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MAGNETIC FIELD PRODUCED BY A SINGLE CURRENT LINE NEAR OR WITHIN A FERROMAGNETIC CYLINDER OR TUBE

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ABSTRACT

In this paper, we carry out an analytical computation of the vector potential produced in the whole space by a current line located near or within a ferromagnetic cylinder or tube. Whenever possible, we determine the image currents that can be used to derive directly the contribution from the ferromagnetic material. Most of these cases are relevant to the determination of the magnetic field produced by accelerator magnets, and we give the expressions of the corresponding multipole field coefficients.

CONTENTS

FREQUENTLY USED SYMBOLS AND ABBREVIATIONS

ROMAN ALPHABET

a	Distance between the z-axis axis and the current line axis (m).
$\mathbf{a}$	Complex coordinates of N.
$\vec{a}$	$\overrightarrow{ON}$ vector.
$\mathbf{A}$	Complex vector potential.
$\vec{A}$, $\overrightarrow{A_k}$	Vector potential.
A_r, A_θ, A_z	Components of the vector potential in a cylindrical coordinate system (T.m).
A_z^{im}	z-component of vector potential associated with an image current line.
A_{1z}, A_{2z}, A_{3z}, A_{4z}	z-components of vector potentials in different regions of space (T.m).
$\mathbf{b} = \mathbf{a} - \mathbf{r}$	Complex coordinates of vector $\overrightarrow{MN}$
$\vec{b}$	$\overrightarrow{MN}$ vector.
$\vec{B}$	Magnetic flux density vector.
B_{n1}, B_{n2}	Normal components of the magnetic flux density on both sides of the boundary separating two regions of space with different magnetic permeability (T).
$\mathbf{B}^{line}$, $\mathbf{B}^{yoke}$, $\mathbf{B}^{tot}$	Complex magnetic flux density.
$\overrightarrow{B^{line}}$	Magnetic flux density vector produced by a current line.
$A_n^{line}, A_n^{yoke}, A_n^{tot}, B_n^{line}, B_n^{yoke}, B_n^{tot}$	Multipole field coefficients.
C_0, C_0^{in}, C_0^{co}, C_0^{ex}	Real constants.
C_n, C_n^{in}, C_n^{co}, C_n^{ex}	Real constants.
D_0, D_0^{in}, D_0^{co}, D_0^{ex}	Real constants.
D_n, D_n^{in}, D_n^{co}, D_n^{ex}	Real constants.
dl	Amplitude of $\vec{dl}$ (m).
$\vec{dl}$	Elementary distance vector.
E_0, $E_{-\alpha^2}$, E_{α^2}	Real constants.
F_0, $F_{-\alpha^2}$, F_{α^2}	Real constants.
$\vec{H}$	Magnetic field vector.
$\overrightarrow{H_1}$, $\overrightarrow{H_2}$	Magnetic field vector in two different regions of space.
H_{t1}, H_{t2}	Tangential components of $\overrightarrow{H_1}$ and $\overrightarrow{H_2}$ on both sides of the boundary separating two regions of space with different magnetic permeability (T).
I	Intensity of a current line (A).
$\vec{j}$	Current density vector per unit volume.

$\vec{j}_s$	Current density vector per unit area.
k	Integer number.
K_z, K_θ	Real constants.
m	Integer number.
M	Given point of space.
n	Integer number.
N	Point corresponding to the position of the current line ($-I$) in the (O,x,y) plane.
$\overrightarrow{n_{12}}$	Normal vector at the interface between two regions of space.
O	Origin of coordinate system.
r	Radius (m).
$\vec{r}$	$\overrightarrow{OM}$ vector.
$R(r)$, $R_0(r)$, $R_k(r)$	Part of A_z which depends on r.
R_y	Salient radius of ferromagnetic yoke.
$\mathbf{s} = x + iy = re^{i\Theta}$	Complex coordinates of M.
$\vec{u}_r$, $\vec{u}_\theta$, $\vec{u}_z$	Unit vectors of cylindrical coordinate system.
x	Abscissa (m).
y	Ordinate (m).
z	z-coordinate (m).

GREEK ALPHABET

α	Real constant.
$\eta = R_{y1} / R_{y2}$	Ratio of radii.
θ	Azimuth (rad).
$\Theta(\theta), \Theta_0(\theta), \Theta_{\alpha^2}(\theta), \Theta_{-\alpha^2}(\theta)$	Part of A_z which depends on θ.
$\mu_0 = 4\pi 10^{-7}$ H/m	Magnetic permeability of vacuum.
μ_r	Relative magnetic permeabilty of ferromagnetic yoke.

$\pi = 3.14159265358979323846264 3$

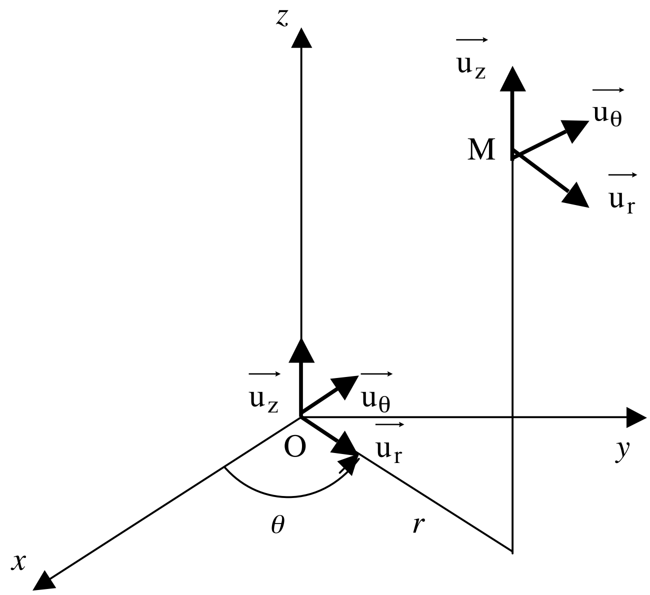

Figure 1. Cylindrical coordinate system.

1 INTRODUCTION

The goal of this note is to derive analytical expressions of the multipole field coefficients produced by a current line located near or within a ferromagnetic cylinder or tube. Such expressions may be used for several purposes, including the computation of the field distortions produced by bus bars outside or within accelerator magnets' iron yokes or near the beam tube in a machine such as LHC.

2 COORDINATE SYSTEM

The coordinate systems used throughout the paper are defined in Figure 1.

3 CASE OF A CURRENT LINE ALONE IN FREE SPACE

3.1 SINGLE CURRENT LINE AT THE COORDINATE SYSTEM ORIGIN

Let us first consider a current line of intensity, (-*I*), parallel to the z-axis, and crossing the (O,*x*,*y*) plane at O, as shown in Figure 2.

