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OBJET : Transfer matrix of a constant gradient alpha magnet for ELSA extension

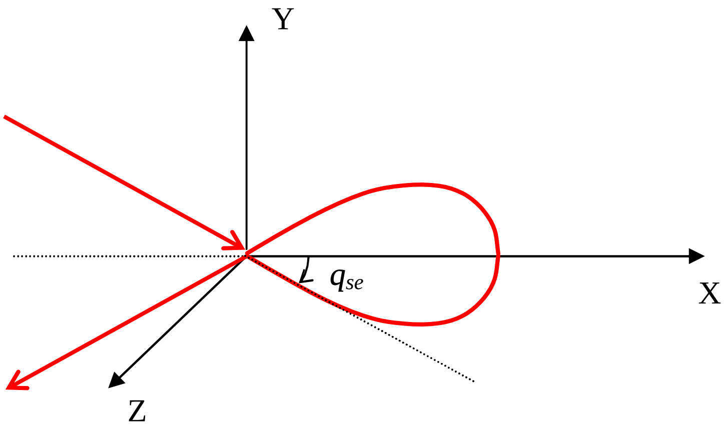
TRANSFER MATRIX OF A CONSTANT GRADIENT ALPHA MAGNET FOR ELSA EXTENSION

In order to bunch the ELSA beam, a set of two constant gradient alpha magnets will be used. To simulate the beam transport trough the ELSA extension, this element has to be included in the TraceWIN program.

In this paper, the motion equation of a particle in such a device is linearised to get the transfer matrix. The matrix has been validated with a step-by-step integration of the motion of a particle in the magnetic field.

1. Magnetic field and fixed coordinate system

The trajectory of the synchronous particle (ideal trajectory) is, by definition, in the plan (X, Y). The frame origin is the point where the particle enters the magnet. The X direction is the direction of the magnetic field gradient. At any position, the main trajectory makes an angle q_s with the X axis. The entrance angle for negative particle q_{se} is: -40.71°. With this particular angle, the synchronous particle exits the magnet at the same position as the entering particle (and with the opposite angle).



In this frame, the magnetic field is:

$$\vec{B} = \begin{pmatrix} B_X & = k \cdot Z \\ B_Y & = 0 \\ B_Z & = k \cdot X \end{pmatrix}$$

2. Trajectory of the synchronous particle

The synchronous particle with charge q moves in the (X, Y) plane.
Its equations of motion with time t are:

$$\begin{cases} \dfrac{dp_X}{dt} = q \cdot v_Y \cdot B_Z \\ \dfrac{dp_Y}{dt} = -q \cdot v_X \cdot B_Z \end{cases}$$

p is the particle momentum, v its velocity,

giving:

$$\begin{cases} \dfrac{dp_X}{ds} = q \cdot \dfrac{p_Y}{p_0} \cdot k \cdot X \\ \dfrac{dp_Y}{ds} = -q \cdot \dfrac{p_X}{p_0} \cdot k \cdot X \\ \dfrac{dX}{ds} = \dfrac{p_X}{p_0} \\ q_s = \arcsin\left(\dfrac{p_Y}{p_0}\right) = \arccos\left(\dfrac{p_X}{p_0}\right) \end{cases}$$

$p_0 = b_0 g_0 m_0 c$ is the particle momentum modulus,
s is the curvilign abscissa, with $ds = v_0 \cdot dt$.

These equations have to be solved using as initial conditions:

$$\begin{cases} p_X = p_0 \cos q_{se} \\ p_Y = p_0 \sin q_{se} \\ X = 0 \\ q_s = q_{se} \end{cases}$$

The maximum *penetration of the particle* X_M in the magnet can be calculated in the following way:

$$\frac{dp_Y}{dX} = -q \cdot k \cdot X \qquad \Rightarrow \qquad p_Y = p_{Y0} - \frac{1}{2} \cdot q \cdot k \cdot X^2 .$$

At maximum penetration, one has: $p_Y = p_0$, giving:

$$X_M = \sqrt{\frac{2 \cdot (p_{Y0} - p_0)}{q \cdot k}} .$$

The length *L* of the trajectory is obtained from the integration of:

$$dt = \frac{g_0 m_0 \cdot dX}{p_X} .$$

One obtains:

$$L = 2 \cdot p_0 \cdot \int_0^{X_M} \left(p_{X0}^2 + q \cdot k \cdot p_{Y0} \cdot X^2 - 0.25 \cdot q^2 k^2 \cdot X^4\right)^{-\frac{1}{2}} \cdot dX .$$

This integral cannot be solved analytically but can be easily calculated numerically.

