STRUCTURE AND EVOLUTION OF THE SOLAR CORONAL MAGNETIC FIELD

J.J. ALY†* and T. AMARI‡

†AIM - Unité Mixte de Recherche CEA - CNRS - Université Paris VII - UMR n^0 7158, Centre d'Études de Saclay, F-91191 Gif sur Yvette Cedex, France
‡CNRS, Centre de Physique Théorique de l'Ecole Polytechnique, F-91128 Palaiseau Cedex, France

(Received 2 January 2007; in final form 25 May 2007)

Abstract

We review some of the results that we have obtained in the last decade on two problems related to the structure and evolution of the solar corona: How to reconstruct the magnetic field of an active region from its values measured at the photospheric level, and how to determine the evolution of the coronal field driven by the stressing of its footpoints on the photosphere and/or by flux emergence through that surface, our main goal being to elucidate the nature of the mechanisms triggering large scale eruptive processes like coronal mass ejections (CME). The first part of the paper is devoted to a first approach in which the coronal field is assumed to be force-free at any time (but during the late development of an eruptive event), its evolution being thus considered to be quasi-static. After presenting some general properties of this type of fields, we use this approximate model as a general framework for the reconstruction problem. To get a well posed problem, we introduce the Grad-Rubin formulation in which only a part of the photospheric data are taken into account. We present some mathematical results on this problem (existence and uniqueness of solutions), and report our method to treat it numerically in an efficient way. Thus we turn to the quasi-static boundary driven evolution problem. We find that a continuous injection of energy into a simple field (arcade or tube) by footpoints shearing leads in the ideal case to an expansion of the field which is at least as fast as $e^{(t/T)^2}$ at large time t, and to its eventual partial or total opening with the formation of a current sheet. The second part of the paper is concerned with a dynamic approach to the evolution problem. The full set

*Corresponding author. Email: jean-jacques.aly@cea.fr

of equations of the resistive magnetohydrodynamics is used and solved numerically in two different classes of models. In the first one, the evolution is driven by changing boundary conditions (describing shear, converging motions, flux cancellation, ...) imposed at the photospheric level. In the second one, both the corona and the subphotospheric layer (top of the convection zone) are simultaneously considered, and the rising of a twisted tube below the photosphere and its emergence through that surface and subsequent evolution in the corona are followed. In all cases, a catastrophic behavior is found to follow a slow quasi-static phase. It is characterized by a rapid expansion of the field and a release of energy by reconnection. Moreover, a twisted flux rope is always observed to form during the evolution. Depending on the conditions, it is created either in equilibrium during the slow phase, then appearing as a favorable site for the support of a prominence, or during the global disruption phase. The energy of the configuration stays below that of the corresponding open field except when the driving of the evolution is ensured by flux cancellation on the boundary. In that case – to which we refer as the Flux Cancellation Model (FCM) of CME – the open field energy decreases up to a critical point at which it becomes close to the value of the magnetic energy of the configuration, and nonequilibrium sets in. The characteristics of the evolution in FCM are found to be similar in simple and complex topologies (in contrast, the Break Out Model of CME works only for a complex topology). However, when the topology is complex, there is a lowering of the confining effect of the overlying field, and the twisted rope is ejected at a faster rate.

Keywords: Solar corona, Eruptive events, Magnetohydrodynamics, Force-free magnetic fields, Numerical simulations

1 INTRODUCTION

The solar corona is the region consisting of a very hot ($T = 10^6$K) and underdense ($n = 10^{15}$ m^{-3}) plasma which extends from the surface of the Sun up to some 2.5 $R_\odot$. The corona is highly inhomogeneous. Decades of observations made by sophisticated instruments located either on the ground or onboard numerous satellites, have shown the presence of a lot of various structures over a wide range of length scales: On the largest scales, coronal holes and helmet streamers, visible by naked eyes during the eclipses, on the intermediate scales, prominences, loops, arcades, ..., and on the smallest scales, elementary flux tubes, bright points, ... Moreover, they have shown the corona to be very unsteady. Small scales structures turn out to be in a state of unceasing evolution on short time scales, and even the largest scale structures, which may exhibit high degree of permanence over long periods, suffer from time to time a global catastrophic restructuring. Very spectacular events are associated with the latter, such as flares, coronal mass ejections (CMEs) and eruptive prominences, during which huge amounts of energy are suddenly released and huge amounts of mass ejected into the interplanetary space. In our current understanding of solar physics, it is the magnetic field which is responsible for the very existence of the corona, its structuring and its quiet or catastrophic evolution. Actually, most of the observations and theoretical arguments strongly support the idea that the corona is a magnetically dominated system. However, we are in a quite strange situation: Although the magnetic field is so important, we are unable to measure it with the standard methods currently at our disposal. We can take measurements of it only at the basis of the corona, on the so-called photosphere. There the much lower temperature allows for the presence of neutral atoms, whose light can reveal the value of the field after an appropriate analysis.

To be more precise, let us describe quickly the general paradigm underlying the largest part of the current research on the physics of the solar corona (see, e.g., Priest (1982)). The story starts in the convection zone of the Sun, that is to say in the external layer of that star in which the transport of the energy is ensured by convective motions rather than by radiative processes. The lower part of this zone (the so-called tachocline) is the location of a dynamo process producing a strong toroidal magnetic field. The confinement of the tubes produced by this process becomes impossible when the intensity of the field reaches some critical value, and they start rising through the convection zone due to buoyancy effects and convective transport, possibly gaining some twist and producing some amount of poloidal field by their interactions with cyclonic motions. At some time, they reach the photosphere and emerge into the corona, producing such structures as the big sunspots with their associated active regions. The emerging magnetic field usually carries electric currents and therefore, it will already have free magnetic energy. This free energy can be increased further after emergence if photospheric motions like turbulence or differential rotation stress the coronal field by moving the footpoints of the field lines, and even by the flux cancellation process which is currently observed at the solar surface (Welsch 2006, Wang and Sheeley 2002). Release of this free energy by various processes like magnetic reconnection thus leads on the smaller scales to a continuous heating of the plasma, which can so maintain its high temperature – this permits the continued existence of the corona –, while it produces big eruptive events on the largest scales. The largest ones are associated with CMEs, and they are characterized by a fast expansion of the field leading to its eventual opening and to an expulsion of a blob of matter. CMEs also transport to large distances from the Sun some of the helicity which has been produced by the dynamo inside and injected into the corona by flux emergence, and they may play a basic role in the magnetic cycle (Low 1994).

To try understanding the details of the mechanisms initiating the production of a big eruption, a large amount of work has been done by several groups around the world on the following problem of magnetohydrodynamics (MHD) – to which we shall refer as the *evolution problem*: How the magnetic field embedded in the low beta highly conducting coronal plasma does evolve as a consequence of the slow perpetual changes which occur in the dense subphotospheric layers in which its lines are anchored. Several types of configurations have thus been considered – single flux tube, interacting flux tubes, bipolar or multipolar arcades with or without embedded flux tubes, ... –, and similarly many types of qualitatively different changes at the photospheric level – shearing of the footpoints, flux cancellation, flux emergence, ... At first, these changes were imposed by just setting adequate boundary conditions on the photosphere, but in some of the most recent contributions, the behaviour of the field and the plasma in at least the upper part of the convection zone has also been included as a part of the computation – with as an ultimate goal the building up of a global model of the solar magnetic machine including in a self-consistent way all the processes (dynamo, tube rising and emergence, corona invasion, plasma heating, large eruptive process, ...). Owing to the large number of possibilities quoted above, it is not a surprise that several different models have been proposed (for very recent reviews, see Forbes et al. (2006), Lin et al. (2003), Mikić and Lee (2006), Zhang and Low (2005)), and it may be reasonably thought that some of the essential features controlling the initiation of an eruptive event are caught by some or the other of these models. But unfortunately it is still difficult to distinguish observationally between them, and no consensus on the actual nature of the triggering mechanism has been reached yet.

One of the reasons for the difficulty to determine the nature of the physical mechanism for large eruptions is the current impossibility, already noted, to make a direct measurement of the magnetic field in the corona. It is clear that a knowledge at any time of such basic features as the topology of the field – with in particular the presence or not of a twisted flux tube –, or the location and intensity of the associated electric currents, would provide precious clues to our understanding of these events. They could give information on the places where reconnection may be expected to take place, on the nature of the instabilities which could possibly develop, etc.. However, it has proven possible, up to some extent, to start bypassing the current observational limitations by developing a new topic in solar physics known as the *reconstruction problem*. The point of departure here is the fact that we are able to measure with a good precision the three components of the magnetic field at the photospheric level with modern magnetographs (e.g., THEMIS, the Imaging Vector Magnetograph, the Advanced Stokes Polarimeter; and many more will be shortly available with SOLIS, HINODE/SOLAR-B, and the programmed missions SOLAR-ORBITER, and Solar Dynamics Observatory). Then we may try to set up reasonable assumptions allowing to extend these values into the corona, and thus to "reconstruct" the coronal field **B**. Earlier models were based either on the rough assumption that **B** is potential – but such a field has no free-energy available for eruptions – or on the more satisfying assumption that **B** is a linear force-free field – but such a field has bad asymptotic properties and can provide an approximation of **B** only in a region of not too large extent. More recently, much efforts have been devoted to a much more realistic model in which **B** is taken to be a nonlinear force-free field. This leads to a difficult mathematical problem, for which several conceptually different practical methods of solution have been developed (for a review, see, e.g., Sakurai (1989), Amari and Démoulin (1992), Amari et al. (1997a), Neukirch (2005)).

The aim of this paper is to present some of the analytical and numerical work we have done ourselves in the last decade or so on both the reconstruction and the evolution three-dimensional problems along with our collaborators, C. Boulbe, A. Bleybel, T. Z. Boulmezaoud, E. Kersalé, J.-F. Luciani, J. A. Linker, Z. Mikic, T. Lepeltier, S. Régnier, and M. Tagger. Therefore, **this is not a general review of these topics**, and important papers by many others are quoted only insofar as they bear a direct relation to our own work. Our presentation is organized as follows. We first give (Sect. 2) a description of the general framework in which our research has been developed, and recall the definitions of some basic physical quantities (like the relative magnetic helicity) that are repeatedly used in the following sections. Then (Sect. 3) we describe the force-free model of the coronal magnetic field, and present some of its general properties (boundedness of a class of generalized energies, asymptotic behavior of the field, ...). Section 4 is devoted to a presentation of our most recent results on the problem of the reconstruction of the field of an active region as a nonlinear force-free field by using data extracted from photospheric measurements. The remaining part of the paper is concerned with the determination of the evolution of the coronal field driven by the changing conditions imposed by the dense photospheric plasma. This evolution is first considered in Section 6 from an analytical point of view in the context of the force-free model (ideal quasi-static evolution). Thus we report the results of our numerical dynamical simulations in which we solve the whole set of the MHD equations, including resistivity, which allows us to study the possibility for the field to reconnect. In Section 7, we assume that the evolution is driven by changing boundary conditions imposed on the basis of the corona and mimicking the effects of the subphotospheric layer. Next this layer is explicitly introduced as a part of the system under consideration (although in a simplified kinematical way) in Section 8, where we consider the rising of a flux tube through the upper part of the convection zone, its emergence through the photosphere, and its subsequent evolution in the corona. Finally we emphasize in our concluding Section 9 some of the problems which we would like to see being elucidated in the near future.

2 THE MHD MODEL

In this section, we describe in some details the general MHD model in which our work has been developed, and we recall some basic definitions.

2.1 Assumptions

Hereafter, we use Cartesian coordinates (x, y, z) and associated standard spherical coordinates (r, θ, φ), with θ and φ measured with respect to the z and x-axis, respectively.

Our general model rests on the following assumptions:

- The corona is represented by the exterior $D = \{r > r_*\}$ of a spherical domain. Actually, we are most often interested in only a part of it – e.g., an active region – in which case we neglect the curvature of the solar surface and take as the relevant domain the upper half-space $D = \{z > 0\}$. In either case, the thin photospheric layer is represented by the surface $S = \partial D$.

- D contains a magnetic field $\mathbf{B}$ of finite energy, i.e.,

$$W[\mathbf{B}] = \int_D \frac{B^2}{8\pi}\, \mathrm{d}V < \infty\,. \tag{1}$$

This field is embedded in a low beta and highly conducting plasma, i.e.,

$$\beta = \frac{\overline{p}}{\overline{B}^2/8\pi} \ll 1 \quad \text{and} \quad R_m = \frac{L\overline{v}}{\overline{d}_m} \gg 1\,, \tag{2}$$

where $\overline{p}$, $\overline{v}$, $\overline{d}_m$, and $\overline{B}$, are typical values of the thermal pressure, velocity, and magnetic diffusivity of the plasma, and of the field strength, respectively, and L is the spatial scale of variation of these quantities.

- The dense plasma in the subphotospheric layers below, out of which the coronal field emerges, is brought into convective and turbulent motions involving kinetic energy densities larger than the magnetic ones, and this drives the corona into a perpetual evolution. This driving is modelled in two different ways:

  1. The value of the ratio of plasma to magnetic energy density in the corona (<< 1) is vastly different from the value in the subphotospheric layers (>> 1). We exploit this by imposing appropriate evolving boundary conditions on S, which mimic the influence of the subphotospheric layers upon the corona. In this approach we neglect the back reaction of the corona onto the photosphere and the layers below it. We speak in that case of a *boundary driven evolution*. For instance, we may require the footpoints on S to move horizontally at some prescribed slow velocity $\mathbf{v}$ – with "slow" meaning here that the typical value $\overline{v}$ of v is much smaller than the typical Alfven speed $\overline{v}_A$ in D (observations indicate that $\overline{v}/\overline{v}_A \simeq 10^{-2} - 10^{-3}$; note that $\overline{v}_A$ is about the speed of all the MHD waves in D as $\beta \ll 1$). An important feature of a boundary driven evolution is that the normal component of the field on S can be considered to be imposed at any time t, i.e., we have $B_n = Q$ on S, with Q a function which can be determined directly from the data on S, independently of the actual field in D (here, we take $\hat{\mathbf{n}} = \hat{\boldsymbol{r}}$ or $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, depending on the choice of D).
  2. A step towards a more realistic model is taken by introducing what we call a *subphotospheric driven evolution*, which amounts to consider as a part of the system the upper part of the convection zone. The latter is represented either by the shell $D_* = \{r_* - h < r < r_*\}$ (when $D = \{r > r_*\}$), or by the layer $D_* = \{-h < z < 0\}$ (when $D = \{z > 0\}$). Up to now, however, we have treated this zone only in a kinematic way, by taking the slow motion of the plasma to be given in the form of convection cells. The back reaction of the corona on the layers below is thus due essentially to the resistive diffusion of the field which is allowed in both D and D_*.

- The evolution of the field and the plasma in D (and D^*) is described by MHD. This assumption is justified for the study of the large scale phenomena in which we are interested, for which the typical spatial and temporal scales are quite large compared to the scales of the microscopic processes coupling together the charged particles, so insuring the validity of a one-fluid approximation.

2.2 The MHD equations

We use the equations of dissipative MHD in the form

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\rho(\mathbf{v}\cdot\nabla\mathbf{v}) + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} - \nabla p + \nabla\cdot(\nu\rho\nabla\mathbf{v}) + \rho\mathbf{g}\,, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times(\eta\mathbf{j})\,, \tag{4}$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v})\,, \tag{5}$$

$$\frac{\partial p}{\partial t} = -\mathbf{v}\cdot\nabla p - \Gamma p(\nabla\cdot\mathbf{v}) + H\,, \tag{6}$$

$$\frac{4\pi}{c}\mathbf{j} = \nabla\times\mathbf{B}\,, \tag{7}$$

$$\nabla\cdot\mathbf{B} = 0\,. \tag{8}$$

The notation used here is standard. In addition to $\mathbf{B}$, $\mathbf{v}$ and p which have been already defined, we use the symbols ρ, ν, η, Γ, H, $\mathbf{j}$, and $\mathbf{g}$, to denote, respectively, the mass density, the kinematic viscosity, the resistivity, the adiabatic index of the plasma, the heating rate, the electric current density, and the gravitational field.

