COMMISSARIAT À L'ÉNERGIE ATOMIQUE

DSM - DAPNIA

DIRECTION DES SCIENCES DE LA MATIÈRE

DEPARTEMENT D'ASTROPHYSIQUE, DE PHYSIQUE DES PARTICULES,
DE PHYSIQUE NUCLÉAIRE ET DE L'INSTRUMENTATION ASSOCIÉE

SERVICE D'ÉTUDE DES ACCÉLÉRATEURS

Nicolas PICHOFF

NOTE	
DATE : 11/09/98 N/RÉF. : DAPNIA/SEA 98/46	OBJET : Intrabeam scattering on Halo formation V/RÉF. :
DE : Nicolas PICHOFF	A : Jean-Michel LAGNIEL

INTRABEAM SCATTERING ON HALO FORMATION

Nicolas PICHOFF
Commissariat à l'Energie Atomique - DSM/DAPNIA/SEA
91191 Gif-sur-Yvette cedex. FRANCE.

Summary

The effect of the coulombian collision between particles of a beam (intra-beam scattering) is investigated. Starting from the basic two-bodies cross section formula in the centre of mass referential, the maximum energy gain that can be catched by a particle along one direction is calculated as a function of the equipartition factor. Then, assuming that particle trajectories are ellipses in (x, x') phase-space (linear force ~ no space charge forces), the intra-beam scattering halo magnitude is calculated and shown to be very small. These calculations are done with different beam distribution functions and equipartition factor. The effect of space-charge is then investigated.

Contents

Summary 2

Contents 3

1. Assumptions and definitions 4

2. Maximum x-slope reachable in one collision by a particle 5
- 2.1 Coulombian collision in the centre-of-mass referential 5
- 2.2 Calculus of x'_{max} 6

3. Halo calculation 7
- 3.1 Probability with θ 7
- 3.2 Projection along x' 8
- 3.3 Probability with ε 9

4. Calculation with a "Water-bag" beam 10
- 4.1 Distribution function 10
- 4.2 Optimisation for numeric calculations 10
- 4.3 Results 12

5. Calculation with other beam distributions 14

6. Discussion about influence of space-charge 16

7. Multiple scattering 18

Conclusion 19

References 20

ANNEXES 21

A1. Coulombian differential cross section calculation 21

A2. Integration over ellipsoids 21
- A2.1 Calculus of the number of Particles 21
- A2.2 Beam density in (x, x') 22
- A2.3 Optimisation of the integration over (y, z) 23

1. Assumptions and definitions

The function f(x,y,z,x',y',z') is the distribution function of a beam in 6D phase-space. The projections of this function in 2D phase or real spaces are assumed to be elliptical.

The variables x_0, y_0, z_0, x'_0, y'_0 and z'_0 are the maximum values of respectively x, y, z, x', y' and z' which can be taken by a beam-particle. We have :

$$\left(\frac{x}{x_0}\right)^2+\left(\frac{x'}{x'_0}\right)^2+\left(\frac{y}{y_0}\right)^2+\left(\frac{y'}{y'_0}\right)^2+\left(\frac{z}{z_0}\right)^2+\left(\frac{z'}{z'_0}\right)^2 \le 1 . \quad \textbf{(1)}$$

We will assume that :

$x_0 = y_0 = r_0$,

$x'_0 = y'_0 = r'_0$, (same transverse size and temperature),

$z'_0 \ge r'_0$ (= if beam is equipartitionned).

We define the equipartition factor :

$$\chi = \frac{z'_0}{r'_0} , \quad \textbf{(2)}$$

which equal 1 in a equipartitionned beam.

For each particle, we define its "emittance" ε_x :

$$\varepsilon_x = \left(\frac{x}{x_0}\right)^2+\left(\frac{x'}{x'_0}\right)^2 , \quad \textbf{(3)}$$

in the beam, $\varepsilon_x \le 1$.

We will assume that ε_x is constant of motion in the particle transport. This is the case in a linear confinement force (with no space-charge force). We will discuss later about this assumption in presence of high space-charge forces (cf. **§6**).

We will calculate the probability, per unit time, for a particle to scatter to ε_x such as $\boldsymbol{\varepsilon_x \in [\varepsilon, \varepsilon+\Delta\varepsilon]}$ (**Figure 1**).

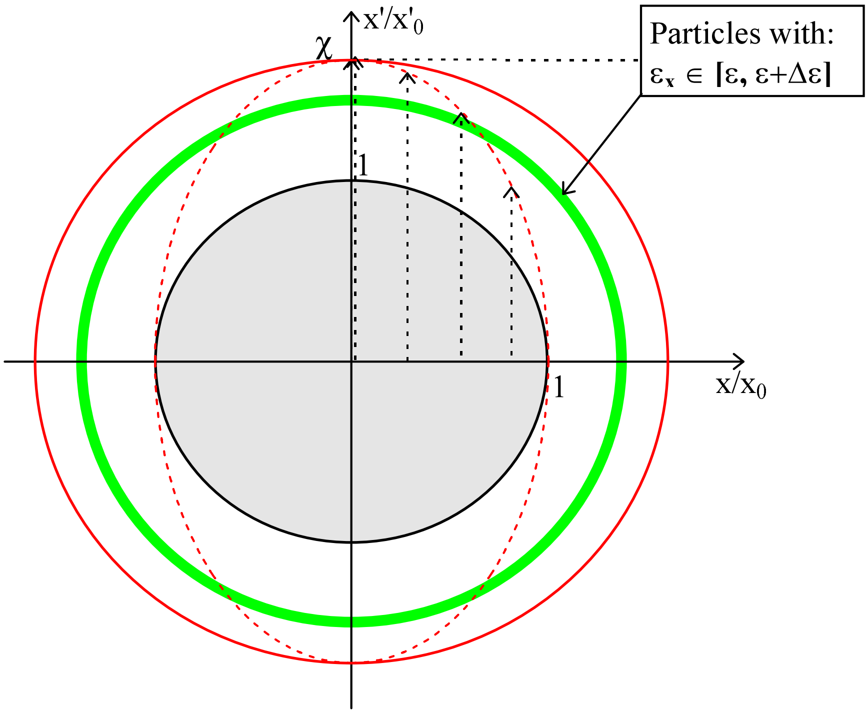

Figure 1 : Particle scattering in the (x,x') phase-space

2. Maximum x-slope reachable in one collision by a particle

Due to the energy conservation in the collision process, the maximum slope in the x direction (x-slope) that a particle can reach in one collision is limited. In this paragraph, the coulombian collision process is examined in order to calculate this maximum x-slope.

2.1 Coulombian collision in the centre-of-mass referential

Two particles with co-ordinates (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2) in the velocity-space, in the beam centre-of-mass referential $\Re_b$, interact. The referential $\Re_c$ is the two-particles centre-of-mass referential. In $\Re_c$, the 2 particles have co-ordinates (X'_1, Y'_1, Z'_1) and ($-X'_1$, $-Y'_1$, $-Z'_1$) defined as :

$$\begin{cases} X'_1 = x'_1 - \dfrac{x'_2 + x'_1}{2} = x'_1 - \overline{x}' \\ Y'_1 = y'_1 - \dfrac{y'_2 + y'_1}{2} = y'_1 - \overline{y}' \\ Z'_1 = z'_1 - \dfrac{z'_2 + z'_1}{2} = z'_1 - \overline{z}' \end{cases}, \qquad \textbf{(4)}$$

The collision happens along the direction pointed by the vector $\vec{u}$:

$$\vec{u} = \overline{x}' \cdot \vec{k}_x + \overline{y}' \cdot \vec{k}_y + \overline{z}' \cdot \vec{k}_z , \qquad \textbf{(5)}$$

$\vec{k}_x$, $\vec{k}_y$ and $\vec{k}_z$ being unit vectors in the x, y and z direction.

As during the collision, the total energy ($\propto X'^2+Y'^2+Z'^2$) is kept constant and the two particle co-ordinates are opposite, both particles are moving in $\Re_c$, during the collision, on a sphere centred on $\Re_c$ of radius $R' = \sqrt{X_1'^2 + Y_1'^2 + Z_1'^2}$. In $\Re_b$, they move on a sphere of same radius but of centre $\left(\overline{x'}, \overline{y'}, \overline{z'}\right)$. On **Figure 2**, is represented in $\Re_b$ the scattering of 2 particles with $y'_1 = y'_2 = 0$. θ is the scattering angle, ϕ is the asymuthal diffusion angle.