3. The moving coordinate system

In the transfer matrix formalism, a beam particle is referenced, at a given curvilign abscissa ***s***, to the synchronous particle in a 6D phase-space, with a 6-coordinate vector:

$$\vec{r} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ j \\ d \end{pmatrix},$$

with:

- ***x*** is the particle transverse position in the deviation plane,
- ***x'*** is the particle transverse angle in the x direction. $x' = \frac{p_x}{p_s} = \frac{dx}{ds}$, $\boldsymbol{p_x}$ the x-component of the particle momentum, and $\boldsymbol{p_s}$ the s-component of the particle momentum,
- **y** is the particle transverse position orthogonal to the x direction,

- ***y'*** is the particle transverse angle in the y direction. $y' = \frac{p_y}{p_s} = \frac{dy}{ds}$, $\boldsymbol{p_y}$ the y-component of the particle momentum,

- *j* is the time difference between the particle arriving in *s* (***t***) and the synchronous particle arriving in s ($\boldsymbol{t^{(S)}}$). This time is normalised by the RF frequency $\boldsymbol{f_{RF}}$.

$$j = 2p \cdot f_{RF} \cdot \left(t - t^{(S)}\right),$$

- *d* is the particle momentum ***p*** relative to the synchronous particle momentum $\boldsymbol{p^{(S)}}$.

$$d = \frac{p - p^{(S)}}{p^{(S)}}.$$

4. Matrix calculation

In the matrix formalism, the particle vector transport from a point (1) to a point (2) is given by:

$$\left(\vec{r}\right)_{(2)} = \left[T\right] \cdot \left(\vec{r}\right)_{(1)},$$

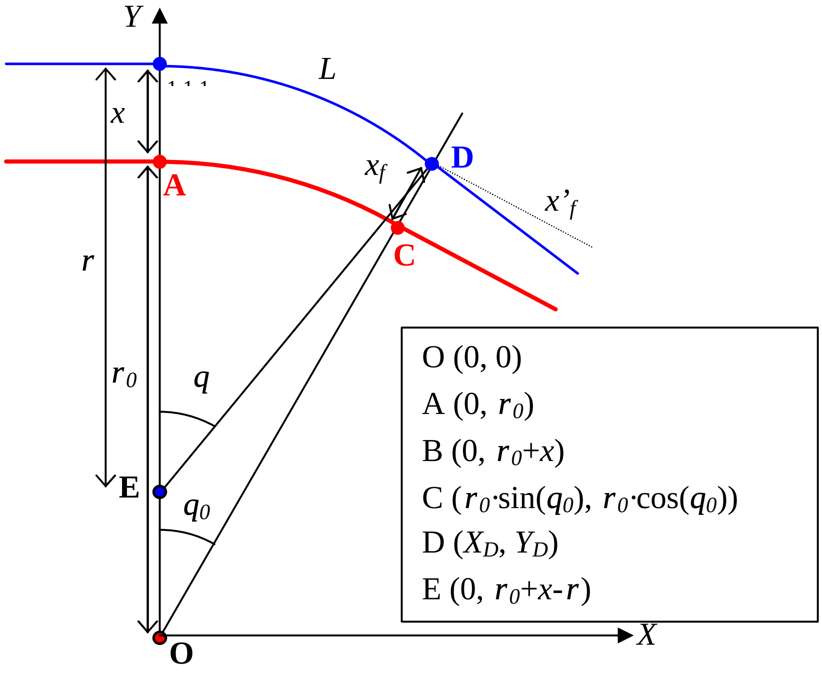
where [*T*] is the transfer matrix from (1) to (2). The coefficients of [*T*] are:

$$T_{i,j} = \frac{\partial r_i)_{(2)}}{\partial r_j)_{(1)}}.$$

i is the line index, *j* is the column index (between 1 and 6).

The matrix coefficients can be calculated by varying the input particle coordinate along each direction independently and looking at the output coordinates.

