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Wave transport in differentially rotating stellar radiation zones

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Abstract

In this work, the complete interaction between low-frequency internal gravity waves and differential rotation in stably strongly stratified stellar radiation zones is examined. First, the modification of the structure of those waves due to the Coriolis acceleration is obtained. Then, theirs feed-back on the angular velocity profile through their induced angular momentum transport is derived. Finally, perspectives are discussed.

Motivation

Internal Gravity Waves (hereafter IGWs) are now considered as an essential transport mechanism in (differentially) rotating stellar radiation zones which are the seat of the mixing during stars evolution (cf. Talon & Charbonnel 2005). Furthermore, they could be excited by turbulent movements induced by adjacent convective regions at low frequencies ($\sim 1\mu$Hz in the Sun) that are of the order of the inertial one (2Ω, Ω being the star's angular velocity). The Coriolis acceleration is thus an essential restoring force for the wave dynamics as the buoyancy one associated to the stable stratification. Moreover, IGWs are excited and propagate in regions that are differentially rotating both in the radial and in the latitudinal directions. This is the reason why we undertake in this work the treatment of the complete interaction between the low-frequency IGWs and the differential rotation, which is chosen to be the more general as possible ($\Omega\,(r,\theta)$). We derive their spatial structure modified by the Coriolis acceleration and theirs feed-back on the angular velocity profile through their induced angular momentum transport.

Low-frequency IGWs in differentially rotating radiation zones

To treat the IGWs dynamics in a differentially rotating star, we have to solve the complete inviscid system formed by the momentum equation

$$\left(\partial_t+\Omega\partial_\varphi\right)\vec{u}+2\Omega\,\widehat{\mathbf{e}}_z\times\vec{u}+r\sin\theta\left(\vec{u}\cdot\vec{\nabla}\Omega\right)\widehat{\mathbf{e}}_\varphi=-\frac{1}{\overline{\rho}}\vec{\nabla}\widetilde{P}-\vec{\nabla}\widetilde{\Phi}+\frac{\widetilde{\rho}}{\overline{\rho}^2}\vec{\nabla}\overline{P},\quad(1)$$

the continuity equation $\left(\partial_t+\Omega\partial_\varphi\right)\widetilde{\rho}+\vec{\nabla}\cdot\left(\overline{\rho}\vec{u}\right)=0$, the energy transport equation which we give here in the adiabatic limit

$$\left(\partial_t+\Omega\partial_\varphi\right)\left(\frac{\widetilde{P}}{\Gamma_1\overline{P}}-\frac{\widetilde{\rho}}{\overline{\rho}}\right)+\frac{N^2}{\overline{g}}u_r=0\quad(2)$$

and the Poisson's equation $\nabla^2\widetilde{\Phi}=4\pi G\widetilde{\rho}$. ρ, Φ, P are respectively the fluid density, gravific potential and pressure. Each of them has been expanded as: $X\left(r,\theta,\varphi,t\right)=\overline{X}\left(r\right)+\widetilde{X}\left(r,\theta,\varphi,t\right)$ where $\overline{X}$ is the mean hydrostatic value of X on the isobar, $\widetilde{X}$ being its wave's associated fluctuation. $N^2=\overline{g}\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln\overline{P}}{\mathrm{d}r}-\frac{\mathrm{d}\ln\overline{\rho}}{\mathrm{d}r}\right)$ is the Brunt-Vaïsälä frequency where $\Gamma_1=\left(\partial\ln P/\partial\ln\rho\right)_S$ (S being the macroscopic entropy) is the adiabatic exponent. $\vec{u}$ is the wave velocity field. Finally, (r,θ,φ) are the usual spherical coordinates with their unit vector basis $\left\{\widehat{\mathbf{e}}_k\right\}_{k=r,\theta,\varphi}$ while $\widehat{\mathbf{e}}_z=\cos\theta\,\widehat{\mathbf{e}}_r-\sin\theta\,\widehat{\mathbf{e}}_\theta$ is the one along the rotation axis. t is the time and G the universal gravity constant.

