Lecture 13: ABELIAN GAUGE THEORIES: THE FRAMEWORK OF QED

With this chapter, we begin the study of a class of geometric models that have played a central role in the theory of fundamental interactions for more than a century: gauge theories. They are characterized by new physics properties and new technical difficulties.

We devote a first chapter to a simplest but physically important example, the Abelian gauge theory, the field theoretical framework for Quantum Electrodynamics (QED). However, since QED is discussed extensively elsewhere, we mainly focus on the formal aspects of Abelian gauge theories.

The set-up of the chapter is the following: we begin with elementary considerations about the massive vector field in perturbation theory. We show that coupling to matter field leads to field theories that are renormalizable in four dimensions only if the vector field is coupled to a conserved current. In the latter case the massless vector limit can be defined and the resulting field theory is gauge invariant.

We discuss the specific properties of gauge invariant theories. We then quantize gauge theories starting directly from first principles. The quantization of gauge theories involves gauge fixing. The formal equivalence between different gauges is established.

In section 13.6, regularization methods are described that overcome the new difficulties one encounters in gauge theories.

The Abelian gauge symmetry, broken by gauge fixing terms, leads to a set of WT identities, which are used to prove renormalizability.

The gauge dependence of correlation functions in a set of covariant gauges is determined.

Renormalization results imply renormalization group equations and we calculate the RG β-function at leading order.

As an introduction to the Standard Model of fundamental interactions at the microscopic scale, we then study the Abelian Higgs model, which also describes, in three dimensions, superconductors in a magnetic field.

13.1 The free massive vector field

The quantization of the free massive vector field is not a trivial extension of the quantization of the scalar field and thus requires a specific discussion.

Notation. In the first part of this section we work in real time with the metric $\{+,-,-,\cdots\}$ where the first component is the time component.

We then use Greek indices for the four space-time components, $\mu = 0,1,2,3$ and Roman indices for the space components $i = 1,2,3$, for example, the vector field is denoted by $A_\mu \equiv \{A_0, A_i\}$.

Space-time coordinates are denoted by $\{t \equiv x_0 = -ix_4, x_i\}$, where x_4 will be later in the section the euclidean or imaginary time.

Finally, time derivative is occasionally indicated by

$$\frac{\partial A_i}{\partial t} \equiv \dot{A}_i \,.$$

The local covariant action. The local $O(1,3)$ invariant classical action for a free massive vector field can be written as

$$\mathcal{A}(A) = -\int \mathrm{d}t\, \mathrm{d}^3x \left[\tfrac{1}{4}\sum_{\mu,\nu} F_{\mu\nu}(t,x)F^{\mu\nu}(t,x) - \tfrac{1}{2}m^2 \sum_{\mu} A_\mu(t,x)A^\mu(t,x)\right] \tag{13.1}$$

with

$$F_{\mu\nu}(t,x) = \partial_\mu A_\nu(t,x) - \partial_\nu A_\mu(t,x). \tag{13.2}$$

One may wonder about the peculiar form of the derivative term, but it is straightforward to verify that the additional covariant term one could think of adding, namely $\sum_{\mu,\nu} \partial_\mu A_\mu \partial^\nu A^\nu$, depending on its sign corresponds either to an A_0 field with a negative metric, or to an unbounded potential.

13.1.1 Quantization

Separating space and time components, the action (13.1) can be rewritten as ($\nabla_\perp \equiv (\partial_1, \partial_2, \partial_3)$ is the space gradient)

$$\mathcal{A}(A) = \int \mathrm{d}t\, \mathrm{d}^3x \left[\tfrac{1}{2} \sum_i \left(\dot{A}_i^2(t,x) - m^2 A_i^2(x) \right) - \tfrac{1}{4} \sum_{i,j} F_{ij}^2(t,x) \right.$$
$$\left. + \tfrac{1}{2} \left(\nabla_\perp A_0(t,x) \right)^2 + \tfrac{1}{2} A_0^2(t,x) - \sum_i A_i(t,x) \partial_i A_0(t,x) \right]. \quad (13.3)$$

The action (13.3) has a peculiar property: it does not involve the time derivative $\dot{A}_0$ and, thus, the time component A_0 of the vector field has no conjugate momentum. Therefore, A_0 is not a dynamical degree of freedom and the corresponding field equation

$$\frac{\delta \mathcal{A}}{\delta A_0(t,x)} = \left(-\nabla_\perp^2 + m^2 \right) A_0(t,x) + \sum_i \partial_i \dot{A}_i(t,x) = 0\,, \quad (13.4)$$

is a constraint equation.

This feature reflects the property that a massive vector field has only three physical degrees of freedom corresponding to a space vector in the particle rest-frame.

Using the A_0 field equation to eliminate A_0 from the action, one obtains the reduced Lagrangian density

$$\mathcal{L}(\mathbf{A}) = \tfrac{1}{2}\sum_{i,j} \dot{A}_i(t,x)\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla_\perp^2 - m^2}\right)\dot{A}_j(t,x) - \tfrac{1}{2}m^2\sum_i A_i^2(t,x) - \tfrac{1}{4}\sum_{i,j} F_{ij}^2(t,x)\,.$$

We denote by E_i (because it becomes the electric field in the massless limit) the momentum conjugated to A_i, given by

$$E_i = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = \dot{A}_i - \partial_i\left(\nabla_\perp^2 - m^2\right)^{-1}\sum_j \partial_j\dot{A}_j\,.$$

The Hamiltonian density $\mathcal{H}$, Legendre transform of the Lagrangian, has the form

$$\mathcal{H}(\mathbf{E},\mathbf{A}) = \tfrac{1}{2}\sum_{i,j} E_i(x)\left(\delta_{ij} - \frac{\partial_i\partial_j}{m^2}\right)E_j(x) + \tfrac{1}{2}m^2\sum_i A_i^2(x) + \tfrac{1}{4}\sum_{i,j} F_{ij}^2(x)\,. \tag{13.5}$$

The differential operator $-\partial_i\partial_j$ being non-negative, the Hamiltonian is positive. The quantization procedure from now on is standard.

13.1.2 Euclidean field theory

After continuation to imaginary time, it leads to an euclidean field integral in which appears the euclidean reduced Lagrangian. This Lagrangian has unpleasant properties: it is non-local and not $O(4)$ space–time symmetric.

However, since the dependence in A_0 of the action (13.3) is quadratic, we can proceed in the following way: we substitute in the field integral representation of the partition function the initial euclidean Lagrangian.

We then perform the Gaussian integral over the time component. As we know this is equivalent to solving the corresponding equation of motion, and we thus recover the reduced Lagrangian.

Finally, the determinant resulting from the integration is field independent and disappears in the normalization of the field integral.

This shows that if we start from the euclidean action,

$$\mathcal{S}(A) = \int \mathrm{d}^4 x \left[\tfrac{1}{4} \sum_{\mu,\nu} F_{\mu\nu}^2(x) + \sum_{\mu} \tfrac{1}{2} m^2 A_\mu^2(x) \right] \tag{13.6}$$

with

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \tag{13.7}$$

we directly obtain the correct quantized action (note that in the continuation to imaginary time we have set $A_0 = -iA_4$).

Two-point connected correlation function. The generating functional $\mathcal{Z}(J)$ of A_μ-field correlation functions is then given by

$$\mathcal{Z}(J) = \int [\mathrm{d}A_\mu] \exp\left[-\mathcal{S}(A) + \int \mathrm{d}^4x\, \mathbf{J}(x) \cdot \mathbf{A}(x)\right]. \tag{13.8}$$

In terms of the Fourier representations of the vector field and the current,

$$A_\mu(x) = \int \mathrm{d}^4k\; \mathrm{e}^{ikx}\, \tilde{A}_\mu(k), \quad J_\mu(x) = \int \mathrm{d}^4k\; \mathrm{e}^{ikx}\, \tilde{J}_\mu(k)$$

the action can be rewritten as

$$\mathcal{S}(A) = (2\pi)^4 \tfrac{1}{2} \int \mathrm{d}^4k \sum_{\mu,\nu} \tilde{A}_\mu(-k) \left[(k^2 + m^2)\delta_{\mu\nu} - k_\mu k_\nu\right] \tilde{A}_\nu(k)$$

and

$$\int \mathrm{d}^4x\, \mathbf{J}(x) \cdot \mathbf{A}(x) = (2\pi)^4 \int \mathrm{d}^4k \sum_\mu \tilde{J}_\mu(-k) \tilde{A}_\mu(k).$$

The Gaussian integration amounts to replacing $\tilde{A}_\mu(k)$ by the solution of the corresponding field equation,

$$\sum_\nu \left[(k^2+m^2)\delta_{\mu\nu} - k_\mu k_\nu\right] \tilde{A}_\nu(k) = \tilde{J}_\mu(-k) \;\Rightarrow\; \tilde{A}_\mu(k) = \sum_\nu \tilde{\Delta}_{\mu\nu}(k)\tilde{J}_\nu(-k)$$

with

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{\delta_{\mu\nu} + k_\mu k_\nu /m^2}{k^2+m^2}.$$

The integration yields the generating functional $\mathcal{Z}(J)$. The generating functional $\mathcal{W}(J)$ of connected functions then is,

$$\mathcal{W}(J) = \ln \mathcal{Z}(J) = \tfrac{1}{2}(2\pi)^4 \int \mathrm{d}^4k \sum_{\mu,\nu} \tilde{J}_\mu(k)\tilde{\Delta}_{\mu\nu}(k)\tilde{J}_\nu(-k). \qquad (13.9)$$

We conclude that $\tilde{\Delta}_{\mu\nu}(k)$ is the vector field propagator in the Fourier representation.

At the pole $k_\mu = ime_\mu$ with $e_\mu^2 = 1$ (the mass-shell), the numerator $\delta_{\mu\nu} - e_\mu e_\nu$ is a projector transverse to the vector $\mathbf{k}$ and only three modes are propagated.

In the rest frame $k_i = 0$, the numerator becomes $\delta_{\mu\nu} - \delta_{\mu 4}\delta_{\nu 4}$. The propagator thus propagates three components belonging to the vector representation of the static group $O(3)$, subgroup of $O(1,3)$.

Power counting. The propagator is such that in some directions it goes to a constant for $|\mathbf{k}|$ large and the field has a power counting dimension $d/2$ in d dimensions. As we have already indicated in section 6.9, field theories involving massive vector fields coupled to matter are renormalizable, in general, only in dimensions $d = 2$ and cannot be used for renormalizable theories in four dimensions.

A directly related disease (because the propagator is a homogeneous function of $\mathbf{k}$ and m) is the impossibility to pass continuously to the massless limit.

Conserved current. If the external source $J_\mu(x)$ has the form of a conserved current,

$$\sum_\mu \partial_\mu J_\mu(x) = 0 \ \Leftrightarrow \ \sum_\mu k_\mu \tilde{J}_\mu(k) = 0\,, \tag{13.10}$$

then expression (13.9) reduces to

$$\mathcal{W}(J) = \tfrac{1}{2} \int \mathrm{d}^4 k \sum_\mu \tilde{J}_\mu(k) \frac{1}{k^2 + m^2} \tilde{J}_\mu(-k).$$

This means that the propagator can be replaced by $\delta_{\mu\nu}/(k^2 + m^2)$ which behaves like the propagator of a scalar particle. In this case both problems of large momentum behaviour and massless limit are solved.

One may now wonder why we have not used at once such a propagator: the reason is that it propagates, in addition to a vector field, a scalar particle with negative metric (like the regulator fields). This is better illustrated by a short calculation.

13.1.3 More general propagators, interpretation

We add to the action (13.6) the action of an uncoupled massive scalar field χ:

$$\mathcal{S}(A,\chi)=\mathcal{S}(A)+\tfrac{1}{2}\int \mathrm{d}^4x\left[\left(\nabla\chi(x)\right)^2+M^2\chi^2(x)\right]. \tag{13.11}$$

The generating functional of correlation functions becomes

$$\mathcal{Z}(J)=\int [\mathrm{d}A_\mu]\,[\mathrm{d}\chi]\exp\left[-\mathcal{S}(A,\chi)+\int \mathrm{d}^4x\,\mathbf{J}(x)\cdot\mathbf{A}(x)\right]. \tag{13.12}$$

In the absence of a source for χ, the integration over χ yields a constant factor multiplying the functional (13.8).