1.1 Matrix first column: variation with x



1.1.1. Calculus of $T_{1,1}$

D is at the intersection of (O, C) and the circle with centre E and radius r. Its coordinates satisfy:

$$\begin{cases} X_D = \tan(q_0)\cdot Y_D \\ X_D^2 + (Y_D - (r_0 + x - r))^2 = r^2 \end{cases}.$$

Giving:

$$(1+\tan^2 q_0)\cdot Y_D^2 - 2\cdot(x-dr)\cdot Y_D + (r_0 + x)\cdot(x - 2\cdot dr - r_0) = 0\,,$$

with: $r = r_0 + dr = r_0 \cdot \left(1 + \dfrac{dr}{r_0}\right).$

As: $r = \dfrac{p}{q\cdot B}$, one has: $\dfrac{dr}{r_0} = -\dfrac{dB}{B_0} = \sin q_s \cdot \dfrac{x}{X_s}$.

The solution of (X) is obtained giving a reduced discriminator:

$$\Delta' = (x-dr)^2 - \frac{(r_0 + x)\cdot(x - dr - r_0)}{\cos^2 q_0}\,,$$

giving:

$$\Delta' = \frac{r_0^2}{\cos^2 q_0} \cdot \left[\left(\frac{x}{r_0} - \frac{dr}{r_0} \right)^2 \cdot \cos^2 q_0 - \left(1 + \frac{x}{r_0} \right) \cdot \left(\frac{x}{r_0} - 2 \cdot \frac{dr}{r_0} - 1 \right) \right].$$

A first order development in $\frac{x}{r_0} << 1$ and $\frac{dr}{r_0} << 1$ gives:

$$\Delta' = \frac{r_0^2}{\cos^2 q_0} \cdot \left[1 + 2 \cdot \frac{dr}{r_0} \right].$$

This gives the coordinates of D:

$$\begin{cases} X_D = r_0 \cdot \sin q_0 \cdot \left(1 + \cos q_0 \cdot \frac{x}{r_0} + (1 - \cos q_0) \cdot \frac{dr}{r_0} \right) \\ Y_D = r_0 \cdot \cos q_0 \cdot \left(1 + \cos q_0 \cdot \frac{x}{r_0} + (1 - \cos q_0) \cdot \frac{dr}{r_0} \right) \end{cases}$$

The final position of the particle in the moving frame is then :

$$\boxed{x_f = \frac{Y_D - Y_C}{\cos q_0} = \left(\cos q_0 + (1 - \cos q_0) \cdot \frac{r_0 \cdot \sin q_s}{X_s} \right) \cdot x = T_{1,1} \cdot x}$$

1.1.2. Calculus of $T_{2,1}$

$$x'_f = \tan(q_0 - q) = \frac{\tan q_0 - \tan q}{1 + \tan q_0 \cdot \tan q},$$

with: $\tan q = \frac{X_D - X_E}{Y_D - Y_E}$.

At first order, one has:

$$\tan q = \tan q_0 \cdot \left(1 + \frac{x - dr}{r_0 \cos q_0} \right),$$

giving :

$$\boxed{x'_f = -\sin q_0 \cdot \left(1 - \frac{r_0 \cdot \sin q_s}{X_s} \right) \cdot \frac{x}{r_0} = T_{2,1} \cdot x}$$

1.1.3. Calculus of $T_{5,1}$

$$j = \frac{2p \cdot f_{RF}}{b_0 c}\left(L - r_0 q_0\right) = \frac{2p \cdot f_{RF}}{b_0 c} r_0 q_0 \cdot \left(\frac{dr}{r_0} + \frac{dq}{q_0}\right).$$