Let M designate a given point of space and let b be the distance between O and M.

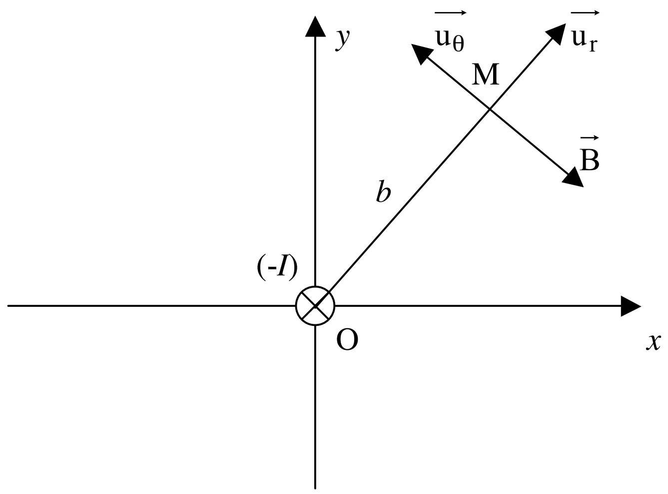

Figure 2. Single current line in free space located at the coordinate system origin.

The magnetic flux density produced by the current line is given by Ampere's theorem, which, in a cylindrical coordinate system, leads to

$$\vec{B} = -\frac{\mu_0 I}{2\pi b}\overrightarrow{u_\theta} \tag{1}$$

Moreover, the vector potential, $\vec{A}$, is defined as

$$\vec{B} = \overrightarrow{\mathrm{rot}\,\vec{A}} \tag{2}$$

In a cylindrical coordinate system, we get

$$\vec{B} = \left| \begin{array}{l} \left[\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right]\overrightarrow{u_r} \\ \left[\frac{1}{r}\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\overrightarrow{u_\theta} \\ \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right]\overrightarrow{u_z} \end{array} \right. \tag{3}$$

Since the problem is infinite along the z-axis and is invariant when rotating around this axis, the vector potential does not depend on θ and z, and Eq. (3) reduces to

$$\vec{B} = \left| \begin{array}{l} 0.\overrightarrow{u_r} \\ -\frac{\partial A_z}{\partial r}\overrightarrow{u_\theta} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\overrightarrow{u_z} \end{array} \right. \tag{4}$$

Then, by identifying Eqs. (1) and (4), we derive

$$-\frac{\mu_0 I}{2\pi b} = -\frac{\partial A_z}{\partial r} \tag{5}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,A_\theta) = 0 \tag{6}$$

The integration of Eq. (5) simply gives

$$A_z = \frac{\mu_0 I}{2\pi}\ln b + K_z \tag{7}$$

where K_z is a constant which can be set to zero.

Similarly, the integration of Eq. (6) yields

$$A_\theta = \frac{K_\theta}{b} \tag{8}$$

where K_θ is a constant which can also be set to zero.

Equations (7) and (8) fully determined A_z and A_θ, while A_r can assume any expression. For the sake of simplicity, let us choose

$$A_r(r) = A_\theta(r) = 0 \qquad \text{for all } r \tag{9}$$

Then, we finally have

$$\overrightarrow{\mathrm{A}} = A_z(r)\,\overrightarrow{\mathrm{u_z}} \tag{10}$$

where A_z is given by Eq. (7).

3.2 SINGLE CURRENT LINE IN FREE SPACE, OUTSIDE THE ORIGIN OF THE COORDINATE SYSTEM

3.2.1 EXPRESSION OF THE VECTOR POTENTIAL

Let us consider a current line, of intensity, (-*I*), parallel to the *z*-axis, and crossing the (O,*x*,*y*) plane, at a point N, different from O. The geometry and the notations are those of Figure 3.

Let M designate a given point of space, let *b* the distance between the current line and the point M and *r* be the distance between the *z*-axis and the point M, and let *a* designate the distance between the *z*-axis and the current line.

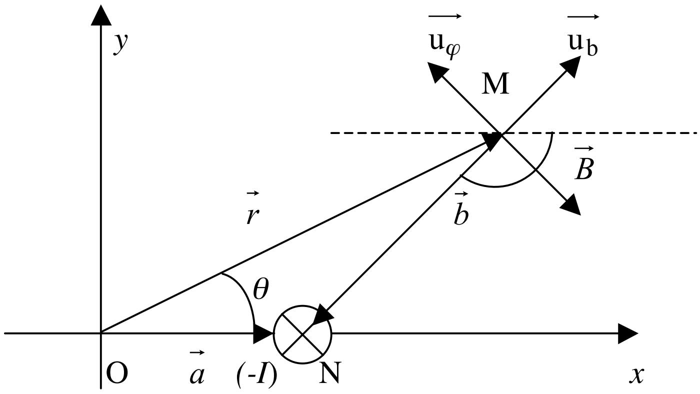

Figure 3. Single current line in free space outside the coordinate system origin.

Furthermore, let **a** and **s** be the coordinates in the complex plane of N and M. In this case, if **b** is vector $\overrightarrow{\mathrm{MN}}$, we have

$$\mathbf{b} = \mathbf{a} - \mathbf{s} \tag{11}$$

with

$$\mathbf{a} = a \qquad \text{and} \qquad \mathbf{s} = r e^{i\theta} \tag{12}$$

To pursue, let us rely on the complex formalism that has been developed for two-dimensional fields {Chapter 7 of Ref. [1]} and let **A** be the complex potential such that

$$\mathrm{Re}(\mathbf{A}) = A_z \tag{13}$$

From Eqs. (7) and (11), we can write

$$\mathbf{A} = \frac{\mu_0 I}{2\pi} \ln \mathbf{b} \qquad \text{for } r,\ r < a \tag{14}$$

and

$$\mathbf{A} = \frac{\mu_0 I}{2\pi} \ln(-\mathbf{b}) \qquad \text{for } r,\ r > a \tag{15}$$

Note that Eqs. (14) and (15) satisfy Eq. (13). In the case of accelerator magnets, Eq. (14) is useful for beam optics computation, while Eq. (15) is useful for magnetic computation.

Let us now distinguish the two cases corresponding to Eqs. (14) and (15).