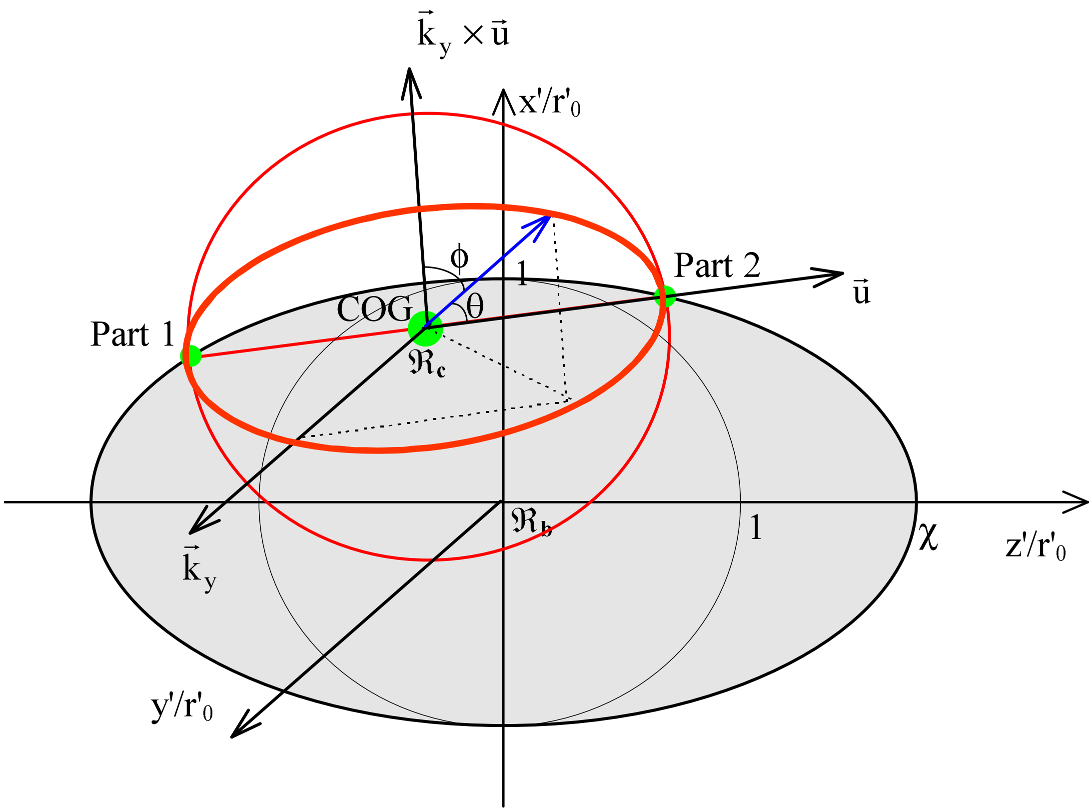

Figure 2 : Particle diffusion in the (x',y',z') phase-space

2.2 Calculus of x'_{max}

It can be easily demonstrated that the maximum x-slope x'_{max} that can be reached by a particle through a coulombian collision with an other beam-particle arises when :

$y'_1 = y'_2 = 0$, (not necessary if $\chi = 1$)

$$x_1' = x_2' = \frac{1}{\sqrt{1+\chi^2}} \cdot x_0',$$

$$z_1' = -z_2' = \frac{\chi^2}{\sqrt{1+\chi^2}} \cdot x_0',$$

$\theta = 90°$.

Then, we have :

$$\boxed{x'_{max} = \sqrt{1+\chi^2} \cdot x'_0}. \qquad \textbf{(6)}$$

An equipartionned beam will have a smaller intrabeam-scattering halo along the x direction than the one with a partition factor $\chi > 1$[1]. But in the last case, it is less dangerous in term of longitudinal halo. Now, let's calculate the probabilities of halo formation that way.

3. Halo calculation

In this paragraph, we will calculate the probability, per unit time, for a beam-particle to reach an emittance between ε and ε+Δε via one coulombian collision with an other particle of the beam. This is equivalent to the collision of 2 beams (which are in fact the same), with a cross-section depending on the relative velocity of the particles.

3.1 Probability with θ

The cross section, differential to the scattering angle θ, of a colombian collision in $\Re_c$ is (cf. **Annexe 1**) :

$$\boxed{\frac{d\sigma}{d\theta}(\theta, V) = -\frac{2\pi \cdot r_0^2}{(2V \cdot \beta)^4} \cdot \frac{\cos \theta/2}{\sin^3 \theta/2}}, \qquad \textbf{(7)}$$

$r_0 = \dfrac{q^2}{4\pi\varepsilon_0 mc^2}$ is the classical radius of proton,

ε_0 is the vacuum permitivity,

q and m are respectively the charge and the mass of particle,

βc is the beam velocity in the lab referential,

V is the half relative slope of the 2 particles (slope of one particle in $\Re_c$).

The probability for one particle at (x,y,z,x'_1,y'_1,z'_1) to scatter:

- on a particle with slopes between x'_2 and $x'_2+dx'_2$, y'_2 and $y'_2+dy'_2$ and z'_2 and $z'_2+dz'_2$,
- to an angle between θ and θ+dθ,
- per unit time, is[2] :

$$\boxed{\frac{dP_\theta}{d\theta} \cdot d\theta = \frac{d\sigma}{d\theta}(\theta, V) \cdot v_1 \cdot \beta c \cdot f(x, y, z, x'_2, y'_2, z'_2) \cdot dx'_2 \cdot dy'_2 \cdot dz'_2 \cdot d\theta}, \qquad \textbf{(8)}$$

with $V = \sqrt{X_1'^2 + Y_1'^2 + Z_1'^2}$,

and $v_1 = \sqrt{x_1'^2 + y_1'^2 + z_1'^2}$.

---

[1] If $\chi<1$, we should use $\chi=1$ in **(6)** as the maximum slope comes from an collision along y axis.

[2] $\dfrac{dP_\theta}{d\theta} \cdot d\theta$ depends on $x,y,z,x'_1,y'_1,z'_1,x'_2,y'_2,z'_2,dx'_2,dy'_2,dz'_2,\theta$ and dθ. It is in s^{-1}.

3.2 Projection along x'

Given $W'_1 = \sqrt{Y_1'^2 + Z_1'^2}$, the projected velocity of the scattering particle in the plan transverse to X (in $\mathcal{R}_c$), we define θ_0 as :

$$\tan\theta_0 = \frac{X'_1}{W'_1}. \qquad \textbf{(9)}$$

A new vector base can be used to describe the collision :

$$\begin{cases} \vec{S} = \vec{k}_x \\ \vec{W} = \dfrac{Y'_1}{W'_1}\cdot\vec{k}_y + \dfrac{Z'_1}{W'_1}\cdot\vec{k}_z \\ \vec{T} = \vec{k}_x \times \vec{W} = \dfrac{Y'_1}{W'_1}\cdot\vec{k}_z - \dfrac{Z'_1}{W'_1}\cdot\vec{k}_y \end{cases} . \qquad \textbf{(10)}$$

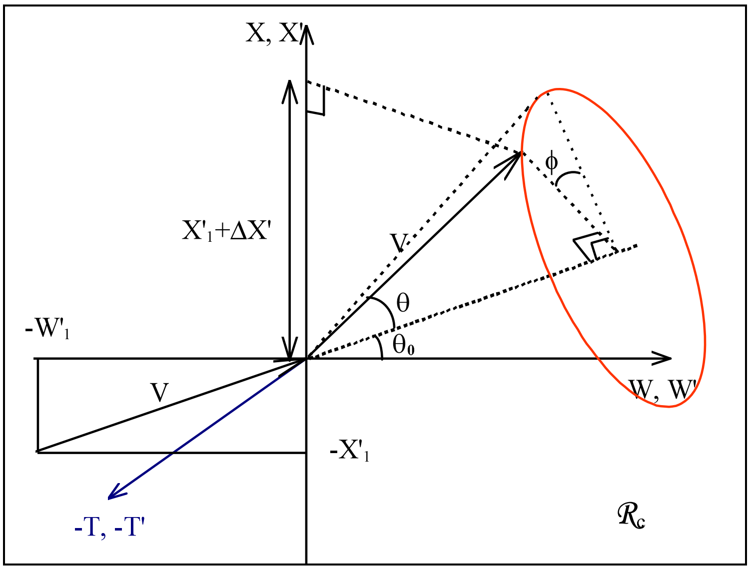

Figure 3 : Scattering of a particle in the (X, W, T) space.