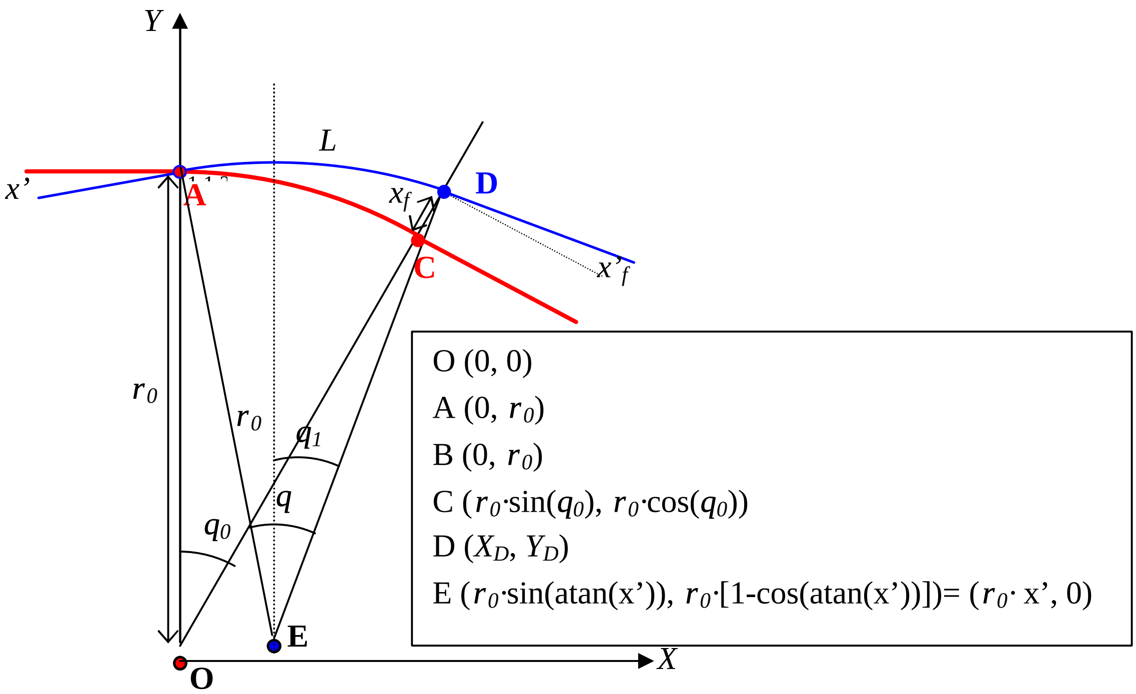
with: $$dq = \cos^2 q_0 \cdot d(\tan q) = \cos^2 q_0 \cdot (\tan q - \tan q_0) = \sin q_0 \cdot \left(\frac{x - dr}{r_0}\right).$$

One obtains finally:

$$\boxed{j = \frac{2p \cdot f_{RF}}{b_0 c} \cdot \left(\sin q_0 + (q_0 - \sin q_0) \cdot \frac{r_0 \sin q_s}{X_s}\right) \cdot x = T_{5,1} \cdot x}.$$

The other terms ($T_{3,1}$, $T_{4,1}$, $T_{6,1}$) are equal to zero.

1.2 Matrix second column: variation with x'



1.1.4. Calculus of $T_{1,2}$

D is at the intersection of (O, C) and the circle with centre E and radius r. Its coordinates satisfy:

$$\begin{cases} X_D = \tan q_0 \cdot Y_D \\ (X_D - r_0 \cdot x')^2 + Y_D^2 = r_0^2 \end{cases}.$$

Giving:

$$\left(1+\tan^2 q_0\right)\cdot Y_D^2 - 2\cdot r_0 \cdot x' \cdot \tan q_0 \cdot Y_D - r_0^2 \cdot \left(1-x'^2\right) = 0 \,.$$

The solution of (X) is obtained giving a reduced discriminator:

$$\Delta' = r_0^2 \cdot \tan^2 q_0 \cdot x'^2 + \frac{r_0^2 \cdot \left(1-x'^2\right)}{\cos^2 q_0} \,.$$

A first order development in $x' << 1$ gives:

$$\Delta' = \frac{r_0^2}{\cos^2 q_0} \,.$$

This gives the coordinates of D:

$$\begin{cases} X_D = r_0 \cdot \sin q_0 \cdot \left(1+\sin q_0 \cdot x'\right) \\ Y_D = r_0 \cdot \cos q_0 \cdot \left(1+\sin q_0 \cdot x'\right) \end{cases}$$

The final position of the particle in the moving frame is then :

$$\boxed{x_f = \frac{Y_D - Y_C}{\cos q_0} = r_0 \cdot \sin q_s \cdot x' = T_{1,2} \cdot x'}$$

1.1.5. Calculus of $T_{2,2}$

$$x'_f = \tan\left(q_0 - q_1\right) = \frac{\tan q_0 - \tan q_1}{1+\tan q_0 \cdot \tan q_1}$$

with: $\tan q_1 = \dfrac{X_D - X_E}{Y_D - Y_E}$.

