A wavelet-based method for multifractal analysis of 3D vector-valued random fields: application to turbulent velocity and vorticity 3D numerical data

Pierre Kestener[1,2] and Alain Arneodo[1]

[1] Laboratoire de Physique, École Normale Supérieure de Lyon
[2] CEA Saclay, DSM/DAPNIA/SEDI, Gif-sur-Yvette

Laboratoire de Physique,
École Normale Supérieure de Lyon
46 allée d'Italie,
69364 Lyon cedex 07, France

Commissariat à l'Energie Atomique,
Centre de Saclay,
DSM/DAPNIA/SEDI,
91191 Gif-sur-Yvette

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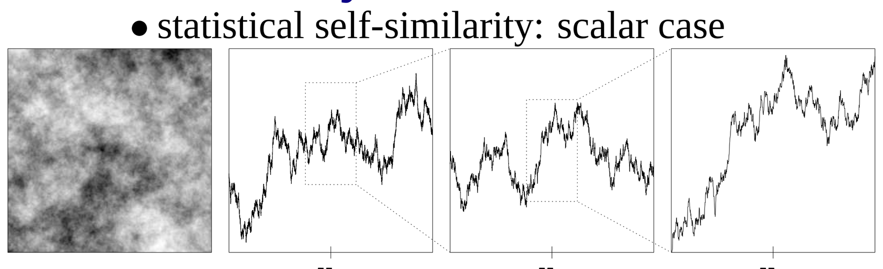
1 Introduction: analyzing singular signals.

The multifractal formalism was introduced in the mid 1980s to provide a statistical description of the fluctuations of regularity of singular measures found in chaotic dynamical systems [1] or in modelling the energy cascading process in turbulent flows [2]. Box-counting algorithms were successfully adapted to resolve multifractal scaling for isotropic self-similar fractals [3]. As to self-affine fractals, Parisi and Frisch [4] developed, in the context of turbulence velocity data analysis, an alternative multifractal description based on the investigation of the scaling behavior of the moments of the velocity increments, the so-called structure functions: $S_p(l) =<(\delta v_l)^p> \sim l^{\zeta_p}$ (p integer > 0), where $\delta v_l(x) = v(x+l) - v(x)$ is an increment of the recorded signal over a distance l. Then from the local behavior $\delta v_l(x) \sim l^{h(x)}$, the $D(h)$ singularity spectrum is defined, in full analogy with the $f(\alpha)$-spectrum for singular measure, as the Hausdorff dimension of the set of points x where the local roughness (or Hölder) exponent $h(x)$ of v is h. In principle, $D(h)$ can be attained by Legendre transforming the SF scaling exponents ζ_p [4]. A natural way of performing a unified multifractal analysis of both singular measures and multi-affine functions [5], consists in using the *continuous wavelet transform* (WT). In the early nineties, a wavelet-based statistical approach was proposed as a unified multifractal description of singular measures and multi-affine functions [5].

Applications of the so-called *wavelet transform modulus maxima* (WTMM) method have already provided insight into a wide variety of problems, e.g., fully developed turbulence, econophysics, meteorology, physiology and DNA sequences [6]. Later on, the WTMM method was generalized to 2D for multifractal analysis of rough surfaces, with very promising results in the context of the geophysical study of the intermittent nature of satellite images of the cloud structure [7] and the medical assist in the diagnosis in digitized mammograms [7]. Recently the WTMM method has been further extended to 3D analysis and applied to dissipation and enstrophy 3D numerical data issue from isotropic turbulence direct numerical simulations (DNS) [8]. Thus far, the multifractal description has been mainly devoted to scalar measures and functions. In the spirit of a preliminary theoretical study of self-similar vector-valued measures by Falconer and O'Neil [9], our objective here is to generalize the WTMM method to vector-valued random fields with the specific goal to achieve a comparative 3D vectorial multifractal analysis of DNS velocity and vorticity fields.

