DAPNIA-03-160

RENORMALIZATION OF GAUGE THEORIES AND MASTER EQUATION

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ABSTRACT

The evolution of ideas which has led from the first proofs of the renormalizability of non-abelian gauge theories, based on Slavnov–Taylor identities, to the modern proof based on the BRS symmetry and the *master equation* is briefly recalled. The content and consequences of the master equation are explained. This lecture has been delivered at the **Symposium in the Honour of Professor C. N. Yang**, Stony-Brook, May 21-22 1999.

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1 Introduction

It is a rare privilege for me to open this conference in honour of Professor Yang. His scientific contributions have been for me an essential source of inspiration. The most obvious example, Yang–Mills fields or gauge theories, will be illustrated by my talk. But there are other important aspects of Pr. Yang's work which have also directly influenced me: Professor Yang has consistently shown us that a theorist could contribute to quite different domains of physics like Particle Physics, the Statistical Physics of phase transitions or integrable systems.... Moreover his work has always emphasized mathematical elegance.

Finally by offering me a position at the ITP in Stony-Brook in 1971, Pr. Yang has given me the opportunity to start with the late Benjamin W. Lee a work on the renormalization of gauge theories, which has kept me busy for several years and played a major role in my scientific career.

Let me add a few other personal words. The academic year 1971–1972 I spent here at the ITP has been one of the most exciting and memorable of my scientific life. One reason of course is my successful collaboration with Ben Lee. However, another reason is the specially stimulating atmosphere Professor Yang had managed to create at the ITP, by attracting talented physicists, both ITP members and visitors, by the style of scientific discussions, seminars and lectures.

My interest in Yang–Mills fields actually dates back to 1969, and in 1970 I started a work, very much in the spirit of the original paper of Yang and Mills, on the application of massive Yang–Mills fields to Strong Interaction dynamics. Although in our work massive Yang–Mills fields were treated in the spirit of effective field theories, we were aware of the fact that such quantum field theories were not renormalizable.

In the summer of 1970 I presented the preliminary results of our work in a summer school in Cargèse, where Ben Lee was lecturing on the renormalization of spontaneous and linear symmetry breaking. This had the consequence that one year later I arrived here at the ITP to work with him.

Ben had just learned, in a conference I believe, from 't Hooft's latest work on the renormalizability of non-abelian gauge theories both in the symmetric and spontaneously broken phase and was busy proving renormalizability of the abelian Higgs model. We immediately started our work on the much more involved non-abelian extension.

Our work was based on functional integrals and other powerful functional methods, in contrast to less reliable and much less transparent manipulations of Feynman diagrams, and a generalization of so-called Slavnov–Taylor identities, consequence of the properties of the Faddeev–Popov (FP) determinant arising in the quantization of gauge theories. In a series of four papers (1972–1973), we examined most aspects of the renormalization of gauge theories.

Notation. In this lecture, we will always use an euclidean formalism, and thus will not distinguish betweeen space and time.

2 Classical gauge action and quantization

The principle of gauge invariance, which promotes a global (or rigid) symmetry under a Lie group G to a local (gauge) symmetry, provides a beautiful geometric method to generate interactions between relativistic quantum particles.

The basic field is a Yang–Mills or gauge field (mathematically a connection) $\mathbf{A}_\mu(x)$, related to infinitesimal parallel transport, and written here as a matrix belonging to the Lie algebra of the symmetry group

$$\mathbf{A}_\mu(x) = \sum_\alpha A_\mu^\alpha(x)\mathbf{t}_\alpha\,, \tag{2.1}$$

where the matrices $\mathbf{t}_\alpha$ are the generators of the Lie algebra of G in some representation.

Acting on the gauge field, a gauge transformation characterized by space-dependent group element $\mathbf{g}(x)$, takes an affine form:

$$\mathbf{A}_\mu(x) \mapsto \mathbf{g}(x)\mathbf{A}_\mu(x)\mathbf{g}^{-1}(x) + \mathbf{g}(x)\partial_\mu\mathbf{g}^{-1}(x). \tag{2.2}$$

In particular, from the point of view of global transformations ($\mathbf{g}(x)$ constant), the field $\mathbf{A}_\mu(x)$ transforms by the adjoint representation of the group G.