2.3 Magnetic energy and helicity

Finally, we introduce some important global physical quantities characterizing a magnetic field in D. First, we set

$$W_k^{(\nu)}[\mathbf{B}] = \int_D \frac{B_k^2}{8\pi}\left(\frac{r}{r_0}\right)^{\nu} \mathrm{d}V\,, \qquad k = r, \perp, \,, \tag{9}$$

where B_r and $\mathbf{B}_\perp$ are, respectively, the radial and orthoradial components of $\mathbf{B}$, r_0 is a reference length (we take $r_0 = r_*$ when $D = \{r > r_*\}$), and ν is an arbitrary number for which the integral is well defined. This is certainly the case when $\nu \leq 0$, as we get for $\nu = 0$ and a blank index the total energy of $\mathbf{B}$ ($W^{(0)}[\mathbf{B}] = W[\mathbf{B}]$) which is finite for all the fields we consider (condition (1)). Quite naturally, we shall refer to $W_r^{(\nu)}[\mathbf{B}]$, $W_\perp^{(\nu)}[\mathbf{B}]$, and $W^{(\nu)}[\mathbf{B}]$, as the radial, orthoradial and total ν-energies of $\mathbf{B}$.

An other very important physical quantity is the magnetic helicity. For considerations in which both the regions D and D_* are included, the field $\mathbf{B}$ can be considered as isolated, and H is defined by

$$H[\mathbf{B}] = \frac{1}{2}\int_{D\cup D_*} \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V\,, \tag{10}$$

where $\mathbf{A}$ is an arbitrary vector potential for $\mathbf{B}$. It is worth noticing that the finite energy condition (1) is not sufficient to ensure the convergence of the integral in the right-hand side of Eq. (10) as the integration domain is here unbounded. Rather, one should assume the integrability of $B^{3/2}$ (Aly 1992, Laurence and Avellaneda 1993). H is physically meaningful under that condition as it does not depend on the gauge $\mathbf{A}$. This is no longer the case if we restrict our attention to the sole coronal field in D as now $B_n \neq 0$ on S, and we need to appeal in that situation to the concept of *relative magnetic helicity* $H[\mathbf{B}]$, first introduced in Berger and Field (1984). If we denote as $\mathbf{B}_\pi$ the unique finite energy potential field (i.e., $\nabla \times \mathbf{B}_\pi = 0$) determined by the boundary condition $B_{\pi n} = B_n$ on S (with $B_n = B_z$ assumed to have a fast decrease to zero at infinity when $D = \{z > 0\}$), and by $\mathbf{A}_\pi$ one of its vector potential, H may be defined by (Finn and Antonsen 1985)

$$H[\mathbf{B}] = \frac{1}{2} \int_D (\mathbf{A} + \mathbf{A}_\pi) \cdot (\mathbf{B} - \mathbf{B}_\pi) \, \mathrm{d}V \,, \tag{11}$$

where we still impose $B^{3/2}$ integrable.

The importance of the concept of helicity stems from the fact that this quantity is conserved in ideal MHD if we assume that the plasma velocity vanishes on S at the places where $B_n \neq 0$. If the latter condition is not fulfilled – i.e., if the field has some moving footpoints on S – the change of the helicity is fully controlled by the flux distribution and the velocity on S.

3 THE FORCE-FREE MODEL

In this section, we introduce the *force-free model of the solar corona*, which is a first approximation of the general model, and we present some of its properties.

3.1 Assumptions

The force-free model of the solar corona is defined by the following specific assumptions:

- At any time $t \geq 0$, say, the finite energy magnetic field $\mathbf{B}$ in D is *force-free*. This means that the electric current density $\mathbf{j}$ is aligned with $\mathbf{B}$ everywhere in D, but possibly on some regular surfaces Σ_j and Σ_c accross which, respectively, $\mathbf{j}$, and $\mathbf{B}$ and $\mathbf{j}$, suffer a discontinuity. Moreover, the force acting on the surface current which has to flow on Σ_c vanishes – i.e., Σ_c appears to be a current sheet in equilibrium. The reasons for introducing this enlarged definition of a force-free field will appear clearly later on.

- $\mathbf{B}$ is driven by slow motions imposed upon its footpoints on S into an unceasing evolution starting from a given equilibrium configuration $\mathbf{B}_0$ at time $t = 0$ – then we are here in the context of a *boundary driven quasi-static evolutionary model*. As noted above, this assumption implies that the flux distribution Q on S can be determined at any time t by the data on S. We assume that the resulting function Q is sufficiently regular and has a fast decrease at infinity in the case where $D = \{z > 0\}$.

- The low beta plasma in D is perfectly conducting.

To justify the main assumption of the model – the use of the quasi-static force-free approximation –, we note that the evolution time $t_{\text{ev}} = L/\overline{v}$ associated to the changing boundary conditions on S is much longer than the time t_{eq} which is needed for the system to reach an equilibrium. The latter indeed is of the order of the time $t_A = L/\overline{v}_A$ it takes for a MHD wave to travel accross the whole structure, and we have noted in the previous section that $\overline{v} \ll \overline{v}_A$ in the solar conditions. Then it is legitimate to consider that the system is in equilibrium at each time t. Moreover, we can see from the equation of equilibrium (the momentum balance equation (3) with $\mathbf{v}$ set to zero) and Ampère's law (7) that the relative contribution $\delta B/B$ to the magnetic field due to the currents flowing perpendicular to the lines is of the order of $\beta \ll 1$, and can thus be neglected – implying indeed that the currents are aligned with the field.

3.2 Equations

The colinearity of $\mathbf{j}$ and $\mathbf{B}$ outside the singular surfaces Σ_j and Σ_c, if any, is expressed by the equation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}\,, \tag{12}$$

which can be also cast into the form

$$-\nabla \frac{B^2}{2} + \mathbf{B} \cdot \nabla \mathbf{B} = \mathbf{0} \tag{13}$$

by applying standard formulae of vector calculus. Alternatively, we can introduce an additional function α to get the equivalent equation

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}\,. \tag{14}$$

It results at once from this equation and the constraint $\nabla \cdot \mathbf{B} = 0$ that

$$\mathbf{B} \cdot \nabla \alpha = 0\,, \tag{15}$$

which states that α keeps a constant value along any field line.

No specific conditions have to be added when there is a surface Σ_j, but we need to impose for having a current sheet Σ_c in equilibrium the jump conditions

$$[\![B_n]\!] = 0\,, \tag{16}$$

$$[\![B_s^2]\!] = 0\,, \tag{17}$$

on the normal and tangential components B_n and B_s of $\mathbf{B}$. These relations are obtained at once from the well known jump conditions of general MHD.

In addition to the equation of equilibrium, we still have in the force-free model an equation of evolution for the field, which is just the ideal version of Eq. (4), i.e.,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) . \tag{18}$$

We remind the reader that an important consequence of this equation is the frozen-in law, which implies in particular that a magnetic line connecting two elements of matter on S at some time, still connects the same elements (which have possibly moved to new positions on S) at all the subsequent times.

Finally, we note that the equations of the force-free model can be formally considered as the limit when $R_m \to \infty$, $\beta \to 0$, and $M_A = \bar{v}/\bar{v}_A \to 0$ of the equations of general MHD – or equivalently we can consider them to be the zeroth-order approximation of the latter in an expansion in terms of the small parameters R_m^{-1}, β, and M_A.

3.3 Energy theorems for force-free fields in D

In this subsection and the following one, we consider all the finite energy force-free fields whose flux distribution is described by the given function Q (i.e., $B_n = Q$ on S). Their set will be denoted by $\mathcal{H} = \mathcal{H}[Q]$. $\mathcal{H}$ contains in particular two fields, which will play an important role hereafter, and which are the only two fields in $\mathcal{H}$ which can be unambiguously computed without further specifications from the only datum Q. The first one has been already introduced in Subsection 2.3: It is just the well-known potential field $\mathbf{B}_\pi$, which satisfies the current-free condition $\nabla \times \mathbf{B}_\pi = 0$ in D. The second one is the so-called *open field* $\mathbf{B}_\sigma$. $\mathbf{B}_\sigma$ is constructed by first introducing the unique potential field $\mathbf{P}$ such that $\nabla \times \mathbf{P} = 0$ and $\nabla \cdot \mathbf{P} = 0$ in D, and $P_n = |Q|$ on S, and it is obtained from $\mathbf{P}$ by setting $\mathbf{B}_\sigma(\mathbf{r}) = \chi(\mathbf{r})\mathbf{P}(\mathbf{r})$, with $\chi(\mathbf{r}) = 1$ (resp., -1) if $\mathbf{r}$ is connected to the part S^+ of S where $B_n > 0$ (resp., the part S^- where $B_n < 0$) by a magnetic line of $\mathbf{P}$. $\mathbf{B}_\sigma$ satisfies the right boundary condition $B_{\sigma n} = Q$ on S, but it is potential only in $D \setminus \Sigma_c$, where Σ_c is the surface separating the part of D where $\chi(\mathbf{r}) = 1$ from that one where $\chi(\mathbf{r}) = -1$. The field reverses accross Σ, which thus appears as a current sheet, and the latter is in equilibrium as conditions (16)-(17) are clearly automatically satisfied. Note that B_π and B_σ decrease at infinity as r^{-3} and r^{-2}, respectively.

We now present some general energy estimates, depending only on Q, which apply to an arbitrary field $\mathbf{B} \in \mathcal{H}$. For such a field (Aly 1984, 1988, 2006a):

- The energy $W[\mathbf{B}]$ satisfies the inequality (this is a standard result)

$$W[\mathbf{B}_\pi] \leq W[\mathbf{B}] . \tag{19}$$

- For $\nu \leq 1$, the ν-energies of $\mathbf{B}$ satisfy the estimate

$$(1-\nu)W_r^{(\nu)} + (1+\nu)W_\perp^{(\nu)} = F_S^{(\nu)}[\mathbf{B}] \leq W_*^{(\nu)}[Q] \quad \text{for} \;\; \nu < 1\,, \tag{20}$$

$$2W_\perp^{(1)} \leq F_S^{(1)}[\mathbf{B}] \leq W_*^{(1)}[Q]\,, \tag{21}$$

where $F_S^{(\nu)}[\mathbf{B}]$ is a functional of the restriction of $\mathbf{B}$ to S, and $W_*^{(\nu)}$ is a number which can be explicitly computed from Q. Their expressions depend on the choice of D. They assume the simplest forms when D is the exterior of a sphere, in which case

$$F_S^{(\nu)}[\mathbf{B}] = \frac{1}{8\pi}\int_S (Q^2 - B_s^2)\,\mathrm{d}S \;\leq\; \frac{1}{8\pi}\int_S Q^2\,\mathrm{d}S \;=\; W_*^{(0)}[Q]\,, \tag{22}$$

with $\mathbf{B}_s$ the tangential component of $\mathbf{B}$ on S. Therefore, for a force-free field in D, the mere integrability of B^2 implies the ones of $r^\nu B^2$ for $\nu < 1$ and of $rB_\perp^2$.

- If we denote as $W[\mathbf{B}](r)$ the part of the energy $W[\mathbf{B}]$ located outside the sphere of radius $r \geq r_*$, then

$$W[\mathbf{B}](r) \leq \frac{r_*}{r} W[\mathbf{B}] \leq \frac{r_*}{r} W_*^{(0)}\,. \tag{23}$$

These inequalities have an important physical consequence: They preclude the existence of a time-sequence of fields along which a finite amount of energy would be transported at infinity and lost. Moreover, Eq. (23) contains some information on the decay of $\mathbf{B}$ at infinity. It imposes indeed $\delta[\mathbf{B}] \geq 2$, where the exponent $\delta[\mathbf{B}] = \sup\{s \mid \lim_{r\to\infty} r^{2s-3} W[\mathbf{B}](r) = 0\}$ ($\delta = \gamma$ if $B \simeq_{r\to\infty} r^{-\gamma}$). Then a $\delta[\mathbf{B}]$ in the range $[3/2, 2[$, allowed for an arbitrary field of finite energy, is no longer possible if the field is force-free. The estimate $\delta \geq 2$ cannot be improved (just note that the open field $\mathbf{B}_\sigma$ has $\delta[\mathbf{B}_\sigma] = 2$).

- A field with $\delta[\mathbf{B}] > 2$ is a closed field. It is such that $B^{3/2}$ is integrable over D, and thus it has a well-defined helicity $H[\mathbf{B}]$. On the contrary, a field with $\delta = 2$ is an open field, i.e., it possesses a bundle of lines carrying on a positive amount of flux to infinity, and has current-sheets. For such a field, $B^{3/2}$ is not integrable and the helicity is not defined. This is the case in particular for the helicity of the open field $\mathbf{B}_\sigma$, and actually it is quite easy to construct sequences of (non force-free) fields $\mathbf{B}_k$ converging to $\mathbf{B}_\sigma$ for which $H[\mathbf{B}_k]$ converges to an arbitrarily fixed real value (finite or infinite).

- It results from Eq. (22) that

$$\int_S B_s^2\,\mathrm{d}S \;\leq\; \int_S Q^2\,\mathrm{d}S \tag{24}$$

when D is taken to be the exterior of a ball. Eq. (24) also holds true when $D = \{z > 0\}$, actually with an equality sign (in that case $B_s^2 = B_x^2 + B_y^2$).

3.4 A conjecture on the least upper bound on the energy

From the considerations of the previous subsection, we have $W[\mathbf{B}] \leq W_*^{(0)}$ for any field in $\mathcal{H}$. This leads us to introduce the so-called *least upper bound* on the energy $\overline{W} = \sup_{\mathcal{H}} W[\mathbf{B}] \leq W_*^{(0)}$, which is the best upper bound one can put on the energy (i.e., it is a upper bound, and there are fields in $\mathcal{H}$ with an energy as close as we want to it). A natural question thus immediately arises: Is it possible to determine exactly the value of $\overline{W}$. There is yet no definite answer to that question, but the following conjecture has been proposed (Aly 1984):

> **Conjecture** C. *The least upper bound on the energy of the fields in $\mathcal{H}$ is equal to the energy of the open field $\mathbf{B}_\sigma \in \mathcal{H}$, and it is reached by only that field, i.e.,*
>
> $$\forall \mathbf{B} \in \mathcal{H}: \quad \mathbf{B} \neq \mathbf{B}_\sigma \Rightarrow W[\mathbf{B}_\pi] \leq W[\mathbf{B}] < \overline{W} = W[\mathbf{B}_\sigma]\,. \tag{25}$$

The importance of C for solar physics stems from the fact that it precludes a spontaneous opening of a field in the way initially proposed by Barnes and Sturrock (1972), thus causing much trouble to the so-called *storage model* of eruptive events (Lin et al. 2003). A first rationale to put it forward was to note that a field $\overline{\mathbf{B}}$ in $\mathcal{H}$ for which $W[\overline{\mathbf{B}}] = \overline{W}$ (if there is one!) is likely to be a field fully determined by Q, and then has to coincide with either $\mathbf{B}_\pi$ or $\mathbf{B}_\sigma$, whence $\overline{\mathbf{B}} = \mathbf{B}_\sigma$ as $\mathbf{B}_\pi$ is known to be an energy minimizer. A second rationale was to note that C is true (but maybe for the uniqueness statement) at least for a particular class of functions Q (defined by $Q = \pm B_0$, with B_0 a constant) for which it is readily checked that $W[\mathbf{B}] \leq W_*^{(0)} = W[\mathbf{B}_\sigma]$.