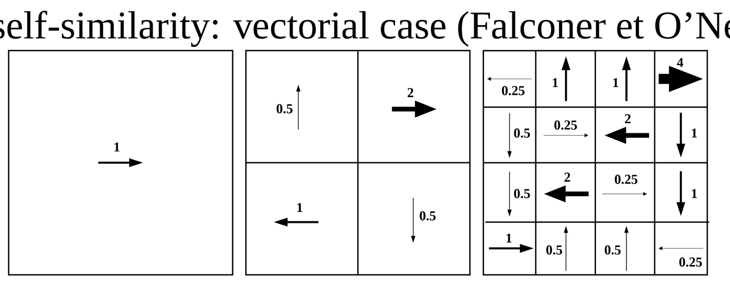
Fractal objects: self-similarity

- statistical self-similarity: scalar case



$$f(\mathrm{x}_0 + \lambda \mathrm{u}) - f(\mathrm{x}_0) \sim \lambda^{h(\mathrm{x}_0)} (f(\mathrm{x}_0 + \mathrm{u}) - f(\mathrm{x}_0))$$

- statistical self-similarity: vectorial case (Falconer et O'Neil, 1995):

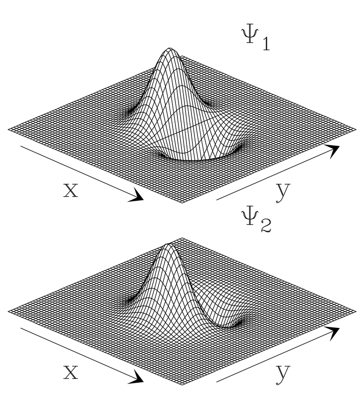


2 Generalizing the wavelet-based multifractal formalism to vector-valued random fields

2.1 Tensorial wavelet transform (2D case)

1. **Tensorial wavelet transform of field** $\mathrm{V} = (V_1, V_2)$ **:**

$$\mathbb{T}_\psi[\mathrm{V}](\mathrm{b}, a) = (\mathrm{T}_{\psi_i}[V_j](\mathrm{b}, a)) = \begin{pmatrix} T_{\psi_1}[V_1] & T_{\psi_1}[V_2] \\ T_{\psi_2}[V_1] & T_{\psi_2}[V_2] \end{pmatrix}$$



$T_{\psi_i}[V_j](\mathrm{b}, a) = a^{-3} \int d^3\mathrm{r}\ \psi_i(a^{-1}(\mathrm{r} - \mathrm{b})) V_j(\mathrm{r}), j = 1, 2$

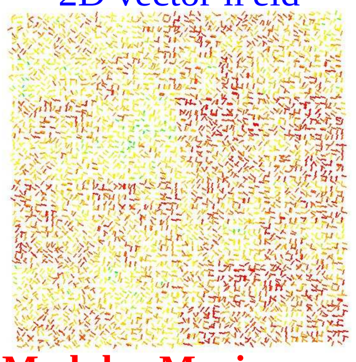
2. **Direction of greatest variation of vector field :** $|\mathbb{T}_\psi[\mathrm{V}]| = \sup_{\mathrm{C} \neq 0} \frac{||\mathbb{T}_\psi[\mathrm{V}].\mathrm{C}||}{||\mathrm{C}||}$

3. **Singular value decomposition of WT tensor:** $\mathbb{T}_\psi[\mathrm{V}] = (G).\begin{pmatrix} \sigma_{\max} & 0 \\ 0 & \sigma_{\min} \end{pmatrix}.(D)^T$

4. **Tensorial wavelet transform** : $\mathrm{T}_{\psi,\max}[\mathrm{V}](\mathrm{b}, a) = \sigma_{\max} \mathrm{G}_{\sigma_{\max}}$

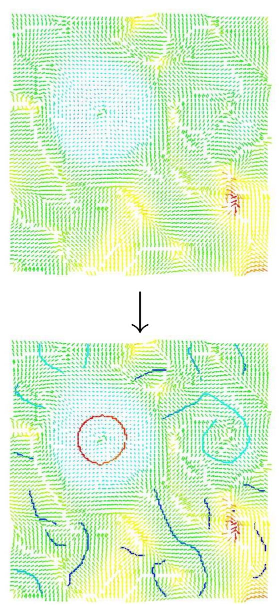
2.2 Multi-scale edge detection

2D vector field



Tensorial Wavelet Transform

$\mathrm{T}_{\psi,\max}[\mathrm{V}](\mathrm{b}, a) = \sigma_{\max} \mathrm{G}_{\sigma_{\max}}$