To solve this system, three main approximations can be assumed:

- *the JWKB approximation*: waves which are studied here are low-frequency ones such that $\sigma\ll N$ (σ is the wave frequency in an inertial reference frame; see Talon & Charbonnel 2005 and Pantillon et al. 2007 for a detailed discussion of their spectrum). Then, the JWKB approximation can be adopted.

- *the Traditional approximation*: stellar radiation zones are stably strongly stratified regions. Then, in the case where the angular velocity (Ω) is reasonably weak compared to the break-down one, $\Omega_K=\sqrt{GM/R^3}$ (M and R being respectively the star's mass and radius), we are in a situation where the centrifugal acceleration can be neglected to the first order and where $2\Omega\ll N$. This allows to adopt the Traditional approximation where the latitudinal component (along $\widehat{\mathbf{e}}_\theta$) of the rotation vector $\vec{\Omega}=\Omega\,\widehat{\mathbf{e}}_z=\Omega_V\widehat{\mathbf{e}}_r+\Omega_H\widehat{\mathbf{e}}_\theta$ (with $\Omega_V=\Omega\cos\theta$ and $\Omega_H=-\Omega\sin\theta$) can be neglected for all latitudes.

Let us present a brief local analysis of this approximation in the simplest case of a uniform rotation (see also Lee & Saio 1997). The wave vector $\vec{k}$

and lagrangian displacement $\vec{\xi}$ are expanded as

$$\vec{k} = k_V \widehat{\mathbf{e}}_r + \vec{k}_H \quad \text{and} \quad \vec{\xi} = \xi_V \widehat{\mathbf{e}}_r + \vec{\xi}_H, \tag{3}$$

where $\vec{k}_H = k_\theta \widehat{\mathbf{e}}_\theta + k_\varphi \widehat{\mathbf{e}}_\varphi$, $k_H = |\vec{k}_H|$, $\vec{\xi}_H = \xi_\theta \widehat{\mathbf{e}}_\theta + \xi_\varphi \widehat{\mathbf{e}}_\varphi$, $\xi_H = |\vec{\xi}_H|$ and $\vec{\xi} \propto \exp\left[i\left(\vec{k}\cdot\vec{r} - \sigma t\right)\right]$.

For low-frequency waves in radiation zones, we can writte $\vec{k}\cdot\vec{\xi} = k_V \xi_V + \vec{k}_H \cdot \vec{\xi}_H \approx 0$ since $\vec{\nabla}\cdot\left(\overline{\rho}\,\vec{\xi}\right) \approx 0$ (this is the anelastic approximation that filters out acoustic waves which have higher frequencies), from which we deduce that $\xi_V/\xi_H \approx -k_H/k_V$.

Next, using the results given in Unno et al. (1989), the dispersion relation for the low-frequency gravito-inertial waves is obtained:

$$\sigma^2 \approx N^2 \frac{k_H^2}{k^2} + \frac{\left(2\vec{\Omega}\cdot\vec{k}\right)^2}{k^2}, \tag{4}$$

where the two terms correspond respectively to the dispersion relations of IGWs and of inertial waves. In the case where $2\Omega \ll N$ and $\sigma \ll N$ the previous dispersion relation gives $k_H^2/k^2 \lll 1$. The vertical wave vector is then larger than the horizontal one while the displacement vector is almost horizontal: $|k_H| \lll |k_V|$, $|\xi_V| \lll |\xi_H|$. On the other hand, we get $\left(2\vec{\Omega}\cdot\vec{k}\right)^2 \approx \left(2\Omega_V k_V\right)^2$. The latitudinal component of the rotation vector can thus be neglected in all the sphere.

A global demonstration in spherical geometry is given in Friedlander (1987) who gives the frequency domain of application of this approximation in the case of uniform rotation ($2\Omega < \sigma \ll N$) which is also discussed in Mathis et al. (2008). Its validy domain in the case of a general differential rotation law will be discussed hereafter.

