whose elements are the singular values of $s_1 \geq s_2 \geq \ldots \geq s_R$ of matrix A

Least-squares solution:

$$\underline{x} = (A^TA)^{-1}A^T\underline{b} = V\,\Sigma^{-1}U^T\underline{b}.$$

Hence

$$\underline{x} = \sum_{j=1}^{R} \frac{\underline{\underline{u}}^{(j)}\cdot\underline{\underline{b}}}{s_j}\,\underline{v}^{(j)}$$

Small singular values cause any errors in $\underline{b}$ to "blow up" in the solution. This is why unregularized least-squares hits a problem.

Note the roles of U and V: the data are projected onto the $\underline{u}^{(i)}$, while the $\underline{v}^{(i)}$ form a basis for the solution vector $\underline{x}$.

$$\Omega(r) = \sum x_k\phi_k(r) = \sum_{j=1}^{R} \frac{\underline{\underline{u}}^{(j)}\cdot\underline{\underline{b}}}{s_j}\left(\sum v_k^{(j)}\phi_k(r)\right)$$

Truncated SVD inversion

Since small singular values cause a problem, one egularization method is just to truncate the summation at j=K, say, when the singular values go below some threshold value:

$$\underline{\mathsf{x}}_{\mathsf{TSVD}} = \sum_{j=1}^{K} \frac{\underline{\mathsf{u}}^{(j)} \cdot \underline{\mathsf{b}}}{s_j} \underline{\mathsf{v}}^{(j)}$$

It turns out that the solution of the RLS problem with regularization in standard form is

$$\underline{x}_{\mathrm{RLS}} = \sum_{j=1}^{R} \frac{s_j^2}{\lambda^2 + s_j^2} \frac{\underline{u}^{(j)} \cdot \underline{b}}{s_j} \underline{v}^{(j)}$$

This is like the unregularized solution but each term is multiplied by a "filter"
$f_i = s_i^2 / (\lambda^2 + s_i^2)$.
When $s_i \gg \lambda$, $f_i \approx 1$; when $s_i \ll \lambda$, $f_i \approx 0$.
This is like truncated SVD but with a smoother cut-off.

This can be generalized to the RLS solution with a general smoothing matrix L. One needs the generalized singular value decomposition (GSVD) of the matrix pair (A,L):

A = U diag(α_i) W^{-1} , L = V diag(β_i) W^{-1} ;