3.2.2 Development in the case where $r < a$

This inequality translates in the complex plane into $\left|\mathbf{s}/a\right| < 1$. Let us introduce this ratio into Eq. (14). We get

$$\mathbf{A} = \frac{\mu_0 I}{2\pi}\ln(\mathbf{b}) = \frac{\mu_0 I}{2\pi}\ln(\mathbf{a}-\mathbf{s}) = \frac{\mu_0 I}{2\pi}\left[\ln(\mathbf{a}) + \ln\left(1-\frac{\mathbf{s}}{\mathbf{a}}\right)\right] = \frac{\mu_0 I}{2\pi}\left[\ln(\mathbf{a}) + \ln\left(1-\frac{\mathbf{s}}{a}\right)\right] \tag{16}$$

The inequality $\left|\mathbf{s}/a\right| < 1$ leads us to develop $\ln(1-\mathbf{r}/a)$ in Taylor's series. With Moivre's formula, this development gives

$$\ln\left(1-\frac{\mathbf{s}}{a}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\mathbf{s}}{a}\right)^{\mathrm{n}} = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r\,e^{\mathrm{i}\theta}}{a}\right)^{n} = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^{n}\left[\cos(n\theta)+\mathrm{i}\sin(n\theta)\right] \qquad (17)$$

By combining Eqs. (16) and (17), we get

$$\mathrm{Re}(\mathbf{A}) = -\frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^{n}\cos(n\theta) - \ln(a)\right] \qquad (18)$$

and by combining Eqs. (13) and (18), we finally obtain

$$A_z = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^{n}\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \qquad \text{for } r,\ 0 < r < a \qquad (19)$$

3.2.3 DEVELOPMENT IN THE CASE WHERE $r > a$

A derivation similar to the one in the previous section leads to

$$A_z = \mathrm{Re}\left[\frac{\mu_0 I}{2\pi}\ln(-\mathbf{b})\right] = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^{n}\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) \qquad \text{for } r,\ \mathrm{a} < \mathrm{r} \qquad (20)$$

Note that the vector potential is continue at $r = a$.

3.3 MULTIPOLE FIELD COEFFICIENTS

Let us define **s** as

$$\mathbf{s} = x + \mathrm{i}\,y = r\,\mathrm{e}^{\mathrm{i}\theta} \qquad (21)$$

and, as done in Ref. [1], let us introduce the complex magnetic flux density, $\mathbf{B}^{\mathbf{line}}$, defined as

$$\mathbf{B}^{\mathbf{line}}(\mathbf{s}) = B_{\mathrm{y}}(x,y) + \mathrm{i}\,B_{\mathrm{x}}(x,y) \qquad (22)$$

By combining Eqs. (3), (9) and (19), it is easy to show that

$$\vec{\mathrm{B}}^{\mathrm{line}} = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{r^{n-1}}{a^n}\left|\begin{array}{l}\sin(n\theta)\ \overrightarrow{\mathrm{u_r}}\\ \cos(n\theta)\ \overrightarrow{\mathrm{u_\theta}}\\ 0.\overrightarrow{\mathrm{u}_z}\end{array}\right. \qquad \text{for } r,\ r < a \qquad (23)$$

If we use cylindrical coordinates, Eq. (22) can be rewritten

$$\begin{aligned}\mathbf{B}^{\mathbf{line}}(\mathbf{s}) &= B_{\mathrm{y}}(x,y) + \mathrm{i}\, B_{\mathrm{x}}(x,y) \\ &= (B_{\mathrm{r}}\sin\theta + B_{\theta}\cos\theta) + \mathrm{i}\,(B_{\mathrm{r}}\cos\theta - B_{\theta}\sin\theta)\end{aligned} \tag{24}$$

By combining Eqs. (23) and (24), we get (for $r < a$)

$$\begin{aligned}\mathbf{B}^{\mathbf{line}}(\mathbf{s}) &= \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{r^{n-1}}{a^n}\left[\cos(n-1)\theta + \mathrm{i}\sin(n-1)\theta\right] \\ &= \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{\mathbf{s}^{n-1}}{a^n}\end{aligned} \tag{25}$$

Introducing the multipole field coefficients $B_{\mathrm{n}}^{\mathrm{line}}$ and $A_{\mathrm{n}}^{\mathrm{line}}$ defined as

$$\mathbf{B}^{\mathbf{line}}(\mathbf{s}) = \sum_{n=1}^{\infty}(B_{\mathrm{n}}^{\mathrm{line}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{line}})\left(\frac{\mathbf{s}}{R_{\mathrm{ref}}}\right)^{n-1} \qquad \text{for } r,\, r < a \tag{26}$$

we simply have

$$B_{\mathrm{n}}^{\mathrm{line}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{line}} = \frac{\mu_0 I}{2\pi R_{\mathrm{ref}}}\left(\frac{R_{\mathrm{ref}}}{a}\right)^{n} \tag{27}$$

which is identical to the expression given in Ref. [1] in the case of a line of current alone in free space.