- *the quasi-adiabatic approximation*: Following Press (1981) and Zahn et al. (1997), we adopt the quasi-adiabatic approximation to treat the thermal damping of IGWs. Let us recall here that this damping is responsible for the net transport of angular momentum which is due to bias in the wave's Doppler schift by differential rotation between retrograde ($m > 0$) and prograde waves ($m < 0$)[1] that transport respectively a negative and a positive flux of angular momentum (see Eq. 16 and Goldreich & Nicholson 1989).

Under those approximations, and assuming the anelastic one ($\vec{\nabla}\cdot(\overline{\rho}\vec{u}) = 0$), the wave's velocity field is then obtained (the details of the derivation are given

[1]The wave phase is expanded as $\exp\left[i\left(m\varphi + \sigma t\right)\right]$.

in Mathis 2008):

$$\vec{u}\left(\vec{r},t\right)=\sum_{k=\{r,\theta,\varphi\}}\left[\sum_{\sigma,m,j}u_{k;j,m}\left(\vec{r},t\right)\right]\widehat{\mathbf{e}}_k \tag{5}$$

where

$$u_{r;j,m}(\vec{r},t)=\frac{\widehat{\sigma}}{N}\frac{\lambda_{j,m}^{1/2}\left(r;\widehat{\nu}\right)}{r}w_{j,m}\left(r,\theta;\widehat{\nu}\right)\sin\left[\Psi_{j,m}\left(r,\varphi,t\right)\right]D_{j,m}\left(r,\theta;\widehat{\nu}\right), \tag{6}$$

$$u_{\theta;j,m}(\vec{r},t) = -\frac{\widehat{\sigma}}{r}\mathcal{G}_{j,m}^{\theta}\left(r,\theta;\widehat{\nu}\right)\cos\left[\Psi_{j,m}\left(r,\varphi,t\right)\right]D_{j,m}\left(r,\theta;\widehat{\nu}\right), \tag{7}$$

$$u_{\varphi;j,m}(\vec{r},t) = \frac{\widehat{\sigma}}{r}\mathcal{G}_{j,m}^{\varphi}\left(r,\theta;\widehat{\nu}\right)\sin\left[\Psi_{j,m}\left(r,\varphi,t\right)\right]D_{j,m}\left(r,\theta;\widehat{\nu}\right). \tag{8}$$

The "local" frequency ($\widehat{\sigma}$) [2] which accounts for the Doppler shift by the differential rotation and the "spin parameter" (see Lee & Saio 1997) are defined:

$$\widehat{\sigma}\left(r,\theta\right)=\sigma+m\Omega\left(r,\theta\right)\quad\text{and}\quad\widehat{\nu}\left(r,\theta\right)=\frac{2\Omega\left(r,\theta\right)}{\widehat{\sigma}\left(r,\theta\right)}=R_o^{-1}, \tag{9}$$

where R_o is the Rossby number. Unlike the case of uniform rotation, variables do not separate neatly anymore in the case of general differential rotations $\Omega\left(r\right)$ and $\Omega\left(r,\theta\right)$. The velocity components are thus expressed in terms of the 2D dynamical pressure ($P/\overline{\rho}$) eigenfunctions $w_{j,m}$ which are solutions of the following eigenvalue equation:

$$\mathcal{O}_{\widehat{\nu};m}\left[w_{j,m}\left(r,x;\widehat{\nu}\right)\right]=-\lambda_{j,m}\left(r;\widehat{\nu}\right)w_{j,m}\left(r,x;\widehat{\nu}\right) \tag{10}$$

where we define the General Laplace Operator (GLO)

$$\begin{aligned}\mathcal{O}_{\widehat{\nu};m} = &\ \frac{1}{\widehat{\sigma}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\left(1-x^2\right)}{\widehat{\sigma}\mathcal{D}\left(r,x;\widehat{\nu}\right)}\frac{\mathrm{d}}{\mathrm{d}x}\right]-\frac{m}{\widehat{\sigma}^2\mathcal{D}(r,x;\widehat{\nu})}\left(1-x^2\right)\frac{\partial_x\Omega}{\widehat{\sigma}}\frac{\mathrm{d}}{\mathrm{d}x}\\ &-\frac{1}{\widehat{\sigma}}\left[\frac{m^2}{\widehat{\sigma}\mathcal{D}\left(r,x;\widehat{\nu}\right)\left(1-x^2\right)}+m\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\widehat{\nu}x}{\widehat{\sigma}\mathcal{D}\left(r,x;\widehat{\nu}\right)}\right)\right]\end{aligned} \tag{11}$$