To gauge transformations are associated *covariant* derivatives, whose form depends on the group representation, for example,

$$\mathbf{D}_\mu = \partial_\mu + \mathbf{A}_\mu\,.$$

They transform linearly under a gauge transformation:

$$\mathbf{D}_\mu \mapsto \mathbf{g}(x)\mathbf{D}_\mu\,\mathbf{g}^{-1}(x). \tag{2.3}$$

The curvature $\mathbf{F}_{\mu\nu}(x)$ associated to the gauge field can be obtained from the covariant derivative by

$$\mathbf{F}_{\mu\nu}(x) = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu]\,.$$

It thus transforms linearly as

$$\mathbf{F}_{\mu\nu}(x) \mapsto \mathbf{g}(x)\mathbf{F}_{\mu\nu}(x)\,\mathbf{g}^{-1}(x).$$

The pure Yang–Mills action is the simplest gauge invariant action. It can be written as

$$\mathcal{S}(\mathbf{A}_\mu) = -\frac{1}{4e^2}\int \mathrm{d}^d x\ \mathrm{tr}\,\mathbf{F}_{\mu\nu}^2(x),$$

where e is here the gauge coupling constant.

Matter fields that transform non-trivially under the group are coupled to the gauge field because gauge invariance dictates that derivatives must be replaced by covariant derivatives. For fermions the action takes the typical form

$$\mathcal{S}_{\mathrm{F}}(\bar{\psi},\psi) = -\int \mathrm{d}^d x\, \bar{\psi}(x)\, (\not{\mathbf{D}} + M)\, \psi(x),$$

and for the boson fields:

$$\mathcal{S}_{\mathrm{B}}(\phi) = \int \mathrm{d}^d x \left[(\mathbf{D}_\mu \phi)^\dagger\, \mathbf{D}_\mu \phi + V(\phi) \right],$$

in which $V(\phi)$ is a group invariant function of the scalar field ϕ.

Quantization. The classical action results from a beautiful construction, but the quantization apparently completely destroys the geometric structure. Due to the gauge invariance, the degrees of freedom associated with gauge transformations have no dynamics and, therefore, a straightforward quantization of the classical action does not generate a meaningful perturbation theory (though non-perturbative calculations in lattice regularized gauge theories can be performed). It is thus necessary to *fix* the gauge, a way of expressing that some dynamics has to be provided for these degrees of freedom. For example, motivated by Quantum Electrodynamics, one may add to the action a covariant non-gauge invariant contribution

$$\mathcal{S}_{\mathrm{gauge}} = \frac{1}{2\xi e^2} \int \mathrm{d}^d x\ \mathrm{tr}\, (\partial_\mu \mathbf{A}_\mu)^2 . \tag{2.4}$$

However, simultaneously, and this is a specificity of non-abelian gauge theories, it is necessary to modify the functional integration measure of the gauge field to maintain formal unitarity. In the case of the gauge (2.4), one finds

$$[\mathrm{d}\mathbf{A}_\mu(x)] \mapsto [\mathrm{d}\mathbf{A}_\mu(x)] \det \mathbf{M}\,, \tag{2.5}$$

where $\mathbf{M}$ is the operator

$$\mathbf{M}(x,y) = \partial_\mu \mathbf{D}_\mu \delta(x-y).$$

This (Faddeev–Popov) determinant is the source of many difficulties. Indeed, after quantizing the theory one has to renormalize it. Renormalization is a theory of deformations of local actions. However, the determinant generates a non-local contribution to the action. Of course, using a well-known trick, it is possible to rewrite the determinant as resulting from the integration over unphysical spin-less fermions $\mathbf{C}, \bar{\mathbf{C}}$ (the "ghosts") of an additional contribution to the action

$$\mathcal{S}_{\mathrm{ghosts}} = \int \mathrm{d}^d x\, \bar{\mathbf{C}}(x) \partial_\mu \mathbf{D}_\mu \mathbf{C}(x).$$

After this transformation the action is local and renormalizable in the sense of power counting. However, in this local form all traces of the original symmetry seem to have been lost.

3 Renormalization

The measure (2.5) is the invariant measure for a set of non-local transformations, which for infinitesimal transformations takes the form

$$\delta \mathbf{A}_\mu(x) = \int \mathrm{d}y \, \mathbf{D}_\mu \mathbf{M}^{-1}(x,y)\omega(y),$$

the field $\omega(x)$ parametrizing the transformation. Using this property it is possible to derive a set of Ward–Takahashi (Slavnov–Taylor) identities between Green's functions and to prove renormalizability of gauge theories both in the symmetric and spontaneously broken Higgs phase. The non-local character of these transformations and the necessity of using two different representations, one non-local but with invariance properties, the other one local and thus suitable for power counting analysis, explains the complexity of the initial proofs.