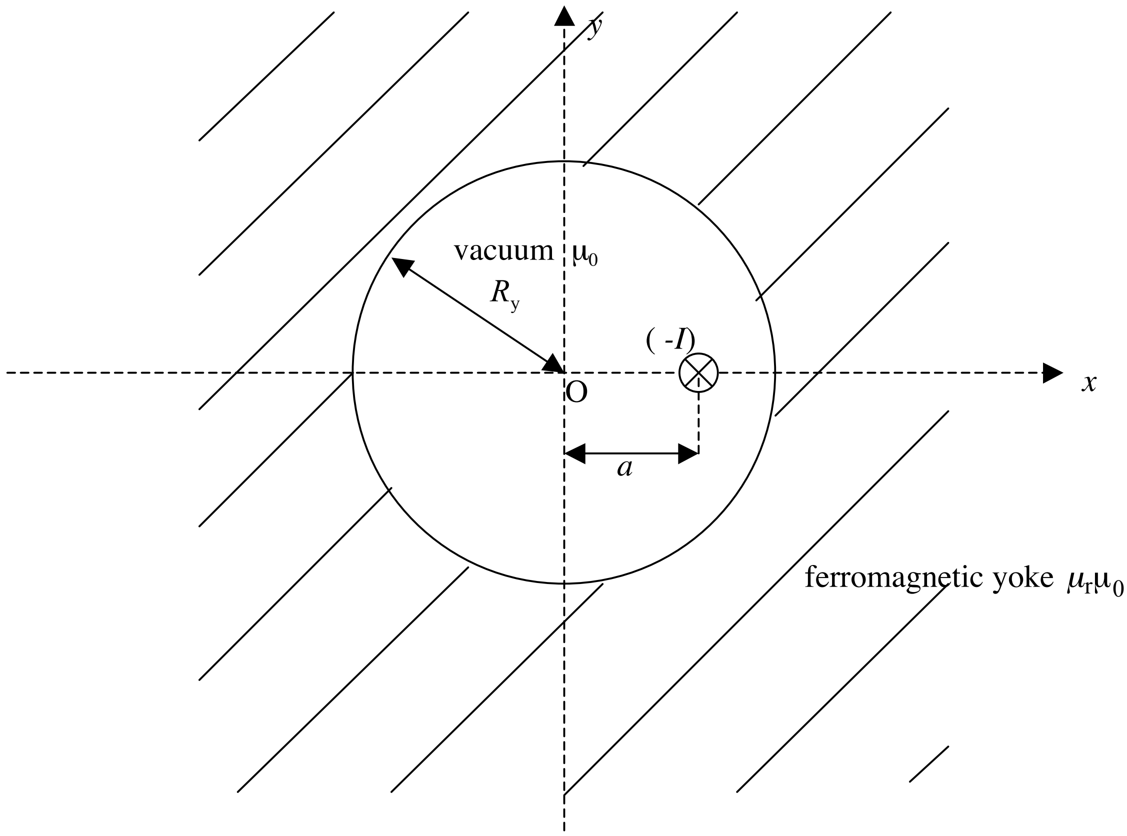

Figure 4. Current line inside a cylindrical ferromagnetic yoke.

4 CASE OF A CURRENT LINE LOCALIZED INSIDE A CYLINDER CONSTITUTED OF VACUUM, WHOSE OUTSIDE IS FERROMAGNETIC

4.1 GEOMETRY OF THE PROBLEM

Let us assume that the current line, (–*I*), is located within a cylindrical hollow space of *z*-axis and of radius, R_y, surrounded by a ferromagnetic yoke of relative magnetic permeability μ_r, as shown in Figure 4.

The method used thereafter is inspired from Ref. [2]. The vector potential will be noted $\overrightarrow{A_i}$, with $i = 1$ in the area 1 where $r < a$, $i = 2$ in the area 2 where $a < r < R_y$, and $i = 3$ in the area 3 where $r > R_y$. It will be determined by superimposing the solution found in the case when the current line is alone in free space and the one found in the case when the ferromagnetic yoke is alone.

4.2 EFFECT OF THE FERROMAGNETIC YOKE WITHOUT THE CURRENT LINE

4.2.1 GENERAL EQUATIONS

Let us consider the ferromagnetic medium alone, that is to say, without the current line (-*I*), and let us determine the vector potential in the whole space.

The problem can be solved using Laplace's equation in cylindrical coordinates

$$\Delta \overrightarrow{\mathrm{A}} + \mu_r \mu_0 \vec{\mathrm{j}} = \vec{0} \tag{28}$$

Without currents, Eq.(28) reduces to

$$\Delta \overrightarrow{\mathrm{A}} = \vec{0} \tag{29}$$

However, as explained in the previous section, the geometry of the problem is such that the vector potential can be taken parallel to the *z*-axis

$$\overrightarrow{\mathrm{A}} = A_z \overrightarrow{\mathrm{u_z}} \tag{30}$$

and, consequently, Eq. (29) can be rewritten

$$\Delta A_z = 0 \tag{31}$$

Then, in cylindrical coordinates, Laplace's equation becomes

$$r \frac{\partial}{\partial r}\left(r \frac{\partial A_z}{\partial r} \right) + \frac{\partial^2 A_z}{\partial \theta^2} = 0 \tag{32}$$

The method of separation of variables leads us to search the solution of Eq. (32) under the form

$$A_z = R(r)\Theta(\theta) \tag{33}$$

where a simple consideration of geometry imposes that $\Theta(\theta)$ is a 2π periodic function of θ.

Combining Eqs. (32) and (33) yields

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=0 \tag{34}$$

and

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \tag{35}$$

The left member of the Eq. (35) is only a function of r, whereas the right member only depends on θ. Then, we must have

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=K \tag{36}$$

where K is a real constant that does not depend on r and θ.

Three cases have now to be taken into consideration depending on the value of K $\{$see differential equations in Ref. [3]$\}$.

4.2.2 STUDY OF THE CASE $K = 0$

Equation (36) becomes

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=0 \tag{37}$$

It follows that Θ is a linear function of the form

$$\Theta_0 = E_0 + F_0\,\theta \tag{38}$$

where E_0 and F_0 are two real integration constants. The Θ_0 function must be periodical with a period of 2π. Hence, we necessary have: $F_0 = 0$, and the solution of Eq. (37) is

$$\Theta_0(\theta) = E_0 \tag{39}$$

For the radial dependence, we simply have

$$R_0(r) = C_0 + D_0\ln(r) \tag{40}$$

where C_0 and D_0 are two real integration constants.

4.2.3 STUDY OF THE CASE $K < 0$

Then, we can write

$$K = -\alpha^2 \tag{41}$$

where α is real, and Eq. (36) becomes

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = -\alpha^2 \tag{42}$$

It follows that Θ is a hyperbolic function of the form

$$\Theta_{-\alpha^2}(\theta) = E_{-\alpha^2} ch(\alpha\theta) + F_{-\alpha^2} sh(\alpha\theta) \tag{43}$$

where $E_{-\alpha^2}$ and $F_{-\alpha^2}$ are two real integration constants.

The expression given by Eq. (43) cannot be periodical with a period of 2π, except when $\alpha = 0$ or $E_{-\alpha^2} = F_{-\alpha^2} = 0$, which is equivalent to the previous case.

4.2.4 STUDY OF THE CASE $K > 0$

Then, we can write

$$K = \alpha^2 \tag{44}$$

where α is real, and Eq. (36) becomes

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = \alpha^2 \tag{45}$$

It follows that Θ is a trigonometric function of the form

$$\Theta_{\alpha^2}(\theta) = E_{\alpha^2}\cos(\alpha\theta) + F_{\alpha^2}\sin(\alpha\theta) \tag{46}$$

where E_{α^2} and F_{α^2} are two real integration constants.

Moreover, the periodicity of Θ imposes that, for any integer, m, different from zero, we must have

$$\begin{aligned}\Theta_{\alpha^2}(\theta) &= E_{\alpha^2}\cos(\alpha\theta) + F_{\alpha^2}\sin(\alpha\theta)\\ &= \Theta_{\alpha^2}(\theta + 2m\pi)\\ &= E_{\alpha^2}\cos\left[\alpha(\theta + 2m\pi)\right] + F_{\alpha^2}\sin\left[\alpha(\theta + 2m\pi)\right]\end{aligned} \tag{47}$$

Thus

$$\cos(\alpha\theta) = \cos\left[\alpha(\theta + 2m\pi)\right] \quad \text{and} \quad \sin(\alpha\theta) = \sin\left[\alpha(\theta + 2m\pi)\right] \quad \text{for all } m \tag{48}$$

This condition can only be satisfied if ($m\alpha$) is an integer. Since m is an integer, it imposes that α be an integer too. In the following, α will be noted k, where k assumes integer values. Let us simplify the notation by indexing Θ by k and by writing it under the form

$$\Theta_k(\theta) = E_k \cos(k\theta) + F_k \sin(k\theta) \tag{49}$$

where E_k and F_k are the real integration constants relative to k.