In the integral (13.12), we now change variables, $\mathbf{A}\mapsto\mathbf{A}'$ with (note the i factor),

$$\mathbf{A}(x)=\mathbf{A}'(x)+\frac{i}{m}\nabla\chi(x). \tag{13.13}$$

This change, called gauge transformation, leaves $F_{\mu\nu}$ invariant. If the source satisfies the conservation equation (13.10), the source term is not modified. Only the vector field mass term is affected:

$$\tfrac{1}{2}m^2\mathbf{A}^2(x) = \tfrac{1}{2}m^2{\mathbf{A}'}^2(x) + im\mathbf{A}'(x)\cdot\nabla\chi(x) - \tfrac{1}{2}\big(\nabla\chi(x)\big)^2.$$

This main effect is to cancel the χ kinetic term. One obtains

$$\mathcal{S}(A',\chi) = \mathcal{S}(A') + \int \mathrm{d}^4x\big(im\mathbf{A}'(x)\cdot\nabla\chi(x) + \tfrac{1}{2}M^2\chi^2(x)\big).$$

After an integration by parts of $\mathbf{A}'\cdot\nabla\chi$, one integrates over $\chi(x)$. This amounts to replacing $\chi(x)$ by the solution of the $\chi(x)$ field equation,

$$\chi(x) = \frac{im}{M^2}\nabla\cdot\mathbf{A}'(x).$$

One obtains the new action (renaming now $\mathbf{A}'\mapsto\mathbf{A}$),

$$\mathcal{S}_\xi(A) = \mathcal{S}(A) + \frac{1}{2\xi}\int\big(\nabla\cdot\mathbf{A}(x)\big)^2\mathrm{d}^4x \quad \text{with} \quad \xi = M^2/m^2\,. \qquad (13.14)$$

The addition of this term is called gauge fixing. By varying ξ from 0 to $+\infty$, one reproduces a set of gauges: $\xi = 0$ corresponds to Landau's gauge, $\xi = 1$ is Feynman's gauge. In the limit $\xi = \infty$ (the unitary gauge), one recovers the initial unitary field theory.

The corresponding propagator is then

$$[\tilde{\Delta}_\xi]_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2 + m^2} + \frac{(\xi - 1) k_\mu k_\nu}{(k^2 + m^2)(k^2 + \xi m^2)}. \tag{13.15}$$

For all finite values of ξ the propagator behaves at large momentum like a scalar field propagator. However, the new propagator has an unphysical pole at $k^2 = -\xi m^2$. The propagator can be rewritten as

$$[\tilde{\Delta}_\xi]_{\mu\nu}(k) = \frac{\delta_{\mu\nu} + k_\mu k_\nu / m^2}{k^2 + m^2} - \frac{k_\mu k_\nu}{m^2 (k^2 + \xi m^2)}, \tag{13.16}$$

showing that it propagates, in addition to the vector particle, an unphysical scalar particle. In the mass-shell limit $k^2 \to -m^2$, the ξ dependence cancels.

The massless limit. For all finite values of ξ, the propagator $[\tilde{\Delta}_\xi]_{\mu\nu}(k)$ behaves at large momentum as a scalar propagator and has the zero mass limit,

$$[\tilde{\Delta}_\xi]_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2} + (\xi - 1)\frac{k_\mu k_\nu}{(k^2)^2}. \tag{13.17}$$

However, for values of $\xi \neq 1$, the term proportional to $1/k^4$ may generate IR divergences in interacting theories in four dimensions, a problem that requires some care.

13.2 Interaction with fermion matter

We conclude from the preceding analysis that only vector fields coupled to conserved currents and, thus, associated with continuous symmetries, provide suitable candidates for the construction of theories renormalizable in four dimensions. We now give an explicit, an physically relevant, example.

We start from a free action for a massive Dirac fermion,

$$\mathcal{S}_{\mathrm{F}}(\bar{\psi}, \psi) = -\int \mathrm{d}^4 x\, \bar{\psi}(x)\, (\not{\partial} + M)\, \psi(x), \tag{13.18}$$

and want to add an $O(4)$ invariant coupling to a vector field.

As we have already noted in section 11.8.3, the action (13.18) for a free Dirac fermion has a $U(1)$ symmetry associated with the conservation of the fermion number (Λ is constant)

$$\psi(x) = \mathrm{e}^{i\Lambda}\, \psi'(x), \quad \bar{\psi}(x) = \mathrm{e}^{-i\Lambda}\, \bar{\psi}'(x). \tag{13.19}$$

To this symmetry corresponds a current whose expression is obtained by calculating the variation of the action under a space-dependent group transformation (see appendix A12.1). If Λ is space-dependent, the variation of the action is

$$\delta \mathcal{S}_{\mathrm{F}} = -i \int \mathrm{d}^4 x \, \bar{\psi}(x) \not{\partial} \Lambda(x) \psi(x)$$

and, thus, the corresponding conserved current $J_\mu(x)$ is

$$J_\mu(x) = -i \bar{\psi}(x) \gamma_\mu \psi(x).$$

The only $O(4)$ symmetric interaction term which is renormalizable, from the point of view of power counting, is proportional to

$$\int \mathrm{d}^4 x \, \bar{\psi}(x) \not{A}(x) \psi(x) \, .$$

This interaction term has precisely the form of a vector field linearly coupled to the conserved current $J_\mu(x)$.

The action of a fermion interacting with a vector field then takes the form

$$\mathcal{S}(A,\bar{\psi},\psi)=\mathcal{S}(A)-\int \mathrm{d}^4x\,\bar{\psi}(x)\,(\not{\partial}+M+ie\not{A})\,\psi(x), \tag{13.20}$$

where $\mathcal{S}(A)$ is the vector free action (13.6) and the parameter e is the current–vector field coupling constant.

The vector field propagator. The transformations of section 13.1, which have led to the propagator (13.15), rely on a change of variables in the field integral of the form (13.13):

$$\mathbf{A}(x)=-\frac{1}{e}\nabla\Lambda(x)+\mathbf{A}'(x). \tag{13.21}$$

One verifies that the induced variation of $\mathcal{S}_{\mathrm{F}}(A,\bar{\psi},\psi)$ can be cancelled by a change of the fermion variables of the form of the transformation (13.19) with a space-dependent function $\Lambda(x)$:

$$\psi(x)=\mathrm{e}^{i\Lambda(x)}\,\psi'(x),\quad \bar{\psi}(x)=\mathrm{e}^{-i\Lambda(x)}\,\bar{\psi}'(x). \tag{13.22}$$

Therefore, the algebraic transformations that allow to pass from a unitary but non-renormalizable action to an non-unitary but renormalizable action remain justified.

The action of massive QED with fermion matter in a renormalizable gauge then can be written as

$$\mathcal{S}_\xi(A,\bar{\psi},\psi) = \int \mathrm{d}^4x \Big[\tfrac{1}{4}\sum_{\mu,\nu} F^2_{\mu\nu}(x) + \tfrac{1}{2}m^2 \mathbf{A}^2(x) + \tfrac{1}{2\xi}\big(\nabla\cdot\mathbf{A}(x)\big)^2 - \bar{\psi}(x)\,(\not{\partial} + M + ie\not{A})\,\psi(x)\Big]. \tag{13.23}$$

This is the action for a massive photon interacting with charged fermion matter.

The equivalence remains valid for the action and transverse A-field correlation functions. Eventually, we shall have to discuss the problem of matter correlation functions. However, we first want to examine the special properties of the massless vector field theory.

Higher spins. The strategy that has allowed transforming a unitary non-renormalizable theory into a renormalizable non-unitary theory does not work for higher spin fields even when the fields are coupled to conserved currents like the spin two energy–momentum tensor or the spin $3/2$ field coupled to the supersymmetry current.

13.3 Massless vector field: Abelian gauge symmetry

In the massless vector field limit $m = 0$, the classical action (13.20) reduces to

$$\mathcal{S}(A, \bar{\psi}, \psi) = \int \mathrm{d}^4 x \left[\tfrac{1}{4} \sum_{\mu,\nu} F_{\mu\nu}^2(x) - \bar{\psi}(x) \left(\partial\!\!\!/ + M + ie A\!\!\!/ \right) \psi(x) \right]. \qquad (13.24)$$

The action then has a remarkable property: it is invariant under $U(1)$ gauge transformations as defined by equations (13.21, 13.22)). Gauge transformations are local (*i.e.*, with space-dependent parameters) group transformations.

Such a symmetry is called a $U(1)$ gauge symmetry and the vector field is then called a gauge field. From the physics viewpoint, the quantum field theory describes the interaction of a charged fermion with the electromagnetic field, which is the basis of QED.

By constructing a renormalizable theory for a vector field in four dimensions we have been led naturally to introduce a new geometric structure, an Abelian (because the $U(1)$ is Abelian or commutative) gauge theory, which is the quantum version of Maxwell's electromagnetism.

13.3.1 Gauge symmetry

Gauge symmetry has a geometric interpretation that is worth describing. The invariance of the fermion part of the action can be seen as a consequence of the replacement of the derivative ∂_μ of the free fermion theory by the covariant derivative $\partial_\mu + ieA_\mu$, which allows the transformation of the gauge field to cancel the term coming from the derivative. We are thus reminded of the concepts of covariant derivative, affine connection, curvature and parallel transport introduced for Riemannian manifolds.

In the corresponding terminology, $A_\mu(x)$ is the affine connection.

The differential operator $\mathrm{D}_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative and, for later purpose, we introduce the notation $\not{\mathrm{D}} = \sum_\mu \gamma_\mu \mathrm{D}_\mu$.

The curvature tensor (which is the electromagnetic field in QED) is

$$F_{\mu\nu} = \frac{1}{ie}\left[\mathrm{D}_\mu, \mathrm{D}_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

Finally, the parallel transporter $U(C_{xy})$ (a notion also essential for lattice gauge theories), where C_{xy} is an oriented, piecewise differentiable, curve with end-points x, y, is an element of the $U(1)$ group. Since the group is Abelian, it can be expressed explicitly in terms of a line integral as

$$U(C_{xy}) = \exp\left[-ie \oint_{C_{xy}} \sum_\mu A_\mu(s)\mathrm{d}s_\mu\right]. \tag{13.25}$$

One immediately verifies that the transformation of $U(C_{xy})$ induced by a gauge transformation (13.21) of A_μ, has the form

$$U(C) = U'(C)\exp\left[i\oint_{C_{xy}}\sum_\mu \partial_\mu\Lambda(s)\mathrm{d}s_\mu\right] = U'(C_{xy})\exp\left[i\big(\Lambda(y)-\Lambda(x)\big)\right],$$

which is such that $\bar\psi(y)U(C_{xy})\psi(x)$ is gauge invariant.

Note that the parallel transporter around a closed curve is gauge invariant. According to Stokes theorem, for a simple closed curve, $U(C)$ can be rewritten as

$$U(C) = \exp\left[\tfrac{1}{2}\int_D \sum_{\mu,\nu} \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu\, F_{\mu\nu}(x)\right],$$

that is, in terms of a surface integral over a surface D that has the curve C as a boundary: $\partial D = C$. This expression relates the curvature tensor $F_{\mu\nu}$ to parallel transport around a closed curve, remark that also will be useful in the construction of lattice gauge theories.

Charged scalar fields. With the geometric ideas of parallel transport in mind, it is straightforward to construct a gauge action for a charged scalar field. One starts from the $U(1)$ invariant action

$$\mathcal{S}_{\mathrm{B}}(\phi) = \int \mathrm{d}^4 x \left[|\nabla\phi(x)|^2 + U(|\phi(x)|^2) \right],$$

in which $\phi(x)$ is a complex field, and replaces the derivative ∂_μ by a covariant derivative. The explicit form of the covariant derivative depends on the charge assigned to the field $\phi(x)$. If one assumes that ϕ couples to A_μ with a coupling constant e_{B}, one obtains,

$$\mathcal{S}_{\mathrm{B}}(\phi, A_\mu) = \int \mathrm{d}^4 x \left[|\mathbf{D}\phi(x)|^2 + U(|\phi(x)|^2) \right], \qquad (13.26)$$

where $\mathbf{D}$ is the covariant derivative, a vector with components

$$\mathrm{D}_\mu = \partial_\mu + i e_{\mathrm{B}} A_\mu \,. \qquad (13.27)$$

Quantization of charge. If we introduce additional charged fields, we have to assign them charges which characterize their transformation properties under $U(1)$. A delicate question arises here. Since the $U(1)$ group has the same Lie algebra as the group of translations, properties depending only on infinitesimal group transformations do not require a quantization of charge. In particular, in perturbation theory WT identities are true even if the charges are not rationally related and the necessity of a quantization of charges, therefore, never appears. The conventional wisdom is that QED does not imply charge quantization.