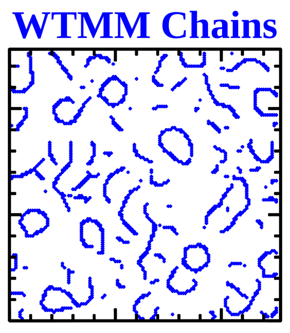


Modulus Maxima $\sigma_{\max}$ chains of tensorial wavelet transform at scale a:

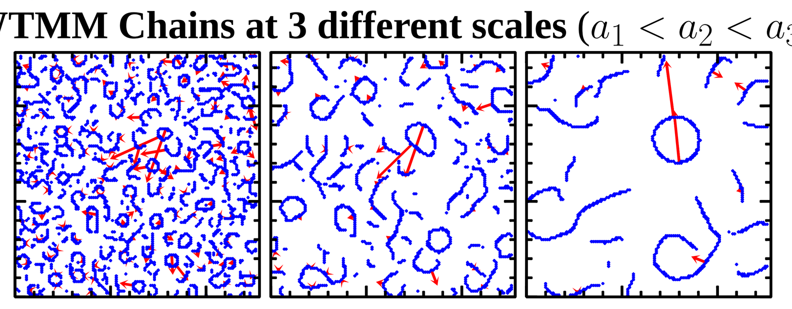
$$\left\{ (b, a) / \frac{\partial \sigma_{\max}}{\partial G_{\max}} = 0 \quad \text{and} \quad \frac{\partial^2 \sigma_{\max}}{\partial G^2_{\max}} < 0 \right\}$$

2.3 Wavelet transform skeleton

WTMM Chains

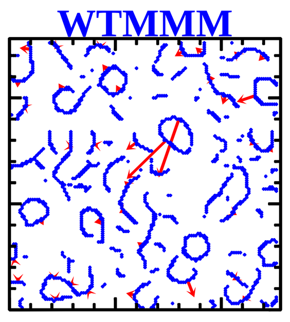


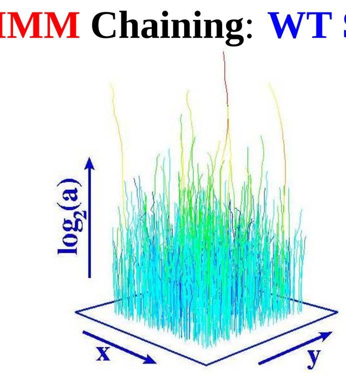
WTMM Chains at 3 different scales ($a_1 < a_2 < a_3$)



WTMMM Chaining: **WT Skeleton**

WTMMM

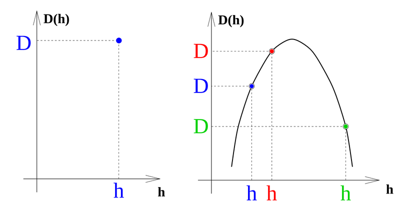


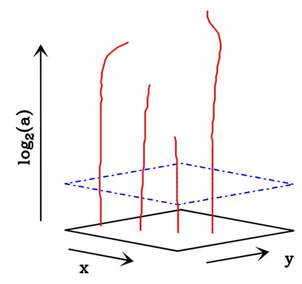


2.4 Multifractal formalism

Singularity Spectrum:

$$D(h) = d_H\{\mathrm{r} \in R^d, h(\mathrm{r}) = h\}$$





Legendre transform:

$$D(h) = \min_q \left(qh - \tau(q) \right)$$

Analogy with statistical physics : compute partition functions

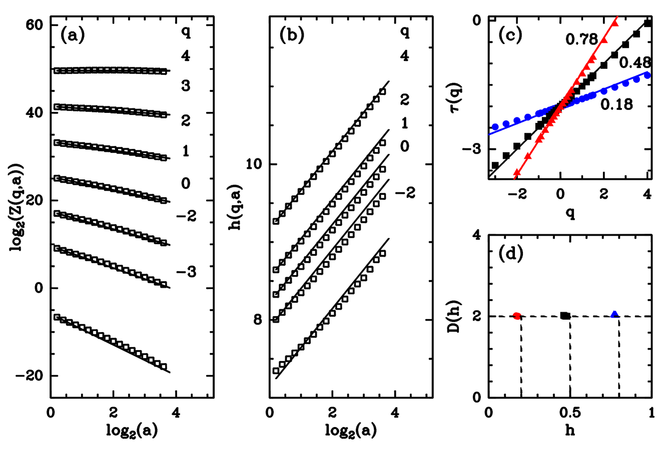
$$\mathcal{Z}(q, a) = \sum_{\mathcal{L}(a)} (\mathcal{M}_\psi(\mathrm{r}, a))^q \sim a^{\tau(q)}$$