Though the problem of renormalizing gauge theories could then be considered as solved, one of the remaining problems was that the proofs, even in the most synthetic presentation like in Lee–Zinn-Justin IV, were quite complicated, and more based on trial and error than systematic methods.

Returning to Saclay I tried to systematize the renormalization program of quantum field theories with symmetries. I abandoned the idea of a determination of renormalization constants by relations between Green's functions, for a more systematic approach based on loop expansion and counter-terms.

The idea is to proceed by induction on the number of loops. Quickly summarized:

One starts from a regularized local lagrangian with some symmetry properties. One derives, as consequence of the symmetry, identities (generally called Ward–Takahashi (WT) identities) satisfied by the generating functional Γ of proper vertices or one-particle irreducible (1PI) Green's functions. By letting the cut-off go to infinity (or the dimension to 4 in dimensional regularization), one obtains identities satisfied by the sum $\Gamma_{\text{div.}}$ of all divergent contributions at one loop order. At this order Γ_{div} is a local functional of a degree determined by power counting. By subtracting Γ_{div} from the action, one obtains a theory finite at one-loop order. One then reads off the symmetry of the lagrangian renormalized at one-loop order and repeats the procedure to renormalize at two-loop order. The renormalization program is then based on determining general identities, valid both for the action and the 1PI functional, which are stable under renormalization, that is stable under all deformations allowed by power counting. One finally proves the stability by induction on the number of loops.

Unfortunately, this program did not apply in an obvious way to non-abelian gauge theories, because it required a symmetry of the local quantized action, and none was apparent. WT identities were established using symmetry properties of the theory in the non-local representation

In the spring of 1974, my student Zuber drew my attention to a preliminary report of a work of Becchi, Rouet and Stora who had discovered a strange fermion-type (like supersymmetry) symmetry of the complete quantized action including the ghost contributions. There were indications that this symmetry could be used to somewhat simplify the algebra of the proof of renormalization. Some time later, facing the daunting prospect of lecturing about renormalization of gauge theories and explaining the proofs to non-experts, I decided to study the BRS symmetry. I then realized that the BRS symmetry was the key allowing the application of the general renormalization scheme and in a summer school in Bonn (1974) I presented a general proof of renormalizability of gauge theories based on BRS symmetry and the *master equation*.

4 BRS symmetry

The form of the BRS transformations in the case of non-abelian gauge transformations is rather involved and hides their simple origin. Thus, we first give here a presentation which shows how BRS symmetry arises in apparently a simpler context.

The origin of BRS symmetry: constraint equations. Let φ^α be a set of dynamical variables satisfying a system of equations

$$E_\alpha(\varphi) = 0\,, \tag{4.1}$$

where the functions $E_\alpha(\varphi)$ are smooth, and $E_\alpha = E_\alpha(\varphi)$ is a one-to-one map in some neighbourhood of $E_\alpha = 0$ which can be inverted into $\varphi^\alpha = \varphi^\alpha(E)$. This implies, in particular, that the equation (4.1) has a unique solution $\varphi^\alpha_{\mathrm{s}}$. We then consider some function $F(\varphi)$ and we look for a formal representation of $F(\varphi_{\mathrm{s}})$, which does not require solving equation (4.1) explicitly.

One formal expression is

$$\begin{aligned} F\left(\varphi_{\mathrm{s}}\right) &= \int \Bigl\{\prod_\alpha \mathrm{d}E^\alpha\, \delta(E_\alpha)\Bigr\} F\bigl(\varphi(E)\bigr) \\ &= \int \Bigl\{\prod_\alpha \mathrm{d}\varphi^\alpha\, \delta\left[E_\alpha(\varphi)\right]\Bigr\} \mathcal{J}(\varphi)\, F(\varphi) \end{aligned} \tag{4.2}$$

with

$$\mathcal{J}(\varphi) = \det \mathbf{E}\,, \quad E_{\alpha\beta} \equiv \frac{\partial E_\alpha}{\partial \varphi^\beta}.$$

We have chosen $E_\alpha(\varphi)$ such that $\det \mathbf{E}$ is positive.