In this base, it can be demonstrated the relationship :

$$\Delta X' = X'_1\cdot(\cos\theta - 1) + W'_1\cdot\sin\theta\cdot\cos\phi\ . \qquad \textbf{(11)}$$

Let's determine the probability, per unit time, for one particle at position (x,y,z,x'_1,y'_1,z'_1) in the phase-space to scatter on a particle with slopes between x'_2 and $x'_2+dx'_2$, y'_2 and $y'_2+dy'_2$ and z'_2 and $z'_2+dz'_2$, with an x-angle ΔX' between ΔX' and ΔX'+dΔX'. It is equal to the sum over θ on [0, 2π] of the probability, per unit time, to scatter between θ and θ+dθ (given by **(8)**) multiplied by the probability to scatter with an asymuthal angle between ϕ and $\phi+d\phi$ (which is equal to $d\phi/2\pi$) such that :

$$\Delta X' = X'_1 \cdot (\cos\theta - 1) + W'_1 \cdot \sin\theta \cdot \cos\phi \quad \text{and} \quad \frac{d\Delta X'}{d\phi} = -W'_1 \cdot \sin\theta \cdot \sin\phi \,.$$

This gives :

$$\frac{dP_{\Delta X'}}{d\Delta X'} \cdot d\Delta X' = \int \frac{dP_\theta}{d\theta} \cdot \frac{d\phi}{2\pi} \cdot d\theta \,.$$

With :

$$d\phi = \frac{d\Delta X'}{-W'_1 \cdot \sin\theta \cdot \sin\phi} \,,$$

and

$$\phi = \mathrm{acos}\left(\frac{\Delta X' - X'_1 \cdot (\cos\theta - 1)}{W'_1 \cdot \sin\theta} \right),$$

we finally get[3] :

$$\boxed{\frac{dP_{\Delta X'}}{d\Delta X'} \cdot d\Delta X' = \frac{-d\Delta X'}{2\pi W'_1} \cdot \int_{\theta_{min}}^{\theta_{max}} \frac{dP_\theta}{d\theta} \cdot \frac{1}{\sin\theta \cdot \sin\left(\mathrm{acos}\left(\frac{\Delta X' - X'_1 \cdot (\cos\theta - 1)}{W'_1 \cdot \sin\theta} \right) \right)} \cdot d\theta}, \quad \textbf{(12)}$$

with :

$$\theta_{min} = \mathrm{asin}\left(\frac{X'_1 + \Delta X'}{V} \right) - \theta_0 \quad \text{and} \quad \theta_{max} = \pi - 2 \cdot \theta_0 - \theta_{min} \,. \quad \textbf{(13)}$$

3.3 Probability with ε

The probability for one particle at (x,y,z,x'_1,y'_1,z'_1) to scatter:

- on a particle with slopes between x'_2 and $x'_2+dx'_2$, y'_2 and $y'_2+dy'_2$ and z'_2 and $z'_2+dz'_2$,
- to an emittance between ε and ε+dε,
- per unit time, is[4] :

$$\boxed{\frac{dP_\varepsilon}{d\varepsilon} \cdot d\varepsilon = \frac{dP_{\Delta X'}}{d\Delta X'}\left(\Delta X' = x'_0 \sqrt{\varepsilon - (x/x_0)^2} - x'_1 \right) \cdot \frac{x_0'^2}{2 \cdot (x'_1 + \Delta X')} \cdot d\varepsilon}, \quad \textbf{(14)}$$

$\frac{dP_{\Delta X'}}{d\Delta X'} \cdot d\Delta X'$ is given by equation **(12)**.

---

[3] $\frac{dP_{\Delta X'}}{d\Delta X'} \cdot d\Delta X'$ depends on $x,y,z,x'_1,y'_1,z'_1,x'_2,y'_2,z'_2,dx'_2,dy'_2,dz'_2,\Delta X'$ and $d\Delta X'$. It is in s^{-1}.

[4] $\frac{dP_\varepsilon}{d\varepsilon} \cdot d\varepsilon$ depends on $x,y,z,x'_1,y'_1,z'_1,x'_2,y'_2,z'_2,dx'_2,dy'_2,dz'_2,\varepsilon$ and $d\varepsilon$. It is in s^{-1}.

The probability, per unit time, for one particle at (x,y,z,x'_1,y'_1,z'_1) to scatter to an emittance between ε and ε+dε is given by the integration of **(14)** over x'_2, y'_2 and z'_2[5] :

$$\boxed{\frac{dP'_\varepsilon}{d\varepsilon}\cdot d\varepsilon = \iiint\limits_{x'_2\, y'_2\, z'_2} \frac{dP_\varepsilon}{d\varepsilon}\cdot d\varepsilon}. \qquad \textbf{(15)}$$

The number of beam particle scattering, per unit time, to an emittance between ε and ε+dε is finally[6] :

$$\boxed{\frac{dN_\varepsilon}{d\varepsilon}\cdot d\varepsilon = \iiiint\!\!\iint\limits_{x\, y\, z\, x'_1\, y'_1\, z'_1}\left(\frac{dP'_\varepsilon}{d\varepsilon}\cdot d\varepsilon\right)\cdot f(x,y,z,x'_1,y'_1,z'_1)\cdot dz'_1\cdot dy'_1\cdot dx'_1\cdot dz\cdot dy\cdot dx}. \qquad \textbf{(16)}$$

4. Calculation with a "Water-bag" beam

4.1 Distribution function

We consider a beam with a distribution function which is constant (f_0) in the hyper-ellipsoid of radii x_0, y_0, z_0, x'_0, y'_0 and z'_0, and which is null outside.

$$f(x,y,z,x',y',z') = \begin{cases} f_0 & \text{if } \left(\frac{x}{x_0}\right)^2 + \left(\frac{x'}{x'_0}\right)^2 + \left(\frac{y}{y_0}\right)^2 + \left(\frac{y'}{y'_0}\right)^2 + \left(\frac{z}{z_0}\right)^2 + \left(\frac{z'}{z'_0}\right)^2 \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(17)}$$

If N is the number of particles in the bunch, we have :

$$f_0 = \frac{6\cdot N}{\pi^3\cdot x_0\cdot y_0\cdot z_0\cdot x'_0\cdot y'_0\cdot z'_0}. \qquad \textbf{(18)}$$

4.2 Optimisation for numeric calculations

We have to solve a 10 dimensions integral, this means that if a 10 points Gaussian quadrature is used, this will require 10^{10} integration points !! An effort has to be done in order to reduce this number of integrals.

4.2.1 Suppression of integral over y and z

The expression in integrals depends on y and z only trough the distribution function. The integral order can be changed in order to put the integrals over y and z inside those over x and the slopes. In that case, we calculate the integration between to particles at positions

[5] $\frac{dP'_\varepsilon}{d\varepsilon}\cdot d\varepsilon$ depends on x,y,z,x'_1,y'_1,z'_1,ε and dε. It is in s^{-1}.

[6] $\frac{dN_\varepsilon}{d\varepsilon}\cdot d\varepsilon$ depends on ε and dε. It is in s^{-1}.

(x,x'_1,y'_1,z'_1) and (x,x'_2,y'_2,z'_2) over y and z. This is in fact the integration with variables y/y_0 and z/z_0 over a circle with square-radius:

$$r_{yz}^2 = \min\left(1-\left(\frac{x}{x_0}\right)^2-\left(\frac{x'_1}{x'_0}\right)^2-\left(\frac{y'_1}{y'_0}\right)^2-\left(\frac{z'_1}{z'_0}\right)^2, 1-\left(\frac{x}{x_0}\right)^2-\left(\frac{x'_2}{x'_0}\right)^2-\left(\frac{y'_2}{y'_0}\right)^2-\left(\frac{z'_2}{z'_0}\right)^2\right), \quad \textbf{(19)}$$

min(x,y) being the minimum value between x and y.

The two integrals over y and z can suppressed and replaced by :

$$\pi \cdot r_{yz}^2 \cdot y_0 \cdot z_0 . \quad \textbf{(20)}$$

This reduces the integral number to 8.

No other integral-number reduction can be done, except if the beam is equipartionned ($y'_0 = z'_0$). In that last case, the integral number can be reduced to 6.