In this case, the solution of the radial part, $R_k(r)$, of the equation

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R_k}{\partial r}\right) = k^2 \tag{50}$$

is

$$R_k(r) = C_k r^k + D_k r^{-k} \tag{51}$$

where C_k and D_k are real integration constants relative to k.

4.2.5 General solutions

4.2.5.1 Expressions of the solutions

The most general solution of Laplace's equation, taking into account the effect of the ferromagnetic medium, is a linear superimposition of all particular solutions

$$A_z = \sum_{k=1}^{\infty}\left[E_k \cos(k\theta) + F_k \sin(k\theta)\right](C_k r^k + D_k r^{-k}) + E_0\left[C_0 + D_0 \ln(r)\right] \tag{52}$$

By symmetry, the vector potential must be invariant when θ becomes $-\theta$. This is only possible when $F_k = 0$. Hence, the general solution can be simplified as

$$A_z = \sum_{k=1}^{\infty} E_k \cos(k\theta)\left(C_k r^k + D_k r^{-k}\right) + E_0\left[C_0 + D_0 \ln(r)\right] \tag{53}$$

To pursue, let us distinguish the cases $r < R_y$ and $r > R_y$.

4.2.5.2 *Study of the case* $r < R_y$

Starting from Eq. (53), the vector potential can be rewritten under the form

$$A_z = \sum_{n=1}^{\infty} C_n^{in}\cos(n\theta)r^n + D_n^{in}\cos(n\theta)r^{-n} + C_0^{in} + D_0^{in}\ln(r) \tag{54}$$

where C_n^{in}, D_n^{in}, C_0^{in} and D_0^{in} are real constants defined as

$$C_n^{in} = E_n\,C_n,\ \ D_n^{in} = E_n\,D_n,\ \ C_0^{in} = E_0\,C_0 \text{ and } D_0^{in} = E_0\,D_0 \tag{55}$$

The index 'in' indicates that the vector potential due to the ferromagnetic yoke is computed inside the cylindrical hollow space of radius R_y.

In this case, one can eliminate the terms D_n^{in} and D_0^{in} in Eq. (54) because they induce a divergence of the vector potential when *r* tends towards 0. Then, Eq. (54) becomes

$$A_z = \sum_{n=1}^{\infty} C_n^{in}\cos(n\theta)r^n + C_0^{in} \qquad \text{for } r,\ r < R_y \tag{56}$$

4.2.5.3 *Study of the case* $r > R_y$

Starting again from Eq. (53), the vector potential can be rewritten under the form

$$A_z = \sum_{n=1}^{\infty} C_n^{ex}\cos(n\theta)r^n + D_n^{ex}\cos(n\theta)r^{-n} + C_0^{ex} + D_0^{ex}\ln(r) \tag{57}$$

where C_n^{ex}, D_n^{ex}, C_0^{ex} and D_0^{ex} are real constants defined as

$$C_n^{ex} = E_n\,C_n,\ \ D_n^{ex} = E_n\,D_n,\ \ C_0^{ex} = E_0\,C_0 \text{ and } D_0^{ex} = E_0\,D_0 \tag{58}$$

The index 'ex' indicates that the vector potential due to the ferromagnetic yoke is computed outside the cylindrical hollow space of radius R_y.

Towards infinity, the vector potential cannot rise faster than ln (*r*), which imposes that the C_n^{ex} terms be eliminated. It follows that Eq. (57) can be rewritten

$$A_z = \sum_{n=1}^{\infty} D_n^{ex}\cos(n\theta)r^{-n} + C_0^{ex} + D_0^{ex}\ln(r) \qquad \text{for } r,\ r > R_y \tag{59}$$

4.3 LINEAR SUPERIMPOSITION OF THE VECTOR POTENTIALS DUE TO THE CURRENT LINE AND TO THE FERROMAGNETIC YOKE

To derive the solutions of the problem shown in Figure 4, we simply have to superimpose the vector potential due to the current line alone with the one due to the ferromagnetic yoke alone.

4.3.1 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $r < a < R_y$

From Eqs. (19) and (56), the vector potential can be written

$$A_{1\mathrm{z}} = \sum_{n=1}^{\infty} C_{\mathrm{n}}^{\mathrm{in}} \cos(n\theta) r^n + C_0^{\mathrm{in}} - \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \cos(n\theta) + \frac{\mu_0 I}{2\pi} \ln(a) \tag{60}$$

which can be simplified into

$$A_{1z} = \sum_{n=1}^{\infty}\left[C_{\mathrm{n}}^{\mathrm{in}} r^n - \frac{\mu_0 I}{2\pi n}\left(\frac{r}{a}\right)^n\right]\cos(n\theta) + C_0^{\mathrm{in}} + \frac{\mu_0 I}{2\pi}\ln(a) \tag{61}$$

4.3.2 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $a < r < R_y$

In this case, from Eqs. (20) and (56), the vector potential can be rewritten

$$A_{2\mathrm{z}} = \sum_{n=1}^{\infty}\left[C_{\mathrm{n}}^{\mathrm{in}} r^n - \frac{\mu_0 I}{2\pi n}\left(\frac{a}{r}\right)^n\right]\cos(n\theta) + C_0^{\mathrm{in}} + \frac{\mu_0 I}{2\pi}\ln(r) \tag{62}$$

4.3.3 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $r > R_y$

In this case, from Eqs. (20) and (59), the vector potential can be rewritten

$$A_{3\mathrm{z}} = \sum_{n=1}^{\infty}\left[D_{\mathrm{n}}^{\mathrm{ex}} r^{-n} - \frac{\mu_0 I}{2\pi n}\left(\frac{a}{r}\right)^n\right]\cos(n\theta) + C_0^{\mathrm{ex}} + \left(D_0^{\mathrm{ex}} + \frac{\mu_0 I}{2\pi}\right)\ln(r) \tag{63}$$

4.3.4 METHOD OF DETERMINATION OF THE CONSTANTS OF INTEGRATION

We finally have to determine the coefficients $C_{\mathrm{n}}^{\mathrm{in}}$, $D_{\mathrm{n}}^{\mathrm{ex}}$, C_0^{in}, C_0^{ex} et D_0^{ex}, which appear in Eqs. (61) through (63). These coefficients can be found by writing the boundary conditions at $r = R_{\mathrm{y}}$.