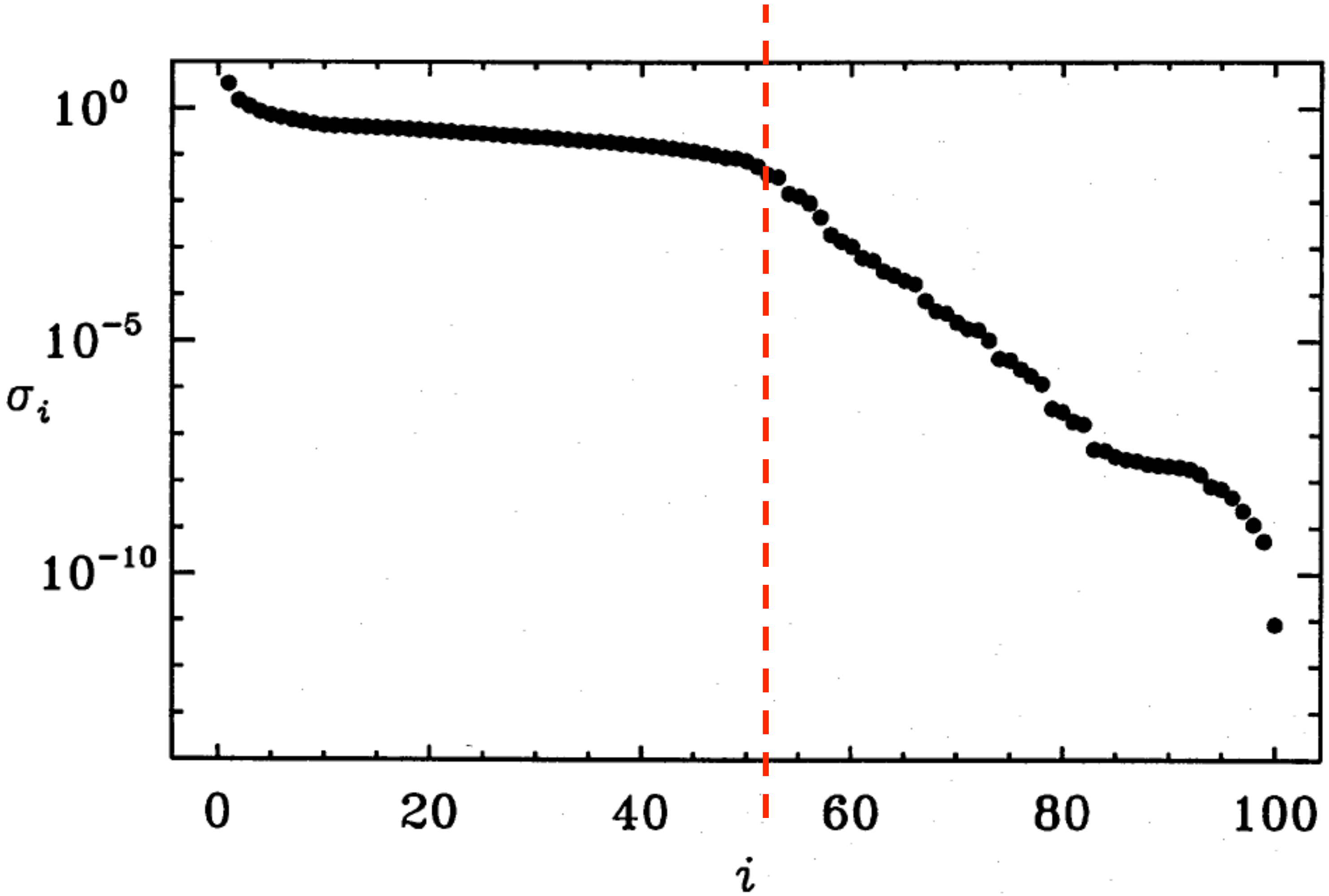
Then

$$\underline{x}_{RLS} = \sum_{j=1}^{R} f_j \frac{\underline{u}^{(j)} . \underline{b}}{\alpha_j} \underline{w}^{(j)}$$

where (roughly) $f_i = \gamma_i^2/(\lambda^2 + \gamma_i^2)$ with $\gamma_i = \alpha_i / \beta_i$.

See Christensen-Dalsgaard et al. (1993), MNRAS **264**, 541 for the details.

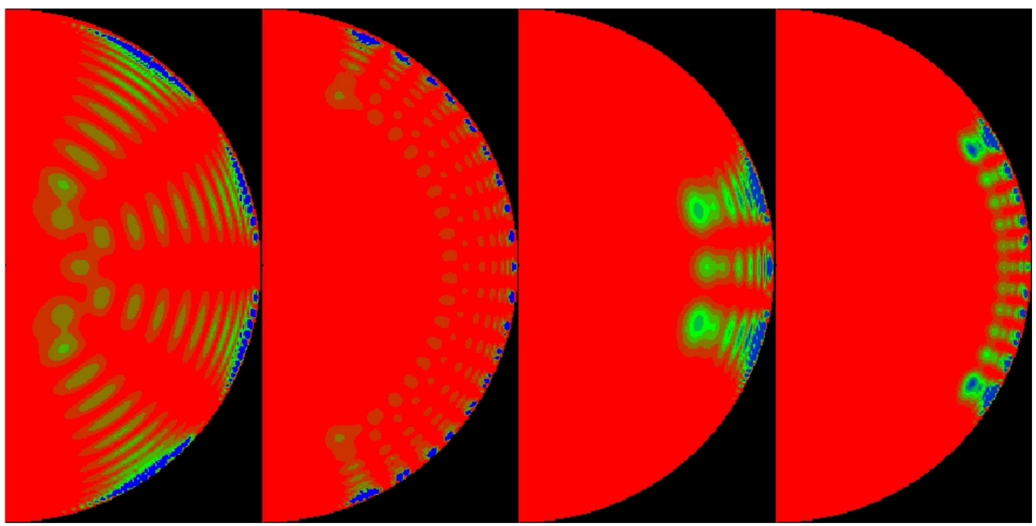
Singular values for 1-D rotation problem
with 834 p-modes and 100-point radial mesh



2-D rotation inversion

The 1-D rotation example developed in the last lecture is straightforwardy generalized to the case of Ω = Ω (r, θ).

$$\omega_{nlm} = \omega_{nl0} + m \int \int K_{nlm}(r, \theta)\, r\, dr\, d\theta$$



e.g. Schou et al. (1994) ApJ 433, 389

a-coefficients

Commonly the results of the analysis of the observations are not individual nlm frequencies.