4.2.2 Integration edges

The quantity to calculate is of the form :

$$\int_x \int_{x'_1} \int_{y'_1} \int_{z'_1} \int_{x'_2} \int_{y'_2} \int_{z'_2} \int_\theta g(x, x'_1, y'_1, z'_1, x'_2, y'_2, z'_2).$$

The integration edges for each integral over particle co-ordinates are the following:

$x \rightarrow \pm x_m = x_0,$

$x'_1 \rightarrow \pm x'_{1m} = x'_0\sqrt{1-\left(\frac{x}{x_0}\right)^2},$

$y'_1 \rightarrow \pm y'_{1m} = y'_0\sqrt{1-\left(\frac{x}{x_0}\right)^2-\left(\frac{x'_1}{x'_0}\right)^2},$

$z'_1 \rightarrow \pm z'_{1m} = z'_0\sqrt{1-\left(\frac{x}{x_0}\right)^2-\left(\frac{x'_1}{x'_0}\right)^2-\left(\frac{y'_1}{y'_0}\right)^2},$

$x'_2 \rightarrow \pm x'_{2m} = x'_0\sqrt{1-\left(\frac{x}{x_0}\right)^2},$

$y'_2 \rightarrow \pm y'_{2m} = y'_0\sqrt{1-\left(\frac{x}{x_0}\right)^2-\left(\frac{x'_2}{x'_0}\right)^2},$

$z'_2 \rightarrow \pm z'_{2m} = z'_0\sqrt{1-\left(\frac{x}{x_0}\right)^2-\left(\frac{x'_2}{x'_0}\right)^2-\left(\frac{y'_2}{y'_0}\right)^2}.$

In order to have integration over intervals [-1, 1] which can be calculated easily with Gaussian quadrature, we use the variable transformations :

$xx = x/x_m \quad \rightarrow \quad dx = x_m \cdot dxx,$

$xx'_1 = x'_1/x'_{1m} \quad \rightarrow \quad dx'_1 = x'_{1m} \cdot dxx'_1,$

$yy'_1 = y'_1/y'_{1m} \quad \rightarrow \quad dy'_1 = y'_{1m} \cdot dyy'_1,$

$zz'_1 = z'_1/z'_{1m} \quad \rightarrow \quad dz'_1 = z'_{1m} \cdot dzz'_1,$

$xx'_2 = x'_2/x'_{2m} \quad \rightarrow \quad dx'_2 = x'_{2m} \cdot dxx'_2,$

$yy'_2 = y'_2/y'_{2m} \quad \rightarrow \quad dy'_2 = y'_{2m} \cdot dyy'_2,$

$zz'_2 = z'_2/z'_{2m} \quad \rightarrow \quad dz'_2 = z'_{2m} \cdot dzz'_2,$

$$\theta\theta = \frac{\theta - \dfrac{\theta_{max} + \theta_{min}}{2}}{\dfrac{\theta_{max} - \theta_{min}}{2}} \quad \rightarrow \quad d\theta = [(\theta_{max} - \theta_{min})/2] \cdot d\theta\theta.$$

4.3 Results

All results presented here have been obtained from the numerical solution of **(16)** using the water-bag distribution function.

On **Figure 4**, is represented the beam density in the phase space (x/x_0, x'/x'_0) as a function of $\sqrt{\varepsilon_x} = \sqrt{\dfrac{x^2}{x_0^2} + \dfrac{x'^2}{x_0'^2}}$. The beam characteristics are approximately those of the APT beam at 6.7 MeV and 100 MeV. In red is represented the beam core density, and in other colours are the tails induced by the intrabeam scattering on 1 meter transport by the beam core. Different equipartition factor have been used. On **Figure 5**, is represented, in the same conditions, the beam density in the phase space (z/z_0, z'/z'_0) as a function of $\sqrt{\varepsilon_z} = \sqrt{\dfrac{z^2}{z_0^2} + \dfrac{z'^2}{z_0'^2}}$.

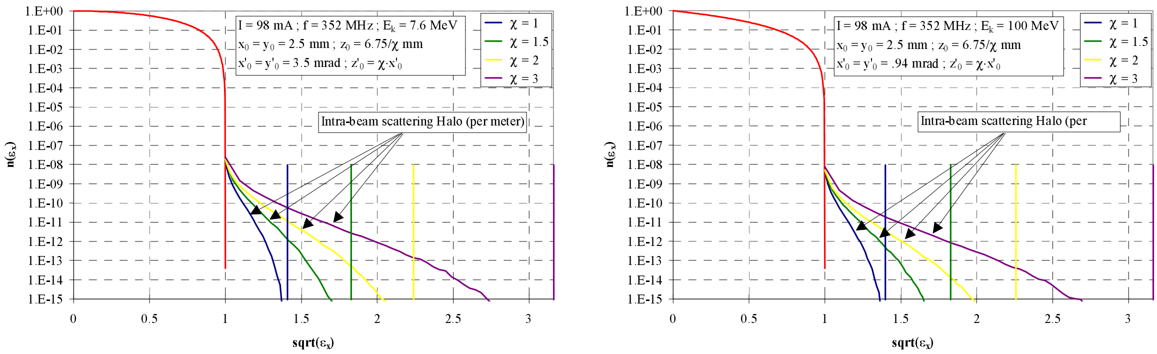

Figure 4 : Intra-beam scattering halo (per meter) of the typical APT beam at 6.7 MeV (a.) and 100 MeV (b.) in (x/x_0, x'/x'_0).

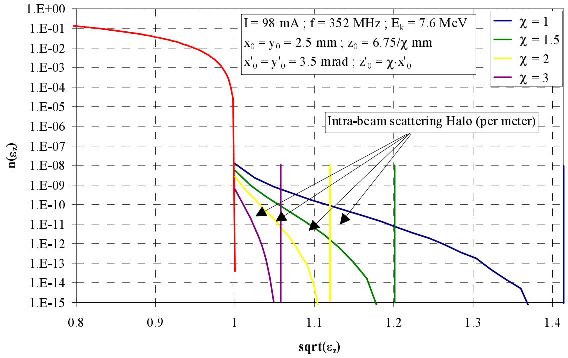

Figure 5 : Intra-beam scattering halo (per meter) of the typical APT beam at 6.7 MeV in (z/z_0, z'/z'_0).

Curves obtained at 100 MeV are approximately 3 times lower than those obtained at 7.6 MeV. This is the difference in velocity (from 6.7 to 100 MeV) of the beam particles. It can concluded that halo creation per unit time is nearly the same whatever the beam energy.

The transit-time of the beam in a 1.7 GeV linac (APT) with a mean energy-gain of 1MeV/m is ~8 µs. With the assumption of a halo formation per unit time being the same whatever the beam energy, the density in the (x, x') phase space is finally obtained and represented on **Figure 6**.

The projection of (x/x_0, x'/x'_0) phase-space along x' and the projection of (x/x_0, y/y_0) space along y are both the same. As beam distributions have a polar symmetry in (x/x_0, x'/x'_0) as well as in (x/x_0, y/y_0), and have the same projection, they have the same density. Then the density in (x/x_0, y/y_0) is the same as the density in (x/x_0, x'/x'_0): **Figure 6** is also represented the transverse density profile of the beam.