Efforts to prove C have first concentrated onto a weaker formulation – conjecture C_c hereafter – in which $\mathcal{H}$ is replaced by its subset $\mathcal{H}_c$ constituted of the fields having all their lines connected to S ($\overline{W}$ being accordingly replaced by the least upper bound $\overline{W}_c \leq \overline{W}$ over that subset). The validity of C_c is supported by a general argument (Aly 1991, Sturrock 1991) showing that a field $\overline{\mathbf{B}}$ for which $W[\overline{\mathbf{B}}] = \overline{W}_c$ (once again, if there is one!) has necessarily all its lines open, for otherwise it would be possible to deform it in $\mathcal{H}_c$ into a field with a larger energy, and then $\overline{\mathbf{B}} = \mathbf{B}_\sigma$. Another argument is provided by the analytical and numerical studies of time sequences $\{\mathbf{B}_t\}$ of fields belonging to $\mathcal{H}_c$ (see Sects. 5-6 below). Such sequences can be made to start from an arbitrary field $\mathbf{B}_0$, to have an increasing energy, and to converge to the open field $\mathbf{B}_\sigma$, whence $W[\mathbf{B}_0] \leq W[\mathbf{B}_t] \leq W[\mathbf{B}_\sigma]$. In particular, these inequalities can be directly checked to be verified (Aly 2006a) by the explicit sequences exhibited by Lynden-Bell and Boily (1994) (in which however Q and then $\mathbf{B}_\sigma$ change in time). Unfortunately, there is also one result going against the validity of C_c: Choe and Cheng (2002) have reported the construction of numerical configurations constituted of two interwinding tubes and having an energy larger than $W[\mathbf{B}_\sigma]$. What differentiates these configurations from the previously considered ones is still unclear.

The later consideration of fields having lines disconnected from S have lead many authors to conclude that the general C is not valid – a possibility previously pointed out on intuitive grounds by Aly (1991). In particular, Hu et al. (2003), Flyer et al. (2004), Wolfson et al. (2007) have constructed numerically parametrized sequences of axisymmetric fields containing one or several embedded toroidal flux ropes. And for some values of the parameters, they have found that the energy can exceed that of $\mathbf{B}_\sigma$. This result deserves some theoretical investigations to be fully understood. In particular, it would be quite helpful to determine the limit of a sequence of fields whose energy approaches $\overline{W}$, and to obtain a physical characterization of it.

An interesting related problem has been discussed in the literature. It amounts to compare the energy of a field $\mathbf{B}$ in $\mathcal{H}$ with that of a *partially open field* $\mathbf{B}'_\sigma[\mathbf{B}]$ in the same set, obtained from $\mathbf{B}$ by opening a bundle of lines while conserving the topology of the other lines. This problem was first introduced by (Wolfson and Low 1992), who found numerically that, in a special axisymmetric setting, it is possible to have $W[\mathbf{B}] > W[\mathbf{B}'_\sigma]$. However, it was noted (Aly 1993, Lepeltier 1994) that, after some adaptation, the arguments supporting C_c suggest that one should rather have $W[\mathbf{B}] \leq W[\mathbf{B}'_\sigma]$, and the latter inequality was conjectured to hold for all the configurations of the type studied by (Wolfson and Low 1992). More recently, the problem has been reconsidered by Hu (2004), who formulated independently the same conjecture and presented new numerical calculations in support of it. Finally, it has been argued in Aly (2006a) that the inequality $W[\mathbf{B}] \leq W[\mathbf{B}'_\sigma]$ should hold quite generally, at least for fields having all their lines connected to S – conjecture C'_c.

4 RECONSTRUCTION OF THE FIELD OF AN ACTIVE REGION

In this section, we discuss the reconstruction problem of the magnetic field of an active region. We thus assume that measurements effected by a vector magnetograph at the photospheric level have provided us with a magnetic field $\mathbf{B}_0$ defined on $S = \{z = 0\}$ and having zero flux through that plane, and we try to use this given function to construct in the half-space $D = \{z > 0\}$ an approximate representation $\mathbf{B}$ of the actual field. We remind the reader that there is actually a well-known indeterminacy in the measured field – the sign of its component transverse to the line of sight cannot be determined – but we assume here that the brut data have been submitted to an adequate treatment to get rid of this 180^0 ambiguity (see, e.g., Metcalf et al. (2006) for a description of the various methods proposed so far, and a comparison of their merits).

4.1 General statements

We first note that the values $\mathbf{B}(x, y, 0)$ taken on S by a finite energy force-free field $\mathbf{B}$ in D are strongly constrained by a series of integral relations. For instance (Aly 1989), we have

$$\int_S \left(-\frac{B^2}{2}\hat{\mathbf{z}} + B_z\mathbf{B}\right) \mathrm{d}S = \mathbf{0}\,, \tag{26}$$

$$\int_S \mathbf{r} \times \left(-\frac{B^2}{2}\hat{\mathbf{z}} + B_z\mathbf{B}\right) \mathrm{d}S = \mathbf{0}\,, \tag{27}$$

which just express the obvious fact that the total force and torque exerted by **B** on S have to vanish. Also, the restriction to S of the force-free function α, which can be computed from **B** on S by using the relation

$$\alpha = \frac{1}{B_z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right), \tag{28}$$

is subject to the infinite set of constraints

$$\int_S f(\alpha)\Theta(\alpha - \tau) B_z \, \mathrm{d}S = 0 \qquad \forall \tau \in \mathbb{R}, \tag{29}$$

where Θ denotes the standard Heaviside step function, and f an arbitrary function. Eq. (29) is a consequence of Eq. (15), and it contains in particular the requirement that the ranges of values taken by α respectively on S^+ and S^- should be identical. It is worth noticing that this set of relations provides necessary but not sufficient conditions for $\mathbf{B}(x, y, 0)$ to be the trace on S of a finite energy force-free field in D (a set of sufficient conditions is not known).

Therefore we certainly should not define the reconstruction problem as consisting in the determination of a finite energy force-free field **B** in D matching exactly on S the observed field $\mathbf{B}_0$. For a solution of this problem to be possible, $\mathbf{B}_0$ should indeed satisfy exactly at least all the relations above, what it will never do in practice. Rather, we propose to define the reconstruction problem as follows: To set up a well posed problem for **B** in D by keeping as much as possible of the two incompatible requirements – that **B** should be exactly force-free in D, and that it should match on S the field $\mathbf{B}_0$ provided by the observations. The requirement of well-posedness means that the problem should have a solution, and that this solution should be unique and continuously depending on the data. Clearly, this is a quite natural condition to impose if we want to get a meaningful problem.

Of course, there are a priori many different ways to try to set up such a well posed problem, and we next recall briefly some of the methods which have been attempted so far and are still under study (for a more detailed presentation, we refer the interested reader to the recent review paper by Neukirch (2005)):

- A first possibility amounts to keep the strict force-free assumption in D and the requirement that the field should decay at infinity, but to disregard a part of the data available on S – the latter being possibly used a posteriori to evaluate the accuracy of the reconstruction by comparing their values with the computed ones. A method of this type is the so-called *Grad-Rubin method*, first considered in a different context by Grad and Rubin (1958) and introduced into the framework of solar physics by Sakurai (1981). It amounts to retain as relevant data the normal component B_{0z} on S, and the function α_0 (computed from $\mathbf{B}_0$ by Eq. (29)) on that part S^+ of S where $B_{0z} > 0$, say.
- A second possibility is to require **B** to be strictly force-free in D and to match exactly the field $\mathbf{B}_0$ on S, which forces us to give up with imposing a condition on **B** at large

distances. This approach is adopted in the *progressive extension methods* (Wu et al. 1990, Cuperman et al. 1991, Demoulin et al. 1992, Song et al. 2006), in which the equations for the force-free field are written in the form (14)-(15), with the terms containing a derivative with respect to z being singled out in the left-hand sides. This form suggests strongly to compute $\mathbf{B}$ by setting a Cauchy problem in which the values of $(\mathbf{B}, \alpha)$ at height $z + \mathrm{d}z$ are computed from those at z, $\mathbf{B}_0$ providing the initial conditions at $\{z = 0\}$ necessary to start the computation. The use of some regularizing scheme is needed for this calculation for avoiding the growing up of errors inherent to the resolution of this type of Cauchy problems, and this method may be a priori expected to give acceptable results only up to some not too large height. Note however that this way of proceeding has given good results in the case (not considered here) where the force-free field is assumed to be linear (Amari et al. 1997a, 1998).

- A third possibility amounts to introduce an approximate problem chosen "as close as possible" to the original one. For instance, the field $\mathbf{B}$ may be only required to be "as force-free as possible" in D and/or to match "as closely as possible" the observed values $\mathbf{B}_0$ on S. This idea may be practically implemented by setting an *optimization problem* of the following type: To determine a field $\mathbf{B}$ minimizing the functional

$$O[\mathbf{B}] = \int_D (a|(\nabla \times \mathbf{B}) \times \mathbf{B}|^2 + b|\nabla \cdot \mathbf{B}|^2)\, \mathrm{d}V + \int_S c|\mathbf{B} - \mathbf{B}_0|^2\, \mathrm{d}S\ , \tag{30}$$

with possibly $a = 0$ and/or $b = 0$ and/or $c = 0$ (in which case the admissible $\mathbf{B}$ are imposed to satisfy $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$ and/or $\nabla \cdot \mathbf{B} = 0$ and/or $\mathbf{B} = \mathbf{B}_0$ on S). The problem with $c = 0$ has been much developed in the last few years (Wheatland et al. 2000, Wiegelmann and Neukirch 2003). Note that it is not possible to achieve individual control on the resulting $\nabla \cdot \mathbf{B}$ and Lorentz force in that setting.

As a last general remark, we note that the constraints on $\mathbf{B}(x, y, 0)$ that we have indicated at the beginning of this section (and any other which could be derived) may be used to replace the field $\mathbf{B}_0$ by a field $\mathbf{B}'_0$ satisfying some of them and choosen to be as close as possible, in some sense, to $\mathbf{B}_0$. It is intuitively clear that such a *reprocessing* of the data should make more easy the reconstruction process by any method. This has been checked to be the case by Wiegelmann et al. (2006), who actually only used the two constraints (26)-(27).

In our own work on the reconstruction problem, we have mainly tried to develop the Grad-Rubin approach by working out new iterative schemes able to determine efficiently the field in D, and in accordance with the spirit of this paper, we concentrate from now on on a presentation of our achievements. We just note that different technical tools have been used by other authors to attack the same problem (Sakurai 1981, Wheatland 2004, Inhester and Wiegelmann 2006, Yan and Sakurai 2000).

4.2 The Grad-Rubin problem and the Grad-Rubin iterative scheme

To make the presentation of our results clearer, we first describe in some details the Grad-Rubin method, which is intuitively grounded on the following simple remark. Consider the form (14)-(15) of the force-free equations. On the one hand, we may note that if we consider α as a given function in D, Eqs. (14) and (8) constitute an elliptic div-curl system which can be solved uniquely if we prescribe B_z on S along with an appropriate asymptotic condition. On the other hand, we see that Eq. (15) with $\mathbf{B}$ assumed to be known in D is an hyperbolic equation for α which determines this function in D by a simple transport along the lines if we prescribe α on S^+. (This decomposition of the system of equations shows their mixed *elliptic-hyperbolic* type, which can be also exhibited by a standard calculation of their characteristics). Then the two retained data may be expected to be sufficient indeed for determining a unique $\mathbf{B}$.

Let us now write formally the problem we have to deal with – to which we shall refer from now on as the *Grad-Rubin boundary value problem*. For its elliptic part, we have

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in} \quad D\,, \tag{31}$$

$$B_z = B_{0z} \quad \text{on} \quad S \quad \text{and} \quad \lim_{r\to\infty} B = 0\,, \tag{32}$$

while its hyperbolic part reads

$$\mathbf{B} \cdot \nabla\alpha \;=\; 0 \quad \text{in } D\,, \tag{33}$$

$$\alpha \;=\; \alpha_0 \quad \text{on } S^+\,. \tag{34}$$

The principle of a quite natural scheme to solve it has been proposed in Grad and Rubin (1958). It is an iterative scheme in which the elliptic and hyperbolic parts are successively solved at each step. More precisely, we look for a sequence $(\mathbf{B}^{(n)}, \alpha^{(n)})$ which is a solution to

$$\mathbf{B}^{(n)}.\nabla\alpha^{(n)} \;=\; 0 \quad \text{in} \quad D\,, \tag{35}$$

$$\alpha^{(n)} \;=\; \alpha_0 \quad \text{on} \quad S^+\,, \tag{36}$$

and

$$\nabla \times \mathbf{B}^{(n+1)} = \alpha^{(n)}\mathbf{B}^{(n)} \quad \text{and} \quad \nabla \cdot \mathbf{B}^{(n+1)} = 0 \quad \text{in} \quad D\,, \tag{37}$$

$$B_z^{(n+1)} = B_{0z} \quad \text{on} \quad S \quad \text{and} \quad \lim_{r\to\infty} B^{(n+1)} = 0\,. \tag{38}$$

The iteration process is initialized by choosing $\mathbf{B}^{(0)} = \mathbf{B}_\pi$ – i.e., we start with the potential field associated to B_{0z}.

It is worth noticing that a solution to this problem may be expected a priori to contain singular surfaces Σ_j once the field to be reconstructed has a *complex topology*. By this expression, we mean that the magnetic mapping $\mathbf{M} : S^+ \to S^-$ associating to the footpoint of any line of

$\mathbf{B}$ on S^+ its footpoint on S^-, has discontinuities along specific curves $\mathcal{L}$ of S^+ (a field with a continuous map $\mathbf{M}$ being said to have a *simple topology*), which implies the presence of separatrix surfaces $\mathcal{S}$ in D cutting S^+ along $\mathcal{L}$. In the general case where α_0 does not vanish in the neighborhood of $\mathcal{L}$, two magnetic lines starting from infinitely close points located on either sides of $\mathcal{L}$ will separate at some point of $\mathcal{S}$, thus transporting the same initial value of α to distant points on S^-. As a result and even if the function α_0 is very smooth, different values of α can be obtained on either sides of $\mathcal{S}$ – we have thus indeed to deal with a surface Σ_j.

4.3 Existence and uniqueness of solutions to the Grad-Rubin problem

The Grad-Rubin problem set in some domain D has now received some attention from the mathematical point of view, and we collect in this section some general results relevant to the reconstruction problem. Unfortunately, existence and uniqueness results are not yet available for the problem in a half-space, for which there is just a simple result which can be stated as follows: If we write the boundary condition on α in the form $\alpha = \lambda g$ on S^+, with $\lambda \geq 0$ a parameter and g a prescribed function, then the Grad-Rubin problem has no solutions when λ is chosen too large, i.e., when $\lambda \geq \lambda_c[Q, g]$ (Aly 1984, 1988).

But some rigorous results have been established – actually by proving the convergence of the Grad-Rubin scheme when some conditions are fulfilled – for the Grad-Rubin problem set in other types of domains D:

- When D is bounded and α_0 not too large, a solution exists (Bineau 1972) in a Hölder functional space under some restrictive additional assumptions (in particular, $\mathbf{B}$ should have a simple topology, the presence of surfaces of discontinuity Σ_j of $\nabla \times \mathbf{B}$ being a priori excluded in the adopted functional setting). Moreover, this solution is unique, and it depends continuously on the boundary conditions – the problem appears to be well-posed.
- Bineau's theorem relative to a bounded domain D has been extended to more general spaces ($(\alpha, \mathbf{B}) \in L^\infty \times \mathbf{H}^1(D)$) by Boulmezaoud and Amari (2000). In this new setting, solutions with singular surfaces Σ_j on which $\mathbf{B}$ is not continuously differentiable are allowed, and the reconstruction of fields with a complex topology does not lead to any particular problem.
- For D an exterior domain – e.g., the exterior of a sphere – and for boundary conditions submitted to restrictions quite similar to those imposed in Bineau (1972), the existence of a solution belonging to a Hölder functional space has been recently proven by Kaiser et al. (2000) in the case where α is not too large.

It is worth noticing that all the results obtained for fields occupying bounded domains are of practical interest for us here, as a solution has eventually to be calculated on a computer, with the problem being set in a bounded numerical box. Moreover, the reconstruction problem in the whole corona should certainly be addressed in the near future, which would make the last stated result directly relevant to solar physics.

4.4 Numerical approach

Finally, we describe the numerical results which have been presented in details in Amari et al. (2006). As noted just above, the effective determination of the field of an active region is not done in the half-space D, but rather in a bounded box $D_b = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, which has just to be choosen large enough for the effects of the boundary to be as small as possible. Accordingly, the asymptotic condition $(32)_b$ has to be replaced by a boundary condition, and we prescribe here conditions of type $(32)_a$ and (34) on the whole ∂D_b.