with

$$\mathcal{D}\left(r,x;\widehat{\nu}\right)=1-\widehat{\nu}^2x^2+\widehat{\nu}\left(\partial_x\Omega/\widehat{\sigma}\right)x\left(1-x^2\right) \tag{12}$$

[2] Note that $\widehat{\sigma}$ can vanish that corresponds to corotation resonance. In layer(s) where this happens (which are called critical layers), a careful treatment of the complete fluid dynamics equations has to be undertaken that is out of the scope of the present paper (see Booker & Bretherton 1967).

and $x = \cos\theta$. $\mathcal{O}_{\widehat{\nu};m}$ is the generalization of the classical Laplace tidal operator (Laplace 1799), the eigenfunctions $w_{j,m}$ being thus a generalization of the Hough functions (Hough 1898, Ogilvie & Lin 2004). $\lambda_{j,m}\left(r;\widehat{\nu}\right)$ are its eigenvalues; here, we focus on positive ones that correspond to propagative waves (cf. Ogilvie & Lin 2004). The GLO is a differential operator in x only and the $w_{j,m}$ form a complete orthogonal basis

$$\int_{-1}^{1} w_{i,m}^{*}\left(r,x;\widehat{\nu}\,\right) w_{j,m}\left(r,x;\widehat{\nu}\,\right) \mathrm{d}x = \mathcal{C}_{i,m}\delta_{i,j}, \tag{13}$$

where $\mathcal{C}_{i,m}$ is the normalization factor and $\delta_{i,j}$ is the usual Kronecker symbol. The dispersion relation is then given by

$$k_{V;j,m}^{2}\left(r\right) = \frac{\lambda_{j,m}\left(r;\widehat{\nu}\,\right) N^2}{r^2} \tag{14}$$

where $k_{V;j,m}$ is the vertical component of the wave vector ($\lambda_{j,m}$ has the dimension of $\left[t^2\right]$). That leads to the following expressions for the JWKB phase function

$$\Psi_{j,m}\left(r,\varphi,t\right) = \sigma t + \int_{r}^{r_c} k_{V;j,m}\,\mathrm{d}r^{'} + m\varphi \tag{15}$$

(r_c is the radius of the basis (or the top) of the adjacent convective region that excites the waves) and for the damping term

$$D_{j,m} = \exp\left[-\frac{\tau_{j,m}\left(r,\theta;\widehat{\nu}\right)}{2}\right] \text{ where } \tau_{j,m} = \int_{r}^{r_c} K \frac{\lambda_{j,m}^{3/2}\left(r;\widehat{\nu}\,\right) N^3}{\widehat{\sigma}} \frac{\mathrm{d}r^{'}}{{r^{'}}^{3}}, \tag{16}$$

K being the thermal diffusivity. On the other hand, the latitudinal and azimuthal eigenfunctions are defined

$$\mathcal{G}_{j,m}^{\theta}\left(r,x;\widehat{\nu}\,\right) = \frac{1}{\widehat{\sigma}^2}\frac{1}{\mathcal{D}\left(r,x;\widehat{\nu}\,\right)\sqrt{1-x^2}}\left[-\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x} + m\widehat{\nu}x\right] w_{j,m} \tag{17}$$

$$\begin{aligned} \mathcal{G}_{j,m}^{\varphi}\left(r,x;\widehat{\nu}\,\right) &= \frac{1}{\widehat{\sigma}^2}\frac{1}{\mathcal{D}\left(r,x;\widehat{\nu}\,\right)\sqrt{1-x^2}} \\ &\times \left[-\left(\widehat{\nu}x - \left(1-x^2\right)\frac{\partial_x\Omega}{\widehat{\sigma}}\right)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x} + m\right] w_{j,m}. \end{aligned} \tag{18}$$