At first order, one has:

$$\tan q = \frac{X_D - X_E}{Y_D - Y_E} \,,$$

giving :

$$\boxed{x'_f = \cos q_0 \cdot x' = T_{2,2} \cdot x'}$$

1.1.6. Calculus of $T_{5,2}$

$$j = \frac{2p \cdot f_{RF}}{b_0 c}\left(L - r_0 q_0\right) = \frac{2p \cdot f_{RF}}{b_0 c} r_0 \cdot dq \ .$$

with : $\quad dq = \cos^2 q_0 \cdot d\left(\tan q\right) = \cos^2 q_0 \cdot \left(\tan q - \tan q_0\right),$

as: $\quad \tan q = \tan\left(q_1 + q_0\right) = \tan q_0 \cdot \left(1 + \frac{1 - \cos q_0}{\sin q_0 \cdot \cos q_0} \cdot x'\right),$

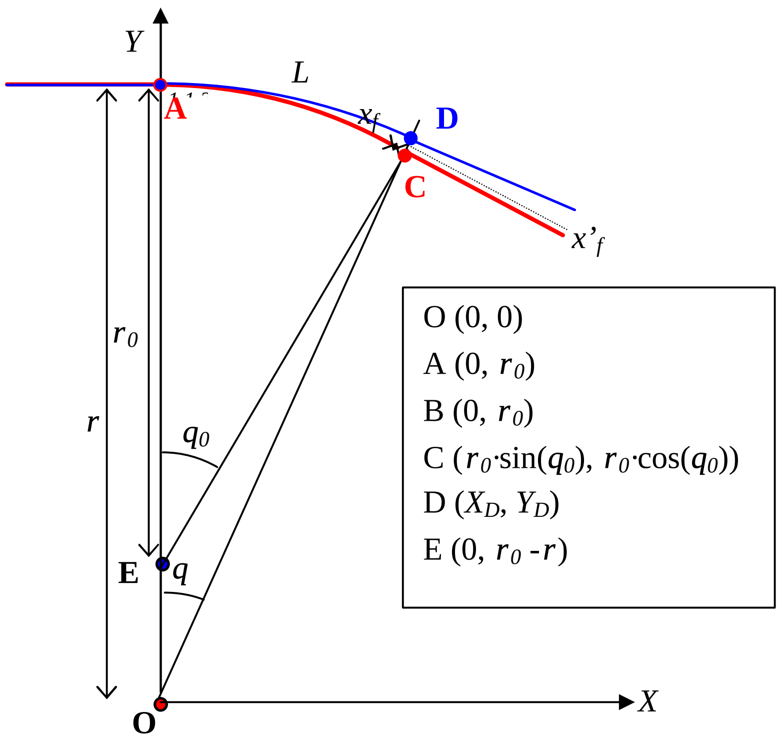
one has: $\quad dq = \left(1 - \cos q_0\right) \cdot x' \ .$

One obtains finally:

$$\boxed{j = \frac{2p \cdot f_{RF}}{b_0 c} \cdot r_0 \cdot \left(1 - \cos q_0\right) \cdot x' = T_{5,2} \cdot x'}.$$

The other terms ($T_{3,2}$, $T_{4,2}$,$T_{6,2}$) are equal to zero.

1.3 Matrix sixth column: variation with d



1.1.7. Calculus of $T_{1,6}$

D is at the intersection of (O, C) and the circle with centre E and radius r. Its coordinates satisfy:

$$\begin{cases} X_D = \tan(q_0) \cdot Y_D \\ X_D^2 + (Y_D - (r_0 - r))^2 = r^2 \end{cases} .$$

Giving:

$$(1 + \tan^2 q_0) \cdot Y_D^2 + 2 \cdot dr \cdot Y_D - r_0 \cdot (r_0 + 2 \cdot dr) = 0$$

with: $r = r_0 + dr = r_0 \cdot \left(1 + \frac{dr}{r_0}\right)$.

As: $r = \frac{p}{q \cdot B}$, one has: $\frac{dr}{r_0} = d$.

The solution of (X) is obtained giving a reduced discriminator:

$$\Delta' = dr^2 + \frac{r_0 \cdot (r_0 + 2 \cdot dr)}{\cos^2 q_0} .$$

A first order development in $\frac{dr}{r_0} << 1$ gives:

$$\Delta' = \frac{r_0^2}{\cos^2 q_0} \cdot \left[1 + 2 \cdot \frac{dr}{r_0}\right]$$

This gives the coordinates of D:

$$\begin{cases} X_D = r_0 \cdot \sin q_0 \cdot \left(1 + (1 - \cos q_0) \cdot \frac{dr}{r_0}\right) \\ Y_D = r_0 \cdot \cos q_0 \cdot \left(1 + (1 - \cos q_0) \cdot \frac{dr}{r_0}\right) \end{cases}$$

The final position of the particle in the moving frame is then :

$$\boxed{x_f = \frac{Y_D - Y_C}{\cos q_0} = r_0 \cdot (1 - \cos q_0) \cdot d = T_{1,6} \cdot d}$$

1.1.8. Calculus of $T_{2,6}$

$$x'_f = \tan(q_0 - q) = \frac{\tan q_0 - \tan q}{1 + \tan q_0 \cdot \tan q}$$

wih: $\tan q = \dfrac{X_D - X_E}{Y_D - Y_E}$.