However, note that the only known non-perturbative regularization, based on a lattice approximation, involves group elements in the form of parallel transporters and, therefore, requires charge quantization (see section 13.6).

The question remains open.

13.3.2 The massless vector field as a limit

Geometric and physical considerations single out the theory with a massless vector field: it has an exact gauge symmetry and it describes the physics of QED because the photon is, at least to an excellent approximation, found experimentally to be massless. Therefore, we could have restricted our discussion to the massless case, as we shall do in the non-Abelian case.

However, considering the massless theory as a limit of the massive theory provides us with a simple resolution to several difficulties.

First, we have already seen in section 13.1 that we could not write at once a propagator for a massless vector field. In a gauge invariant theory, this difficulty persists beyond perturbation theory.

Indeed, the gauge symmetry implies that the action does not depend on one of the dynamical variables which is related to a gauge transformation.

As a consequence, the field integral cannot be defined because the volume of the gauge group, which can be factorized, is infinite.

This difficulty reflects the property that in classical electrodynamics only the electromagnetic tensor $F_{\mu\nu}$ is physical. The vector field A_μ is a mathematical entity that enables deriving the classical field equations by varying a local covariant action. It contains redundant degrees of freedom whose evolution is not determined by the field equations.

We show in section 13.4 how the gauge action can be quantized starting from first principles. The procedure is less straightforward and leads to non-covariant gauges, problems whose analysis we want to postpone.

Starting from the massive theory, performing the algebraic transformations indicated in section 13.1 (which lead to gauge fixing) and taking the massless limit, we have been able to define directly the gauge theory.

Finally, the mass of the vector field provides the theory with a natural IR cut-off that allow to analyse the IR problems of the massless theory.

13.4 Canonical quantization: non-covariant gauges

Although we have been able to construct a gauge invariant theory as a limit of a theory of a massive vector field coupled to a conserved current, it is useful to contemplate the difficulties one encounters when one tries to quantize a gauge theory starting from first principles. Moreover, in the case of non-Abelian gauge symmetries, the massless limit is not continuous. Therefore, we show how, starting directly from the classical field equations of a gauge invariant theory, it is possible to recover the field integral representation of the generating functional of correlation functions.

The problem can be solved by several different strategies and we present two of them, corresponding to so-called Coulomb's gauge and temporal gauge.

We again consider the simple action

$$\mathcal{S}(A,J)=\mathcal{S}(A)-\int \mathrm{d}^4x\,\mathbf{J}(x)\cdot\mathbf{A}(x),\quad \mathcal{S}(A)=\tfrac{1}{4}\int \mathrm{d}^4x\sum_{\mu,\nu}F_{\mu\nu}^2(x)\,, \tag{13.28}$$

of a gauge field coupled to a conserved current.

13.4.1 Coulomb's gauge

We first proceed as in the massive case and eliminate the gauge field time component from the action. Taking into account current conservation (note $J_0=iJ_4$), we then obtain the integral of a reduced Lagrangian density:

$$\mathcal{L}(A_i)=\sum_{i,j}\Bigl[\tfrac{1}{2}\dot{A}_i(t,x)\left(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla_\perp^2}\right)\dot{A}_j(t,x)-\tfrac{1}{4}F_{ij}^2(t,x)$$
$$+J_i(t,x)\left(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla_\perp^2}\right)A_j(t,x)\Bigr].$$

In contrast with the massive case, the action depends only on

$$\sum_j \left(\delta_{ij} - \partial_i\partial_j/\nabla_\perp^2\right) A_j(t,x).$$

After Fourier transformation, this implies that the action does not depend on the component of $\tilde{A}_i(t,\hat{k})$ along $\hat{k}$, the space component of the momentum $\mathbf{k}$. We recover the well-known property that a massless vector field has only two physical components. We thus expand the vector $\tilde{A}_i(t,\hat{k})$ on a transverse basis $e_i^a(\hat{k})$, calling $\tilde{\mathfrak{A}}_a$ the corresponding two components (the polarizations):

$$\hat{k}\cdot\mathbf{e}^a(\hat{k}) = 0\,,\quad \mathbf{e}^a(\hat{k})\cdot\mathbf{e}^b(\hat{k}) = \delta_{ab},\quad \tilde{\mathbf{A}}(t,\hat{k}) = \frac{\hat{k}\cdot\tilde{\mathbf{A}}}{\hat{k}^2}\hat{k} + \sum_{a=1}^{2}\mathbf{e}^a(\hat{k})\tilde{\mathfrak{A}}_a(\mathbf{k})\,,$$

and J_a the corresponding sources. The Lagrangian density in these variables becomes

$$\mathcal{L}(\mathfrak{A}_a) = \sum_{a=1,2}\left[\sum_\mu \partial_\mu\mathfrak{A}_a(t,x)\partial^\mu\mathfrak{A}_a(t,x) + J_a(t,x)\mathfrak{A}_a(t,x)\right].$$

The quantization now is straightforward. Eventually, one obtains a field integral over the fields $\mathfrak{A}_a$. However, the corresponding action, once expressed in terms of the initial current J_μ, is non-local. One can reintroduce the components A_i of the gauge field provided one multiplies the integrand by $\delta(\sum_i \partial_i A_i)$. The last step, that is, returning to an integral involving the time component, is the same as in the massive case. The final result is the euclidean generating functional in Coulomb's gauge:

$$\mathcal{Z}_{\text{Coul.}}(\mathbf{J}) = \int \left[\mathrm{d}A_\mu(x) \delta\left(\sum_i \partial_i A_i(x) \right) \right] \mathrm{e}^{-\mathcal{S}(A,J)}, \qquad (13.29)$$

where $\mathcal{S}(A, J)$ is defined in (13.28). Coulomb's gauge, in the Abelian case, has a nice physical interpretation: only physical degrees of freedom propagate, but it leads to non-covariant calculations, and this is a serious drawback.

In particular, the gauge field propagator ($\mathbf{k} \equiv \{k_4, \mathbf{k}_\perp\}$) becomes

$$\begin{cases} \tilde{W}^{(2)}_{44}(k) = \dfrac{1}{\mathbf{k}_\perp^2}, \\ \tilde{W}^{(2)}_{i4}(k) = 0\,, \\ \tilde{W}^{(2)}_{ij}(k) = \dfrac{1}{\mathbf{k}^2}\left(\delta_{ij} - \dfrac{k_i k_j}{\mathbf{k}_\perp^2}\right). \end{cases}$$

The time component does not decrease in the large time direction. Therefore, with this propagator the theory is not explicitly renormalizable by power counting.

We still have to prove that this gauge is equivalent to the covariant gauges introduced in section 13.1, but we postpone this issue and discuss before another quantization scheme.

13.4.2 The temporal (or Weyl) gauge

In the non-Abelian case the quantization in Coulomb's gauge is complicate and, therefore, we now explain another, more easily generalizable, method.

The field equations in real time t that correspond to the action (13.28) are

$$\sum_{\mu} \partial_\mu F^{\mu\nu}(t,x) = J^\nu(t,x), \quad x \in \mathbb{R}^3. \tag{13.30}$$

The extension of the arguments that follow to a gauge theory containing matter fields is straightforward.

The method relies on the observation that the gauge transformed of any solution of equation (13.30) is again a solution. The set of all solutions can thus be obtained by restricting the gauge field to a gauge section.

Here, we considers only the solutions satisfying

$$A_0(t,x) = 0\,, \tag{13.31}$$

in which A_0 is the time component of the field A_μ. Equation (13.30) can then be rewritten, separating time and space components, and taking into account the condition (13.31), as $(1 \le i,j \le 3)$

$$\sum_i \partial_i \dot{A}_i(t,x) = J_0(t,x)\,, \tag{13.32}$$

$$\ddot{A}_i(t,x) - \sum_j \partial_j F_{ji}(t,x) = J_i(t,x)\,. \tag{13.33}$$

The equation (13.33) is the field equation that can be derived from the classical Lagrangian density

$$\mathcal{L}(A_i) = \sum_i \left[\tfrac{1}{2}\dot{A}_i^2(t,x) + J_i(t,x)A_i(t,x)\right] - \tfrac{1}{4}\sum_{i,j} F_{ij}^2(t,x), \quad 1 \le i,j \le 3\,.$$

The conjugated momentum of the field $A_i(t,x)$ is the electric field

$$E_i(t,x) = \dot{A}_i(t,x).$$

The expression of the Hamiltonian density follows,

$$\mathcal{H}\bigl(E(x),A(x)\bigr) = \tfrac{1}{2}\sum_i \left[E_i^2(x) - J_i(x)A_i(x)\right] + \tfrac{1}{4}\sum_{i,j} F_{ij}^2(x). \tag{13.34}$$

The partition function $\mathcal{Z}(J_i)$ is then

$$\mathcal{Z}(J_i) = \int [\mathrm{d}A_i]\, \mathrm{e}^{-\mathcal{S}(A_i,J_i)} = \int [\mathrm{d}A_\mu] \prod_x \delta\bigl(A_4(x)\bigr)\, \mathrm{e}^{-\mathcal{S}(A,J)}, \tag{13.35}$$

since $\mathcal{S}(A_i,J_i)$ is the covariant euclidean action (13.28) taken for $A_4 = 0$.

We still have to implement the constraint (13.32), which is Gauss's law. After quantization, it becomes a constraint on the physically acceptable states $\Psi(\mathbf{A})$. The conjugated momenta E_i as quantum operators are represented by functional differential operators $-i\delta/\delta A_i$. The condition (13.32) thus takes the form

$$\frac{1}{i}\sum_i \partial_i \frac{\delta}{\delta A_i(x)}\Psi(\mathbf{A}) = J_0(t,x)\Psi(\mathbf{A})\,. \tag{13.36}$$

We recognize in the left hand side of equation (13.36) the generator of time-independent gauge transformations of the field $\mathbf{A}(x)$ acting on Ψ.

In the absence of an external source ($J_0(t,x) = 0$), the physical states must be gauge invariant. This condition is consistent with quantum evolution because in the gauge (13.31) the theory has still an invariance under time-independent gauge transformations.

For a general external source, the condition (13.36) tells us how the state transforms. Consistency with quantum evolution then requires the commutation of the operator $\sum_i \partial_i E_i - J_0$ with the Hamiltonian. A short calculation shows that this commutation is implied by current conservation.

We exhibit, for later purpose, a state satisfying the condition (13.36) in the case of two opposite static charges:

$$J_0(t,x) = e\left[\delta(x-x_2) - \delta(x-x_1)\right], \quad J_i(t,x) = 0\,. \tag{13.37}$$

The state,

$$\Psi(\mathbf{A}) = \exp\left[-ie\oint_C \sum_i A_i(s)\mathrm{d}s_i\right], \tag{13.38}$$

in which C is an arbitrary differentiable path joining x_1 to x_2, indeed verifies

$$\begin{aligned}\frac{1}{i}\sum_i \partial_i \frac{\delta}{\delta A_i(x)}\Psi(\mathbf{A}) &= \frac{1}{i}\frac{\delta}{\delta\Lambda(x)}\left.\Psi(\mathbf{A} - \nabla_x\Lambda)\right|_{\Lambda=0} \\ &= e\left[\delta(x-x_2) - \delta(x-x_1)\right]\Psi(\mathbf{A}).\end{aligned}$$

The result is consistent with the equations (13.36,13.37).

The representation (13.38) has the form of a parallel transporter corresponding to time-independent gauge transformations. This representation, as well as its non-Abelian generalization, will be useful in chapter 16, in the discussion of the confinement problem.

The propagator in the temporal gauge. The propagator of the gauge field in the temporal gauge (in the euclidean formalism) reads

$$\tilde{W}^{(2)}_{ij} = \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}_\perp^2}\right) + \frac{1}{k_4^2}\frac{k_i k_j}{\mathbf{k}_\perp^2},$$

in which $\mathbf{k}_\perp$ is the 'space' part of $\mathbf{k}$, that is, its projection on $\mathbb{R}^3$. This propagator, as in the case of Coulomb's gauge, has a large momentum behaviour which is not uniform and thus, in contrast with covariant gauges, leads to theories which are not explicitly renormalizable in four dimensions. Moreover, its longitudinal part has a double pole at $k_4 = 0$ that also requires some regularization.