4.4 RESOLUTION AND SOLUTION OF THE PROBLEM

The boundary conditions at $r = R_{\mathrm{y}}$ are (see Appendix)

$$\left(A_{2\mathrm{z}} = A_{3\mathrm{z}}\right)_{r=R_{\mathrm{y}}} \tag{64}$$

and

$$\left(\frac{\partial A_{2\mathrm{z}}}{\partial r} = \frac{1}{\mu_{\mathrm{r}}}\frac{\partial A_{3\mathrm{z}}}{\partial r}\right)_{r=R_{\mathrm{y}}} \tag{65}$$

By combining Eqs. (62), (63) and (64), we find

$$\sum_{n=1}^{\infty}\left(C_{\mathrm{n}}^{\mathrm{in}}\, {R_{\mathrm{y}}}^{n} - D_{\mathrm{n}}^{\mathrm{ex}}\, {R_{\mathrm{y}}}^{-n}\right)\cos(n\theta) + C_{0}^{\mathrm{in}} - C_{0}^{\mathrm{ex}} - D_{0}^{\mathrm{ex}} \ln R_{\mathrm{y}} = 0 \tag{66}$$

So, by identifying the harmonics terms

$$\begin{cases} C_{\mathrm{n}}^{\mathrm{in}}\, {R_{\mathrm{y}}}^{n} - D_{\mathrm{n}}^{\mathrm{ex}}\, {R_{\mathrm{y}}}^{-n} = 0 \qquad \text{for } n,\ n \geq 1 \\ C_{0}^{\mathrm{in}} = C_{0}^{\mathrm{ex}} + D_{0}^{\mathrm{ex}} \ln R_{\mathrm{y}} \end{cases} \tag{67}$$

The vector potential involves an arbitrary constant. Let us choose $C_{0}^{\mathrm{in}} = 0$. Then

$$C_{0}^{\mathrm{ex}} = -\, D_{0}^{\mathrm{ex}} \ln R_{\mathrm{y}} \tag{68}$$

Similarly, by identifying the harmonics terms, Eq. (65) gives

$$\begin{cases} C_{\mathrm{n}}^{\mathrm{in}}\, {R_{\mathrm{y}}}^{n-1} + \dfrac{1}{\mu_{\mathrm{r}}} D_{\mathrm{n}}^{\mathrm{ex}}\, {R_{\mathrm{y}}}^{-n-1} = -\dfrac{\mu_0 I}{2\pi n}\left(1 - \dfrac{1}{\mu_{\mathrm{r}}}\right)\dfrac{a^{n}}{{R_{\mathrm{y}}}^{n+1}} \qquad \text{for } n,\ n \geq 1 \\ D_{0}^{\mathrm{ex}} = \dfrac{\mu_0 I}{2\pi}(\mu_{\mathrm{r}} - 1) \end{cases} \tag{69}$$

By combining Eqs. (67), (68) and (69), we find

$$\left.\begin{cases} C_{\mathrm{n}}^{\mathrm{in}} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}} + 1}\right)\left(\dfrac{a}{{R_{\mathrm{y}}}^{2}}\right)^{n} \\ D_{\mathrm{n}}^{\mathrm{ex}} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}} + 1}\right)a^{n} \end{cases}\right. \qquad \text{for } n,\ n \geq 1 \tag{70}$$

and

$$C_0^{\mathrm{ex}} = -\frac{\mu_0 I}{2\pi}(\mu_\mathrm{r}-1)\ln R_\mathrm{y} \quad \text{and} \quad D_0^{\mathrm{ex}} = \frac{\mu_0 I}{2\pi}(\mu_\mathrm{r}-1) \tag{71}$$

By inserting these expressions into Eqs. (61), (62) and (63), we finally get

$$A_{1\mathrm{z}} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n + \left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left(\frac{ar}{{R_\mathrm{y}}^2}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r,\ r < a < R_\mathrm{y} \tag{72}$$

$$A_{2\mathrm{z}} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n + \left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left(\frac{ar}{{R_\mathrm{y}}^2}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) \quad \text{for } r,\ a < r < R_\mathrm{y} \tag{73}$$

$$A_{3\mathrm{z}} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{2\mu_\mathrm{r}}{\mu_\mathrm{r}+1}\right)\left(\frac{a}{r}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) + \left(\mu_\mathrm{r}-1\right)\frac{\mu_0 I}{2\pi}\ln\left(\frac{r}{R_\mathrm{y}}\right) \quad \text{for } r,\ r > R_\mathrm{y} \tag{74}$$

Note that, as expected, these expressions are continuous at $r = a$ and R_y.

4.5 IMAGE CURRENT

To analyse the last results, let us consider the vector potential, $A_\mathrm{z}^{\mathrm{im}}$, created by a current line of intensity, $\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)(-I)$, parallel to the z-axis, and located at a distance $\left(\frac{{R_\mathrm{y}}^2}{a}\right)$ from this axis. From Eq. (19), we have

$$A_\mathrm{z}^{\mathrm{im}} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left(\frac{ar}{{R_\mathrm{y}}^2}\right)^n\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln\left(\frac{{R_\mathrm{y}}^2}{a}\right) \quad \text{for } r,\ r < R_\mathrm{y} < \frac{{R_\mathrm{y}}^2}{a} \tag{75}$$

The vector potential is defined with an arbitrary constant. Let us eliminate the term in $\ln\left(\frac{{R_\mathrm{y}}^2}{a}\right)$ from Eq. (75) and write

$$A_\mathrm{z}^{\mathrm{im}} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left(\frac{ar}{{R_\mathrm{y}}^2}\right)^n\cos(n\theta) \quad \text{for } r,\ r < R_\mathrm{y} < \frac{{R_\mathrm{y}}^2}{a} \tag{76}$$

By superimposing Eqs. (19) and (76), one finds the same expression as in Eq. (72).