Rather, the frequencies in each nl multiplet are fitted as a polynomial in m:

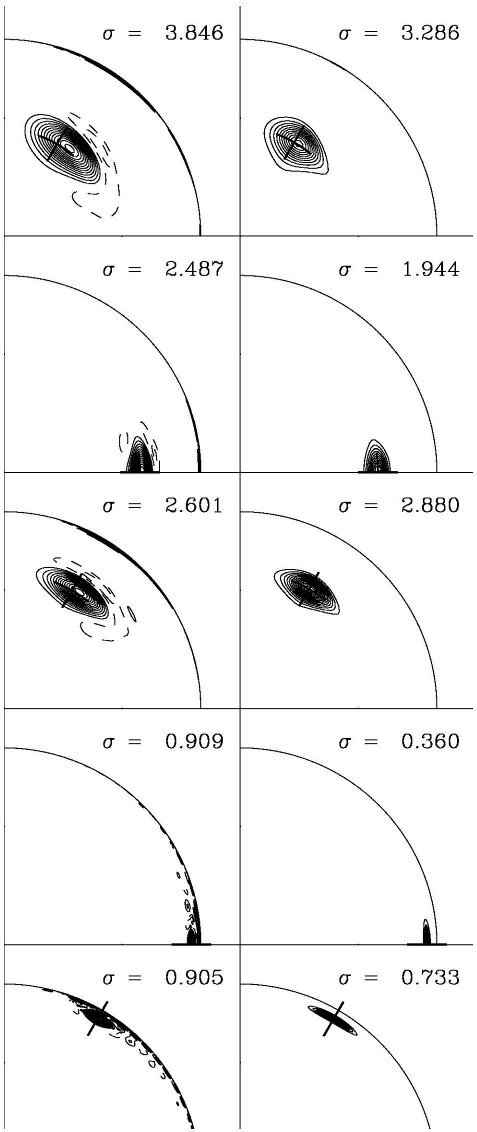
$$\nu_{nlm} = \nu_{nl} + \sum_j a_j(n,l)\mathcal{P}^{(l)}(m)$$

where the P's are even or odd polynomials in m of degree j.

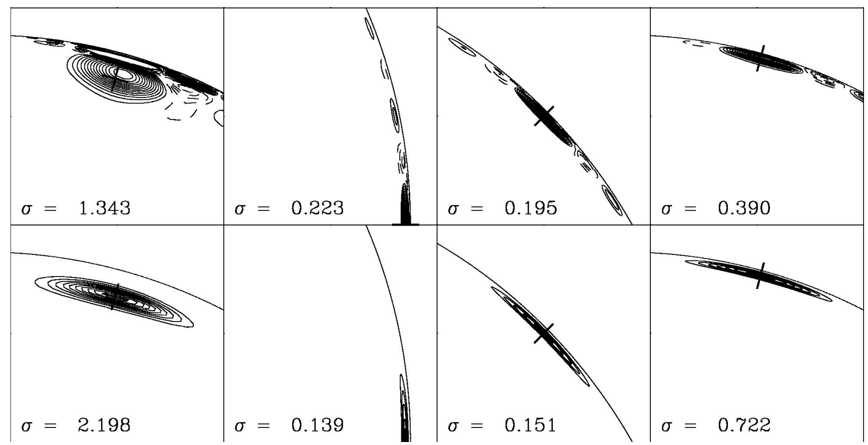
The odd coefficients a_j can be used as the data for rotation inversions.