13.5 Equivalence with covariant gauges

We have obtained different field integral representations of the same theory. We now show that, at least for expectation values of gauge invariant observables, they are formally equivalent to the $O(4)$ covariant representations we have discussed in section 13.1. More generally, we show here the equivalence between the temporal gauge and a class of gauges characterized by a gauge condition of the form

$$\mathbf{n}(\partial) \cdot \mathbf{A}(x) = \nu(x),$$

where the vector $\mathbf{n}$ is a constant or a differential operator and $\nu(x)$ an arbitrary external field. By setting the external field $\nu(x)$ to zero, one enforces the strict gauge condition $\mathbf{n}(\partial) \cdot \mathbf{A}(x) = 0$ but by integrating over it with a Gaussian weight one can generate actions of the form (13.14). This covers all the examples met so far. The method easily generalizes to other gauges.

Derivation. We start from the temporal gauge. We use the second expression (13.35), which involves an integral over a 4-component vector field:

$$\mathcal{Z}(J) = \int [\mathrm{d}A_\mu] \prod_x \delta\left(A_4(x)\right) \mathrm{e}^{-\mathcal{S}(A,J)} . \tag{13.39}$$

We insert the identity

$$\int [\mathrm{d}\Lambda(x)] \prod_x \delta\left[\mathbf{n}(\partial) \cdot \left(\nabla_x \Lambda(x) + \mathbf{A}(x)\right) - \nu(x)\right] = \text{ const. } ,$$

inside expression (13.39),

$$\mathcal{Z} \propto \int [\mathrm{d}A_\mu \mathrm{d}\Lambda] \prod_x \delta(A_4(x))\delta\left[\mathbf{n}(\partial) \cdot \left(\nabla_x \Lambda(x) + \mathbf{A}(x)\right) - \nu(x)\right] \mathrm{e}^{-\mathcal{S}(A,J)} .$$

We change variables, substituting

$$\mathbf{A}(x) \mapsto \mathbf{A}(x) - \nabla \Lambda(x).$$

The change of variables has the form of a gauge transformation. Since the current J_μ is conserved, only the δ-functions are modified:

$$\delta(A_4)\delta\left[\mathbf{n}(\partial)\cdot\big(\nabla_x\Lambda(x)+\mathbf{A}(x)\big)-\nu(x)\right]\mapsto\delta(A_4-\dot\Lambda)\delta\left[\mathbf{n}(\partial)\cdot\mathbf{A}(x)-\nu(x)\right].$$

The integration over Λ can again be performed. It yields

$$\int[\mathrm{d}\Lambda]\prod_x\delta\left(A_4(x)-\dot\Lambda(x)\right)=\ \text{const.}$$

and, therefore,

$$\mathcal{Z}(J)=\int[\mathrm{d}A_\mu]\,\delta\left[\mathbf{n}(\partial)\cdot\mathbf{A}(x)-\nu(x)\right]\mathrm{e}^{-\mathcal{S}(A,J)}\,.$$

Since the result by construction does not depend on $\nu(x)$, one can either set $\nu(x)$ to zero or integrate over $\nu(x)$ with, for example, the Gaussian measure

$$[\mathrm{d}\rho(\nu)]=[\mathrm{d}\nu(x)]\exp\left[-\tfrac{1}{2}\int\mathrm{d}^4x\,\nu^2(x)\right].\tag{13.40}$$

One then obtains

$$\mathcal{Z}(J) = \int [\mathrm{d}A_\mu] \exp\left[-\mathcal{S}_{\text{gauge}}(A) + \int \mathrm{d}^4x\, \mathbf{J}(x) \cdot \mathbf{A}(x)\right]$$

with

$$\mathcal{S}_{\text{gauge}}(A) = \mathcal{S}(A) + \tfrac{1}{2} \int \mathrm{d}^4x\, [\mathbf{n}(\partial) \cdot \mathbf{A}(x)]^2 \,. \tag{13.41}$$

Specializing to $\mathbf{n} = \xi^{-1/2}\nabla_x$, one sees that the result in particular demonstrates the equivalence between the temporal gauge $A_4 = 0$ and the covariant gauges (13.14). By contrast, if one chooses $n_4 = 0$ and $\mathbf{n}_\perp \equiv \partial_\perp$, and sets $\nu = 0$, one recovers Coulomb's gauge.

The propagator. To the action (13.41) corresponds the gauge field propagator

$$\tilde{W}^{(2)}_{\mu\nu}(k) = \frac{1}{k^2}\left[\delta_{\mu\nu} - \frac{(k_\mu n_\nu + k_\nu n_\mu)}{k\cdot n} + \frac{\left(k^2+n^2\right) k_\mu k_\nu}{(k\cdot n)^2}\right].$$

Remark. The strict gauge condition is recovered in the limit $|\mathbf{n}| \to \infty$, which exists for the propagator but not for the action. To write explicitly the limiting action one has to introduce a Lagrange multiplier $\lambda(x)$ which implements the gauge condition as

$$\mathcal{S}_{\text{gauge}}(A) = \mathcal{S}(A) + \int \mathrm{d}^4x\, \lambda(x)\mathbf{n}(\partial)\cdot\mathbf{A}(x)\,.$$

13.5.1 Interpretation: the Faddeev–Popov quantization

The result we have obtained has an interpretation that can be rigorously justified only in the lattice approximation on a finite lattice (see section 13.6).

The problem of a gauge invariant theory is that locality requires an action with redundant degrees of freedom or equivalently that the local gauge invariant action does not provide a dynamics to the degrees of freedom attached to gauge transformations.

Therefore, we supply them with a 'stochastic' dynamics: we express the gauge field A_μ as the sum of a gauge field B_μ projection of A_μ on some gauge section, that is, satisfying some gauge condition, and a gauge transformation Λ:

$$\mathbf{A}(x) = \mathbf{B}(x) + \nabla\Lambda(x). \tag{13.42}$$

We assume that this decomposition is unique. Gauge invariance implies that the gauge action depends only on B_μ and specifies its dynamics.

To $\Lambda(x)$, whose dynamics is undetermined, we impose, for example,

$$\mathcal{E}(\Lambda, x) \equiv \nabla^2 \Lambda(x) + \nabla \cdot \mathbf{B}(x) - \nu(x) = 0\,, \tag{13.43}$$

in which $\nu(x)$ is a stochastic field with some probability distribution.

We impose the equation in the field integral using the general identity (chain rule)

$$\int [\mathrm{d}\Lambda] \prod_x \delta[\mathcal{E}(\Lambda, x)] \det \mathbf{M} = \text{ const. }\,,$$

where $\mathbf{M}$ is the functional derivative of the equation (13.43) with respect to the field $\Lambda(x)$,

$$\langle y|\mathbf{M}|x\rangle = \frac{\delta \mathcal{E}(\Lambda, x)}{\delta \Lambda(y)}\,.$$

However, here $\mathbf{M} = \nabla^2$ and its determinant is field independent. The determinant yields a factor that disappears in the normalization of the field integral.

The field integral reduces to

$$\mathcal{Z} = \int [\mathrm{d}B_\mu \mathrm{d}\Lambda]\, \delta\left[\nabla^2 \Lambda(x) + \nabla \cdot \mathbf{B}(x) - \nu(x)\right] \exp\left[-\mathcal{S}(B)\right]. \tag{13.44}$$

The field measure $[\mathrm{d}B_\mu \mathrm{d}\Lambda]$ is the decomposition of the flat measure $[\mathrm{d}A_\mu]$ into a product of measures on B_μ and Λ. The action $\mathcal{S}(B)$ is the gauge invariant action $\mathcal{S}(A)$ in which equation (13.42) has been used. We now recognize that the whole expression can be rewritten in terms of A_μ as

$$\mathcal{Z} = \int [\mathrm{d}A_\mu]\, \delta\left[\nabla \cdot \mathbf{A}(x) - \nu(x)\right] \exp\left[-\mathcal{S}(A)\right].$$

Moreover, since the result of the field integration does not depend on the dynamics of $\Lambda(x)$, the result does not depend on the field $\nu(x)$ either and we can integrate over $\nu(x)$ with, for example, the Gaussian measure (13.40).

The result remains valid in the presence of sources for gauge invariant operators (polynomials in the fields that are invariant under gauge transformations).

13.6 Perturbation theory: regularization

For most of the section, we consider an action for massive QED with fermions in a covariant gauge, of the form (13.23).

With the propagator (13.15) power counting is the same as for a scalar field. Therefore, one can construct interacting theories renormalizable by power counting in four dimensions. However, because gauge invariance is essential for the physical consistency of the theory, it has to be preserved by the renormalization. In a first step, gauge invariant regularizations are thus required.

Dimensional regularization. We have defined dimensional regularization in section 9.2. This regularization is well suited to perturbative calculations in QED. Examples will be given in section 13.11. It leads to problems only in the case of chiral gauge theories, due to the γ_5 problem (see section 15.5) because γ_5 has no proper dimensional continuation.

13.6.1 Momentum cut-off regularization

In this chapter the special problems generated by chiral fermions are not met and we can calculate with dimensional regularization. However, for later purpose it is instructive to also discuss momentum cut-off or Pauli–Villars's regularization, especially in the case of fermion matter. Moreover, momentum regularization allows proving bare RG equations and thus discussing the flow of effective low energy parameters, in particular, the effective low energy charge.

It is convenient to decompose the regularization of a matter action in presence of a gauge field into two steps, first regularization of matter in a gauge background, then regularization of the integral over the gauge field.

Charged fermions in a gauge background. From the point of view of momentum regularization, a specific problem arises because in a gauge theory only covariant derivatives can be used. A regularized action of fermions in a gauge background then takes the form ($\not{\mathrm{D}} \equiv \sum_\mu \gamma_\mu \mathrm{D}_\mu$),

$$\mathcal{S}_{\mathrm{F}}(\bar{\psi}, \psi, A) = \int \mathrm{d}^4 x \, \bar{\psi}(x) \, (M + \not{\mathrm{D}}) \prod_r \left(1 - \not{\mathrm{D}}^2 / M_r^2\right) \psi(x), \quad \frac{M_r}{\Lambda} = O(1) \, .$$

Note that up to this point the regularization, unlike dimensional or lattice regularizations, preserves a possible chiral symmetry for $M = 0$.

However, the higher order derivatives of the regularization, while they improve the large momentum behaviour of the fermion propagator, generate new, more singular, gauge interactions and it is no longer clear whether the theory can be rendered finite.

Correlation functions in the gauge background then are generated by

$$\mathcal{Z}(\bar{\eta},\eta;A)=\int\left[\mathrm{d}\psi(x)\mathrm{d}\bar{\psi}(x)\right]\exp\Big[-\mathcal{S}_{\mathrm{F}}(\bar{\psi},\psi,A)$$
$$+\int\mathrm{d}^4x\,\left(\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right)\Big], \qquad (13.45)$$

where $\bar{\eta},\eta$ are Grassmann sources.

Integrating over fermions explicitly, one obtains

$$\mathcal{Z}(\bar{\eta},\eta;A)=\mathcal{Z}_0(A)\exp\left[-\int\mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\eta}(y)\Delta_{\mathrm{F}}(A;y,x)\eta(x)\right],$$
$$\mathcal{Z}_0(A)=\mathcal{N}\det\left[(M+\not{D})\prod_r\left(1-\not{D}^2/M_r^2\right)\right],$$

where $\mathcal{N}$ is a gauge field-independent normalization and $\Delta_{\mathrm{F}}(A;y,x)$ the fermion propagator in an external gauge field.

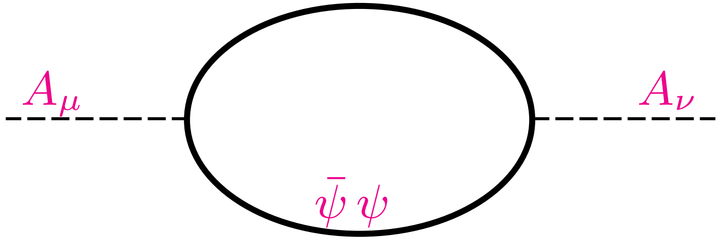

Fig. 13.1 – One-loop contribution to the gauge field two-point function.

The fermion determinant. The first problem arises from the determinant that generates closed fermion loops in a gauge background (like in Fig. 13.1).