Slavnov–Taylor identity. The measure

$$\mathrm{d}\rho(\varphi) = \mathcal{J}(\varphi) \prod_\alpha \mathrm{d}\varphi^\alpha\,, \tag{4.3}$$

has a simple property. The measure $\prod_\alpha \mathrm{d}E_\alpha$ is the invariant measure for the group of translations $E_\alpha \mapsto E_\alpha + \nu_\alpha$. It follows that $\mathrm{d}\rho(\varphi)$ is the invariant measure for the translation group non-linearly realized on the new coordinates φ_α (provided ν_α is small enough):

$$\varphi^\alpha \mapsto \varphi'^\alpha \quad \text{with} \quad E_\alpha(\varphi') - \nu_\alpha = E_\alpha(\varphi). \tag{4.4}$$

This property is, in gauge theories, the origin of the Slavnov–Taylor symmetry.

The infinitesimal form of the transformation can be written more explicitly as

$$\delta\varphi^\alpha = [E^{-1}(\varphi)]^{\alpha\beta}\nu_\beta\,. \tag{4.5}$$

BRS symmetry. Let us again start from identity (4.2) and first replace the δ-function by its Fourier representation:

$$\prod_\alpha \delta\left[E_\alpha(\varphi)\right] = \int \prod_\alpha \frac{\mathrm{d}\lambda^\alpha}{2i\pi}\, \mathrm{e}^{-\lambda^\alpha E_\alpha(\varphi)}\,. \tag{4.6}$$

The λ-integration runs along the imaginary axis. From the rules of fermion integration, we know that we can also write the determinant as an integral over Grassmann variables c^α and $\bar{c}^\alpha$:

$$\det \mathbf{E} = \int \prod_\alpha \left(\mathrm{d}c^\alpha \mathrm{d}\bar{c}^\alpha\right) \exp\left(\bar{c}^\alpha E_{\alpha\beta} c^\beta\right). \tag{4.7}$$

Expression (4.2) then takes the apparently more complicated form

$$F(\varphi_\mathrm{s}) = \mathcal{N} \int \prod_\alpha \left(\mathrm{d}\varphi^\alpha \mathrm{d}c^\alpha \mathrm{d}\bar{c}^\alpha \mathrm{d}\lambda^\alpha\right) F(\varphi) \exp\left[-S(\varphi, c, \bar{c}, \lambda)\right], \tag{4.8}$$

in which $\mathcal{N}$ is a constant normalization factor and $S(\varphi, c, \bar{c}, \lambda)$ the quantity

$$S(\varphi, c, \bar{c}, \lambda) = \lambda^\alpha E_\alpha(\varphi) - \bar{c}^\alpha E_{\alpha\beta}(\varphi) c^\beta\,. \tag{4.9}$$

While we seem to have replaced a simple problem by a more complicated one, in fact in many situations (and this includes the case where equation (4.1) is a field equation) it is easy to work with the integral representation (4.8).

Quite surprisingly, the function S has a symmetry, which actually is a consequence of the invariance of the measure (4.3) under the group of transformations (4.5). This BRS symmetry, first discovered in the quantization of gauge theories by Becchi, Rouet and Stora (BRS), is a fermionic symmetry in the sense that it transforms commuting variables into Grassmann variables and vice versa. The parameter of the transformation is a Grassmann variable, an anti-commuting constant $\bar{\varepsilon}$. The variations of the various dynamic variables are

$$\begin{cases} \delta\varphi^\alpha = \bar{\varepsilon} c^\alpha \,, & \delta c^\alpha = 0 \,, \\ \delta \bar{c}^\alpha = \bar{\varepsilon}\lambda^\alpha \,, & \delta\lambda^\alpha = 0 \end{cases} \tag{4.10}$$

with

$$\bar{\varepsilon}^2 = 0 \,, \qquad \bar{\varepsilon} c^\alpha + c^\alpha \bar{\varepsilon} = 0 \,, \qquad \bar{\varepsilon}\bar{c}^\alpha + \bar{\varepsilon}\bar{c}^\alpha = 0 \,.$$

The transformation is obviously *nilpotent* of vanishing square: $\delta^2 = 0$.