For reconstructing a field, we use two different implementations of the Grad-Rubin scheme, the code XTRAPOL and FEMQ. They do differ firstly by the way they do address the issue of the $\nabla \cdot \mathbf{B} = 0$ constraint, which still represents a serious challenge in the development of any numerical MHD scheme. In XTRAPOL (a code based on a finite difference scheme initially introduced in Amari et al. (1999a)), we use a vector potential $\mathbf{A}$ for the magnetic field $\mathbf{B}$ ($\mathbf{B} = \nabla \times \mathbf{A}$) and a staggered mesh such that the approximate solution lies in the functional space that is in the kernel of the div operator. In FEMQ on the contrary, the introduction of $\mathbf{A}$ is avoided. Rather, the method consists in working in a functional space corresponding to $\nabla \cdot \mathbf{B} \neq 0$ Q1 finite elements, with $\nabla \cdot \mathbf{B}$ being minimized in the least square sense for the associated div-curl system corresponding to the elliptic Biot-Savart problem. The second difference between the two implementations stems from the way the force-free function α (which is constant along the field lines) is computed at each iteration. This is done by following the characteristics (field lines) in XTRAPOL, and by solving a hyperbolic linear system in FEMQ. Both methods have advantages and drawbacks. In XTRAPOL the solution satisfies $\nabla \cdot \mathbf{B} = 0$ to machine accuracy while in FEMQ $\nabla \cdot \mathbf{B}$ is just minimized. On the other hand the use of a potential vector in XTRAPOL and of $\mathbf{B}$ as the main variable in FEMQ makes the electric current to be given by second derivatives in the former case, but by derivatives of only the first order in the latter one.

To test and compare these different methods, we have applied them to the reconstruction of the particular semi-analytic exact solutions derived in Low and Lou (1990) from its values of B_z on S and of α on S^+. The latter has also been used with the same error diagnostics and the same choices of parameters in Schrijver et al. (2006). We have also considered other more local diagnostics such as cuts of α at various heights, and the divergence of $\mathbf{B}$ in infinite norm. Diagnostics of this type are indeed important since global diagnostics alone cannot reveal the failures of a reconstruction method which is not based on a well posed formulation.

Table 1 shows the results of a test case (called FF1 in Amari et al. (2006)) corresponding to one particular set of parameters. It can be seen that a characteristic error smaller than 0.01 is achieved, while Figure 1 shows accordingly a good agreement between the computed field lines and those of the exact solution. We have also computed an other case (called FF2 in Amari et al.

(2006)), corresponding to an extreme nonlinear situation, for which the numerical results give some feedback to solar data. They show indeed that increasing numerical resolution leads to an improvement of the error diagnostics and to a better fitting of the approximate solution. This result, although limited, may imply that in order to handle some of the active region current distributions, high enough resolution vector magnetographs are necessary.

In addition to these experimental cases, our methods have been applied successfully to the reconstruction of the actual coronal magnetic field above observed active regions (Bleybel et al. 2002, Régnier et al. 2002, Régnier and Amari 2004). Finally, we note that with the coming of SOLIS and HINODE, we should have soon at our disposal full disk measurements and therefore synoptic vector magnetic maps. Methods able to handle this new type of data will then be needed, and we have undertaken an effort in that direction by developing a new approach. It consists in using finite elements for $\mathbf{B}$ which belong to a functional space in which $\nabla \cdot \mathbf{B} = 0$ on tetrahedral meshes. Until now, this method has been applied only to laboratory toroidal configurations (Boulbe et al. 2006), but the same numerical code should be used soon to handle the reconstruction of the field of both a localized active region and the full Sun (Amari et al., in preparation).

5 BOUNDARY DRIVEN EVOLUTION OF B: THE ANALYTIC QUASI-STATIC APPROACH

Next we consider the problem of the boundary driven evolution of a coronal structure in the quasi-static approximation.

5.1 The boundary driven evolution problem

As stated in the Introduction, the boundary driven evolution problem amounts to determine the evolution of the field $\mathbf{B}$ in D which is driven by imposed boundary changes on S. This problem has been set mainly to try understanding the nature of the mechanisms which may be the triggers of the large-scale eruptive events which occur sporadically in the solar corona. Among the various relevant issues that it is worth addressing in this context, we may list the following ones:

- Is an evolution describable in terms of a stable equilibrium sequence, or on the contrary does it lead either to an unstable state, or to a state beyond which an equilibrium is no longer possible?
- Does an evolution lead to a fast opening of the field in a part of the corona?
- Is there any constraint on the magnetic energy and helicity during the evolution?

- Is there a change of topology during the evolution leading, e.g., to the formation of a twisted flux tube? We remind the reader that there is observational evidence in favor of the presence of such a structure during the disruption phase of an eruptive event (Gary and Moore 2004).
- Is a twisted flux rope (TFR) created as an equilibrium structure or only produced by reconnection during a nonequilibrium phase of the evolution?

To begin the study of that problem, we consider it in this section in the framework of the force-free model, i.e., we assume that an evolution can be described by a sequence of force-free equilibria. This simplification has the advantage of allowing analytic developments, and it is certainly reasonable at least to study the evolution of a structure which is not too far from a potential configuration, in which case we may be quite safely ensured of the validity of the justifications reported in Section 3, which were based on the orders of magnitude of $\overline{v}$ and L. The model will, however, break down if the system starts to evolve on time-scales which are comparable to the Alfven time-scale and/or if there is a spontaneously formation of structures with length scales which are much smaller than the characteristic length L of the system itself.

We start with the case which has been studied in greatest details, that of an axisymmetric arcade occupying the exterior of a sphere (but similar results hold true for an axisymmetric arcade in a half-space). Thus we quote some results on the evolution of a flux tube and on the evolution of a configuration with a complex topology.

5.2 Ideal evolution of an axisymmetric arcade

Let us assume that the field in the domain $D = \{r > r_*\}$ is at any time t axisymmetric about the z-axis, and mirror symmetric with respect to the equatorial plane $\{z = 0\}$. It can thus be given the representation

$$\mathbf{B} = \frac{1}{r\sin\theta}(\nabla A \times \hat{\boldsymbol{\varphi}} + G\hat{\boldsymbol{\varphi}}) \tag{39}$$

in terms of the flux functions $A(r, \theta, t)$ and $G(r, \theta, t)$, with $A(r, \pi - \theta, t) = A(r, \theta, t)$ and $G(r, \pi - \theta, t) = G(r, \theta, t)$. At the initial time $t = 0$, $\mathbf{B}$ is taken to be potential ($\nabla \times \mathbf{B} = \mathbf{0}$) and to have an arcade structure, i.e., its lines (which are drawn in poloidal planes) all emerge from the northern hemisphere, say, and re-enter into the sphere in the southern hemisphere, after bridging over the equator, which imposes $A \geq 0$. For $t \geq 0$, the footpoints of $\mathbf{B}$ on S are submitted to purely azimuthal stationnary motions, i.e., we impose on S a velocity field of the form

$$\mathbf{v} = v(\theta)\hat{\boldsymbol{\varphi}} = r\sin\theta\Omega(\theta)\hat{\boldsymbol{\varphi}}\,, \tag{40}$$

with $v(\pi - \theta) = -v(\theta)$ in order to be coherent with the symmetry assumptions on $\mathbf{B}$. As a result of this imposed shearing, the field evolves through a sequence of force-free configurations, which imposes that

$$
\begin{aligned}
G(r, \theta, t) &= F[A(r, \theta, t), t], \quad (41) \\
-\Delta^* A &= (F\dot{F})(A, t), \quad (42)
\end{aligned}
$$

where $\Delta^* = \partial_r^2 + (1-\mu^2)\partial_\mu^2$ ($\mu = \cos\theta$) is a standard elliptic operator. The second relation is the well known Grad-Shafranov equation. The function F appearing above is not given a priori in this problem. Rather, it has to be determined in such a way that the field, which keeps a dipolar topology due to the frozen-in law, has the connectivity imposed by the boundary motions. This requires that the total variation $\delta\varphi(a, t)$ of φ when going along a line on which $A = a$ is equal to the difference in the φ-positions of its footpoints imposed by the velocity field, i.e.,

$$
\delta\varphi(a, t) = F(a, t) \int_{\mathcal{L}_a(t)} \frac{\mathrm{d}l}{r \sin\theta |\nabla A|} = [\Omega^-(a) - \Omega^+(a)]t = \zeta(a)t, \quad (43)
$$

where $\mathcal{L}_a$ is the curve traced out in a meridional plane by the toroidal surface on which the line is drawn, $\mathrm{d}l$ is an element of length along it, and $\Omega^{+/-}$ is the angular velocity at its upper /lower footpoint.

To determine the evolution of the field, we thus have to compute at each time t in the plane domain $\Pi = \{r > r_*, \varphi = 0\}$ a flux function A whose level contours have an arcade topology, which solves the Grad-Shafranov equation (42) with a right-hand side determined by the condition (43), and which takes on the boundary the initial value $A(r_*, \theta, 0)$. Note that this problem is both nonlinear and nonlocal. In particular, inserting the expression of F given by Eq. (43) into Eq. (42) results in a strange type of equations first introduced in plasma physics by Grad and Hogan (1970) and named by them *Queer Differential Equations*, or *Generalized Differential Equations*.

5.3 Characteristics of the evolution

It can be shown (see Aly (1995) and important improvements in Aly (2006b)) that there are two different phases in the evolution of the field (these results are not obtained by computing an exact or approximate solution of the problem, but by deriving directly from the equations a series of exact relations which constrain the evolution of the system and allow at least its qualitative description).

- There is a first phase where the poloidal structure does not change much. The strength of the toroidal field increases linearly with time at the (field line dependent) rate $|\zeta(a)|$, and free magnetic energy accumulates at a rate quadratic in t.

- The second phase starts roughly when the strength of the toroidal field becomes of the order of that of the poloidal field. It is characterized by a fast increasing expansion of the poloidal structure at a rate that becomes very large compared to the driving rate $|\zeta(a)|$. Using in particular Eqs. (21) and (43), it is possible to derive the estimate

$$\bar{r}_a(t) \geq r_* \mathrm{e}^{(t/T_a)^2 - 1/2} \tag{44}$$

for the radius $\bar{r}_a$ of the smallest circle of center O in Π which contains the region Π_a where $A > a$ (clearly, Π_a is bounded by the line $\mathcal{L}_a$ and an arc of circle of radius r_*). Simultaneously, the toroidal component of the field decreases to zero (at least as $e^{-(t/T_a)^2}$), and the bulk electric currents concentrate into a layer of decreasing thickness. Eventually, the field opens, while the current layer collapses to an infinitely thin current sheet. The opening is total (the asymptotic state coinciding with the open field introduced in Sect. 3.4) if there is an interval $]a_1, \bar{a} = \max A[$ on which $\zeta(a) > 0$ (meaning that the field is effectively sheared arbitrarily close to the equator), but only partial if $\zeta(a) = 0$ on some interval $[a_1, \bar{a}]$ ($a_1 < \bar{a}$). In that case, the equatorial current sheet extending to infinity starts above the surface of the Sun, and there is a separatrix separating a region of unsheared closed lines from a region of open ones.

One of the arguments in favor of the conjecture C_c quoted in Sect. 3.4 is linked as follows to the results above. Let us give an arbitrary arcade equilibrium $\mathbf{B}$, and consider the evolution generated for $t \geq 0$ by a shearing function of the form $\delta\varphi(a,t) = (t/t_0)\delta\varphi(a,t_0)$, with $t_0 > 0$ an arbitrary number and $\delta\varphi(a,t_0)$ computed from $\mathbf{B}$ with the help of Eq. (43). Then we get obviously for $t \geq t_0$ a sequence of fields $\{\mathbf{B}_t\}$ along which the energy increases and converges eventually to the open field energy (this last statement being supported by inequality (23), which precludes a loss of energy at infinity), whence $W[\mathbf{B}] \leq W[\mathbf{B}_t] \leq W[\mathbf{B}_\sigma]$ indeed.

5.4 Discussion

In the previous analysis, a continuous sequence of regular force-free equilibria $\mathbf{B}(t)$ corresponding to an arbitrary initial arcade $\mathbf{B}(0) = \mathbf{B}_\pi$ and to an arbitrary shearing function ζ, has been implicitly assumed to exist for $0 \leq t < \infty$, i.e., we have assumed that a field can stay in force-free equilibrium all along when it is indefinitely sheared. Of course, this assumption has to be proved to get a satisfactory theory, and this could possibly be done by working with the well known variational formulation of the problem, which amounts to look at each time t for a field which extremizes the magnetic energy among all the arcade fields which can be obtained from $\mathbf{B}(0)$ by a continuous deformation compatible with the imposed displacements of the footpoints on S up to t (see, e.g., Aly (2006b)). Actually, we could even look for a field making the energy an absolute minimum rather than a mere extremum, as this would give at the same time the existence of the sequence and its nonlinear stability with respect to axisymmetric perturbations (note that this possibility is compatible with our initial condition as $\mathbf{B}_\pi$ is actually an energy minimizer at $t = 0$; see Eq. (19)).

Although we currently feel that the existence of $\{\mathbf{B}(t)\}$, $0 \leq t < \infty$, is a reasonable assumption, we have to note that the possibility for the sequence to exist only for the finite interval

$0 \leq t < T$ has been considered in the literature. For instance, it has been recently conjectured by Zhang et al. (2006) that there is a upper bound on the absolute value $|H|$ of the helicity which can be injected into a force-free field when the boundary condition $B_n = Q$ is enforced on S, i.e., $\overline{H} = \sup_{\mathcal{H}} |H[\mathbf{B}]| < \infty$ in our notations. As the stationary motions we have imposed on the boundary inject helicity at the constant rate $\dot{H}[Q, \zeta]$, this would imply indeed that a field can stay in equilibrium at most up to a time $T \leq \overline{H}/\dot{H}$. If the field does exist as an equilibrium only for $0 \leq t < T$, the problem arises of what happens when $t \to T$. A possibility is that the field opens totally ($\mathbf{B}(t) \to \mathbf{B}_\sigma$) or partially at this finite time (Uzdensky 2002). An argument in favor of this behavior is provided by the existence of the approximate analytical solution due to Lynden-Bell and Boily (1994) and already considered in Subsection 3.4. This solution describes a continuous transition in $D = \{r > r_*\}$ from a potential dipolar field to an open field of the split monopolar type through a sequence of force-free configurations, with the total amount of injected helicity being finite. It is however quite peculiar as the boundary motions become singular when opening does occur (this is possible because these motions are not stationary as in our formulation above), and then it may at least be doubted that this example represents the most general situation (Aly 2006b). In any case, we remind the reader that the helicity of the open field is not defined (as noted in Sect. 3.3), and this leaves the possibility for an opening being obtained with the injection of either a finite or an infinite amount of helicity, depending on the choice of the boundary motions. To discuss further this last point, it may be interesting to anticipate on the presentation of our numerical results in the next part of the paper. For an evolution driven by either converging motions, flux cancellation or flux diffusion, the total magnetic helicity remains constant, while the configuration experiences a global disruption (Amari et al. 2003a,b). It must be noted, however, that B_n is not fixed during the flux change, and then the cases that we have computed do not correspond exactly to the problem above. Moreover, although the total magnetic helicity of the configuration remains constant, the "partial helicity" of the flux rope may be expected to increase.