2-D Rotational Averaging Kernels

RLS OLA



Close-Up

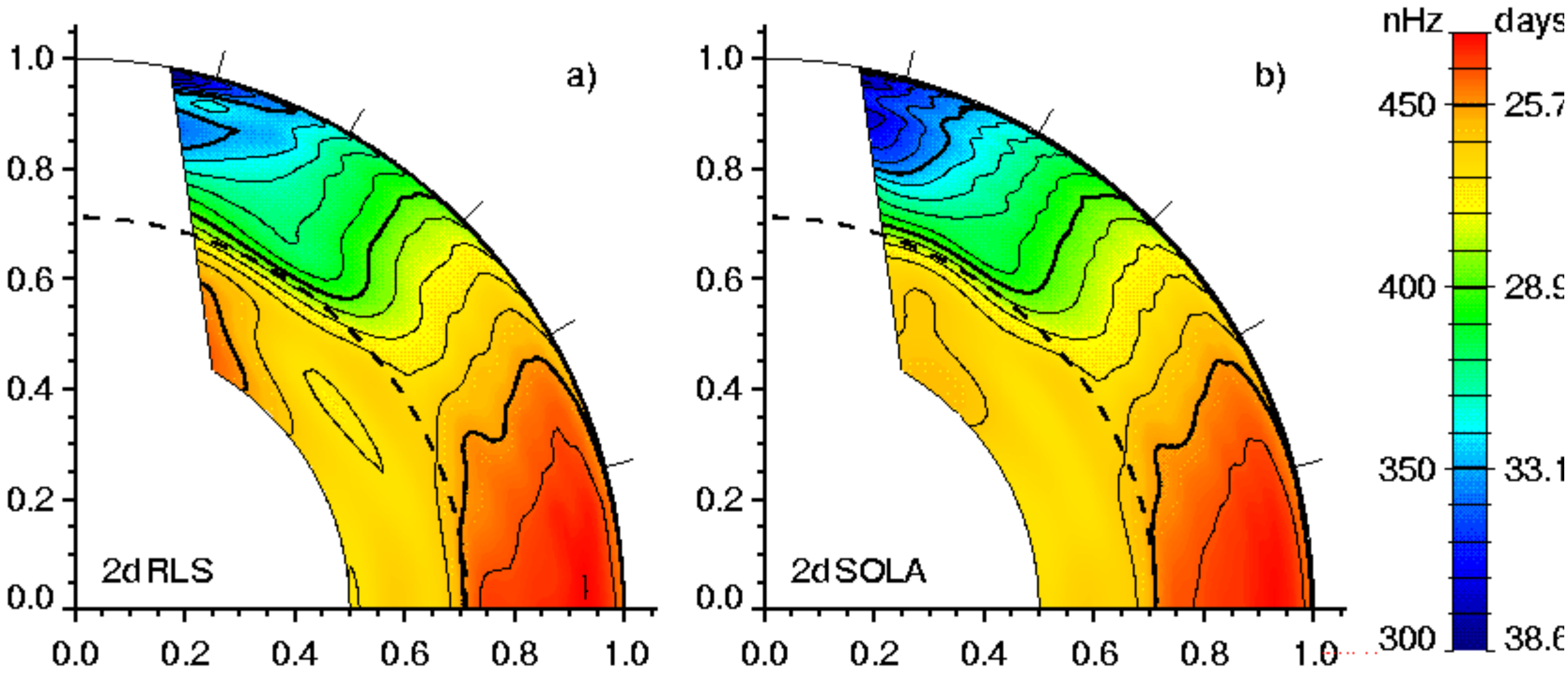


RLS

OLA

(1 s.d. uncertainties on inversion are indicated in nHz, for a typical MDI dataset)

Inferred rotation inside the Sun



from MDI data (Schou et al. 1998, ApJ 505, 390)

Transforming between variable pairs

A key to computing kernels for other variable pairs is how to use hydrostatic equilibrium to transform between δρ/ρ and δu/u, where u=p/δ .

Let $\psi(r)$ be a solution of

$$\left(\frac{\psi'}{r^2\rho}\right)' + \frac{4\pi G\rho\psi}{r^2 p} = \left(\frac{F(r)}{r^2\rho}\right)'$$

with ψ=0 at r=0 and r=R.
Then

$$\left\langle F(r)\frac{\delta\rho}{\rho}\right\rangle \equiv \left\langle -p\,(\psi/p)'\,\frac{\delta u}{u}\right\rangle$$

integration
from 0 to R

logarithmic
derivatives

$(c^2, \rho) \rightarrow (u, \gamma_1)$

$$K_{\gamma_1,u} \equiv K_{c^2,\rho}$$

$$K_{u,\gamma_1} \equiv K_{c^2,\rho} - p\left(\frac{\psi}{p}\right)'$$

with $F \equiv K_{\rho,c^2}$;

$(\gamma_1, \rho) \rightarrow (u, Y)$

$$K_{Y,u} \equiv \gamma_{,Y}\, K_{\gamma_1,\rho}$$

$$K_{u,Y} \equiv \gamma_{,p}\, K_{\gamma_1,\rho} - p\left(\frac{\psi}{p}\right)'$$

with $F \equiv (\gamma_{,p} + \gamma_{,\rho})K_{\gamma_1,\rho} + K_{\rho,\gamma_1}$; and

$(u, \gamma_1) \rightarrow (u, Y)$

$$K_{Y,u} \equiv \gamma_{,Y}\, K_{\gamma_1,u}$$

$$K_{u,Y} \equiv \gamma_{,p}\, K_{\gamma_1,u} + K_{u,\gamma} - p\left(\frac{\psi}{p}\right)'$$

with $F \equiv (\gamma_{,p} + \gamma_{,\rho})K_{\gamma_1,u}$.

Further reading:

Gough, D. (1985). *Inverting helioseismic data.*
Solar Physics **100**, 65-99.

Christensen-Dalsgaard, J. et al. (1990).
A comparison of methods for inverting helioseismic data.
Mon. Not. R. astr. Soc. **242**, 353-369.

Further reading:

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Inversion methods in helioseismology and solar tomography
J. Comp. Applied Math. 109, 1-39.

Thompson, M. J. et al. (2003).
The internal rotation of the Sun
Ann. Rev. Astron. Astrophys. 41, 599-643.