Using $\ln\det = \operatorname{tr}\ln$, one finds

$$\ln \mathcal{Z}_0(A) = \operatorname{tr}\ln\left(M + \not{D}\right) + \sum_r \operatorname{tr}\ln\left(1 - \not{D}^2/M_r^2\right) - (A=0).$$

In four dimensions, we can then use the anticommutation of γ_5 with $\not{D}$,

$$\det(\not{D} + M) = \det\gamma_5(\not{D} + M)\gamma_5 = \det(M - \not{D}),$$
$$\ln \mathcal{Z}_0(A) = \tfrac{1}{2}\operatorname{tr}\ln\left(M^2 - \not{D}^2\right) + \sum_r \operatorname{tr}\ln\left(1 - \not{D}^2/M_r^2\right) - (A=0).$$

We see that, from the point of view of power counting, the regularization does not improve the determinant and, thus, not one-loop diagrams of the form of fermion closed loops with external gauge fields, a problem that requires an additional regularization. This analysis signals a difficulty in constructing in general a regularized gauge invariant expression for the determinant of operators of the form $\not{D} + M$ in the continuum and at fixed dimension (see section 15.5).

Auxiliary fields method. The fermion determinant can be regularized by adding to the action an unphysical boson regulator fields with spin $1/2$ and, therefore, a propagator analogous to Δ_{F} but with different masses

$$\mathcal{S}_{\mathrm{B}}(\bar{\phi}, \phi; A) = \int \mathrm{d}^4 x\, \bar{\phi}(x) \left(M_0^{\mathrm{B}} + \not{D}\right) \prod_{r=1} \left[1 - \not{D}^2/(M_r^{\mathrm{B}})^2\right] \phi(x),\ M_r^{\mathrm{B}} = O(\Lambda).$$

The integration over the boson fields $\bar{\phi}, \phi$ adds to $\ln \mathcal{Z}_0$ the quantity

$$\delta \ln \mathcal{Z}_0(A) = -\tfrac{1}{2} \operatorname{tr} \ln \left[(M_0^{\mathrm{B}})^2 - \not{D}^2\right] - \sum_{r=1} \operatorname{tr} \ln \left[1 - \not{D}^2/(M_r^{\mathrm{B}})^2\right] - (A = 0).$$

Expanding in inverse powers of $\not{D}$, one adjusts the masses M_r^B to cancel as many powers as possible. However, the unpaired initial fermion mass M is the source of a problem. The corresponding determinant can only be regularized with an unpaired boson with mass $M_0^{\mathrm{B}} = O(\Lambda)$.

In the chiral limit $M = 0$ we have two options: either we give a chiral charge to the boson field and the mass M_0^{B} breaks chiral symmetry, or we leave it invariant in a chiral transformation. Then, after a chiral gauge transformation, we obtain the determinant of the operator

$$\mathrm{e}^{i\theta(x)\gamma_5}\,\not{D}\,\mathrm{e}^{i\theta(x)\gamma_5}\,(\not{D} + M_0^{\mathrm{B}})^{-1}.$$

For $\theta(x)$ constant, $\mathrm{e}^{i\gamma_5\theta}$ anticommutes with $\not{D}$ and cancels. Otherwise, a non-trivial contribution remains. The method thus indicates potential difficulties with chiral gauge transformations.

Since the problem reduces to the study of a determinant in an external background, one can examine it directly, as we will, starting with section 15.5. One verifies whether it is possible to define some regularized form in a way consistent with chiral symmetry. One then inserts the one-loop renormalized diagrams in the general diagrams regularized by the preceding cut-off methods.

The gauge field propagator. After integration over the gauge field, diagrams constructed from $\Delta_{\mathrm{F}}(A;y,x)$ belong to loops with gauge field propagators (figure 13.2), and, therefore, can be rendered finite if the gauge field propagator can be improved.

For the free gauge action in a covariant gauge usual derivatives can be used because in an Abelian theory the gauge field is neutral. The tensor $F_{\mu\nu}$ is gauge invariant and the action for the scalar $\chi(x)$ of equation (13.11) and thus $\nabla\cdot\mathbf{A}$ is arbitrary. Therefore, the large momentum behaviour of the gauge field propagator can be arbitrarily improved.

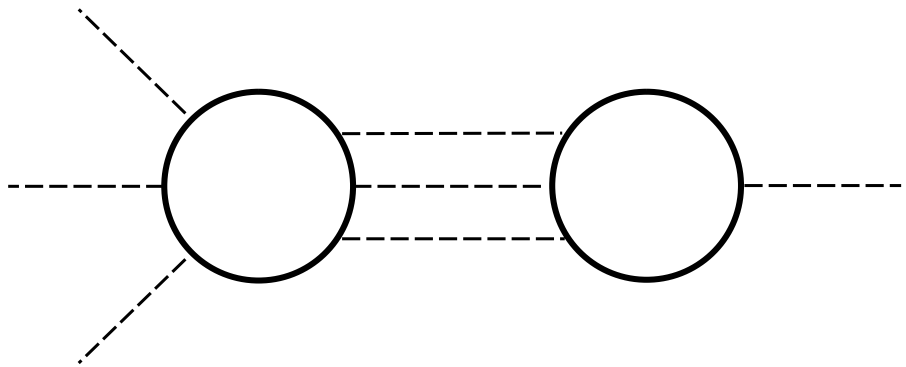

Fig. 13.2 – Example of a multi-loop diagram.

Scalar matter. In the case of scalar matter, a similar analysis holds.

The determinant $\det \mathrm{D}_\mu^2$ generated by integrating over charged scalar fields in a gauge background can be regularized by Schwinger's proper time method. The determinant is then expressed in terms of the evolution operator corresponding to a non-relativistic Hamiltonian in a magnetic field.

For multi-loop diagrams scalar self-interaction vertices can be added, but then the number of matter propagators exceeds the number of gauge field vertices and again the diagrams can be made superficially convergent.

13.6.2 Lattice regularization

The construction of a lattice regularized form of a gauge theory is directly based on the notion of parallel transport and the geometric interpretation of the gauge field as a connection. Since on the lattice points are split, the gauge field has to be replaced by link variables corresponding to parallel transport along links of the lattice (see section 13.3.1 and chapter 16 for a detailed discussion).

A link variable $U(x,y)$ is an element of the $U(1)$ group, lattice analogue of the parallel transporter $U(C)$ defined by equation (13.25), the curve C being the link joining the site x to the neighbour site$y$ on the lattice.

It can be parametrized in terms of an angle θ_{xy}, and is such that

$$U(x,y) \equiv U_{xy} = \mathrm{e}^{i\theta_{xy}} = (U_{yx})^{-1} .$$

The measure of integration over the gauge variables is the group invariant measure, that is, the flat measure $\mathrm{d}\theta_{xy}$.

Lattice dependent $U(1)$ group transformations are then the lattice equivalents of the gauge transformations of the continuum theory.

Gauge invariant fermion interaction terms on the lattice, for example, have the form (but lead to subtle issues for the fermion lattice propagator)

$$\bar{\psi}(x + an_\mu)\gamma_\mu U(x + an_\mu, x)\psi(x),$$

where n_μ is the unit vector in μ direction and a the lattice spacing.

Plaquette action. We have shown in section 13.3.1 that the curvature tensor is associated with parallel transport around a closed curve. This indicates that a regularized form of $\int d^4x \sum_{\mu,\nu} F^2_{\mu\nu}$ is the product of link variables around a closed curve on the lattice, the simplest one on an hypercubic lattice being a square, forming a plaquette. This leads to the famous plaquette action (for details see chapter 16),

$$\sum_{\text{all plaquettes}} U_{xy}U_{yz}U_{zt}U_{tx}\,, \quad x, y, z, t \text{ forming a square}\,.$$

À typical gauge invariant lattice action corresponding to the continuum action of a Abelian gauge field coupled to fermions then has the form

$$\mathcal{S}(U,\bar{\psi},\psi) = \beta \sum_{\text{plaquettes}} U_{xy}U_{yz}U_{zt}U_{tx} - \kappa \sum_{\text{links}} \bar{\psi}_y \gamma_{yx} U_{yx} \psi_x - \sum_{\text{sites}} M \bar{\psi}_x \psi_x \,, \tag{13.46}$$

where have denoted by x, y,..., the lattice sites, $\gamma_{xy} \equiv \sum_\mu \gamma_\mu (x-y)_\mu$ and β and κ are the coupling constants.

Note that on the lattice and in a finite volume the gauge invariant action leads to a well-defined partition function because the $U(1)$ group is compact: the volume of the gauge group for a lattice of Ω lattice sites is $(2\pi)^\Omega$.

However, in the continuum limit, the compact character of the group is lost. Therefore, even on the lattice it is necessary to fix the gauge in order to be able to construct a regularized perturbation theory. Since we shall devote chapter 16 to lattice gauge theories, we postpone the discussion of this problem.

13.7 WT identities

In this section, for convenience, we assume dimensional regularization.

In gauge theories WT identities play an essential role because it is necessary to maintain the gauge symmetry in order to ensure that the theory, which is not explicitly unitary, remains equivalent to a unitary theory, at least for the 'physical' observables, that is, gauge invariant observables (including S-matrix elements).

Their derivation is simple. We again take the example of the action (13.23)

$$\mathcal{S}_\xi(A,\bar\psi,\psi)=\int \mathrm{d}^4x\Big[\tfrac14\sum_{\mu,\nu}F^2_{\mu\nu}(x)+\tfrac12 m^2\mathbf{A}^2(x)+\frac{1}{2\xi}\big(\nabla\cdot\mathbf{A}(x)\big)^2 - \bar\psi(x)\left(\not\partial+M+ie\not A\right)\psi(x)\Big]$$

We add sources $J_\mu(x)$, $\eta(x)$ and $\bar\eta(x)$ for the fields $A_\mu(x)$, $\bar\psi(x)$ and $\psi(x)$, respectively. The action becomes

$$\Sigma(A,\bar\psi,\psi)=\mathcal{S}_\xi(A,\bar\psi,\psi)-\int \mathrm{d}^4x\left[\mathbf{J}(x)\cdot\mathbf{A}(x)+\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)\right].$$

We make infinitesimal gauge transformations (13.22, 13.21) on the action Σ. The terms that are not invariant are the A_μ mass term, the gauge fixing term and the sources. The variation of the action is

$$\delta\Sigma=-\frac{1}{e}\int \mathrm{d}^4x\,\Lambda(x)\left\{\left(\nabla^2/\xi-m^2\right)\nabla\cdot\mathbf{A}(x)+\nabla\cdot\mathbf{J}(x)\right.$$
$$\left.+ie\left[\bar\eta(x)\psi(x)-\bar\psi(x)\eta(x)\right]\right\}.$$

This leads, following the usual arguments, to an equation for the generating functionals $\mathcal{Z}(J,\bar\eta,\eta)$ and thus $\mathcal{W}(J,\bar\eta,\eta)$ which has the form,

$$\left\{\left(m^2-\nabla^2/\xi\right)\sum_\mu \partial_\mu\frac{\delta}{\delta J_\mu(x)}-ie\left[\bar\eta(x)\frac{\delta}{\delta\bar\eta(x)}-\eta(x)\frac{\delta}{\delta\eta(x)}\right]\right\}\mathcal{W}(J,\bar\eta,\eta)$$
$$=\nabla\cdot\mathbf{J}(x). \qquad (13.47)$$

This equation implies identities satisfied by correlation functions.

The contribution linear in $\mathbf{J}$ yields a constraint for the gauge field two-point function, which after Fourier transformation reads

$$\sum_\mu k_\mu \tilde{W}^{(2)}_{\mu\nu}(k) = \xi \frac{k_\nu}{k^2 + \xi m^2} \,. \tag{13.48}$$

For $\mathbf{J} = 0$, from the coefficients of powers of η and $\bar{\eta}$, one infers relations between correlation functions (in the Fourier representation),

$$\begin{aligned}(k^2/\xi + m^2) \sum_\mu k_\mu \tilde{W}^{(2n+1)}_\mu (k; p_1, \ldots, p_n; q_1, \ldots, q_n) \\ = e \sum_i \Big[\tilde{W}^{(2n)}(p_1, \ldots, p_i + k, \ldots, p_n; q_1, \ldots, q_n) \\ - \tilde{W}^{(2n)}(p_1, \ldots, p_n; q_1, \ldots, q_i + k, \ldots, q_n) \Big] ,\end{aligned} \tag{13.49}$$

in which k is the gauge field momentum, and p_i and q_i the momenta of ψ and $\bar{\psi}$ fields, respectively.

The presence of additional external gauge fields does not modify the identities.