Let us now consider the results of the previous subsection from a physical rather than a mathematical point of view. Clearly, they can describe what happens in the real corona only up to some point. The predicted large-time behavior is indeed in conflict with the basic assumptions of the force-free model. Firstly, there is a violation of the quasi-static approximation when the field is stretched out to large distances and expansion becomes faster than some fraction of the Alfven speed. And secondly, ideal MHD is violated in the thin region where the electric currents concentrate. To care for the first problem, we need to add inertial effects. This may be expected on intuitive grounds to have the simple following consequences. The domain D divides into two parts. An outer one which is strictly static, as the information about the motion of the footpoints has not yet reached it. And an inner extending one, in which the field expands at a speed which has initially about the value predicted by the quasi-static model (it is just a little bit smaller), but which saturates at some stage at a fraction of the Alfven velocity. This retardation effect makes the magnetic energy becoming larger than the force-free energy (possibly, it could even exceed $W[\mathbf{B}_\sigma]$), and it is the corresponding energy excess which allows for the acceleration of the plasma. Then the often expressed opinion that conjecture C_c precludes the opening of the field and acceleration of the plasma appears to be a quite pessimistic view. If we give up with the strict storage paradigm of eruptive processes (see, e.g., Lin et al. (2003)) and substitute for it a model in which energy is continuously injected into a structure, there is no longer any problem. At some stage, the field starts expanding at a fast rate, following the tendency exhibited by the force-free model, and it still eventually opens, the excess magnetic energy due to the retardation effect accounting for the kinetic energy of the ejected plasma.

The problem posed by current concentration is more interesting. Clearly we have to introduce resistivity to take care of it, and this opens the possibility for the field to reconnect at some stage, with one or several isolated twisted ropes being produced. The details of such a process are beyond the present capability of an analytic approach, and they have to be computed numerically (see the following sections). However, we can at least address in general terms the following question: Is reconnection energetically favorable at time t? If we assume that this process implies only a localized violation of the frozen-in law and then a quasi-conservation of magnetic flux, we can follow the approach used for a Cartesian arcade in Aly (1990). It consists to reformulate the variational problem quoted above by enlarging the set of allowable fields: We now also admit complex topology fields which can be obtained from those with arcade topology by reconfiguration conserving the differential poloidal and toroidal magnetic fluxes. The asymptotic results quoted in the previous subsection imply that a transition from an arcade to an allowed more complex field is energy releasing when the shear is sufficiently large: reconnection is thus to be expected. There is however an energy barrier between the two configurations, and a finite amplitude perturbation is required for the transition to actually occur.

5.5 Extensions to more general configurations

Arguments of the type of those used above for treating the evolution of an axisymmetric arcade can also be used to deal with the problem of the twisting of a flux tube in D (exterior of a sphere, or half-space). In the latter problem, a tube which is initially a part of a potential field is driven into an evolution by imposing to its footpoints on S rotational motions conserving Q, say. It is found that after a quiet phase during which the overall shape of the tube does not change much, there is a phase of very fast expansion leading to a partial or total opening of the field. It is worth noticing here, however, that there is an important difference between this fully 3D case and the previous axisymmetric one: In an axisymmetric situation, the expansion of a line implies for an obvious geometrical reason an expansion of all the lines above it, while in 3D there is the possibility for an expanding tube to make its way between the overlying lines without pushing them outward. This makes a precise characterization of the 3D opening difficult to obtain.

An arcade and a tube are examples of configurations with a simple topology, and the question immediately arises of what happens when a complex configuration – e.g., a quadrupolar axisymmetric field – is driven into an evolution by boundary motions. In fact, it turns out that all the arguments used for simple fields still apply. There is however a well known but important difference: Once the evolution starts, current sheets need to form on the separatrices as a consequence of the joint requirements that the field be in equilibrium and its topology be conserved. Then we still get in the ideal case an expansion and an opening of the field, but this result appears to be of little application, as even in the corona the plasma resistivity is tiny, but nonvanishing. Therefore it would be desirable to use a more realistic description of the physics in the vicinity of, for example, separatrix layers, because they would be preferred locations for reconnection to occur. This still needs to be done for the analytical point of view considered here.

Finally, we report on work (Aly, in preparation) on a problem which does not seem to have been considered up to now, but whose solution would be important to better understand important points like the linear or nonlinear stability of a sequence of evolving equilibria $\{\mathbf{B}(t)\}$ or the formation of current sheets when the fields $\mathbf{B}(t)$ have no separatrices (Parker problem; see, e.g., Parker (1994)). It can be formulated as follows in general terms: Given an equilibrium in some domain D, is there another equilibrium which can be obtained from it by a continuous ideal deformation preserving the positions of the footpoints on the boundary S. Owing to its difficulty, it has yet been attacked only for the Parker specific model (and a variant of it for which explicit sequences $\{\mathbf{B}(t)\}$ are available) in which an initially uniform vertical field $\mathbf{B}_0$ contained in a cylinder of height h, say, is driven into an evolution by motions imposed on the two horizontal parts, $S^\pm$, of its boundary. In the case where the driving motions are incompressible and vanish on the boundaries of $S^\pm$, the most interesting result obtained so far is the estimate

$$\int_D (B^2 - B_0^2)\,\mathrm{d}V = \int_D |\mathbf{B} - \mathbf{B}_0|^2\,\mathrm{d}V \leq \frac{B_0^2}{3h} \int_{S^+} |\mathbf{r} \cdot \nabla \mathbf{r}^h - \mathbf{r}^h|^2\,\mathrm{d}S\ , \tag{45}$$

where $\mathbf{r}^h(\mathbf{r})$ is the magnetic mapping associating to the point $\mathbf{r} = (x, y)$ on the lower boundary S^+ the point $\mathbf{r}^h(\mathbf{r}) = (x^h(x, y), y^h(x, y))$ of the upper one, S^-, and $\mathbf{B}$ is an arbitrary equilibrium compatible with this mapping. A consequence of that formula is that at low shear, two possible equilibria would have about the same energy, which may be taken as an indication of uniqueness when $\mathbf{r}^h(\mathbf{r})$ is close to the identity mapping. A strict result of uniqueness is obtained for the identity mapping $\mathbf{r}^h(\mathbf{r}) = \mathbf{r}$, in which case one gets indeed $\mathbf{B} = \mathbf{B}_0$ (Aly 2005). This is a quite modest result (however relevant to the treatment of Parker problem by Ng and Bhattacharjee (1998)), but it may be hoped that the somewhat involved method introduced to obtain it will be applicable to more interesting cases soon.

6 BOUNDARY DRIVEN EVOLUTION OF B: THE NUMERICAL APPROACH

We now turn to the numerical approach to the boundary driven evolution problem.

6.1 Model

In our numerical calculations, we consider the evolution of the 3D field of an active region, represented theoretically by the half-space $D = \{z > 0\}$, and practically by a finite box of large size D_b. Our code deals with the nondimensional form of the system of dissipative MHD equations (3)-(8) in D_b, which are discretized on a nonuniform mesh, and solved by using our semi-implicit scheme (Amari et al. 1999b). As for the choice of the various parameters, we use small values for the dissipation coefficients: $\nu = 10^{-2} - 10^{-3}$ for the kinematic viscosity and $\eta = 10^{-4}, 10^{-5}, 0$ for the resistivity (for our mesh resolution, this gives Lundquist numbers of order $10^4, 10^5$). Note that these values are the values of the true physical parameters. In addition, there are also the usual viscosity and resistivity of numerical origin, with the latter playing a nonnegligible role when we impose $\eta = 0$. The term H can be neglected in equation (6), and the plasma β is taken to be 10^{-3} (i.e., of the order of the very small values observed in the corona) or 0 – without any differences being actually found between the results. When we choose $\beta = 0$, we need to fix arbitrarily a mass density profile, and of course to neglect the gravity term in equation (3). Here, we choose $\rho = B^2$, which insures a constant Alfven velocity, or $\rho = 1$. Alternative choices of density profiles (exhibiting for instance a slower decrease with distance) do not lead to noticeably different results.

In the simulations described below, we start from a force-free equilibrium constructed by first shearing a potential field and thus letting the system relax viscously for a long period of time. We consider two types of evolution. The first one – *flow driven evolution* – is triggered by horizontal motions imposed on the lower boundary of D_b, while the second one – *flux driven evolution* – results from diffusive motions on the latter. In either case, the evolution is imposed by fixing the form of the tangential component, $\mathbf{E}_s$, of the electric field, which leads to a well posed MHD boundary value problem.

6.2 Flow driven Evolution

Imposing the velocity field to be horizontal on the photosphere in order to describe twisting and shearing motions, we have from Ohm's law

$$-c\hat{\mathbf{z}} \times \mathbf{E}_s = \hat{\mathbf{z}} \times (\mathbf{v} \times \mathbf{B})_s = B_z \mathbf{v}_s = \nabla_s f + \nabla_s g \times \hat{\mathbf{z}} \quad \text{on } S \,, \tag{46}$$

assuming the resistive effects to be negligible (here, $\nabla_s = \hat{\mathbf{x}}\,\partial_x + \hat{\mathbf{y}}\,\partial_y$). The last equality is a Helmholtz decomposition of $B_z \mathbf{v}_s$ into an irrotational part and a solenoidal one, which introduces two functions $f(x, y, t)$ and $g(x, y, t)$ uniquely determined (when appropriate asymptotic conditions are assumed) from the equations

$$\nabla_s^2 f = \nabla_s \cdot (B_z \mathbf{v}_s) \,, \tag{47}$$

$$\nabla_s^2 g = -[\nabla_s \times (B_z \mathbf{v}_s)] \cdot \hat{\mathbf{z}} \,. \tag{48}$$

Moreover, we have from Faraday's induction law

$$\frac{\partial B_z}{\partial t} = -c\nabla \cdot (\mathbf{E}_s \times \hat{\mathbf{z}}) = -\nabla \cdot (B_z \mathbf{v}_s) = -\nabla_s^2 f \, . \tag{49}$$

In some of our simulations, we consider motions which keep B_z invariant on S, which can be easily achieved by imposing $f = 0$. A different (but of course mathematically equivalent) procedure was actually used in our earlier paper (Amari et al. 1996) where we imposed directly a velocity field of the form $\mathbf{v}_s = \nabla_s \phi \times \hat{\mathbf{z}}$ on S, with ϕ a potential related to g (Amari et al. 2003a).

One of the main results in Amari et al. (1996) is that large scale twisting motions of a bipolar configuration leads to a transition towards *very fast expansion*, a phenomenon that appears to be generic for simple and complex topologies (Amari et al. 1997b). This phenomenon was later confirmed using different numerical schemes by Török and Kliem (2003), Aulanier et al. (2005). Imposing a more localised twist on the bipolar configuration (which results in the formation of a confined TFR) leads to a transition to nonequilibrium, and, in presence of finite conductivity, to reconnection of the twisted tube with the overlying confining arcade (Amari and Luciani 1999). There is thus a splitting of the single flux rope into two ropes (Amari and Luciani 2000), as was also found by Baty (2000) to occur in cylindrical geometry during the reconnection process following the development of the ideal kink instability. Although magnetic helicity is conserved during the whole evolution, the relaxed state after reconnection is different from a Taylor state (Amari and Luciani 2000), i.e., a linear force-free state with a value of the constant α determined by the total helicity. Both the confined and unconfined evolutions can be interpreted as a way of redistributing magnetic helicity towards the boundary of the domain: Helicity is indeed transported to infinity in the unconfined case (relevant to CMEs and interplanetary magnetic clouds), and transferred to the overlying confining arcade (which represents an *artificial boundary* of the domain) in the confined case (relevant to confined eruptions).

Since magnetic fields emerge with twist/shear (Leka et al. 1996, Liu et al. 2005), it is worth considering the effects of converging motions on a pre-sheared configuration represented by a force-free equilibrium. We have thus constructed a set of force-free states with increasing magnetic energies and helicities by applying large scale twisting motions as described above, but by staying below the limit at which very fast expansion occurs. For equilibria obtained by twisting motions for which $f \neq 0$, the evolution of the field due to the converging motions exhibits two different phases (Amari et al. 2003a). In the first one, the evolution is almost quasi-static. The magnetic topology remains arcade-like, with a shear along the inversion line increasing, magnetic energy being stored, and helicity keeping its initial value. At some critical stage, however, this quiet phase is stopped and the configuration experiences a transition to a dynamic and strongly dissipative phase, during which reconnection leads to the formation of a TFR, however not in equilibrium. These results extend and complement the 2D results found earlier by Priest and Forbes (1991), Forbes and Priest (1995), Forbes (1991), Inhester et al. (1992). In these papers it was shown that when a flux distribution on the boundary evolves in a suitable way a catastrophic nonequilibrium transition can occur which implies the ejection of a plasmoid.

But these conclusions were limited by the presence in the system of an unanchored flux rope, and the unsolved issue of nonequilibrium in 3D. In 3D the system dynamically reconnects as soon as it looses equilibrium, with no secondary intermediate nonequilibrium bifurcation being produced.

Therefore a global disruption may occur in a magnetic structure with a nonzero helicity contents when it is driven into an evolution by the converging motions which have been shown by some observations to be actually present on the photosphere. Since magnetic helicity keeps a constant value during the quasi-static phase of evolution, it needs to have been produced during a previous phase. In our simulations described here, helicity is obtained by twisting motions, but we do not claim that this process actually occurs in the corona – observational evidence for adequate transverse photospheric velocities still being needed. It could be as well the result of processes taking place before the emergence of the structure as also estimated recently by Démoulin et al. (2002), Nindos and Zhang (2002). We should also note that we cannot exclude that some amount of helicity be produced by the converging motions themselves, if they are less symmetric than the ones we have considered in our model.

Helical structures associated with prominences ejected as part of the CMEs are sometimes observed, and it is clear that twisted ropes are good candidates for the support of cool material. It is still an open problem, however, whether a rope does exist prior to the disruption, thus possibly playing a role in its triggering. Previous 3D results have shown that both a sheared complex topology configuration of the multiarcade type (Antiochos et al. 1999) or a twisted flux rope (Amari et al. 2000) in a not necessarily bipolar configuration are candidates for the initiation of a CME. These results complement the earlier ones, by showing an example of an evolving bipolar configuration suffering a major disruption, but without the presence of a TFR in equilibrium. A rope is created, but only as a result of reconnection during the global disruption, and it is then part of a process far away from equilibrium.

6.3 Flux driven evolution: cancellation, diffusion

We now want to model the effects of photospheric flux variations, either flux cancellation or flux dispersion, during the last stages in the life of an active region. For that, we start from a class of force-free equilibria having different magnetic helicities and energies, constructed as above, and we drive an evolution by imposing a particular form of the tangential component $\mathbf{E}_s$ of the electric field on S, which can be Helmholtz decomposed according to

$$c\mathbf{E}_s = \nabla_s \phi + \nabla_s \psi \times \hat{\mathbf{z}}\,. \tag{50}$$

Specifically, we set

$$\psi(x, y, t) = \kappa_b B_z(x, y, 0, t)\,, \tag{51}$$

with $\kappa_b > 0$ a constant having the dimension of a magnetic diffusivity, and

$$\phi(x, y, t) = 0\,. \tag{52}$$

Calculations are conducted without adding any other conditions on S. In particular, the tangential component $\mathbf{B}_s$ of $\mathbf{B}$ is not required to satisfy any a priori constraint.

Once more, the evolution is divided into two phases – quasi-static and dynamic, respectively –, during which magnetic energy decreases at different rates. The topology changes from an arcade type to a flux rope type at some $t_{fl} < t_c$, i.e., a TFR appears spontaneously during the first slow quasi-static phase and stays in equilibrium. Rope formation is associated with a reconnection process occuring at the inversion line on S, a process already used in Amari et al. (1999c) to obtain a TFR in equilibrium. Nonequilibrium develops at t_c, and leads to a confined disruption for small initial helicity, and to an unconfined major disruption for large initial helicity. Moreover, for all the values of the initial helicity, the energy of the field remains always below that of the open field having the same distribution of B_z on the boundary plane.

These results are actually relevant for understanding the observed persistence of CMEs in the late phase of dispersion of an active region. They show indeed that the dispersion process (insofar as it can be modelled by boundary flux diffusion, as first proposed by Leighton (1964)), can trigger eruptive events that may be either confined or unconfined, depending on the value of the initial helicity. Moreover, they shed some light on the question of the necessity or not of the presence of a TFR in the preerupting configuration. It appears that such a rope can be formed during the diffusion driven evolution and stay in equilibrium for a while (see also Amari et al. (2000)). This is in contrast with the results of Amari et al. (2003a) and of Antiochos et al. (1999) (where a quadrupolar configuration is studied), as it was found therein that in an evolution driven by some types of boundary motions the flux rope can only be produced by reconnection during the disruption itself. This rope may be the site of the formation of a prominence – the lines have a shape favorable to mass support against the Sun gravitational field – (Aulanier and Démoulin 1998, Lionello et al. 2002), and this could explain why prominences form again in the same place between CMEs during the active region dispersion phase.