When acting on functions of $\{\varphi, c, \bar{c}, \lambda\}$, the BRS transformation can be represented by a Grassmann differential operator

$$\mathcal{D} = c^\alpha \frac{\partial}{\partial \varphi^\alpha} + \lambda^\alpha \frac{\partial}{\partial \bar{c}^\alpha} \,. \tag{4.11}$$

The operator $\mathcal{D}$ has the form of a cohomology operator since

$$\mathcal{D}^2 = 0 \,, \tag{4.12}$$

and is the source of a BRS cohomology.

BRS symmetry in gauge theories. For simplicity, we consider form now on only pure gauge theories, but the generalization in presence of matter fields is simple.

In gauge theories, the role of the φ variables is played by the group elements which parametrize gauge transformations and the equation (4.1) is simply the gauge fixing equation. In the example of the gauge choice (2.4), it reads

$$\partial_\mu \mathbf{A}_\mu(x) = \nu(x),$$

where ν is a stochastic field with gaussian measure.

The corresponding quantized action can be written as

$$\begin{aligned} \mathcal{S}(\mathbf{A}_\mu, \bar{\mathbf{C}}, \mathbf{C}, \lambda) = \int \mathrm{d}^4 x \ \mathrm{tr} \Big[& -\frac{1}{4e^2} \mathbf{F}^2_{\mu\nu} + \frac{\xi e^2}{2} \lambda^2(x) + \lambda(x) \partial_\mu \mathbf{A}_\mu(x) \\ & + \mathbf{C}(x) \partial_\mu \mathbf{D}_\mu \bar{\mathbf{C}}(x) \Big]. \end{aligned} \tag{4.13}$$

The form of BRS transformations that leave the action invariant,

$$\begin{cases} \delta \mathbf{A}_\mu(x) = -\bar{\varepsilon} \mathbf{D}_\mu \mathbf{C}(x) \,, & \delta \mathbf{C}(x) = \bar{\varepsilon} \mathbf{C}^2(x), \\ \delta \bar{\mathbf{C}}(x) = \bar{\varepsilon}\lambda(x), & \delta\lambda(x) = 0 \,, \end{cases} \tag{4.14}$$

is more complicated only because they are expressed in terms of group elements instead of coordinates.

Introducing the BRS differential operator

$$\mathcal{D} = \int \mathrm{d}^d x \,\mathrm{tr}\left[-\mathbf{D}_\mu \mathbf{C}(x)\frac{\delta}{\delta \mathbf{A}_\mu(x)} + \mathbf{C}^2(x)\frac{\delta}{\delta \mathbf{C}(x)} + \lambda(x)\frac{\delta}{\delta \bar{\mathbf{C}}(x)}\right], \tag{4.15}$$

one can also express the BRS symmetry of the quantized action by the equation

$$\mathcal{D}\mathcal{S}(\mathbf{A}_\mu, \bar{\mathbf{C}}, \mathbf{C}, \lambda) = 0\,. \tag{4.16}$$

Moreover, one shows quite generally that the non gauge contrbution $\mathcal{S}_{\mathrm{gauge}}$, which results from quantization, is BRS exact. Here,

$$\mathcal{S}_{\mathrm{gauge}} = \mathcal{D}\int \mathrm{d}^d x\ \mathrm{tr}\,\bar{\mathbf{C}}(x)\left[\partial_\mu \mathbf{A}_\mu(x) + \xi e^2 \lambda(x)\right]. \tag{4.17}$$

5 Renormalization and master equation

To BRS symmetry now correspond a set of WT identities that can be used to prove the renormalizability of gauge theories.

WT identities are based on change of variables of the form of BRS transformations. As a consequence, they involve the composite operators (non-linear local functions of the fields) $\mathbf{D}_\mu \mathbf{C}(x)$ and $\mathbf{C}^2(x)$, which appear in the r.h.s. of the BRS transformation (4.14) and which necessitate additionnal renormalizations. To discuss renormalization, it is thus necessary to add to the action two sources $\mathbf{K}_\mu$, $\mathbf{L}$ for them:

$$\mathcal{S}(\mathbf{A}_\mu, \mathbf{C}, \bar{\mathbf{C}}, \lambda) \mapsto \mathcal{S}(\mathbf{A}_\mu, \mathbf{C}, \bar{\mathbf{C}}, \lambda) + \int \mathrm{d}^4 x \,\mathrm{tr}\left(-\mathbf{K}_\mu(x)\mathbf{D}_\mu \mathbf{C}(x) + \mathbf{L}(x)\mathbf{C}^2(x)\right).$$

The sources for BRS transformations, $\mathbf{K}_\mu$ and $\mathbf{L}$, have been later renamed antifields. No other terms are required because the two composite operators are BRS invariant.