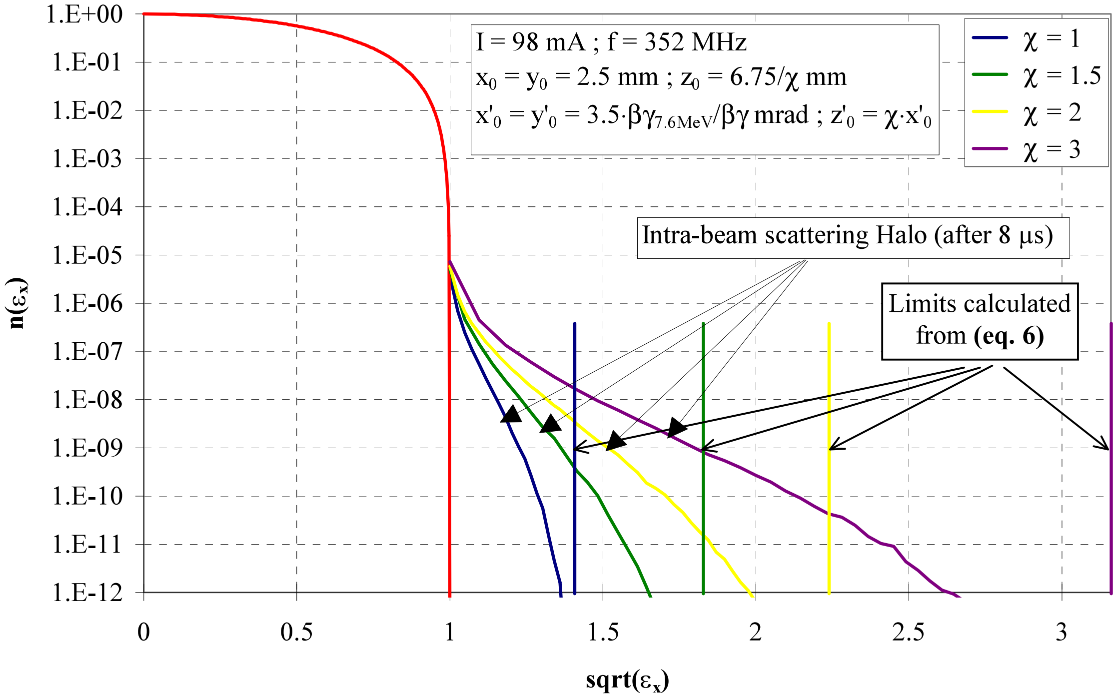

Figure 6 : Intrabeam scattering halo after 8 µs of the typical APT beam

5. Calculation with other beam distributions

We consider a beam with a distribution function which is :

$$f(x,y,z,x',y',z') = \begin{cases} f_0 \cdot \left(r^2 = \left(\frac{x}{x_0}\right)^2 + \left(\frac{x'}{x'_0}\right)^2 + \left(\frac{y}{y_0}\right)^2 + \left(\frac{y'}{y'_0}\right)^2 + \left(\frac{z}{z_0}\right)^2 + \left(\frac{z'}{z'_0}\right)^2 \right)^{\frac{n}{2}} & \text{if } r \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(21)

When n=0, the Water-bag distribution **(17)** is obtained, when $n \rightarrow \infty$, the distribution tends to a "3D K-V beam" (which has not the properties of an homogeneous beam) (**Figure 7**).

If N is the number of particles in the bunch, we have :

$$f_0 = \frac{(6+n) \cdot N}{\pi^3 \cdot x_0 \cdot y_0 \cdot z_0 \cdot x'_0 \cdot y'_0 \cdot z'_0}. \quad \textbf{(22)}$$

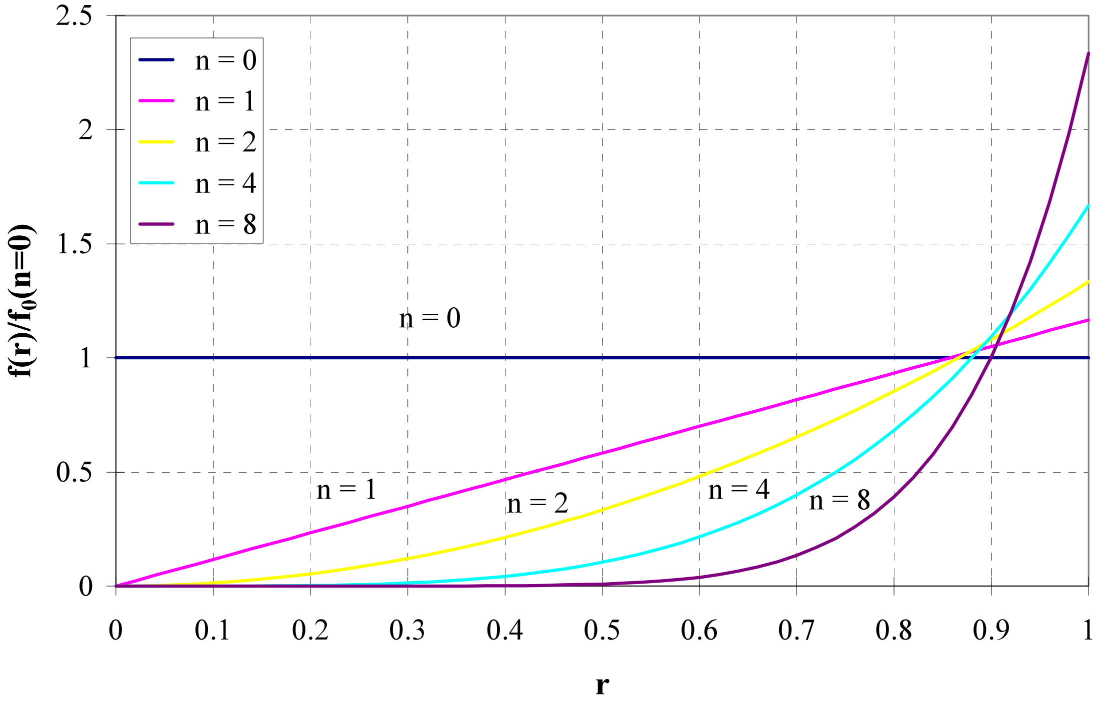

Figure 7 : Distribution functions from (21) for different n values
Normalised with $f_0(n=0)$ with the same number of particles.

As with the water-bag beam, the expression in integrals depends on y and z only trough the distribution function. The integral sequence can be changed in order to put the integrals over y and z inside those over x and the slopes. In that case, we calculate the integration between to particles at positions (x,x'_1,y'_1,z'_1) and (x,x'_2,y'_2,z'_2) over y and z. This is in fact the integration of the distribution function with variables y/y_0 and z/z_0 over a circle with square-radius (see **Annexe**):

$$r_{yz}^2 = \min\left(1-\left[\left(\frac{x}{x_0}\right)^2+\left(\frac{x'_{1,2}}{x'_0}\right)^2+\left(\frac{y'_{1,2}}{y'_0}\right)^2+\left(\frac{z'_{1,2}}{z'_0}\right)^2\right]^{\frac{n+2}{2}}\right), \qquad \textbf{(23)}$$

r_{yz} is the minimum between index 1 or 2.

Then the two integrals over y and z can suppressed and replaced by (**Annexe**) :

$$\frac{2\pi\cdot r_{yz}^2\cdot y_0\cdot z_0}{n+2}. \qquad \textbf{(24)}$$

Figure 8 represents the intrabeam scattering halo evolution for different values of n. Halo relative intensity increases with n, but the order of magnitude is not changing a lot.

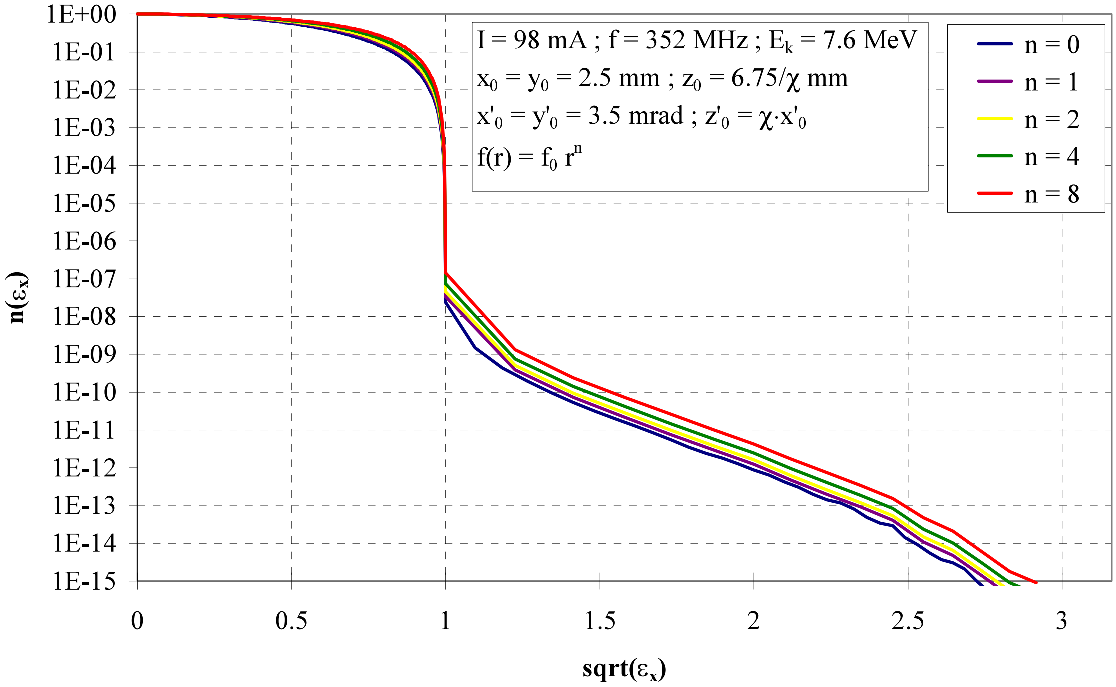

Figure 8 : Intra-beam scattering halo for different core distributions

6. Discussion about influence of space-charge

When the beam is space-charge driven, 2D phase-space beam distributions are no longer ellipses, but tend to rectangles. In that case, particle emittance as defined with **(eq. 3)** are not constant of motion but should be replaced by the hamiltonian which depends on the beam distribution trough the space-charge potential.