The equation (13.47) is a linear first-order partial differential equation. The Legendre transformation is simple and yields an equation for the generating functional of vertex functions $\Gamma(A,\bar{\psi},\psi)$:

$$\left(\nabla^2/\xi - m^2\right)\nabla\cdot\mathbf{A}(x)+\sum_{\mu}\partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)}+ie\left[\psi(x)\frac{\delta\Gamma}{\delta\psi(x)}-\bar{\psi}(x)\frac{\delta\Gamma}{\delta\bar{\psi}(x)}\right]=0\,. \tag{13.50}$$

It can be verified that the equations (13.47) and (13.50) have the same content as the quantum equations of motion of the χ-field of section 13.1.

The general solution of equation (13.50) can be written as

$$\Gamma(A,\bar{\psi},\psi)=\Gamma_{\text{sym.}}(A,\bar{\psi},\psi)+\tfrac{1}{2}\int\left[m^2\mathbf{A}^2(x)+\left(\nabla\cdot\mathbf{A}(x)\right)^2/\xi\right]\mathrm{d}^4x\,,$$

where $\Gamma_{\text{sym.}}$ is gauge invariant.

13.8 Renormalization

We perform a loop expansion of the functional Γ. Because the equation (13.50) is linear, the tree approximation satisfies the inhomogeneous equation while all higher order terms satisfy the homogeneous equation. Denoting by Γ_ℓ the ℓ-loop contribution to Γ, we find for $\ell > 0$,

$$\sum_\mu \partial_\mu \frac{\delta\Gamma_\ell}{\delta A_\mu(x)} + ie\left[\psi(x)\frac{\delta\Gamma_\ell}{\delta\psi(x)} - \bar{\psi}(x)\frac{\delta\Gamma_\ell}{\delta\bar{\psi}(x)}\right] = 0\,.$$

Therefore, the generating functional Γ_ℓ of ℓ-loop vertex functions is gauge invariant. The divergent part Γ_ℓ^{div} of Γ_ℓ, which is given by the singular part of the Laurent expansion in $\varepsilon = 4 - d$, thus is also gauge invariant.

The conclusion is that the action can be completely renormalized by adding gauge invariant counter-terms.

As in the example of the linear symmetry breaking in section 12.3, one can say that the terms which break the gauge invariance, the gauge field mass term and the gauge fixing term, are not renormalized since they are not modified by counterterms.

The full renormalized action can then be written as

$$\mathcal{S}_{\mathrm{r}}(A_\mu,\bar\psi,\psi)=\int \mathrm{d}^4x\left[\tfrac{1}{4}Z_A\sum_{\mu,\nu}F_{\mu\nu}^2+\tfrac{1}{2}m^2\mathbf{A}^2+\tfrac{1}{2\xi}\left(\nabla\cdot\mathbf{A}\right)^2\right.$$
$$\left.-\,Z_\psi\bar\psi\left(\not{D}+M_0\right)\psi\right], \tag{13.51}$$

where Z_A is the gauge field, Z_ψ and M_0 are the ψ field and mass renormalization constants.

We now introduce the unrenormalized (bare) fields $\psi^0,\bar\psi^0$, A_μ^0 and charge e_0,

$$\psi^0=Z_\psi^{1/2}\psi\,,\quad \bar\psi^0=Z_\psi^{1/2}\bar\psi\,,\quad A_\mu^0=Z_A^{1/2}A_\mu\,,\quad e_0=Z_e^{1/2}e\,.$$

We note that the covariant derivative is not renormalized:

$$\partial_\mu + ie_0 A^0_\mu = \partial_\mu + ieA_\mu \,.$$

As a consequence, we obtain the relation

$$Z_A Z_e = 1 \,. \tag{13.52}$$

Gauge invariance relates the renormalization of the charge and the gauge field.

Moreover, since the quantities $F_{\mu\nu}(x)$ and $\bar\psi(x)\psi(x)$ are gauge invariant, their correlation functions are the same in the unitary and ξ gauges. Therefore, Z_A and M_0 (but not Z_ψ) can be chosen independent of ξ.

13.9 Gauge dependence: the fermion two-point function

In most of the section, we assume dimensional regularization.

Some insight into the physical properties of the theory can be gained from a study of the the gauge dependence of correlation functions. In covariant gauges of the action (13.23) this amounts to studying the dependence on the parameter ξ of correlation functions.

As an example, we discuss the fermion two-point function:

$$W^{(2)}_{\xi}(u,v) \equiv \left\langle \bar{\psi}(u)\psi(v) \right\rangle = \mathcal{Z}^{-1} \int [\mathrm{d}A_{\mu}\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi}]\bar{\psi}(u)\psi(v)\,\mathrm{e}^{-\mathcal{S}_{\xi}(A,\bar{\psi},\psi)}, \tag{13.53}$$

where the partition function $\mathcal{Z}$ is gauge independent and, thus, ξ independent.

We now invert the algebraic transformations of section 13.1.3. We introduce an uncoupled scalar field χ and add a mass term to action:

$$\mathcal{S}_\xi(A,\bar\psi,\psi) \mapsto \mathcal{S}_\xi(A,\bar\psi,\psi,\chi) = \mathcal{S}_\xi(A,\bar\psi,\psi) + \tfrac{1}{2}\xi m^2 \int \mathrm{d}^d x\, \chi^2(x).$$

The direct integration over χ yields a constant factor that cancels in the ratio (13.53).

Instead, we change variables, shifting $\chi(x)$ as,

$$\chi(x) \mapsto \chi(x) + \frac{i}{\xi m} \nabla\cdot\mathbf{A}(x).$$

After an integration by parts the action becomes

$$\mathcal{S}(A,\bar\psi,\psi,\chi) = \mathcal{S}_\xi(A,\bar\psi,\psi) + \tfrac{1}{2}\xi m^2 \int \mathrm{d}^d x\, \chi^2(x) - im \int \mathrm{d}^d x\, \mathbf{A}(x)\cdot\nabla\chi(x) \\ - \frac{1}{2\xi}\int \mathrm{d}^d x \big(\nabla\cdot\mathbf{A}(x)\big)^2.$$

The last term cancels the gauge fixing term and the second term in the right hand side can be eliminated by the gauge transformation,

$$\begin{aligned} &\mathbf{A}(x) \mapsto \mathbf{A}(x) + i\nabla\chi(x)/m\,, \\ &\psi(x) \mapsto \psi(x)\,\mathrm{e}^{ie\chi(x)/m}, \quad \bar{\psi}(x) \mapsto \bar{\psi}(x)\,\mathrm{e}^{-ie\chi(x)/m}\,. \end{aligned}$$

The field integral representation of the two-point function then factorizes and leads to the relation

$$W^{(2)}_{\xi}(u,v) = W^{(2)}_{\infty}(u,v)\mathcal{Z}_{\chi}^{-1}\int [\mathrm{d}\chi]\,\mathrm{e}^{-\mathcal{S}_{\xi}(\chi)}, \tag{13.54}$$

in which $W^{(2)}_{\infty}(u,v)$ is calculated with the unitary action (13.20) of massive QED and

$$\mathcal{S}_{\xi}(\chi) = \int \mathrm{d}^d x\left[-\tfrac{1}{2}\left(\nabla\chi(x)\right)^2 - \tfrac{1}{2}\xi m^2\chi^2(x)\right] + ie\chi(v) - ie\chi(u).$$

The χ integration can be performed by noting that

$$ie\chi(v) - ie\chi(u) = \int \mathrm{d}^d x\, J(x)\chi(x) \text{ with } J(x) = ie[\delta(x-v) - \delta(x-u)].$$

Thus, in terms of the χ propagator Δ_χ,

$$\begin{aligned}\int [\mathrm{d}\chi]\, \mathrm{e}^{-\mathcal{S}_\xi(\chi)} &= \exp\left[-\tfrac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, J(x)\Delta_\chi(\xi, x-y)J(y)\right] \\ &= \exp\left[-e^2\left(\Delta_\chi(\xi,0) - \Delta_\chi(\xi, u-v)\right)\right]. \qquad (13.55)\end{aligned}$$

The χ propagator has the Fourier representation

$$\Delta_\chi(\xi, x) = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d p\, \mathrm{e}^{ipx}}{p^2 + \xi m^2}$$

and, thus,

$$\Delta_\chi(\xi,0) - \Delta_\chi(\xi, x) = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d p\left(1 - \mathrm{e}^{ipx}\right)}{p^2 + \xi m^2}\,.$$

We then infer the ratio of the bare two-point functions for two different values of ξ. For example,

$$W_{\xi}^{(2)}(u,v)=\exp\left[\frac{\xi e^2}{(2\pi)^d}\int \mathrm{d}^d p \frac{\mathrm{e}^{ip(u-v)}-1}{p^2\,(p^2+\xi m^2)}\right] W_{(\xi=0)}^{(2)}(u,v). \tag{13.56}$$

Only the second term in the exponential is divergent while the first one contains the unphysical χ poles.

Introducing the fermion renormalization constant Z_ψ, we obtain the ratio of renormalization constants for two values of ξ (up to a finite renormalizations, required in the limit $m\to 0$ to avoid IR divergences),

$$Z_\psi(\xi)=Z_\psi(0)\exp\left[-\frac{\xi e^2}{(2\pi)^d}\int\frac{\mathrm{d}^d p}{p^2\,(p^2+\xi m^2)}\right]. \tag{13.57}$$

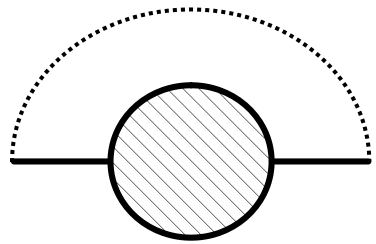

Fig. 13.3 – Order e^2 contribution. The full lines correspond to the fermion two-point function, the dotted line to the unphysical scalar field propagator.

The relation between renormalized functions then becomes

$$W_{\xi}^{(2)}(u,v) = \exp\left[\frac{\xi e^2}{(2\pi)^d}\int \mathrm{d}^d p \frac{\mathrm{e}^{ip(u-v)}}{p^2\,(p^2+\xi m^2)}\right] W_{(\xi=0)}^{(2)}(u,v). \tag{13.58}$$

Unitarity. An expansion in powers of ξe^2 of the factor in equation (13.58) immediately shows that only the first term, which is ξ independent, has the physical fermion pole (see figure 13.3 for the e^2 contribution) .

The method we have used can be generalized to all $\bar{\psi}, \psi, F_{\mu\nu}$ correlation functions.

Again one proves factorization and, after renormalization, only the first term in an expansion in powers of $e^2\xi$ of the ratio between correlation functions, which is ξ-independent, contributes in the mass-shell limit.

Therefore, the properly normalized S-matrix elements are gauge independent and equal to the matrix elements of the unitary theory. The full S-matrix, in the subspace of physical states, is thus unitary.

Gauge invariant operators. We have examined the gauge dependence of S-matrix elements. From the point of view of correlation functions, the only gauge independent quantities are the expectation values of products of gauge invariant operators, that is, local polynomials in the field invariant under the transformations (13.19, 13.21).

The simplest such operators are $F_{\mu\nu}$, which selects the transverse part of the gauge field, $\bar{\psi}(x)\psi(x)$ or more generally $\bar{\psi}(x)\Gamma_A\psi(x)$ in which the matrix Γ_A is any element of the algebra of γ matrices.

The mechanism which makes the correlation functions of ψ gauge dependent while $\bar{\psi}(x)\psi(x)$ is gauge independent can be seen in equation (13.56).

When in the product $\bar{\psi}(x)\psi(y)$, y approaches x the additional gauge dependent renormalization needed to make the product $\bar{\psi}(x)\psi(x)$ finite cancels the gauge dependent part of the fermion field renormalization.

To study the renormalization properties of gauge invariant operators one can add to the action sources for them. The form of WT identities is not modified. The arguments of section 13.7 are still valid: the counterterms are gauge invariant and can be chosen gauge independent. This also proves that gauge invariant operators mix under renormalization only with gauge invariant operators of lower or equal canonical dimensions.