As it has been emphasized by Mathis et al. (2008) and references therein, the Traditional approximation has to be carefully used since it modifies the mathematical properties of the adiabatic wave operator. Here, in the case of a general differential rotation law, it is applicable in spherical shell(s) such that $\mathcal{D} > 0$ everywhere ($\forall\, r$ and $\forall\, \theta \in [0,\pi]$). There, the adiabatic wave operator is elliptic that corresponds to regular (elliptic) gravito-inertial waves (see Dintrans & Rieutord 2000 for a detailed classification of such waves). In the other spherical shell(s), where both $\mathcal{D}<0$ and $\mathcal{D}>0$, the adiabatic wave operator is hyperbolic and the Traditional approximation can not be applied because of the adiabatic wave's velocity field (and wave operator) singularity where $\mathcal{D}=0$. Regularization is there allowed by thermal and viscous diffusions that lead to shear layers, the attractors, where strong dissipation occurs that can induce transport and mixing. In Fig. 2, we illustrate for a given chosen theoretical angular velocity profile (cf. Fig. 1) how those two types of spherical shells (respectively where the Traditional approximation is allowed or forbidden) could appear.

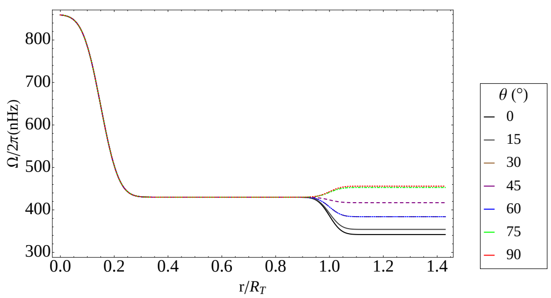

Figure 1: Synthetic internal rotation profile as it may be in the Sun (cf. García et al. 2007): $\Omega_{\rm syn}\left(r,\theta\right) = \Omega_s + \Omega_s A_c\left[1-{\rm Erf}\left((r-R_c)/l_c\right)\right] + 1/2\left[1+{\rm Erf}\left((r-R_T)/l_T\right)\right]\left(A+B\cos^2\theta+C\cos^4\theta-\Omega_s\right)$, where $\Omega_s = 430\,{\rm nHz}$, $A_c=1/2$ (such that $\Omega_{\rm syn}\left(0,\theta\right)=2\Omega_s$), $R_c=0.15R_T$, $l_c=0.075R_T$, $R_T=0.71R_\odot$ (the position of the Tachocline), $l_T=0.05R_T$, $A=456\,{\rm nHz}$, $B=-42\,{\rm nHz}$ and $C=-72\,{\rm nHz}$ (we assume here a Tachocline that is thicker than in reality).

Transport of angular momentum

Since the complete wave's velocity field is derived, we focus on the induced transport of angular momentum. The vertical and horizontal Lagrangian angular momentum fluxes are respectively defined:

$$\mathcal{F}_V^{\rm AM}\left(r,\theta\right) = \overline{\rho}r\sin\theta\int_\sigma\left<u_r u_\varphi+2\Omega\cos\theta u_r\xi_\theta\right>_\varphi{\rm d}\sigma$$