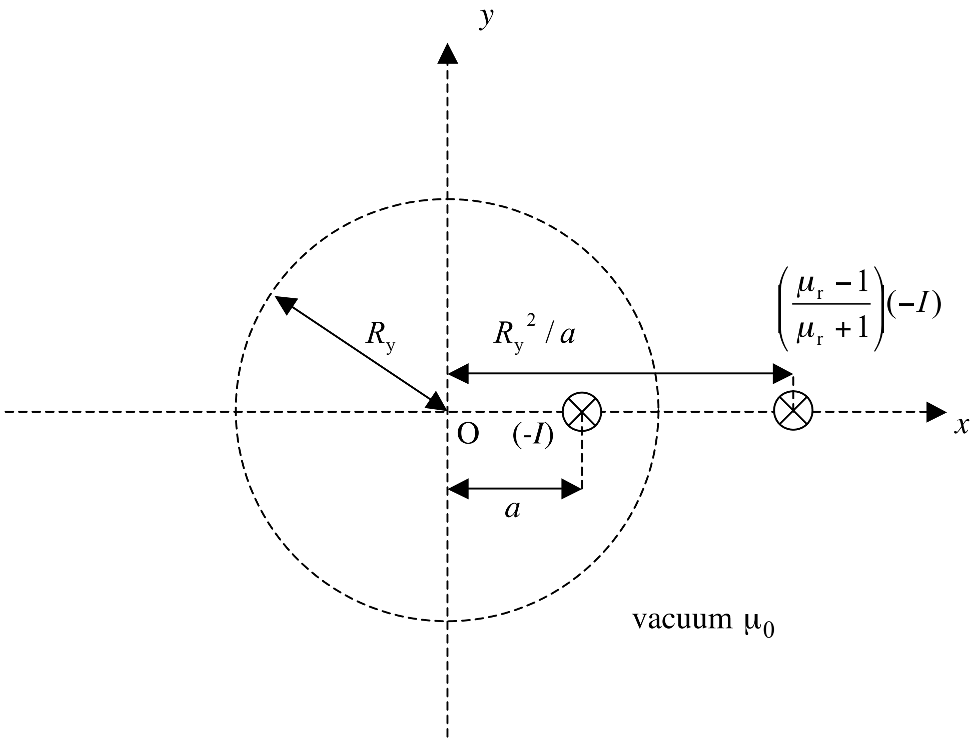

Figure 5. Image current for the problem of Figure 4 seen from inside the cylindrical hollow space.

Hence, the contribution of the ferromagnetic yoke can be directly calculated with the help of this 'image' current line. This result is in agreement with the classical results published in the literature (p. 72–73 of Ref. [1] and p. 53 of Ref. [4]). The image current method is illustrated in Figure 5.

4.6 MULTIPOLE FIELD COEFFICIENTS

Since the contribution from the ferromagnetic yoke is equivalent to that of a single current line $\left(\frac{\mu_r - 1}{\mu_r + 1}\right)(-I)$ at a distance $\left(\frac{R_y^{\,2}}{a}\right)$ from the z-axis, it follows from section 3.3 that the resulting complex magnetic flux density, $\mathbf{B}^{\mathbf{yoke}}$, is

$$\mathbf{B}^{\mathbf{yoke}}(\mathbf{s}) = \frac{\mu_0 I}{2\pi R_{\mathrm{ref}}}\left(\frac{\mu_r - 1}{\mu_r + 1}\right)\sum_{n=1}^{\infty}\left(\frac{a R_{\mathrm{ref}}}{R_y^{\,2}}\right)^{n}\left(\frac{\mathbf{s}}{R_{\mathrm{ref}}}\right)^{n-1} \tag{77}$$

Then, the corresponding multipole field coefficients, $B_n^{\mathrm{yoke}} + \mathrm{i}\,A_n^{\mathrm{yoke}}$, defined from

$$\mathbf{B}^{\mathbf{yoke}}(\mathbf{s}) = \sum_{n=1}^{\infty}\left(B_n^{\mathrm{yoke}} + \mathrm{i}\,A_n^{\mathrm{yoke}}\right)\left(\frac{\mathbf{s}}{R_{\mathrm{ref}}}\right)^{n-1} \qquad \text{for } r,\ r < a \tag{78}$$

are given by

$$B_{\mathrm{n}}^{\mathrm{yoke}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{yoke}} = \frac{\mu_0 I}{2\pi R_{\mathrm{ref}}} \left(\frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}} + 1} \right) \left(\frac{a R_{\mathrm{ref}}}{R_{\mathrm{y}}{}^2} \right)^n \tag{79}$$

Furthermore, by adding the effects of the current line and of the ferromagnetic yoke, it follows that the total complex magnetic flux density $\mathbf{B}^{\mathbf{tot}}$ is

$$\mathbf{B}^{\mathbf{tot}}(\mathbf{s}) = \frac{\mu_0 I}{2\pi R_{\mathrm{ref}}} \sum_{n=1}^{\infty} \left[1 + \left(\frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}} + 1} \right) \left(\frac{a}{R_{\mathrm{y}}} \right)^{2n} \right] \left(\frac{R_{\mathrm{ref}}}{a} \right)^n \left(\frac{\mathbf{s}}{R_{\mathrm{ref}}} \right)^{n-1} \tag{80}$$

while the corresponding multipole field coefficient, $B_{\mathrm{n}}^{\mathrm{tot}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{tot}}$ are given by

$$B_{\mathrm{n}}^{\mathrm{tot}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{tot}} = \frac{\mu_0 I}{2\pi R_{\mathrm{ref}}} \left[1 + \left(\frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}} + 1} \right) \left(\frac{a}{R_{\mathrm{y}}} \right)^{2n} \right] \left(\frac{R_{\mathrm{ref}}}{a} \right)^n \tag{81}$$

By comparison with Eq. (27), we find

$$B_{\mathrm{n}}^{\mathrm{tot}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{tot}} = \left[1 + \left(\frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}} + 1} \right) \left(\frac{a}{R_{\mathrm{y}}} \right)^{2n} \right] \left(B_{\mathrm{n}}^{\mathrm{line}} + \mathrm{i}\, A_{\mathrm{n}}^{\mathrm{line}} \right) \tag{82}$$

This result is consistent with the results given in Refs. [1] and [4].