When the beam is space-charge dominated, it can be assumed that iso-hamiltonian curves in (x/x_0, x'/x'_0) are nearly rectangles [1] (as represented in **Figure 9**). In that case, in order to get the same amplitude, a particle needs to be scattered to an angle larger than the one that would have been necessary without space-charge force (ellipses as iso-hamiltonian).

This situation can be seen an other way: When there are space-charge forces, more external focusing forces are needed to transport the beam with the same size. In that case, the confinement potential well is deeper, and the kinetic energy (get by a collision) needed to get the same amplitude is higher (this is the same phenomenon than with the scattering on residual gas [2]). This will decrease the intra-beam scattering halo.

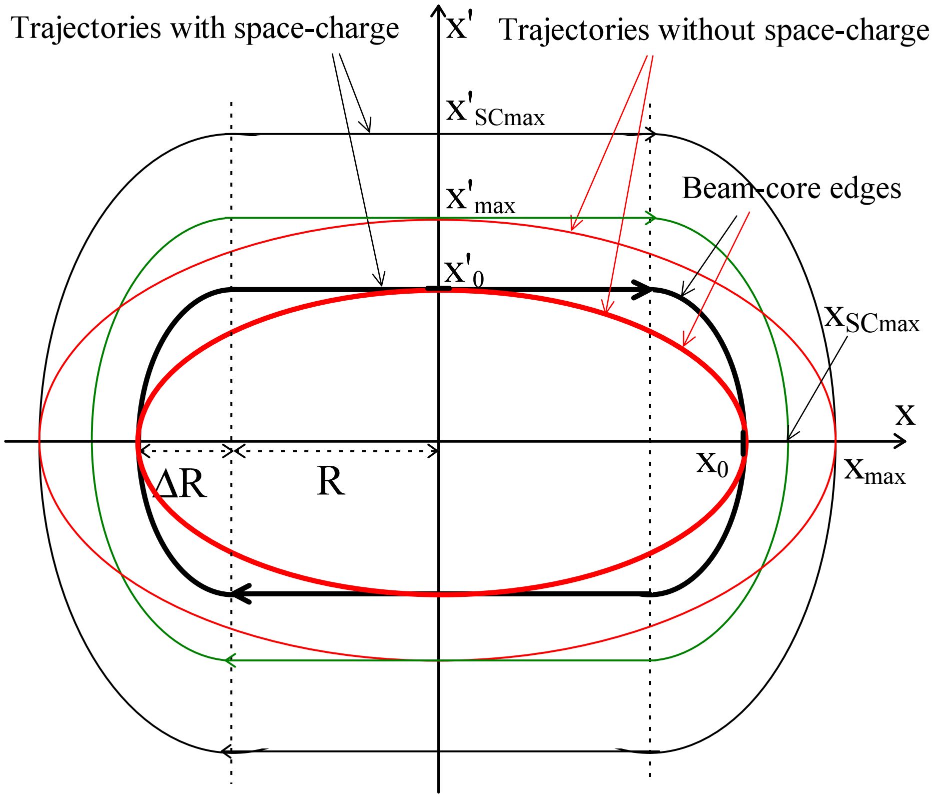

Figure 9 : Beam shape and particle trajectories around a beam with :
- No space-charge forces (ellipses, in red),
- Space charge forces (quasi-rectangles, in black and green).

The scattering angle necessary to get an amplitude x_{max} is larger with space-charge than without space charge.

If k_0 is the external focusing phase advance per meter, and η is the depress-tune factor of the beam, then :

$$x'_0 = \eta k_0 \cdot x_0,$$

R is the homogenous-part beam size, and ΔR is its border size, then :

$$x_0 = R + \Delta R,$$

If particles outside the beam feel only external forces, we can assumed that :

$$\Delta R = \frac{x'_0}{k_0}, \text{ which gives } R = x_0 - \frac{x'_0}{k_0} = x_0(1-\eta).$$

Two particles scattering to the same angle x'_{max}, will not reach the same amplitude depending on whether we take into account space-charge (x_{SCmax}) or not (x_{max}). We have :

$$\frac{x_{SC\,max}}{x_{max}} = \frac{R + x'_{max}/k_0}{x'_{max}/k_0} = \frac{R + \eta x_{max}}{x_{max}} = \eta + (1-\eta) \cdot \frac{x_0}{x_{max}}.$$

The space charge decrease the particle amplitude by a factor between 1 (close to the core when $x_{max} = x_0$) to η (far from the core when $x_{max}/x_0 \to \infty$).

The amplitude density is increased because scattering x-slope range for which particle have amplitude between x and x+dx is larger with space-charge than without space-charge. This amplification factor is : $\frac{d\,x_{SC\,max}}{d\,x_{max}} = \eta + (1-\eta)\cdot\frac{x_0}{x_{max}} < 1$.

Halo density calculated without space-charge is overestimated the one with space-charge effects. Curves like on **Figure 4** can be corrected (**Figure 10**) in order to take into account space-charge effects by making the transformations :

$$\begin{cases} n(x) = \dfrac{n(x)}{\left(\eta-(1-\eta)\cdot\dfrac{x_0}{x}\right)} \\ x = x\cdot\left(\eta-(1-\eta)\cdot\dfrac{x_0}{x}\right) \end{cases}.$$

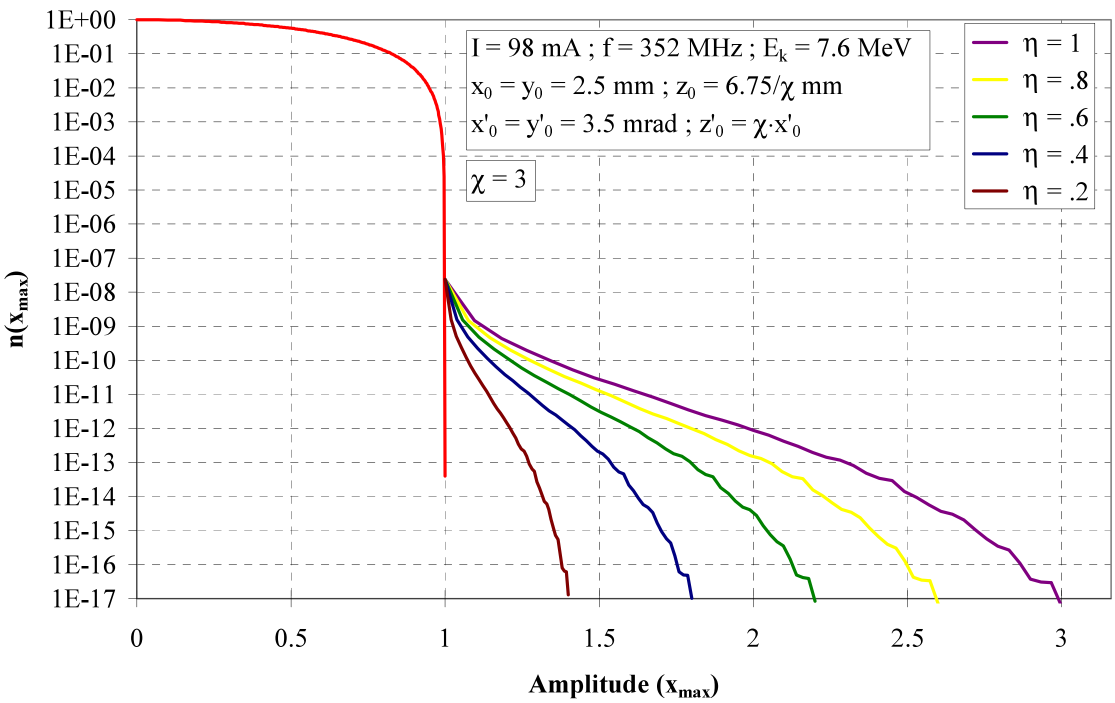

Figure 10 : Space-charge influence on the particle amplitude distribution.