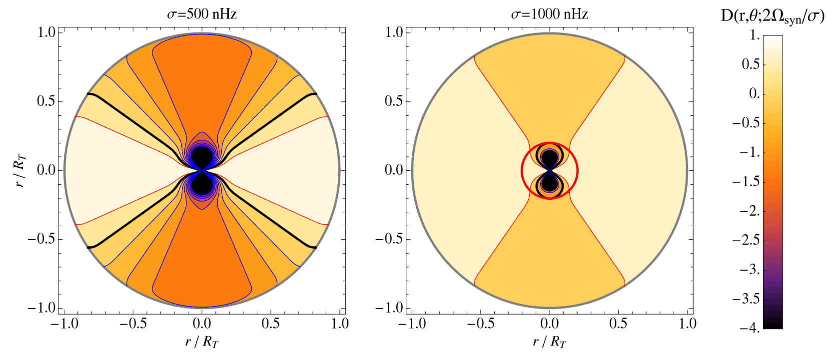

Figure 2: $\mathcal{D}\left(r,\theta;2\Omega_{\rm syn}/\sigma\right)$ as a function of r and θ (cf. Eq. 12) for $\sigma=500$ nHz $(\mathbf{Left})$ and $\sigma=1000$ nHz $(\mathbf{Right})$ for axisymmetric waves $(m=0)$. The critical surface $\mathcal{D}\left(r,\theta;2\Omega_{\rm syn}/\sigma\right)=0$ is given by the thick black line and the iso-$\mathcal{D}$ lines such that $\mathcal{D}\left(r,\theta;2\Omega_{\rm syn}/\sigma\right)>0$ and $\mathcal{D}\left(r,\theta;2\Omega_{\rm syn}/\sigma\right)<0$ are respectively given by the red and the blue lines. The Traditional Approximation (T. A.) applies in spherical shell(s) such that $\mathcal{D}>0$ everywhere ($\forall\, r$ and $\forall\,\theta\in[0,\pi]$); there, waves are regular at all latitudes. In other spherical shell(s), where both $\mathcal{D}>0$ and $\mathcal{D}<0$, the T. A. does not apply due to the singularity at $\mathcal{D}=0$. Therefore, for $\Omega_{\rm syn}$, the T. A. does not apply for $\sigma=500$ nHz while it applies for $\sigma=1000$ nHz in the external spherical shell with the inner border given by the thick red circle.

$$\text{and}\quad \mathcal{F}_H^{\rm AM}\left(r,\theta\right)\;=\;\overline{\rho}r\sin\theta\int_{\sigma}\left<u_\theta u_\varphi\right>_\varphi{\rm d}\sigma \tag{19}$$

where $\left<\cdots\right>_\varphi=\left(1/2\pi\right)\int_0^{2\pi}\cdots{\rm d}\varphi$, where the lagragian wave displacement is defined such that: $\vec{u}=\left(\partial_t+\Omega\partial_\varphi\right)\vec{\xi}-r\sin\theta\left(\vec{\xi}\cdot\vec{\nabla}\Omega\right)\widehat{\mathbf{e}}_\varphi$ and where we sum over the excited spectrum. Using Eqs. (7-8), we get $\mathcal{F}_H^{\rm AM}=0$. Then, following the methodology given in Zahn et al. (1997), Pantillon et al. (2007) and Mathis et al. (2008), we get the vertical action of angular momentum which is conserved in the adiabatic limit

$$\begin{aligned}\mathcal{L}_V^{\rm AM}\left(r,x;\widehat{\nu}\,\right)&=r^2\mathcal{F}_V^{\rm AM}\\&=-r_c^2\int_\sigma\sum_{m,j}\left\{\frac{\widehat{m}_{j,m}\left(r_c,x;\widehat{\nu}_c\right)}{\widehat{\sigma}_{\rm CZ}}\mathcal{F}^{\rm E}_{V;j,m}\left(r_c,x;\widehat{\nu}_c\right)D^2_{j,m}\right\}{\rm d}\sigma.\end{aligned}\tag{20}$$

r_c is the radius of the basis (or the top) of the adjacent convective region that excites the waves while $\widehat{\nu}_c=2\Omega_{\rm CZ}\left(r_c,\theta\right)/\widehat{\sigma}_{\rm CZ}$ where $\widehat{\sigma}_{\rm CZ}=\sigma+m\Omega_{\rm CZ}\left(r_c,\theta\right)$, $\Omega_{\rm CZ}$ being its angular velocity. On the other hand, $\mathcal{F}^{\rm E}_{V;j,m}\left(r_c,x;\widehat{\nu}_c\right)$ is the monochromatic energy flux injected by turbulent convective movements at $r = r_c$ in the studied radiation zone and

$$\widehat{m}_{j,m}\left(r,x;\widehat{\nu}\right)=\frac{\sin\theta\,\widehat{\sigma}^2\,w_{j,m}\left[\mathcal{G}^{\varphi}_{j,m}-\widehat{\nu}\cos\theta\,\mathcal{G}^{\theta}_{j,m}\right]}{w^2_{j,m}} \tag{21}$$

is the 2D function which describes its conversion into angular momentum flux.