It also appears that the helicity, which keeps a constant value through the diffusion driven evolution, cannot be the only parameter controlling the triggering of an ejection – the initial configuration does not erupt, in spite of the fact that it has the same helicity as the final erupting one. Then having a large enough helicity seems to be a necessary condition for an ejection to occur, but not a sufficient one.

Finally, from the observational point of view, changes at the photospheric level in both j_z (vertical component of the electric current density) and B_z may be measured, as well as changes of the twist of the coronal configuration. For instance, each half-turn of twist observed in the twisted arcade configuration merges to give a flux rope of twist 2π, with the same magnetic helicity. This is an evidence of conversion of mutual helicity to self-helicity with conservation of the total magnetic helicity (see Fig. 2). By the same token, the coronal magnetic helicity contents cannot be explained by this process. Therefore, the amount of magnetic helicity in the pre and post CME configurations depends entirely on that of the initial force-free configuration possibly injected by emergence (although we cannot exclude the possibility of some addition due to differential rotation and boundary motions as in Amari et al. (2003a) since these have been proved not to inject helicity into the system).

To describe an evolution driven by flux submergence through the boundary, it is possible to set (Amari et al. 2000)

$$\nabla_s^2 \psi(x, y, t) \quad = \quad \mu B_z(x, y, 0, t_0) \,, \tag{53}$$

with $\mu < 0$ a constant. Eq. (53) results in a linear variation of B_z on S, and the associated evolution of a bipolar configuration leads to a transition to nonequilibrium when the rapidly decreasing energy of the open field becomes of the order of the magnetic energy of the configuration, unlike for the diffusion case. We have recently extended (Amari et al. 2007) these calculations to the case of a quadrupolar configuration having an X-point in $D = \{z > 0\}$ (see Fig. 3). As for the simple topology case, the evolution leads to the formation of a TFR and its disruption accross the overlying system of magnetic lines which have a weaker tension. This proves that the observed presence of an X-point in an erupting configuration cannot be taken as evidence in favor of the validity of the Break-Out model of Antiochos et al. (1999). Flux cancellation can operate as well in a quadrupolar configuration to initiate an eruptive event.

Let us conclude this subsection by a general comment on the relation between the appearance of a catastrophic phase in the evolution of a field, and the way the magnetic energy compares with the energy of the open field. For most of the cases considered above, the energy $W(t)$ is monotonically increasing and approaches W_σ (which remains constant as B_n is kept fixed), but it fails to reach this bound. In FCM however, flux changes on the boundary and $W_\sigma(t)$ decreases at a fast rate, while the free magnetic energy actually increases. As a result we have $W(t)$ becoming very close to $W_\sigma(t)$ only for this mechanism. Moreover, we see that TFRs are important in this respect since configurations for which $W(t)$ exceeds $W_\sigma(t)$ contain such an object (which would be disconnected from S in 2D). This is an interesting feature since in the 3D FCM we have proposed that a TFR which remains connected to the boundary is created, and allows $W_\sigma(t_c) \simeq W(t_c)$ at the critical time. Flux cancellation as it is currently observed (Welsch 2006, Wang and Sheeley 2002), could therefore be considered as a good explanation for the triggering of a CME, and the open field conjecture could be a key feature of this mechanism.

7 SUBPHOTOSPHERIC DRIVEN EVOLUTION OF B: NUMERICAL APPROACH

7.1 General statements

In the approach followed in the previous section, the details of the physics of the subphotospheric layers are completely omitted, and their influence on the corona is taken into account by merely imposing changing boundary conditions on S. These may be chosen either to fit some observations on the photosphere, or to mimic phenomena which are expected to take place below on theoretical grounds. This state of affairs is not fully satisfying, and it is certainly worth trying to include in the system under study both the corona and the convection zone (or at least its upper part) to get a more consistent picture. Clearly, the most satisfying way to realize this would be to use the results of recent studies dealing with the rising through the convection zone (CZ) of flux ropes either having a minimum amount of twist in order to reach the solar surface (Moreno-Insertis and Emonet 1996, Linton et al. 1998, Fan et al. 1999, Linton et al. 1998, Fan et al. 1999, Abbett et al. 2000, Fan 2001) or having already started to emerge (Magara 2001, Magara and Longcope 2003, Magara 2004, Abbett and Fisher 2003). This turns out however to be too ambitious, due to many technical difficulties, and we have therefore considered in a first step a model in which the corona is described dynamically as in our boundary driven evolution studies, while the convection zone is described kinematically, the motions inside being assumed to be given. Actually, we have considered two models, differing from each other by the types of the motions in the convection zone.They are taken to be purely vertical in the first case, and to be convective in the second case, being organized in cells in which the motions near the photosphere are mostly horizontal.

As in the previous section, we want to model what happens in an active region rather than in the whole corona, and thus we neglect the curvature of the solar surface. The corona is represented by the upper half-space $D = \{z > 0\}$, and the convection zone by the layer $D_* = \{-h < z < 0\}$ – actually by the bounded but large domains D_b and D_{*b}, respectively, in practice. Note that the total helicity in $D \cup D_*$ is approximately conserved owing to the low resistivity of the plasma.

7.2 Rigid emergence model

In the first model (Amari et al. 2004), we start from a TFR located at the basis of D_*, and we impose a nonuniform vertical velocity field $\mathbf{v}^B = v(x, y)\hat{\mathbf{z}}$ in all that domain. Although quite simple, this approach allows to add an important element to our models without the necessity to solve the difficult problem – actually a true numerical challenge! – resulting from the strong variation of some physical quantities accross the thin photosphere (Amari et al. 2004). As usual, we use in our calculations the tangential component

$$\mathbf{E}_s = v_z^B \mathbf{B}_s \tag{54}$$

of the electric field at the boundary S, which is the important quantity controlling the exchanges between D_* and D through S.

The imposed velocity field turns out to be sufficient to get an emergence of magnetic flux and electric current in the corona. Once this emergence has occurred, the subsequent evolution of the field in D exhibits two phases. During the first one, the coronal field in D evolves quasi-statically through a series of equilibria. At some time, however, there is a change in the topology of the lines, which evolves from an arcade type to a flux rope type. Thus there is a second phase during which the configuration experiences a nonequilibrium transition, and the TFR does not stay any longer in equilibrium. The topological transition as well as the development of the nonequilibrium both occur when the net flux on the photosphere has already started decreasing, which happens when a sufficiently large part of the tube has already emerged through the photosphere. The formation of a TFR occurs without reconnection and the magnetic energy of the configuration in D stays between the magnetic energy of the potential field and that of the open field field having the same distribution of flux.

Similar behaviour has been found when $\mathbf{v}^B$ is uniform in the presence of a pre-existing coronal magnetic field (Fan and Gibson 2003, 2004, Fan 2005)

7.3 Photosphere prevents rigid emergence : the Resistive Layer Model

The simple picture presented above is somewhat limited. Indeed, we impose in the CZ a vertical flow which still persists at the photospheric boundary (we just stop it from time to time to check the existence of an accessible nearby equilibrium), while the actual rising flow is certainly expected to be reduced when hitting the sharp and optically thin photospheric layer, with the subsequent appearance of a tangential velocity field. This is actually what is shown by some numerical studies of the emergence through the stiff photosphere of an individual flux rope launched not too deep in the CZ (Fan 2001, Magara and Longcope 2003, Abbett and Fisher 2003, Magara 2004, Manchester IV et al. 2004): There is a large horizontal component of the velocity field and then a cell-like structure of the motions. We are thus lead to address the following question: How it is possible for a magnetic structure to emerge when the flows are almost horizontal at the top of the convection zone. To make a step towards the solution to that problem, we introduce the *Resistive Layer Model* (RLM) (Amari et al. 2005) which allows us to close a subphotospheric MHD model by naturally allowing the transfer of magnetic energy and helicity into the solar corona through non current-free fields. The resistive layer modelizes a turbulent photospheric boundary layer in which the effective resistivity is larger than in the convection zone below and in the corona above. It has been suggested by many observations, like the observations of Ellerman bombs (Pariat et al. 2004), that the photospheric resistivity plays an important role in many solar activity processes.

In this approach, the horizontal component of the electric field, $\mathbf{E}_s$, is continuous during the crossing of the RLM, but its expression changes from an inductive form (related to the convection flow) near the top of the convection zone to a resistive form inside the photospheric layer and again to an inductive form at the basis of the corona, where it acts as the driver of an evolution in which force-free magnetic fields are naturally produced.

We have found that in the case $\eta = 0$ (which was run for a test) the TFR in D_* is strongly deformed, but no flux is transferred through S. Emergence into D occurs only when $\eta \neq 0$, and it implies in that case both the magnetic flux and the electric current, i.e., the emerging field appears to be sheared. This shows that a closure of the MHD model in D_* by a current carrying solution in D can be performed only in the presence of resistivity when the condition $v_z = 0$ is imposed on S. As magnetic flux and electric current continue emerging, a critical time $t_{fl} > 0$ is reached at which the magnetic topology switches from an arcade type to a TFR type. This transition occurs while the rate of increase of the magnetic flux on S starts decreasing. Later on, the configuration inflates much more rapidly as shown by the variation of the kinetic energy. Eventually, it reaches the top of the domain, exhibiting a dynamic transition at some critical time t_c, $0 < t_{fl} < t_c$. Moreover, the energy W_π of the potential field having a distribution of B_z on $\{z = 0\}$ identical to that of $\mathbf{B}$, first increases and thus decreases at a finite rate. But the energy W of the configuration decreases at a much smaller rate than W_π. Unlike the case of the purely vertical CZ flow considered in Amari et al. (2004), the total relative magnetic helicity does not keep increasing at the same rate, but seems to saturate at a value outside the limits of the simulations.

By introducing a resistive layer of width equal to that of the return layer where the convection flow matches the photosphere, it is thus possible to transfer a part of the transverse component of $\mathbf{B}$ from the region where v_z is strong to the corona. In the case where the rising of a TFR is described by a kinematical convection model, the RLM exhibits several features observed in full MHD simulations of the transition between the convection zone and the chromosphere, such as concentration of magnetic flux and transfer of magnetic helicity. However, it shows that a divergence-free velocity field closing up at the boundary implies a tendency for magnetic energy and helicity to saturate, a feature not seen when imposing vertical compressible nonuniform motions (Amari et al. 2004). As in several previous studies (Amari and Luciani 2000, Amari et al. 2003b, 2004), a TFR is produced during an equilibrium phase of the evolution rather than during a major disruption as in Amari et al. (2003a) and Antiochos et al. (1999). Note that the RLM could be used to couple coronal models with more elaborate large scale CZ models such as anelastic spherical harmonics or compressible models. While the results obtained in Amari et al. (2004) show that the TFR may emerge without any reconnection occurring at the photosphere, the RLM shows that the role of resistivity and therefore of reconnection is important in this layer, as it was the origin of the coronal models driven by flux changes (Amari et al. 2000, 2003b). The reconnection location is however different from that one in the work by Antiochos et al. (1999) in which reconnection is necessary in the corona above the sheared arcade to trigger the large scale disruption.

8 CONCLUSION

We have reviewed in this paper some of the analytical and numerical work we have recently done with several collaborators on two important topics of solar physics: The problem of the reconstruction of the field of an active region from measurements made at the photospheric level, and the problem of the MHD evolution of the coronal field driven by changes imposed by the dense plasma constituting the subphotospheric layers – our main motivation for studying these problems being that they are likely to be important for understanding the initiation of large scale eruptive events, and for establishing efficient tools for predicting them. Although some important progress has been made in the study of both problems by us and by several other teams, there are still a lot of interesting issues which are unsolved, and we would like to indicate some of them in form of a conclusion.

Let us first consider the force-free model, which constitutes a convenient approximation which may be thought a priori to give a good description of the corona, but during the most violent phase of an eruptive event. Among the problems we would like to see being solved in a not too far future, we quote the following ones:

- Proving or disproving the conjecture C according to which the least upper bound on the energies of all the force-free fields in the unbounded domain D which share the same flux distribution Q on the boundary is just the energy of the open field. As a first step, it would be interesting to understand from a mathematical point of view the recent numerical results suggesting that the conjecture could be true only for the restricted class of fields having all their lines connected to the boundary (conjecture C_c).
- For the theoretical part of the reconstruction problem: Finding a proof of convergence of the Grad-Rubin scheme valid for fields of arbitrary complexity occupying the half-space.
- For the practical reconstruction problem: Determining which one among the algorithms proposed so far is the more efficient for a fast extrapolation of the boundary measurements to the corona. This point may be first explored by using tests based on a known solution, with the possible addition of some amount of noise, but eventually the algorithms should be compared on the actual new challenging data provided by THEMIS, SOLIS and SOLAR-B.
- For the boundary driven evolution problem: Determining the maximal interval of time $[0, T[$ (with possibly $T = \infty$) for which a sequence of fields solving the axisymmetric and the fully 3D problems do exist. As a first step here, it would be interesting to investigate in more details the conjecture of Zhang et al. (2006) on the existence of a upper bound on the helicity which can be injected into a force-free field.

As for the more complex MHD model, we indicate the following problems:

- Extending some of the analytical results which have been obtained for the boundary driven evolution in the framework of the force-free model (e.g., the fast expansion of a structure and its eventual opening) by including the effects of inertia and/or resistivity.
- Elucidating the role of the instabilities of the kink type in the evolution of a magnetic structure.
- Computing the evolution of various structures of complex topology in order to have a larger set of results for comparison with the observations, the goal being of course to be

able to determine eventually which one of the models – breakout or cancellation, or both, or none – is really able to explain the initiation of an eruptive event.

- Improving the models in which both the corona and the convection zone (or at least the upper part of it) are included. In particular, we would like to give up our kinematic description of the convection zone, by taking into account in a dynamical way the effects of the stratification and the associated buoyancy. Note that the next generation of space missions following SOHO will provide useful hints and constraints for the subphotospheric models.

Finally, we want to stress the importance of developing in parallel the analytical and the numerical approaches, which turn out to be quite complementary. Analytical studies allow indeed to establish from first principles general results providing qualitative informations at least on the gross behavior of simple fields. Numerical simulations on the contrary can deal only with specific cases, but they are able to describe them quantitatively in details, even when intricate phenomena like 3D reconnection occur in the system under consideration.

Acknowledgements

We thank C. Boulbe, A. Bleybel, T. Z. Boulmezaoud, E. Kersalé, J.-F. Luciani, J. A. Linker, T. Lepeltier, Z. Mikic, S. Régnier and M. Tagger, for their collaboration over many years. We also thank the two anonymous referees for their help in improving the presentation of the paper.

References

Abbett, W. P. and Fisher, G. H., A coupled model for the emergence of active region magnetic flux into the solar corona. *Astrophys. J.*, 2003, **582**, 475–485.

Abbett, W. P., Fisher, G. H. and Fan, Y., The three-dimensional evolution of rising, twisted magnetic flux tubes in a gravitationally stratified model convection zone. *Astrophys. J.*, 2000, **540**, 548–562.

Aly, J. J., Some properties of force-free magnetic fields in unbounded regions of space. *Astrophys. J.*, 1984, **283**, 349–362.

Aly, J. J., Some properties of the solutions of a non-linear boundary value problem for a force-free field in an infinite region of space. I. Energy estimates. *Astron. Astrophys.*, 1988, **203**, 183–188.

Aly, J. J., On the reconstruction of the nonlinear force-free coronal magnetic field from boundary data. *Solar Phys.*, 1989, **120**, 19–48.

Aly, J.-J., Quasi-static evolution of a force-free magnetic field. *Comp. Phys. Comm.*, 1990, **59**, 13–20.