Non-gauge invariant correlation functions in the unitary gauge. With the original action (13.20), all correlation functions are 'physical', but the theory is not renormalizable. However, one is able to define some correlation functions, the renormalized gauge invariant correlation functions, which have a large cut-off limit. The relation (13.54), in presence of a cut-off,

$$W_{\infty}^{(2)}(u,v)=\exp\left[-\frac{e^2}{(2\pi)^d}\int \mathrm{d}^d k\,\frac{1-\mathrm{e}^{ik(u-v)}}{m^2k^2}\right]W_{0}^{(2)}(u,v),$$

yields an explanation for this surprising property. For $|u-v|\neq 0$, the leading term in the large cut-off limit in the exponential is

$$\frac{e^2}{(2\pi)^d}\int\frac{\mathrm{d}^d k}{m^2k^2}\propto e^2\frac{\Lambda^{d-2}}{m^2}.$$

Therefore, in the physical representation all non-gauge invariant correlation functions vanish in the infinite cut-off limit.

The technical explanation is the following: although the mass term breaks gauge invariance, this breaking is not sufficient to prevent fluctuations coming from the gauge degrees of freedom to suppress these correlation functions.

13.10 Renormalization group equations

In this section, we derive RG equations in the case of the action (13.23), which corresponds to massive QED with fermions, displaying the dependence of RG functions on the gauge fixing parameter ξ. We denote by $\Gamma^{(l,n)}$ the vertex functions corresponding to l gauge fields, and n fermion pairs ψ and $\bar{\psi}$. In momentum representation, the relation between bare and renormalized vertex functions is

$$\tilde{\Gamma}_{\mathrm{B}}^{(l,n)}(p_i, q_j; \alpha_0, \xi_0, m_0, M_0) = Z_A^{-l/2} Z_\psi^{-n} \tilde{\Gamma}^{(l,n)}(p_i, q_j; \mu, \alpha, \xi, m, M), \tag{13.59}$$

in which μ is the renormalization scale, and we have called α the loop expansion parameter:

$$\alpha = e^2/4\pi \,.$$

Differentiating equation (13.59) with respect to μ at bare parameters fixed, we find the RG equations:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\alpha)\frac{\partial}{\partial\alpha}+\delta(\alpha)\xi\frac{\partial}{\partial\xi}+\eta_m(\alpha)m\frac{\partial}{\partial m}+\eta_M(\alpha)M\frac{\partial}{\partial M}\right.$$
$$\left.-\frac{l}{2}\eta_A(\alpha)-n\eta_\psi(\alpha)\right]\tilde{\Gamma}^{(l,n)}(p_i,q_j;\mu,\alpha,\xi,m,M)=0\,.$$

The equation (13.52) relates Z_A and Z_e, the gauge field and coupling constant renormalization constants,

$$Z_A Z_e = 1\,.$$

Therefore,

$$\alpha = Z_A\,\alpha_0\,. \tag{13.60}$$

Moreover, we have shown in section 13.7 that the parameters m and ξ are not renormalized. It follows that,

$$m_0^2 = m^2 Z_A^{-1},\quad \xi_0 = \xi Z_A\,. \tag{13.61}$$

Finally, in section 13.9, we have shown that the renormalization constant Z_A can be chosen to be independent of ξ (the minimal subtraction scheme satisfies this requirement). The equations (13.60, 13.61) then imply a set of relations between RG functions:

$$\beta(\alpha) = \eta_A(\alpha), \quad \delta(\alpha) = -\beta(\alpha), \quad \eta_m(\alpha) = \beta(\alpha)/2\,.$$

In addition $\beta(\alpha)$ is independent of ξ. The function η_M can also be chosen independent of ξ, only the fermion field renormalization is necessarily gauge dependent. Actually, from equation (13.57) it is even possible to determine the gauge dependence of η_ψ. A short calculation within the minimal subtraction scheme leads to

$$\eta_\psi(\alpha, \xi) = \eta_\psi(\alpha, 0) - \alpha\xi/2\pi\,.$$

13.11 The one-loop β-function: charged fermions

Dimensional regularization and minimal subtraction are used in the calculation.

We calculate the β-function in the case of the action (13.23), which involves a gauge field and charged fermions, at one-loop order. We infer the coupling constant renormalization from the gauge field renormalization, provided by the gauge field two-point function, and the relation (13.52).

The one-loop contribution to the generating functional of vertex functions coming from the fermion integration is

$$\Gamma_{1\,\text{loop}}(A_\mu) = -\operatorname{tr}\ln\left(\not{\partial} + ie\not{A} + M\right).$$

The second derivative with respect to A_μ yields the one-loop contribution to the renormalized vertex two-point function, which we decompose as

$$\tilde{\Gamma}^{(2)}_{\mu\nu}(p) = Z_A\left(\delta_{\mu\nu}p^2 - p_\mu p_\nu\right) + p_\mu p_\nu/\xi + e^2\Sigma_{\mu\nu}(p)\,. \tag{13.62}$$

Then,

$$\Sigma_{\mu\nu}(p) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\operatorname{tr}\left[\gamma_\mu \left(\not{k} + iM\right) \gamma_\nu \left(\not{k} - \not{p} + iM\right)\right]}{(k^2 + M^2)\left[(p-k)^2 + M^2\right]}.$$

One verifies that, as expected, the one-loop contribution is transverse and, thus, ξ is not renormalized. We calculate the one-loop integral by introducing Feynman parameters. After some algebra, and with the help of the identity

$$m \frac{\mathrm{d}}{\mathrm{d}m} \int \frac{\mathrm{d}^d k}{k^2 + m^2} = (d-2) \int \frac{\mathrm{d}^d k}{k^2 + m^2} = -2m^2 \int \frac{\mathrm{d}^d k}{\left(k^2 + m^2\right)^2}, \quad (13.63)$$

one obtains (using $\operatorname{tr}_\gamma \mathbf{1} = 4$),

$$\Sigma_{\mu\nu}(p) = 8(p^2 \delta_{\mu\nu} - p_\mu p_\nu) \int_0^1 \mathrm{d}s\, s(1-s) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2 + s(1-s)p^2 + M^2\right]^2}.$$

In particular, the divergent part of $\Sigma_{\mu\nu}(p)$ is ($\varepsilon = 4 - d$)

$$\Sigma_{\mu\nu}(p) = \frac{1}{6\pi^2 \varepsilon}(p^2 \delta_{\mu\nu} - p_\mu p_\nu) + O(1).$$

This determines the A_μ field renormalization Z_A and thus, from equation (13.52), also Z_e:

$$Z_A = 1 - \frac{e^2}{6\pi^2\varepsilon} + O\left(e^4\right), \quad Z_e = 1 + \frac{e^2}{6\pi^2\varepsilon} + O\left(e^4\right). \tag{13.64}$$

We infer the corresponding β-function at one-loop order ($\alpha = e^2/4\pi$):

$$\beta(\alpha) = -\varepsilon \left[\frac{\mathrm{d}\ln\left(\alpha Z_e\right)}{\mathrm{d}\alpha}\right]^{-1} = -\varepsilon\alpha + \frac{2\alpha^2}{3\pi} + O\left(\alpha^3\right).$$

The sign of the β-function. Note that in the domain of validity of the expansion (13.64) (α small), Z_A satisfies

$$Z_A \le 1\,.$$

This is a consequence of the Källen–Lehmann representation for the two-point function (see section 5.11).

In the case of the gauge field the property remains true because Z_A is related to the transverse part of the two-point function to which unphysical states do not contribute.

Since $Z_A Z_e = 1$ we see that the sign of the β-function in four dimensions is determined by hermiticity for α small enough.

The sign of the β-function indicates that QED is trivial: in the infinite cut-off limit the effective coupling constant at low energy scale goes to zero. This is consistent with the small value of α at low energy.

13.11.1 Furry's theorem

For more general perturbative calculations the following observation is useful. Correlation functions without matter field and an odd number of gauge fields vanish. The proof is based on charge conjugation.

We consider the contribution to the effective gauge field action, $\det(\not\partial + ie\not A + M)$, which is generated by the integration over the fermion fields, and use the property of the charge conjugation matrix C introduced in section A11.2.5:

$$\begin{aligned}\det(\not\partial + ie\not A + M) &= \det\left[{}^T(\not\partial + ie\not A + M)\right] = \det C^{-1}\left[{}^T(\not\partial + ie\not A + M)\right]C \\ &= \det(\not\partial - ie\not A + M)\,.\end{aligned}$$

Therefore, the interaction between gauge fields generated by the fermions is even in A_μ. Note in particular the implication for Feynman diagrams: fermion loops with an odd number of external gauge fields can be omitted.

13.12 The Abelian Higgs model

As an introduction to chapter 14, we now consider a gauge field coupled to a charged scalar field $\phi(x)$ in an unusual phase. In three dimensions, this is also a macroscopic model for a superconductor in a magnetic field.

We start from the gauge invariant action (13.26). The scalar field $\phi(x)$ is complex and the covariant derivative D_μ is defined as in equation (13.27): $\mathrm{D}_\mu = \partial_\mu + ieA_\mu$.

Renormalizability requires a scalar field self-interaction of the $|\phi|^4$ type and a renormalizable action thus has the form

$$\mathcal{S}(A,\phi) = \int \mathrm{d}^4x \Big(\tfrac{1}{4} \sum_{\mu,\nu} F_{\mu\nu}^2 + \sum_\mu |\mathrm{D}_\mu \phi|^2 + r\, |\phi|^2 + \tfrac{1}{6} g\, |\phi|^4 \Big). \qquad (13.65)$$

In the classical limit, for $r > 0$ the $U(1)$ global symmetry is unbroken, the gauge field has two massless components and the scalar field ϕ has two real components with equal mass $\sqrt{r}$.

However, for $r < 0$ the global $U(1)$ symmetry is spontaneously broken and $\phi(x)$ acquires a non-vanishing expectation value $\langle\phi\rangle = v = \sqrt{-3r/2g}$, which for convenience we have chosen real positive (we comment later on the meaning of $\langle\phi\rangle$, which is not gauge invariant).

This is a situation we have discussed in section 12.4 and we have concluded that the SSB of a continuous symmetry implies the presence of a massless state, a Goldstone particle. This result can be derived, in the classical limit, by parametrizing the field ϕ as (see section 12.4)

$$\phi(x) = \frac{1}{\sqrt{2}}\left[v + \rho(x)\right] \mathrm{e}^{i\theta(x)} .$$

As a consequence of the symmetry, the resulting action then depends only on $\nabla\theta$ and the field θ is massless.

However, the transformation $\phi(x) \mapsto v + \rho(x)$ has the form of a gauge transformation. If we perform the corresponding transformation on the gauge field $A_\mu(x)$,

$$\mathbf{A}(x) = \mathbf{A}'(x) + \frac{1}{e}\nabla\theta(x),$$

we eliminate the field θ from the action completely. After this transformation the action $\mathcal{S}(A, \phi)$ indeed reduces to

$$\mathcal{S}(A, \rho) = \int \mathrm{d}^4x \Big[\tfrac{1}{4} \sum_{\mu,\nu} F_{\mu\nu}^2 + \tfrac{1}{2} (\nabla\rho)^2 + \tfrac{1}{2} e^2 \mathbf{A}^2 (\rho + v)^2 + \tfrac{1}{2} r(\rho + v)^2 \\ + \tfrac{1}{24} g(\rho + v)^4 \Big]. \tag{13.66}$$

In the tree approximation the spectrum of the theory now contains one massive vector particle and one massive scalar, generally called the Higgs particle, with masses

$$m(A) = |e|v, \quad m(\rho) = v\sqrt{g/3} = \sqrt{-2r}\,.$$

As a consequence of gauge invariance, no Goldstone boson has been generated. This is a most remarkable property, which is also at the basis of the Meissner effect in superconductivity.

Note that the total number of physical degrees of freedom has not changed between the symmetric phase and the spontaneously broken phase since one degree of freedom of the scalar field has been transferred to the vector field.

From the technical point of view, the field theory has a surprising property: in this so-called unitary gauge corresponding to the action (13.66), the theory contains only physical fields, can be trivially quantized but is not renormalizable (and the ρ integration measure not defined).

By contrast, if one starts from the action (13.65) and quantize it in the same way as in the symmetric phase, the field theory contains unphysical degrees of freedom, but is renormalizable by power counting.

We are reminded of the massive vector field coupled to a conserved current, discussed in the first part of the chapter. Actually, there exists a relation between the massive vector field and the Higgs model: if one takes the formal non-linear σ-model limit of the action (13.65), a limit in which the bare mass of the Higgs field becomes infinite at fixed expectation value v, one recovers the action (13.6) with the identification $m = |e|v$.

Therefore, in order to be able to calculate perturbatively gauge invariant observables and S-matrix elements, we return to the action (13.65). We fix the gauge by adding a term proportional to $(\nabla \cdot \mathbf{A})^2$. This amounts to couple the phase field $\theta(x)$ which plays the role of the $\Lambda(x)$ field of section 13.1.