7. Multiple scattering

Multiple scattering has to be taken into account for angles whose scattering probability[7] is not much smaller than one. This is the case only for very small angles which will not contribute to the far halo density. The multiple scattering will modify the profile obtained on **Figure 6** close to the core, as qualitatively represented on **Figure 11**. It will actually smooth

[7] Integrated over the total linac.

the final beam density profile as if it was folded with a narrow gaussian distribution. An approximate value of the width of the gaussian can be found from the calculation of the time needed to "thermalised" the beam, using the fokker-Planck equation. The ratio between the beam life time and the thermalisation time has been estimated to 2.3 10^{-4} in the case of TRISPAL linac [3]. This would give a ratio of ~10^{-3} for the APT linac. This means that the gaussian width should be of the order of magnitude of 10^{-3} the beam size.

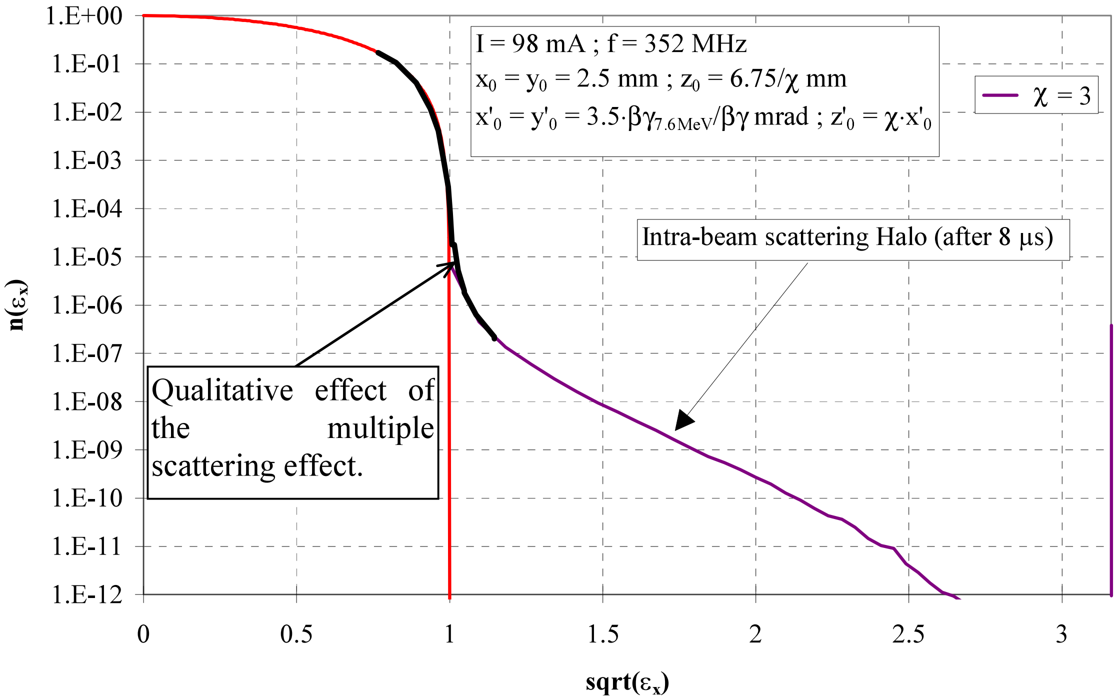

Figure 11 : Effect of the multiple scattering of the final profile

Conclusion

The influence of the intra-beam scattering on halo formation seems to be negligible as well in extension as in density. Moreover, the space-charge reduces it a lot. The equipartition conditions are not made necessary by the intrabeam scattering phenomenon. The largest emittance growth and halo formation of non-equipartionned beam observed in simulations can not be justified by intrabeam scattering. Are other physical effects (coupling resonance ?...) or spurious space-charge model effects [4] explaining these observations ?

References

[1] N. Pichoff, *Stationary distribution of space-charge driven continuous beam*, Note DAPNIA/SEA 98/43.

[2] N. Pichoff et al., *Halo from Coulomb Scattering of Beam Particles on Residual Gas*, Note DAPNIA/SEA 98/45.

[3] N. Pichoff, *Etude Théorique et Expérimentale du Halo d'un Faisceau Intense de particules chargées dans un accélérateur*, Ph.D. report, December 1997.

[4] N. Pichoff et al., *Simulation Results with an Alternate 3D Space Charge Routine, PICNIC*, Proc. of Linac98, Chicago, August 23-28, 1998.

ANNEXES

A1. Coulombian differential cross section calculation

The relation given the coulombian scattering angle θ, of two particles of charge q, of mass m, with a relative velocity v, as a function of the impact parameter b is :

$$\cot an\left(\frac{\theta}{2}\right)=\frac{2\pi\varepsilon_0\, b\, m v^2}{q^2}\,, \qquad \textbf{(A1)}$$

The cross section σ of a collision with an angle greater than θ is :

$$\sigma=\pi b^2=\pi\left(\frac{q^2\cot an\left(\frac{\theta}{2}\right)}{2\pi\varepsilon_0\, m v^2}\right)^2 . \qquad \textbf{(A2)}$$

The differential cross section is then :

$$\frac{d\sigma}{d\theta}=\pi\left(\frac{q^2}{2\pi\varepsilon_0\, m v^2}\right)^2\cdot\frac{\cos\left(\frac{\theta}{2}\right)}{\sin^3\left(\frac{\theta}{2}\right)},$$

which gives finally gives equation **(7)**.

A2. Integration over ellipsoids

A2.1 Calculus of the number of Particles

$$N=\int_{-x_0}^{x_0}dx\int_{-y_0\sqrt{1-X^2}}^{y_0\sqrt{1-X^2}}dy\int_{-z_0\sqrt{1-X^2-Y^2}}^{z_0\sqrt{1-X^2-Y^2}}dz\int_{-x_0'\sqrt{1-X^2-Y^2-Z^2}}^{x_0'\sqrt{1-X^2-Y^2-Z^2}}dx'\int_{-y_0'\sqrt{1-X^2-Y^2-Z^2-X'^2}}^{y_0'\sqrt{1-X^2-Y^2-Z^2-X'^2}}dy'\int_{-z_0'\sqrt{1-X^2-Y^2-Z^2-X'^2-Y'^2}}^{z_0'\sqrt{1-X^2-Y^2-Z^2-X'^2-Y'^2}}f(x,y,z,x',y',z')\,dz'$$

with : $X=\dfrac{x}{x_0}$, $Y=\dfrac{y}{y_0}$, $Z=\dfrac{z}{z_0}$, $X'=\dfrac{x'}{x_0'}$, $Y'=\dfrac{y'}{y_0'}$ and $Z'=\dfrac{z'}{z_0'}$.

* If $f=f_0$, and using : $\displaystyle\int_{-a}^{a}\sqrt{a^2-w^2}\,dw=\frac{\pi}{2}\cdot a^2$

(1)	$\Rightarrow$	$2\cdot z_0'\cdot\sqrt{1-X^2-Y^2-Z^2-X'^2-Y'^2}$,
(2)	$\Rightarrow$	$f_0\cdot y_0'z_0'\cdot\pi\cdot\sqrt{1-X^2-Y^2-Z^2-X'^2}$,
(3)	$\Rightarrow$	$f_0\cdot x_0'y_0'z_0'\cdot\frac{4\pi}{3}\cdot\sqrt{1-X^2-Y^2-Z^2}$,
(4)	$\Rightarrow$	$f_0\cdot z_0x_0'y_0'z_0'\cdot\frac{\pi^2}{2}\cdot\sqrt{1-X^2-Y^2}$,
(5)	$\Rightarrow$	$f_0\cdot y_0z_0x_0'y_0'z_0'\cdot\frac{8\pi^2}{15}\cdot\sqrt{1-X^2}$,
N	$\Rightarrow$	$f_0\cdot x_0y_0z_0x_0'y_0'z_0'\cdot\frac{\pi^3}{6}$.

$$\boxed{N=f_0\cdot x_0y_0z_0x_0'y_0'z_0'\cdot\frac{\pi^3}{6}}, \qquad \textbf{(A3)}$$

* If $f=f\left(r=\sqrt{X^2+Y^2+Z^2+X'^2+Y'^2+Z'^2}\right)$,

As we have a m-dimension integration (m between 1 to 6) of a spherical function, it can be writen :

$$\underbrace{\int_{-R}^{R}\ldots\int}_{m} f\underbrace{(x,\ldots)}_{m}\cdot\underbrace{dx\ldots}_{m}=C_m\cdot\int_0^R r^{m-1}f(r)\cdot dr\,, \qquad \textbf{(A4)}$$

C_m being the result of integration over all angles.