Aly, J. J., How much energy can be stored in a three-dimensional force-free magnetic field. *Astrophys. J.*, 1991, **375**, L61–L64.

Aly, J. J., Some properties of finite energy constant-α force-free magnetic fields in a half-space. *Solar Phys.*, 1992, **138**, 133–162.

Aly, J. J., Eruptive Processes in the Solar Corona. In: *Cosmical Magnetism*, edited by D. Lynden-Bell, pp. 7–15, 1993 (Institute of Astronomy, Cambridge)

Aly, J. J., Nonequilibrium in sheared axisymmetric force-free magnetic fields. *Astrophys. J.*, 1995, **439**, L63–L66.

Aly, J.-J., A uniqueness result for a simple force-free equilibrium submitted to a topological constraint. *Astron. Astrophys.*, 2005, **429**, 15–18.

Aly, J. J., Integral properties of force-free magnetic fields in unbounded domains of space. I. Fields occupying the exterior of a spherical region. *Astron. Astrophys.*, 2006a, submitted.

Aly, J. J., Boundary-driven evolution of an axisymmetric force-free magnetic field in the exterior of a spherical region. *Astrophys. J.*, 2006b, submitted.

Amari, T. and Démoulin, P., Extrapolation of solar magnetic fields. In: *Workshop "Méthodes de Détermination des Champs Magnétiques Solaires et Stellaires"*, edited by M. Faurobert-Sholl, H. Frish and N. Mein, Vol. 212, pp. 187–209, 1992 (Observatoire de Paris, Meudon).

Amari, T. and Luciani, J. F., Confined disruption of a three-dimensional twisted magnetic flux tube. *Astrophys. J.*, 1999, **515**, L81–L84.

Amari, T. and Luciani, J. F., Helicity redistribution in relaxation of astrophysical plasmas. *Phys. Rev. Let.*, 2000, **84**, 1196–1199.

Amari, T., Luciani, J. F., Aly, J. J. and Tagger, M., Very fast opening of a three dimensional twisted magnetic flux tube. *Astrophys. J.*, 1996, **466**, L39–L42.

Amari, T., Aly, J. J., Luciani, J. F., Boulmezaoud, T. Z. and Mikic, Z., Reconstructing the solar coronal magnetic field as a force-free magnetic field. *Solar Phys.*, 1997a, **174**, 129–149.

Amari, T., Luciani, J. F., Aly, J. J. and Mikic, Z., Opening solar magnetic fields: Some analytical and numerical aspects. In *Coronal Mass Ejections*, edited by N. Crooker, J. A. Joselyn and J. Feynman, Geophysical Monograph, Vol. 99, pp. 101–109, 1997b (American Geophysical Union, Washington, DC).

Amari, T., Boulmezaoud, T. Z. and Maday, Y., A regularization method for the extrapolation of the photospheric solar magnetic field. I. Linear force-free field. *Astron. Astrophys.*, 1998, **339**, 252–260.

Amari, T., Boulmezaoud, T. Z. and Mikic, Z., An iterative method for the reconstruction of the solar coronal magnetic field. I. Method for regular solutions. *Astron. Astrophys.*, 1999a, **350**, 1051–1059.

Amari, T., Luciani, J. F. and Joly, P., A preconditioned semi-implicit method for magnetohydrodynamics equations. *SIAM J. Sci. Comput.*, 1999b, **21**, 970–986.

Amari, T., Luciani, J. F., Mikic, Z. and Linker, J., Three dimensional solutions of magnetohydrodynamic equations for prominence magnetic support : Twisted magnetic flux rope. *Astrophys. J.*, 1999c, **518**, L57–L60 .

Amari, T., Luciani, J. F. , Mikic, Z. and Linker, J., A twisted flux rope model for coronal mass ejections and two-ribbon flares. *Astrophys. J.*, 2000, **529**, L49-L52.

Amari, T., Luciani, J. F., Aly, J. J., Mikic, Z. and Linker, J., Coronal mass ejection: Initiation, magnetic helicity and flux ropes. I. Boundary motions driven evolution, *Astrophys. J.*, 2003a, **585**, 1073–1086.

Amari, T., Luciani, J. F., Aly, J. J., Mikic, Z. and Linker, J., Coronal mass ejection: Initiation, magnetic helicity and flux ropes. II. Boundary flux changes driven evolution. *Astrophys. J.*, 2003b, **595**, 1231–1250.

Amari, T., Luciani, J. F. and Aly, J. J., Coronal magnetohydrodynamic evolution driven by subphotospheric conditions. *Astrophys. J.*, 2004, **615**, L165-L168.

Amari, T., Luciani, J. F. and Aly, J. J., Non-current-free coronal closure of subphotospheric MHD models. *Astrophys. J.*, 2005, **629**, L37–L40.

Amari, T., Boulmezaoud, T. Z. and Aly, J. J., Well posed reconstruction of the solar coronal magnetic field. *Astron. Astrophys.*, 2006, **446**, 691–705.

Amari, T., Aly, J. J., Mikic, Z. and Linker, J., Coronal mass ejection initiation and complex topology configurations – Flux cancellation and break-out models. *Astrophys. J.*, 2007, submitted.

Antiochos, S. K., Devore, C. R. and Klimchuk, J. A., A model for solar coronal mass ejections. *Astrophys. J.*, 1999, **510**, 485–493.

Aulanier, G. and P. Demoulin, 3-D magnetic configurations supporting prominences. I. The natural presence of lateral feet. *Astron. Astrophys.*, 1998, **329**, 1125–1137.

Aulanier, G., Démoulin, P. and Grappin, R., Equilibrium and observational properties of line-tied twisted flux tubes. *Astron. Astrophys.*, 2005, **430**, 1067–1087.

Barnes, C. W. and Sturrock, P. A., Force-free magnetic-field structures and their role in solar activity. *Astrophys. J.*, 1972, **174**, 659–670.

Baty, H., Magnetic topology during the reconnection process in a kinked coronal loop. *Astron. Astrophys.*, 2000, **360**, 345–350.

Berger, M. A. and Field, G. B., The topological properties of magnetic helicity. *J. Fluid Mech.*, 1984, **147**, 133–148.

Bineau, M., Existence of force-free magnetic fields. *Comm. Pure and Applied Math*, 1972, **27**, 77–84.

Bleybel, A., Amari, T., van Driel-Gesztelyi, L. and Leka, K. D., Global budget for a flaring active region. I. Equilibrium reconstruction approach. *Astron. Astrophys.*, 2002, **395**, 685–695.

Boulbe, C., Boulmezaoud, T. Z. and Amari, T., An iterative method for solving nonlinear hydromagnetic equations. In: *Numerical Mathematics and Advanced Applications. Proceedings of ENUMATH 2005, the 6th European Conference on Numerical Mathematics and Advanced Applications*, edited by A. Bermudez de Castro, pp. 917–925, 2006 (Springer, Berlin).

Boulmezaoud, T. Z. and Amari, T., On the existence of non-linear force-free fields in three-dimensional domains. *Zeitschrift fur Angewandte Mathematik und Physik*, 2000, **51**, 942–967.

Choe, G. S. and Cheng, C. Z., Energy of force-free magnetic fields in relation to coronal mass ejections. *Astrophys. J.*, 2002, **574**, L179–L182.

Cuperman, S., Demoulin, P. and Semel, M., Removal of singularities in the Cauchy problem for the extrapolation of solar force-free magnetic fields. *Astronon. Astrophys.*, 1991, **245**, 285–288.

Demoulin, P., Cuperman, S. and Semel, M., Determination of force-free magnetic fields above the photosphere using three-component boundary conditions. II - Analysis and minimization of scale-related growing modes and of computational induced singularities. *Astronon. Astrophys.*, 1992, **263**, 351–360 .

Démoulin, P., Mandrini, C. H., van Driel-Gesztelyi, L., Thomson, B. J., Plunkett, S., Kovari, Z., Aulanier, G. and Young, A., What is the source of magnetic helicity shed by CMEs ? The long term helicity budget of AR 7978. *Astron. Astrophys.*, 2002, **382**, 650–665.

Fan, Y., The emergence of a twisted omega-tube in the solar atmosphere. *Astrophys. J.*, 2001, *554*, L111–L114.

Fan, Y., Coronal mass ejections as loss of confinement of kinked magnetic flux ropes. *Astrophys. J.*, 2005, **630**, 543–551.

Fan, Y. and Gibson, S. E., The emergence of a twisted magnetic flux tube into a preexisting coronal arcade. *Astrophys. J.*, 2003, **589**, L105–108.

Fan, Y. and Gibson, S. E., Numerical simulations of three-dimensional coronal magnetic fields resulting from the emergence of twisted magnetic flux tubes. *Astrophys. J.*, 2004, **609**, 1123–1133.

Fan, Y., Zweibel, E. G., Linton, M. G. and Fisher, G. H., The rise of kink-unstable magnetic flux tubes and the origin of δ- configuration sunspots. *Astrophys. J.*, 1999, **521**, 460–477.

Finn, J. M., and Antonsen, T. M., Magnetic helicity: What is it and what is it good for? *Comments Plasma Phys. Controlled Fusion*, 1985, **9**, 111–126.

Flyer, N., Fornberg, B., Thomas, S. and Low, B. C., Magnetic field confinement in the solar corona. I. Force-free magnetic fields. *Astrophys. J.*, 2004, **631**, 1210–1222.

Forbes, T. G., Magnetic reconnection in solar flares. *Geophys. Astrophys. Fluid Dynamics*, 1991, **62**, 16–36.

Forbes , T. G. and Priest, E. R., Photospheric magnetic field evolution and eruptive flares. *Astrophys. J.*, 1995, **446**, 377–389.

Forbes, T. G., et al., CME theory and models. *Space Science Rev.*, 2006, **123**, 251–302.

Gary, G. A. and R. L. Moore, Eruption of a multiple-turn helical magnetic flux tube in a large flare: Evidence for external and internal reconnection that fits the breakout model of solar magnetic eruptions. *Astrophys. J.*, 2004, **611**, 545–556.

Grad, H. and Hogan, J., Classical diffusion in a tokomak. *Phys. Rev. Let.*, 1970, **24**, 1337–1340.

Grad, H. and Rubin, H., Hydromagnetic equilibria and force-free fields. In: *Proc. 2nd Intern. Conf. on Peaceful Uses of Atomic Energy*, Vol. 31, pp. 190–197, 1958 (United Nations, Geneva).

Hu, Y. Q., Energy Buildup of Multipolar Magnetic Fields by Photospheric Shear Motion. *Astrophys. J.*, 2004, **607**, 1032–1038.

Hu, Y. Q., Li, G. Q. and Xing, X. Y., Equilibrium and catastrophe of coronal flux ropes in axisymmetrical magnetic field. *J. Geophys. Research*, 2003, **108**, 1072–1080.

Inhester, B. and Wiegelmann, T., Nonlinear force-free magnetic field extrapolations: Comparison of the Grad Rubin and Wheatland Sturrock Roumeliotis algorithm. *Solar Phys.*, 2006, **235**, 201–221.

Inhester, B., Birn, J. and Hesse, M., The evolution of line-tied coronal arcades including a converging footpoint motion. *Solar Phys.*, 1992, **138**, 257–281.

Kaiser, R.,Neudert, M. and von Wahl, W., On the existence of force-free magnetic fields with small nonconstant α in exterior domains. *Commun. Math. Phys.*, 2000, **211**, 111–136.

Laurence, P. and Avellaneda, M., Woltjer's variational principle. II. The case of unbounded domains. *Geophys. Astrophys. Fluid Dynamics*, 1993, **69**, 201–241.

Leighton, R. B., Transport of magnetic fields on the Sun. *Astrophys. J.*, 1964, **140**, 1547–1562.

Leka, K. D., Canfield, R. C., Mc Clymont, A. N. and van Driel-Gesztelyi, L., Evidence for current-carrying emerging flux. *Astrophys. J.*, 1996, **462**, 547–560.

Lepeltier, T., *Etude de Quelques Equilibres Magnétohydrostatiques*, Ph.D. Thesis, 1994 (Université Paris VII).

Lin, J., Soon, W. and Baliunas, S. L., Theories of solar eruptions: A review. *New Astron. Rev.*, 2003, **47**, 53–84.

Linton, M. G., Dahlburg, R. B., Fisher, G. H. and Longcope, D. W., Nonlinear evolution of kink-unstable magnetic flux tubes and solar δ-spot active regions. *Astrophys. J.*, 1998, **507**, 404–416.

Lionello, R., Mikic, Z., Linker, J. and Amari, T., Magnetic field topology in prominences. *Astrophys. J.*, 2002, **581**, 718–725.

Liu, C., Deng, N., Liu, Y., Falconer, D., Goode, P. R., Denker, C. and Wang, H., Rapid change of δ-spot structure associated with seven major flares. *Astrophys. J.*, 2005, **622**, 722–736.

Low, B. C., Magnetohydrodynamic processes in the solar corona: Flares, coronal mass ejections, and magnetic helicity. *Phys. Plasmas*, 1994, **1**, 1684–1690.

Low, B. C. and Lou, Y., Modeling solar force-free magnetic fields. *Astrophys. J.*, 1990, **352**, 343–352.

Lynden-Bell, D. and C. Boily, Self-similar solutions up to flashpoint in highly wound magnetostatics. *Monthly Notices Royal Astron. Soc*, 1994, **267**, 146–252.

Magara, T., Dynamics of emerging flux tubes in the Sun. *Astrophys. J.*, 2001, **549**, 608–628.

Magara, T., A model for dynamic evolutin of emerging magnetic field in the Sun. *Astrophys. J.*, 2004, **605**, 480–492.

Magara, T. and Longcope, D. W., Injection of magnetic energy and magnetic helicity into the solar atmosphere by an emerging magnetic flux tube. *Astrophys. J.*, 2003, **586**, 630–649.

Manchester IV, W., Gombosi, T. and DeZeeuw, D., Eruption of a buoyantly emerging magnetic flux rope. *Astrophys. J.*, 2004, **610**, 588–596.

Metcalf, T. R., et al., An overview of existing algorithms for resolving the 180^0 ambiguity in vector magnetic fields: Quantitative tests with synthetic data. *Solar Phys.*, 2006, **237**, 267–296.

Mikić, Z. and Lee, M. A., An introduction to theory and models of CMEs, shocks, and solar energetic particles. *Space Science Rev.*, 2006, **123**, 57–80.

Moreno-Insertis, F. and Emonet, T., The rise of twisted magnetic tubes in a stratified medium. *Astrophys. J.*, 1996, **472**, L53–L56.

Neukirch, T., Magnetic field extrapolation. In: *Proceedings of the International Scientific Conference on Chromospheric and Coronal Magnetic Fields*, edited by D. E. Innes, A. Lagg and S. A. Solanki, 2005 (ESA).

Ng, C. S. and Bhattacharjee, A., Photospheric motions and coronal mass ejection productivity. *Phys. Plasmas*, 1998, **5**, 4028–4040.

Nindos, A. and Zhang, H., Photospheric motions and coronal mass ejection productivity. *Astrophys. J.*, 2002, **573**, L133–L136.

Pariat, E., Aulanier, G., Schmieder, B., Georgoulis, M. K., Rust, D. M. and Bernasconi, P. N. Resistive emergence of undulatory flux tubes. *Astrophys. J.*, 2004, **614**, 1099–1112.

Parker, E. N., *Spontaneous Current Sheets in Magnetic Fields, with Applications to Stellar X-Rays*, 1994 (Oxford University Press, Oxford).

Priest, E., *Solar Magnetohydrodynamics*, 1982 (D. Reidel, Norwell, Mass., New York).

Priest, E. R. and Forbes , T. G. Magnetic field evolution during prominence eruptions and two-ribbon flares. *Solar Phys.*, 1991, **126**, 319–350.

Régnier, S. and Amari, T., 3D magnetic configuration of the Hα filament and X-ray sigmoid in NOAA AR 8151. *Astron. Astrophys.*, 2004, **425**, 345–352.

Régnier, S., Amari, T. and Kersalé, E., 3D Coronal magnetic field from vector magnetograms: Non-constant-alpha force-free configuration of the active region NOAA 8151. *Astron. Astrophys.*, 2002, **392**, 1119–1127.