Master equation. Because the composite operators $\mathbf{D}_\mu \mathbf{C}(x)$ and $\mathbf{C}^2(x)$ require renormalizations, the form (4.14) of BRS transformations is not stable under renormalization.

Instead, one discovers that the complete action $\mathcal{S}(\mathbf{A}_\mu, \mathbf{C}, \bar{\mathbf{C}}, \lambda, \mathbf{K}_\mu, \mathbf{L})$, which includes these additional source terms, satisfies after renormalization a quadratic relation, the master equation, which does not involve the explicit form of the BRS transformations (4.14). In component form (generalizing the notation (2.1) to all fields), the master equation reads

$$\int d^4 x \sum_\alpha \left(\frac{\delta \mathcal{S}}{\delta A^\alpha_\mu(x)}\frac{\delta \mathcal{S}}{\delta K^\alpha_\mu(x)} + \frac{\delta \mathcal{S}}{\delta C^\alpha(x)}\frac{\delta \mathcal{S}}{\delta L^\alpha(x)} + \lambda^\alpha(x)\frac{\delta \mathcal{S}}{\delta \bar{C}^\alpha(x)}\right) = 0\,. \tag{5.1}$$

The proof involves first showing that this equation implies a similar equation for the generating functional of proper vertices (the 1PI functional). The latter equation, in turn, implies relations between divergences. In the framework of the loop expansion, these relations imply that the counter-terms that have to be added to the action to cancel divergences, can be chosen such that the renormalized action still preserves the master equation.

What is striking is that the master equation (5.1) contains no explicit reference to the initial gauge transformations. Therefore, one might worry that it does not determine the renormalized action completely, and that the general renormalization program fails in the case of non-abelian gauge theories. However, one slowly discovers that the master equation has remarkable properties. In particular, all its local solutions which satisfy the power counting requirements and ghost number conservation, have indeed the form of an action for a quantized non-abelian gauge theory.

Power counting relies on the canonical dimensions of fields, which are (with the notation $[X]$ for the canonical dimension of X)

$$[\mathbf{A}]=1\,,\quad [\mathbf{C}+\bar{\mathbf{C}}]=2\,,\quad [\lambda]=2\,,\quad [\mathbf{K}_\mu+\mathbf{C}]=3\,,\quad [\mathbf{L}+\mathbf{C}+\bar{\mathbf{C}}]=4\,,$$

where this form takes into account ghost number conservation by displaying only the dimensions of the relevant products.

The action density has dimension 4 and, therefore, is quadratic in λ and linear in $\mathbf{K}_\mu$ and $\mathbf{L}$. The master equation implies that the coefficients $f_{\alpha\beta\gamma}$ of $L^\alpha C^\beta \bar{C}^\gamma$ are constants that satisfy the Jacobi identity, and the coefficients $D_\mu^{\alpha\beta}$ of $K_\mu^\alpha C^\beta$ are affine functions of $\mathbf{A}_\mu$ that satisfy Lie algebra commutations with the $f_{\alpha\beta\gamma}$ as structure constants. Then continuity implies, in the semi-simple example at least, preservation of all geometric properties.

One somewhat surprising outcome of analysis is that the master equation has, for general gauge fixing functions of dimension 2, solutions with quartic ghost interactions, which cannot be obviously related to a determinant. On the other hand the master equation (and this one of its main properties) implies directly that the non gauge invariant part of the quantized action is BRS exact (like in equation (4.17)). This property then ensures gauge independence and unitarity.

Only a few years later, elaborating on a remark of Slavnov, was I able to reproduce a general quartic ghost term as resulting from a generalized gauge fixing procedure (Zinn-Justin 1984).

After the renormalization program was successfully completed, one important problem remained, of relevance for instance to the description of deep-inelastic scattering experiments: the renormalization of gauge invariant operators of dimension higher than 4. Using similar techniques Stern-Kluberg and Zuber were able to solve the problem for operators of dimension 6 and conjecture the general form. Only recently has the general conjecture been proven rigourously by non-trivial cohomology techniques (Barnich, Brandt and Henneaux 1995).

Acknowledgments. To Professor C.N. Yang as a testimony of admiration and gratitude, Stony-Brook, May 21, 1999.

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