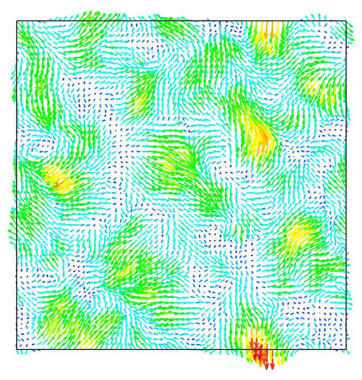
$$\mathcal{H}(q, a) = \sum_{\mathcal{L}(a)} \ln |\mathcal{M}_\psi(\mathrm{r}, a)|\ \mathcal{W}_\psi(\mathrm{r}, a) \sim a^{h(q)}$$

$$\mathcal{D}(q, a) = \sum_{\mathcal{L}(a)} \ln |\mathcal{W}_\psi(\mathrm{r}, a)|\ \mathcal{W}_\psi(\mathrm{r}, a) \sim a^{D(q)}$$

2.5 Monofractal 2D vector fields:

fractional Brownian fields $\mathrm{B}_H(\mathrm{r})$
(Spectral method simulation):

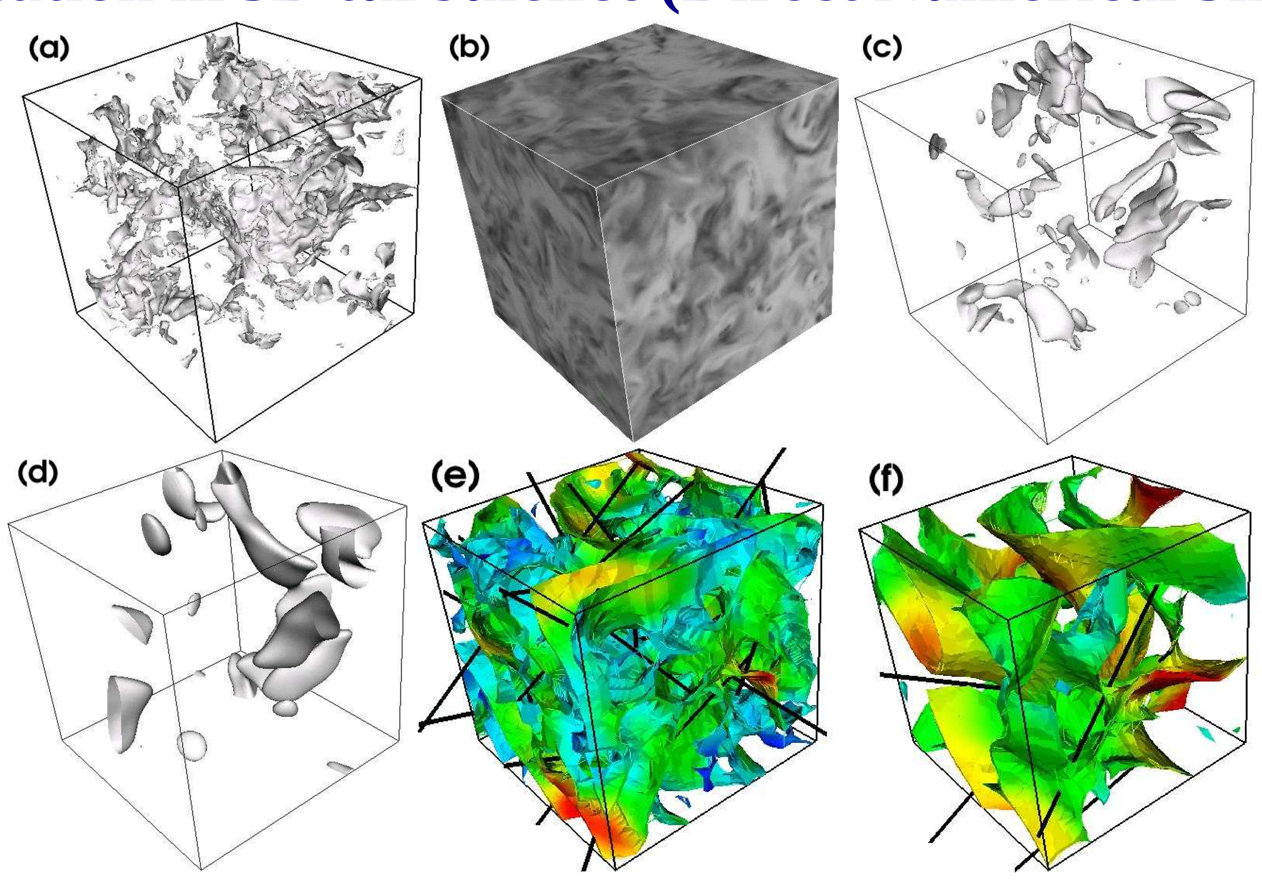




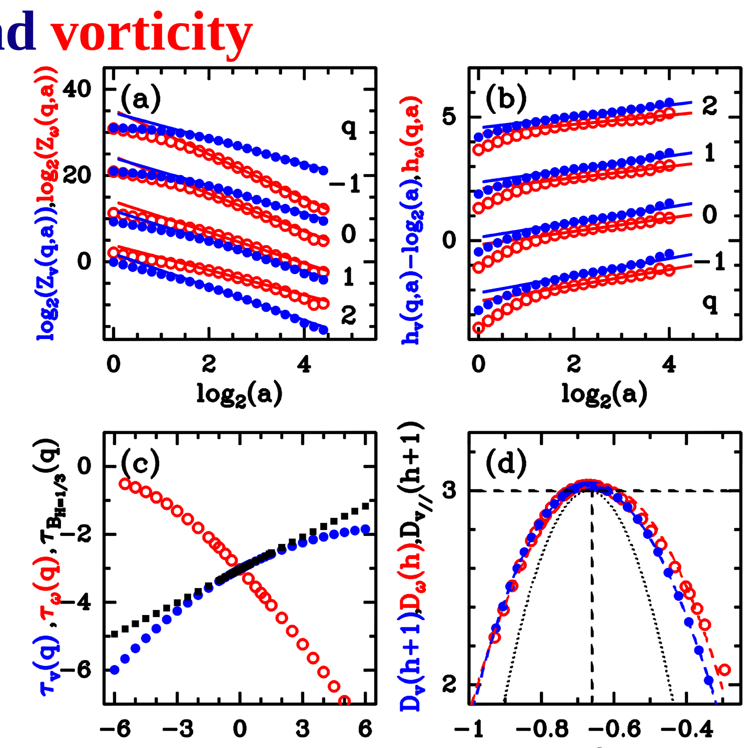
Theoretical predictions :

- **linear** $\tau(q)$: $\tau(q) = qH - 2$
- **degenerated singularity spectrum:** $D(h = H) = 2$

3 Application in 3D turbulence (Direct Numerical Simulation)



Tensorial 3D WTMM method: computing singularity spectrum of velocity and vorticity



- parabolic fit: $\tau(q) = -C_0 - C_1 q - C_2 \frac{q^2}{2}$
- same intermittency coefficient (velocity and vorticity) $C_2 = 0.049 \pm 0.004$

1D increments method:

- longitudinal : $C_2(\delta v_L) \sim 0.025$
  transverse : $C_2(\delta v_T) \sim 0.040$

Preliminary applications to DNS turbulence data have revealed the existence of an intimate relationship between the velocity and vorticity 3D statistics that turn out to be significantly more intermittent than previously estimated from 1D longitudinal velocity increments statistics.

This new methodology looks very promising to many extents. Thanks to the singular value decomposition, one can focus on fluctuations that are locally confined in 2D ($\min_i \sigma_i = 0$) or in 1D (the two smallest σ_i are zero) and then simultaneously proceed to a multifractal and structural analysis of turbulent flows. The investigation along this line of vorticity sheets and vorticity filaments in DNS is in current progress. We are very grateful to E. Lévêque for allowing us to have access to his DNS data and to the CNRS under GDR turbulence.

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