At first order, one has:

$$\tan q = \tan q_0 \cdot \left(1 - \frac{dr}{r_0 \cos q_0}\right),$$

giving :

$$\boxed{x'_f = \sin q_0 \cdot d = T_{5,5} \cdot d}$$

1.1.9. Calculus of $T_{5,6}$

$$j = \frac{2p \cdot f_{RF}}{b_0 c}(L - r_0 q_0) = \frac{2p \cdot f_{RF}}{b_0 c} r_0 q_0 \cdot \left(\frac{dr}{r_0} + \frac{dq}{q_0}\right).$$

with: $dq = \cos^2 q_0 \cdot d(\tan q) = \cos^2 q_0 \cdot (\tan q - \tan q_0) = -\sin q_0 \cdot \dfrac{dr}{r_0}$,

One obtains finally:

$$\boxed{j = \frac{2p \cdot f_{RF}}{b_0 c} \cdot r_0 q_0 \cdot \left(1 - \frac{\sin q_0}{q_0}\right) \cdot d = T_{5,6} \cdot d}.$$

The terms ($T_{3,6}$, $T_{4,6}$) are equal to zero, the term $T_{6,6}$ is equal to 1.

1.4 Matrix fifth column: variation with j

The output position, slope, energy do not depend on the input phase j:
The terms ($T_{1,5}$, $T_{2,5}$, $T_{3,5}$, $T_{4,5}$, $T_{6,5}$) are equal to zero, the term $T_{5,5}$ is equal to 1.

1.5 Matrix third and fourth columns: motion in y (or Z)

The equation of motion along Z direction is :

$$\frac{dp_Z}{dt} = -q \cdot \frac{p_y}{gm} \cdot B_X = -q \cdot \frac{p_y}{gm} \cdot k \cdot Z \ .$$

For the matrix calculation, one uses a first order development of the force, giving:

$$\frac{dp_Z}{dt} = -q \cdot \frac{p_0 \cdot \sin q_0}{gm} \cdot k \cdot Z \; .$$

This equation is the classical one in a quadrupole with gradient: $k \cdot \sin q_0$.

The associated matrix coefficients are:

$$T_{3,3} = T_{4,4} = \cos\left(\sqrt{K}\, r_0 q_0\right),$$

$$T_{3,4} = \frac{\sin\left(\sqrt{K}\, r_0 q_0\right)}{\sqrt{K}},$$

$$T_{4,3} = -\sqrt{K} \cdot \sin\left(\sqrt{K}\, r_0 q_0\right),$$

with: $K = \dfrac{k \cdot \sin q_s}{p_0}$.

The other coefficients are equal to 0.

The final matrix of a fraction of a alpha magnet (on which, X_s and q_s are kept almost constant) :

$$\begin{pmatrix} \cos q_0 + (1-\cos q_0)\cdot\frac{r_0\cdot\sin q_s}{X_s} & r_0\cdot\sin q_s & 0 & 0 & 0 & r_0\cdot(1-\cos q_0) \\ -\frac{\sin q_0}{r_0}\cdot\left(1-\frac{r_0\cdot\sin q_s}{X_s}\right) & \cos q_0 & 0 & 0 & 0 & \sin q_0 \\ 0 & 0 & \cos\left(\sqrt{K}\,r_0 q_0\right) & \frac{\sin\left(\sqrt{K}\,r_0 q_0\right)}{\sqrt{K}} & 0 & 0 \\ 0 & 0 & -\sqrt{K}\cdot\sin\left(\sqrt{K}\,r_0 q_0\right) & \cos\left(\sqrt{K}\,r_0 q_0\right) & 0 & 0 \\ K_j\cdot\left(\sin q_0 + (q_0-\sin q_0)\cdot\frac{r_0\sin q_s}{X_s}\right) & K_j\cdot r_0\cdot(1-\cos q_0) & 0 & 0 & 1 & K_j\cdot r_0\cdot(q_0-\sin q_0) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with: $K = \frac{k\cdot\sin q_s}{X_s}$,

and: $K_j = \frac{2p\cdot f_{RF}}{b_0 c}$.

The matrix of the full element is a product of all matrixes for varying X_s and q_s.