Following Mathis & Zahn (2005), averaging over latitudes Ω and $\mathcal{L}_V^{\rm AM}$ in spherical shell(s) where the Traditional approximation applies and expanding this former as $\mathcal{L}_V^{\rm AM}=\sum_l \mathcal{L}_{V;l}^{\rm AM}\left(r\right)\sin^2\theta P_l\left(\cos\theta\right)$, we get for the mean rotation rate on an isobar ($\left<\Omega\right>_\theta$)

$$\overline{\rho}\frac{\rm d}{{\rm d}t}\left(r^2\left<\Omega\right>_\theta\right)-\frac{1}{5r^2}\partial_r\left(\overline{\rho}r^4\left<\Omega\right>_\theta U_2\right)=\frac{1}{r^2}\partial_r\left(\overline{\rho}\nu_V r^4\partial_r\left<\Omega\right>_\theta\right)-\frac{1}{r^2}\partial_r\left[\left<\mathcal{L}_V^{\rm AM}\right>_\theta\right], \tag{22}$$

and for the first mode of the latitudinal rotation

$$\begin{aligned}&\overline{\rho}\frac{\rm d}{{\rm d}t}\left(r^2\Omega_2\right)-2\overline{\rho}\left<\Omega\right>_\theta\left[2V_2-\frac{1}{2}\frac{{\rm d}\ln\left(r^2\left<\Omega\right>_\theta\right)}{{\rm d}\ln r}U_2\right]\\ &=\frac{1}{r^2}\partial_r\left(\overline{\rho}\nu_V r^4\partial_r\Omega_2\right)-10\,\overline{\rho}\nu_H\Omega_2-\frac{1}{r^2}\partial_r\left[\mathcal{L}^{\rm AM}_{V;2}\left(r\right)\right],\end{aligned} \tag{23}$$

where $\widetilde{\Omega}_2\left(r,\theta\right)=\Omega_2\left(r\right)\left[P_2\left(\cos\theta\right)+1/5\right]$ and $\Omega=\left<\Omega\right>_\theta+\widetilde{\Omega}_2$.

The meridional circulation is expanded in Legendre polynomials as $\vec{\mathcal{U}}_{\rm M}\left(r,\theta\right)=\sum_{l>0}\left\{U_l\left(r\right)P_l\left(\cos\theta\right)\widehat{\bf e}_r+V_l\left(r\right)\partial_\theta P_l\left(\cos\theta\right)\widehat{\bf e}_\theta\right\}$ while $\left(\nu_V,\nu_H\right)$ are respectively the vertical and the horizontal turbulent viscosities and ${\rm d}/{\rm d}t$ is the Lagrangian derivative that accounts for the contractions and the dilatations of the star during its evolution.

Those equations give the evolution of the differential rotation, both in the radial and in the latitudinal directions, in the spherical shell(s) where the Traditional approximation can be applied. This is the first time that an evolution equation for differential rotation (both in r and θ) capturing gravito-inertial waves feedback is derived, with taking into account the modification of IGWs through the Coriolis acceleration and theirs feed-back on the angular velocity profile through the net induced transport of angular momentum due to the differential damping of retrograde and prograde waves.

Conclusion

In this work, a complete formalism to treat the dynamics of regular (elliptic) low-frequency gravito-inertial waves in stably strongly stratified differentially rotating stellar radiation zones from Tachocline(s) where they are excited to their bulk as well as theirs feed-back on the angular velocity profile through the induced angular momentum transport has been derived. Future works must be devoted to its implementation in existing dynamical stellar evolution codes and to its application to different type of stars and evolution stages. This effort will lead to the building of more and more realistic stellar models which will benefit from new constraints provided by the development of asteroseismology both on the ground and in space.

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