Sakurai, T., Calculation of force-free magnetic field with non-constant α. *Solar Phys.*, 1981, **69**, 343–359.

Sakurai, T., Computational modeling of magnetic fields in solar active regions. *Space Science Rev.*, 1989, **51**, 11–48.

Schrijver, C. J., Derosa, M. L., Metcalf, T. R., Liu, Y., McTiernan, J., Régnier, S., Valori,G., Wheatland, M. S. and Wiegelmann, T., Nonlinear force-free modeling of coronal magnetic fields. Part I: A quantitative comparison of methods. *Solar Phys.*, 2006, **235**, 161–190.

Song, M. T., Fang, C., Tang, Y. H., Wu, S. T. and Zhang, Y. A., A new and fast way to reconstruct a nonlinear force-free field in the solar corona. *Astrophys. J.*, 2006, **649**, 1084–1092.

Sturrock, P. A., Maximum energy of semi-infinite magnetic field configurations. *Astrophys. J.*, 1991, **380**, 655-659.

Török, T., and Kliem, B., The evolution of twisting coronal magnetic flux tubes. *Astron. Astrophys.*, 2003, **406**, 1043–1059.

Uzdensky, D. A., Shear-driven field-line opening and the loss of a force-free magnetostatic equilibrium. *Astrophys. J.*, 2002, **574**, 1011–1020.

Wang, Y.-M. and Sheeley Jr., N. R., Coronal White-Light Jets near Sunspot Maximum. *Astrophys. J.*, 2002, **575**, 542–552.

Welsch, B. T., Magnetic flux cancellation and coronal magnetic energy. *Astrophys. J.*, 2006, **638**, 1101–1109.

Wheatland, M., Parallel construction of nonlinear force-free fields. *Solar Phys.*, 2004, **222**, 247–264.

Wheatland, M., Sturrock, P. and Roumeliotis, G., An optimization approach to reconstructing force-free fields. *Astrophys. J.*, 2000, **540**, 1150–1155.

Wiegelmann, T. and Neukirch, T., Computing nonlinear force-free coronal magnetic fields, *Nonlin. Processes Geophys.*, 2003, **10**, 313–322.

Wiegelmann, T., Inhester, B. and Sakurai, T., Preprocessing of Vector Magnetograph Data for a Nonlinear Force-Free Magnetic Field Reconstruction. *Solar Phys.*, 2006, **233**, 215–232.

Wolfson, R. and Low, B. C., Energy buildup in sheared force-free magnetic fields. *Astrophys. J.*, 1992, **391**, 353–358.

Wolfson, R., Larson, J. and Lionello, R., Maximum Energies of Force-Free Coronal Flux Ropes, *Astrophys. J.*, 2007, **660**, 1683–1689.

Wu, S., Sun, M., Chang, H. M., Hagyard, M. and Gary, G., On the numerical computation of nonlinear force-free magnetic fields. *Astrophys. J.*, 1990, **362**, 698–708.

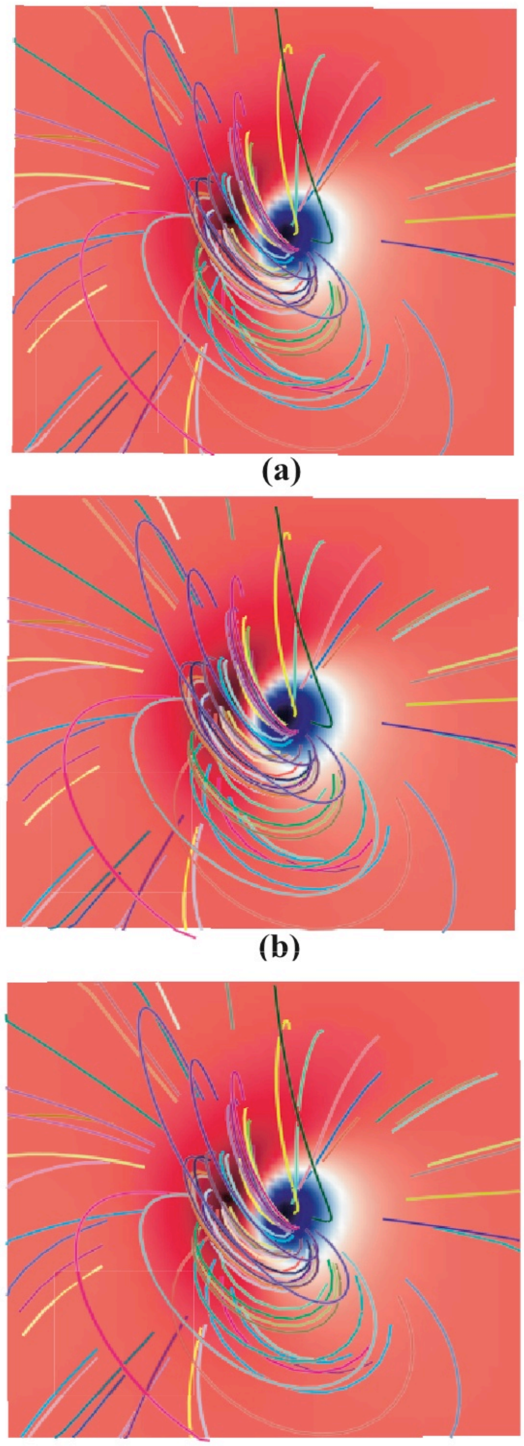
Yan, Y. and Sakurai , T., New boundary integral equation representation for finite energy force-free magnetic fields in open space above the Sun. *Solar Phys.*, 2000, **195**, 89–109.

Zhang, M. and Low, B. C., The hydromagnetic nature of solar coronal mass ejections. *Annual Rev. Astron. Astrophys.*, 2005, **43**, 103–137.

Zhang, M., Flyer, N. and Low, B. C., Magnetic field confinement in the corona: The role of magnetic helicity accumulation. *Astrophys. J.*, 2006, **644**, 575–586.

Table 1: Comparison between the solutions obtained by using our two implementations of the Grad-Rubin methods in XTRAPOL and FEMQ, and the exact solution called FF1. The error diagnostics are defined as in Schrijver et al. (2006) and Amari et al. (2006) by $VC(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{N} \mathbf{u}_i \cdot \mathbf{v}_i / \sqrt{\sum_{i=1}^{N} \mid \mathbf{u}_i \mid^2} \sqrt{\sum_{i=1}^{N} \mid \mathbf{v}_i \mid^2}$, $CS(\mathbf{u},\mathbf{v}) = (1/N) \sum_{i=1}^{N} \mid \mathbf{u}_i \cdot \mathbf{v}_i \mid / \mid \mathbf{u}_i \parallel \mathbf{v}_i \mid$, $NVE(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{N} \mid \mathbf{v}_i - \mathbf{u}_i \mid / \sum_{i=1}^{N} \mid \mathbf{v}_i \mid$, $MVE(\mathbf{u},\mathbf{v}) = (1/N) \sum_{i=1}^{N} \mid \mathbf{v}_i - \mathbf{u}_i \mid / \mid \mathbf{u}_i \mid$, where $\mathbf{u}$ and $\mathbf{v}$ denote the exact solution and a numerical one, respectively, and N is the number of computational nodes.

Model	VC	CS	NVE	MVE	$\parallel \nabla \cdot \mathbf{B} \parallel_{L^\infty}$
Exact Solution 64^3	1	1	0	0	theory: 0 residual 1.510^{-2}
FEMQ 64^3	0.9999	0.9999	0.0097	0.0109	0.056
XTRAPOL 64^3	0.9999	0.9999	0.0208	0.0162	0.5310^{-14}
XTRAPOL 128^3	0.99999	0.99999	0.0043	0.0058	0.1510^{-13}



(a) (b) (c)

Figure 1: Comparison of a selected set of field lines obtained with our two implementations of the Grad-Rubin method in XTRAPOL (a) and FEMQ (b), and the exact solution called FF1 (c).

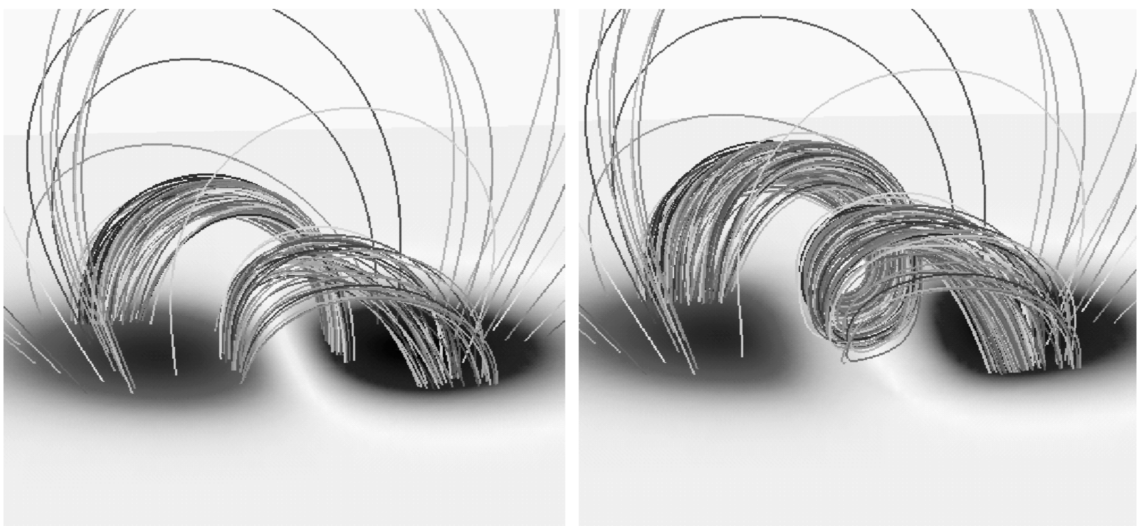


Figure 2: Conversion of mutual helicity into self helicity during the process of flux change occuring at the boundary. The total magnetic helicity is conserved during this process.

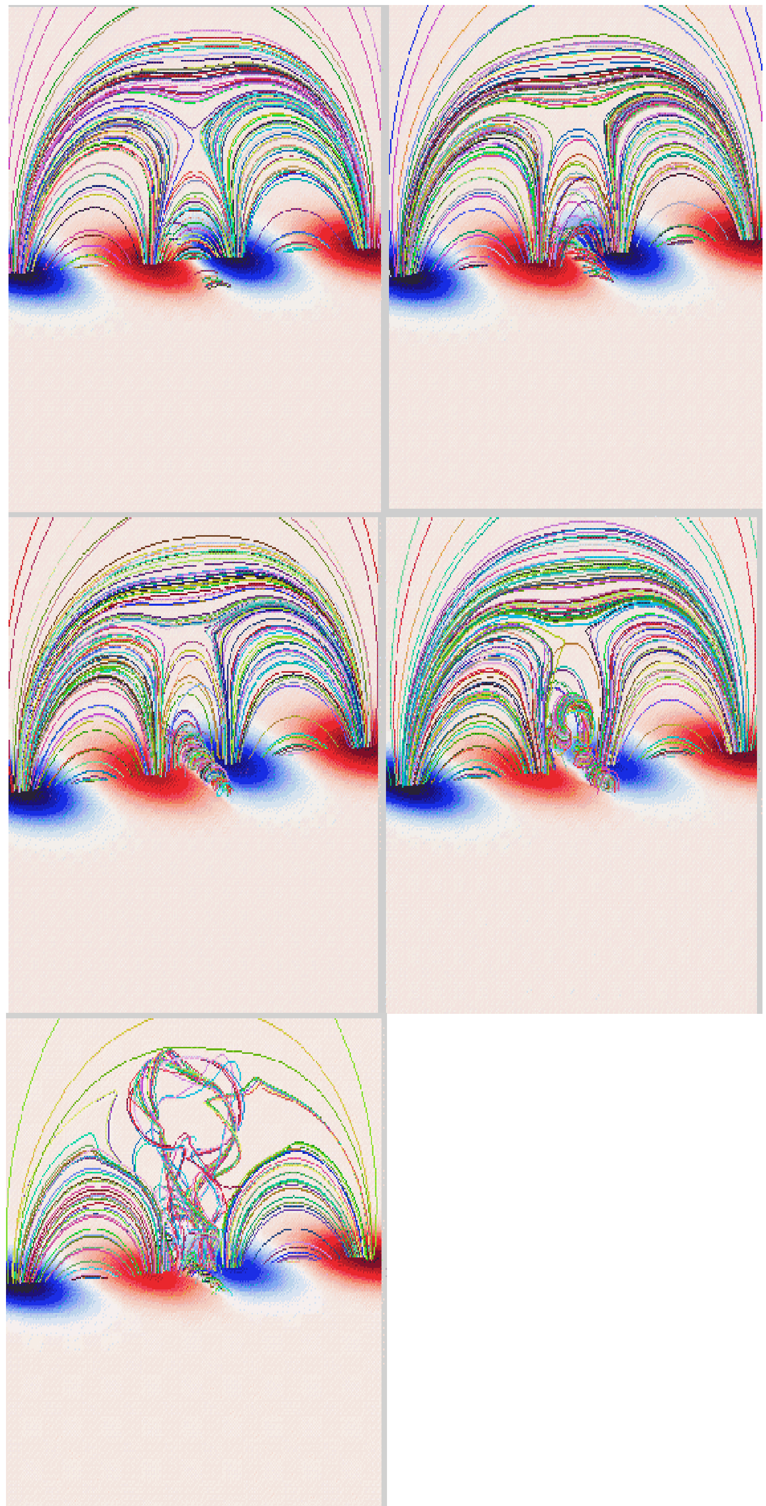


Figure 3: Selected set of field lines for the MHD flux cancellation driven evolution of an initially quadrupolar configuration having an X-Point in $D = \{z > 0\}$.

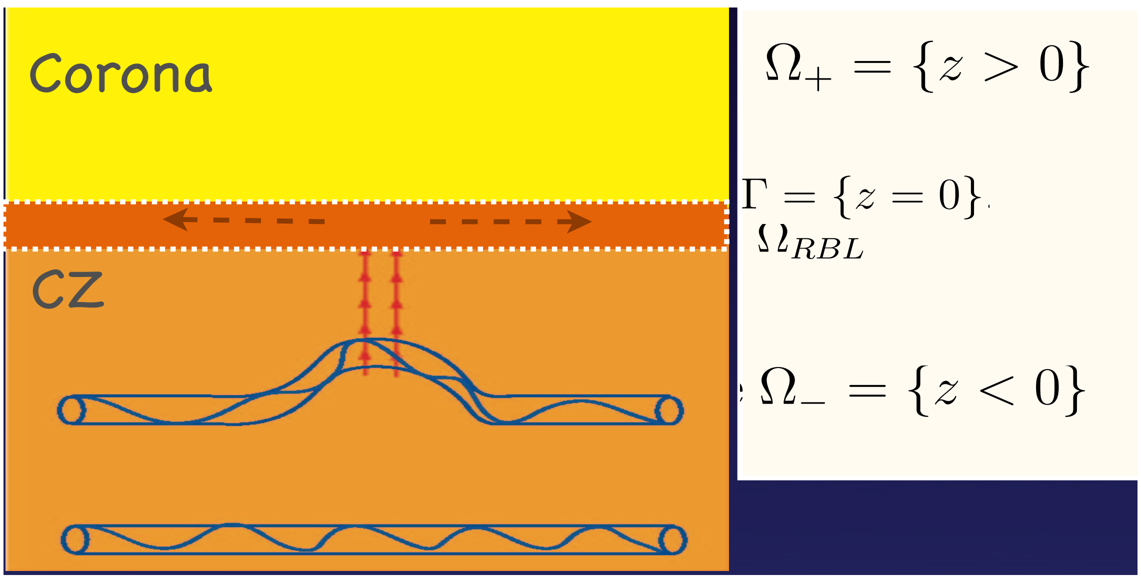


Figure 4: In the Resistive Layer Model a resistive layer is introduced above the convection zone to model the diffusive photospheric layer.