We find with $f(r)=f_0$:

$C_1=2$, $C_2=2\pi$, $C_3=4\pi$, $C_4=2\pi^2$, $C_5=\frac{8\pi}{3}$, $C_6=\pi^3$.

When $f(r)=f_0\,r^n$, we have :

$$\boxed{N=f_0\cdot x_0y_0z_0x_0'y_0'z_0'\cdot\frac{\pi^3}{6+n}}$$

A2.2 Beam density in (x, x')

The beam density in the (x, x') phase space can be obtained by the integration of the density function over y, z, y' and z'.

$$n(x,x')=\int_{-y_0\sqrt{1-X^2-X'^2}}^{y_0\sqrt{1-X^2-X'^2}}dy\int_{-z_0\sqrt{1-X^2-X'^2-Y^2}}^{z_0\sqrt{1-X^2-X'^2-Y^2}}dz\int_{-y_0'\sqrt{1-X^2-X'^2-Y^2-Z^2}}^{y_0'\sqrt{1-X^2-X'^2-Y^2-Z^2}}dy'\int_{-z_0'\sqrt{1-X^2-X'^2-Y^2-Z^2-Y'^2}}^{z_0'\sqrt{1-X^2-X'^2-Y^2-Z^2-Y'^2}}f(x,y,z,x',y',z')\,dz'$$

Which can be written :

$$n(\varepsilon_x)=\int_{-y_0\sqrt{1-\varepsilon_x}}^{y_0\sqrt{1-\varepsilon_x}}dy\int_{-z_0\sqrt{1-\varepsilon_x-Y^2}}^{z_0\sqrt{1-\varepsilon_x-Y^2}}dz\int_{-y_0'\sqrt{1-\varepsilon_x-Y^2-Z^2}}^{y_0'\sqrt{1-\varepsilon_x-Y^2-Z^2}}dy'\int_{-z_0'\sqrt{1-\varepsilon_x-Y^2-Z^2-Y'^2}}^{z_0'\sqrt{1-\varepsilon_x-Y^2-Z^2-Y'^2}}f(x,y,z,x',y',z')\,dz'\,. \qquad \textbf{(A5)}$$

This is the result of a 4-D integration over a hyper-sphere of radius : $\sqrt{1-\varepsilon_x}$.

We have then :

$$n(\varepsilon_x) = C_4 \; y_0 z_0 y_0' z_0' \cdot \int_0^{\sqrt{1-\varepsilon_x}} r^3 \; f\left(\sqrt{\varepsilon_x + r^2}\right) dr \,, \qquad \textbf{(A6)}$$

with : $r = \sqrt{\left(\frac{y}{y_0}\right)^2 + \left(\frac{z}{z_0}\right)^2 + \left(\frac{y'}{y_0'}\right)^2 + \left(\frac{z'}{z_0'}\right)^2}$.

With $f(r) = f_0 \; r^n$, (eq. **A6**) gives :

$$n(\varepsilon_x) = C_4 \; f_0 \; y_0 z_0 y_0' z_0' \cdot \int_{\sqrt{\varepsilon_x}}^{1} \left(u^2 - \varepsilon_x\right) u^{n+1} \; du \,, \qquad \textbf{(A7)}$$

This gives with an integration by part :

$$n(\varepsilon_x) = \frac{2\pi^2}{n+2} \; f_0 \; y_0 z_0 y_0' z_0' \cdot \left(1 - \varepsilon_x - \frac{2}{n+4}\left(1 - \varepsilon_x^{2+\frac{n}{2}}\right)\right). \qquad \textbf{(A8)}$$

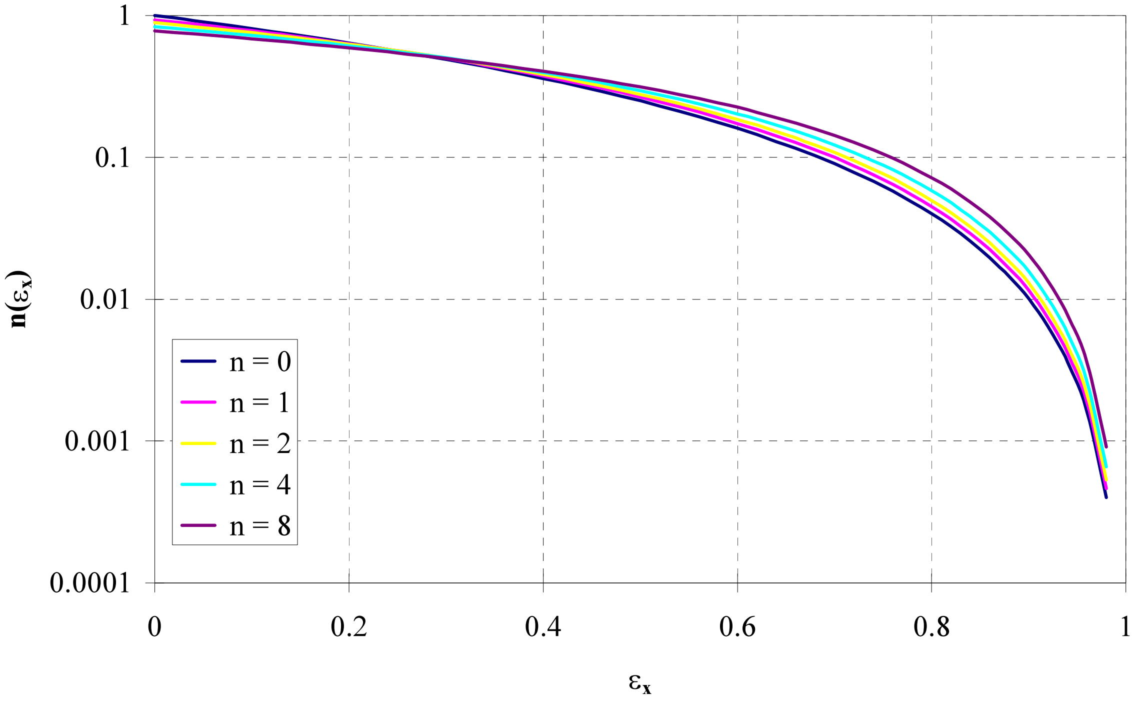

Figure A1 : Density distribution in (x, x') phase-space

A2.3 Optimisation of the integration over (y, z)

The 10-D integration can be reduced to a 8-D one if we succeed in solving the last integration over y and z. We have to solve :

$$I = \int_{-y_0\sqrt{1-R^2}}^{y_0\sqrt{1-R^2}} dy \int_{-z_0\sqrt{1-R^2-Y^2}}^{z_0\sqrt{1-R^2-Y^2}} f(x, y, z, x', y', z')\, dz\,, \qquad \textbf{(A9)}$$

with $R^2 = \max\left\{X_1^2 + X_1'^2 + Y_1'^2 + Z_1'^2, X_2^2 + X_2'^2 + Y_2'^2 + Z_2'^2\right\}$.

By using the variables X and Y,

$$I = 2\pi\, y_0 z_0 \int_0^{\sqrt{1-R^2}} r\, f\left(\sqrt{R^2 + r^2}\right) dr\,. \qquad \textbf{(A10)}$$

With $f(r) = r^n$:

$$I = 2\pi\, f_0\, y_0 z_0 \int_0^{\sqrt{1-R^2}} r\,\left(R^2 + r^2\right)^{\frac{n}{2}}\, dr\,, \qquad \textbf{(A11)}$$

$$I = 2\pi\, f_0\, y_0 z_0 \int_R^1 u^{n+1}\, du = \frac{2\pi\, y_0 z_0}{n+2} \cdot R^{n+2}\,. \qquad \textbf{(A12)}$$