As a final remark, we recall for later purpose that the mechanism of spontaneous symmetry breaking can also be used to give a mass to fermions in a chiral invariant theory (section 12.5).

13.13 Quantization of the Abelian Higgs model

We now start from the action

$$\mathcal{S}(A,\phi)=\int \mathrm{d}^4x\left[\tfrac{1}{4}\sum_{\mu,\nu}F_{\mu\nu}^2+\frac{1}{2\xi}\left(\nabla\cdot\mathbf{A}\right)^2+\tfrac{1}{2}m^2\mathbf{A}^2\right]$$
$$+\int \mathrm{d}^4x\left[\sum_{\mu}|\mathrm{D}_\mu\phi|^2+r|\phi|^2+\tfrac{1}{6}g|\phi|^4\right], \qquad (13.67)$$

in which a mass has been given to the vector field to provide an IR cut-off.

We assume that ϕ has a real expectation value v in the classical limit. We introduce the real and imaginary parts of ϕ and set

$$\phi(x)=\frac{1}{\sqrt{2}}\left[v+\varphi(x)+i\chi(x)\right]. \qquad (13.68)$$

The quadratic part $\mathcal{S}_2$ of the action is then

$$\mathcal{S}_2(A,\phi) = \int \mathrm{d}^4x \left[\tfrac{1}{4} \sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2\xi} \left(\nabla\cdot\mathbf{A}\right)^2 + \tfrac{1}{2}\left(e^2v^2 + m^2\right)\mathbf{A}^2 \right.$$
$$\left. - ev\chi\nabla\cdot\mathbf{A} + \tfrac{1}{2}\left(\nabla\varphi\right)^2 + \tfrac{1}{6} g v^2 \varphi^2 + \tfrac{1}{2}\left(\nabla\chi\right)^2 \right].$$

We see that $\nabla\cdot\mathbf{A}$ is coupled to the Goldstone field χ. The corresponding propagators are

$$\tilde{W}^{(2)}_{\mu\nu}(k) = \frac{\delta_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 + e^2v^2 + m^2} + \xi \frac{k_\mu k_\nu}{k^2(k^2+\xi m^2)},$$
$$\tilde{W}^{(2)}_{\mu\chi}(k) = -\xi \frac{iev k_\mu}{k^2(k^2+\xi m^2)}, \quad \tilde{W}^{(2)}_{\chi\chi}(k) = \frac{1}{k^2} + \frac{\xi e^2 v^2}{k^2(k^2+\xi m^2)},$$
$$\tilde{W}^{(2)}_{\varphi\varphi}(k) = \frac{1}{k^2 + gv^2/3}.$$

The spectrum of the theory contains three physical states and the usual state with negative norm coming from the regulator. We see that in the absence of a mass term for the vector field in the action (13.67), the theory is potentially IR divergent in four dimensions. On the other hand, with the mass term the gauge symmetry is broken and the χ-field corresponds really to a Goldstone mode. Even in the physical gauge, a massless scalar field is then present and coupled.

13.13.1 WT identities and renormalization

It follows from the combined analysis of chapter 12 and section 13.7 that after renormalization the correlation functions satisfy the equivalent of WT identities (13.47) and (13.50). As a consequence the dependence of correlation functions on the parameter ξ can be determined as in section 13.9. In particular, only correlation functions of gauge invariant operators and S-matrix elements are gauge independent.

The explicit form of the WT for correlation functions is now rather complicate. We display here only the identities corresponding to the (A_μ, χ) two-point vertex functions. Denoting by v the expectation value of the renormalized φ-field, one obtains by differentiating equation (13.50) with respect to A_μ:

$$\sum_\nu k_\nu \tilde{\Gamma}^{(2)}_{\mu\nu}(k) + iev\, \tilde{\Gamma}^{(2)}_{\mu\chi}(k) = k_\mu \left(k^2/\xi + m^2\right). \qquad (13.69)$$

Differentiating then with respect to χ, one finds,

$$\sum_\mu k_\mu \tilde{\Gamma}^{(2)}_{\mu\chi}(k) - iev\, \tilde{\Gamma}^{(2)}_{\chi\chi}(k) = 0\,. \qquad (13.70)$$

We parametrize the different functions as

$$\tilde{\Gamma}^{(2)}_{\mu\nu}(k) = a(k^2)\delta_{\mu\nu} - b(k^2)k_\mu k_\nu, \quad \tilde{\Gamma}^{(2)}_{\mu\chi}(k) = iev\, c(k^2)k_\mu, \quad \tilde{\Gamma}^{(2)}_{\chi\chi}(k) = d(k^2).$$

In the tree approximation the values of a, b, c and d are

$$\begin{cases} a(k^2) = e^2 v^2 + m^2 + k^2, & b(k^2) = 1 - 1/\xi\,, \\ c(k^2) = 1\,, & d(k^2) = k^2. \end{cases}$$

From the identity (13.70) follows

$$d(k^2) = k^2 c(k^2).$$

The identity (13.69) leads to

$$a(k^2) - k^2 b(k^2) - e^2 v^2 c(k^2) = k^2/\xi + m^2.$$

In particular, in the $k = 0$ limit, the equation implies

$$a(0) - e^2 v^2 c(0) = m^2. \tag{13.71}$$

The corresponding connected two-point functions have the general form

$$\tilde{W}^{(2)}_{\mu\nu}(k) = \frac{1}{a}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{\xi k_\mu k_\nu}{k^2(k^2+\xi m^2)},$$
$$\tilde{W}^{(2)}_{\mu\chi}(k) = -\frac{\xi i e v k_\mu}{k^2(k^2+\xi m^2)}, \quad \tilde{W}^{(2)}_{\chi\chi}(k) = \frac{1}{ck^2}. \tag{13.72}$$

RG β functions. For completeness we give here the RG β-functions at one-loop in a more general model with N charged scalars:

$$\beta_g = \frac{1}{24\pi^2}\left[(N+4)g^2 - 18ge^2 + 54e^4\right], \quad \beta_{e^2} = \frac{1}{24\pi^2}Ne^4. \tag{13.73}$$

The origin $e^2 = g = 0$ is an IR stable fixed point only for $N \geq 183$.

13.13.2 Decoupling gauge

The quantization method we have used above leads to massless fields and thus IR divergences, even though the physical theory contained only massive fields. By the cleverer choice of a gauge which explicitly breaks the global $U(1)$ symmetry of the action (and, therefore, eliminates Goldstone modes), it is possible to circumvent this difficulty.

In terms of the decomposition (13.42), $\mathbf{A}(x) = \mathbf{B}(x) + \nabla\Lambda(x)$, where $\mathbf{B}(x)$ belongs to a gauge section, we impose the condition

$$\mathcal{E}(\Lambda, x) \equiv \nabla \cdot \big(\mathbf{B}(x) + \nabla\Lambda(x)\big) + \lambda e v \,\mathrm{Im}\left(\phi(x)\,\mathrm{e}^{-ie\Lambda(x)}\right) - \xi^{1/2}\nu(x) = 0, \tag{13.74}$$

in which λ is an arbitrary constant and $\nu(x)$ an external stochastic field.

The important new feature is that the operator (see section 13.5.1)

$$\langle y\,|\mathbf{M}|\,x\rangle \equiv \frac{\delta\mathcal{E}(\Lambda,x)}{\delta\Lambda(y)} = \left[\nabla_x^2 + \lambda e v \ \mathrm{Re}\left(\phi(x)\,\mathrm{e}^{-ie\Lambda(x)}\right)\right]\delta^{(4)}(x-y),$$

functional derivative of equation (13.74) with respect to Λ, now depends on the dynamical fields and the associated determinant $\det\mathbf{M}$ is no longer a constant.

This is a source of new difficulties: one has to introduce spinless fermion fields to express $\det\mathbf{M}$ in a local form,

$$\det\mathbf{M} = \int [\mathrm{d}C\,\mathrm{d}\bar{C}]\exp\left[\int \mathrm{d}^4x\,\mathrm{d}^4y\,C(y)\,\langle y|\mathbf{M}|x\rangle\,\bar{C}(x)\right].$$

The fields C and $\bar{C}$ are two scalar fermion fields. As we have explained on an example in section 11.11, scalar fermions cannot be interpreted as physical particles. They are of a nature similar to Pauli–Villars regulator fields.

As before we integrate over $\nu(x)$ with the distribution (13.40) and use the gauge invariance of the initial action:

$$\mathcal{S}_{\text{sym.}}(B,\phi) = \mathcal{S}_{\text{sym.}}(A, \phi\, \mathrm{e}^{-ie\Lambda}).$$

Changing then variables in the field integral $\phi\, \mathrm{e}^{-ie\Lambda} \mapsto \phi$, we obtain the quantized action

$$\mathcal{S}_{\text{qu}}(A,\phi,\bar{C},C) = \int \mathrm{d}^4 x \left\{ \tfrac{1}{4} \sum_{\mu,\nu} F_{\mu\nu}^2 + \frac{1}{2\xi} \left(\nabla\cdot\mathbf{A} + \lambda\, \mathrm{e}\, v\, \mathrm{Im}\,\phi\right)^2 + \left|D_\mu \phi\right|^2 \right.$$
$$\left. + M^2 \left|\phi\right|^2 + \tfrac{1}{6} g \left|\phi^4\right| - C \left(\nabla^2 + \lambda\, \mathrm{e}\, v\, \mathrm{Re}\,\phi\right) \bar{C} \right\}.$$

The initial gauge symmetry is hardly visible in this action. However, it has a surprising new symmetry, the BRS symmetry, that can be used to prove that renormalization preserves the form of the action. We postpone the analysis of such a problem until section 14.5.4, and discuss here only the tree approximation.

Tree approximation. We now use the parametrization (13.68) and choose

$$\lambda = \xi\sqrt{2}\ .$$

This is the relation only at leading order. The propagators are then

$$\begin{aligned}
\tilde{W}^{(2)}_{\mu\nu} &= \frac{\delta_{\mu\nu}}{k^2 + e^2 v^2} + \frac{(\xi - 1)k_{\mu k_\nu}}{(k^2 + e^2 v^2)(k^2 + \xi e^2 v^2)}, \\
\tilde{W}^{(2)}_{\chi\chi} &= \frac{1}{k^2 + \xi e^2 v^2}, \\
\tilde{W}^{(2)}_{\bar{C}C} &= \frac{1}{k^2 + \xi e^2 v^2}.
\end{aligned} \tag{13.75}$$

The advantages of this gauge (introduced by 't Hooft) are that by construction there is no $A_\mu \chi$ propagator and that all unphysical fields are massive and have the same mass $\xi e^2 v^2$. To prove gauge independence of physical observables, it suffices to show that the pole at $k^2 = -\xi e^2 v^2$ cancels.

The price to pay here is the more complicated form of WT identities which now are mixed with BRS symmetry (see section 14.3.4). We examine this question in next chapter in detail.

13.14 Physical observables. Unitarity of the S-Matrix

The unphysical pole at $k^2 = -\xi m^2$ can be shown to cancel in physical observables (gauge invariant operators, S-matrix) either through a gauge dependence analysis as we have done in section 13.9, or directly by using the whole set of WT identities and showing explicitly that the pole coming from $W^{(2)}_{\mu\nu}$ cancels the contribution coming from $W^{(2)}_{\mu\chi}$ in the intermediate state in generalized unitarity relations. As the expressions (13.72) show, the residues of the pole are related and, therefore, one understands that a cancellation is possible. The proof is not very difficult but tedious and we refer to the literature.

In the limit $m = 0$, we expect also the pole at $k^2 = 0$ to cancel in physical observables. According to relation (13.71), for $k^2 \to 0$ the different propagators behave like

$$\tilde{W}^{(2)}_{\mu\nu} \sim \frac{k_\mu k_\nu}{k^2}\left(\frac{1}{m^2} - \frac{1}{m^2 + e^2 v^2 c(0)}\right), \quad \tilde{W}^{(2)}_{\mu\chi} \sim -iev\frac{k_\mu}{k^2 m^2},$$
$$\tilde{W}^{(2)}_{\chi\chi} \sim \frac{1}{c(0)k^2}.$$

Again a direct argument based on WT identities for connected correlation functions and unitarity relations allows to prove that in the $m^2 = 0$ limit the χ-field decouples from physical observables. Here we do not have an alternative proof based upon gauge dependence. However, we shall construct one in a more general context by using a different gauge.