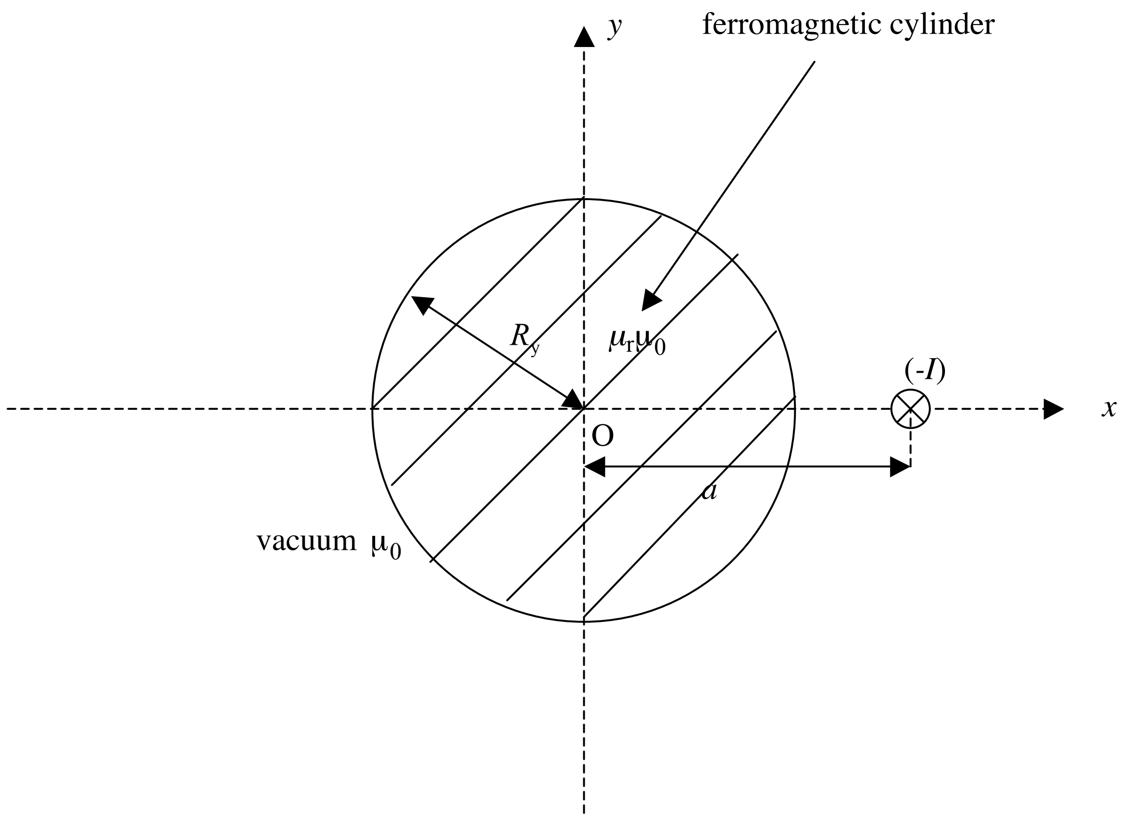

Figure 7. Current line outside a ferromagnetic cylinder.

5 CASE OF A CURRENT LINE LOCALIZED OUTSIDE A FERROMAGNETIC CYLINLINDER

5.1 GEOMETRY OF THE PROBLEM

Let us consider, in the vacuum of permeability, μ_0, a current line, (–*I*), localized outside a ferromagnetic cylinder of radius, R_y, and of relative permeability, μ_r, as represented in Figure 7. The *z*-axis and the current line are parallel and separated by a distance, referred to as *a*.

The method used thereafter is similar to the one used in section 4. The vector potential will be written $\overrightarrow{A_i}$, with $i = 1$ in the area 1 where $r < R_y$, $i = 2$ in the area 2 where $R_y < r < a$, and $i = 3$ in the area 3 where $r > a$.

Once again, the solution will be determined by superimposing the potential vector found in the case where the current line is alone in free space [as given by Eq. (19) for $0 < a < r$ and Eq. (20) for $a < r$] and the one found in the case where the ferromagnetic yoke is alone [as given by Eq. (56) for $r < R_y$ and Eq. (59) for $r > R_y$].

5.2 LINEAR SUPERIMPOSITION OF THE VECTOR POTENTIALS DUE TO THE CURRENT LINE AND TO THE FERROMAGNETIC CYLINDER

5.2.1 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $r < R_y < a$

From Eqs. (19) and (56), the vector potential can be written under the form

$$A_{1z} = \sum_{n=1}^{\infty} \left\{ C_n^{in} r^n - \frac{\mu_0 I}{2\pi n} \left(\frac{r}{a} \right)^n \right\} \cos(n\theta) + C_0^{in} + \frac{\mu_0 I}{2\pi} \ln(a) \tag{83}$$

5.2.2 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $R_y < r < a$

In this case, from Eqs. (19) and (59), we get

$$A_{2z} = \sum_{n=1}^{\infty} \left\{ D_n^{ex} r^{-n} - \frac{\mu_0 I}{2\pi n} \left(\frac{r}{a} \right)^n \right\} \cos(n\theta) + C_0^{ex} + D_0^{ex} \ln(r) + \frac{\mu_0 I}{2\pi} \ln a \tag{84}$$

5.2.3 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $r > a$

In this case, from Eqs. (20) and (59), we get

$$A_{3z} = \sum_{n=1}^{\infty} \left\{ D_n^{ex} r^{-n} - \frac{\mu_0 I}{2\pi n} \left(\frac{a}{r} \right)^n \right\} \cos(n\theta) + C_0^{ex} + \left(D_0^{ex} + \frac{\mu_0 I}{2\pi} \right) \ln(r) \tag{85}$$

5.2.4 METHOD OF DETERMINATION OF THE CONSTANTS OF INTEGRATION

Now, one must determine the coefficients C_n^{in}, D_n^{ex}, C_0^{in}, C_0^{ex} and D_0^{ex}, which appear in Eqs. (83), (84) and (85). These coefficients can be found by writing the boundary conditions at $r = R_y$.

5.2.5 RESOLUTION AND SOLUTION OF THE PROBLEM

The boundary conditions at $r = R_y$ are (see Appendix)

$$\left(A_{1z} = A_{2z} \right)_{r=R_y} \tag{86}$$

and

$$\left(\frac{\partial A_{2z}}{\partial r} = \frac{1}{\mu_r} \frac{\partial A_{1z}}{\partial r} \right)_{r=R_y} \tag{87}$$

Equation (86) yields

$$\sum_{n=1}^{\infty}\left(C_{\mathrm{n}}^{\mathrm{in}}\,{R_{\mathrm{y}}}^{n}-D_{\mathrm{n}}^{\mathrm{ex}}\,{R_{\mathrm{y}}}^{-n}\right)\cos(n\theta)+C_{0}^{\mathrm{in}}-C_{0}^{\mathrm{ex}}-D_{0}^{\mathrm{ex}}\ln R_{\mathrm{y}}=0 \tag{88}$$

Hence, we get

$$\begin{cases} C_{\mathrm{n}}^{\mathrm{in}}\,{R_{\mathrm{y}}}^{n}-D_{\mathrm{n}}^{\mathrm{ex}}\,{R_{\mathrm{y}}}^{-n}=0 & \text{for } n,\ n\geq 1 \\ C_{0}^{\mathrm{in}}=C_{0}^{\mathrm{ex}}+D_{0}^{\mathrm{ex}}\ln R_{\mathrm{y}} \end{cases} \tag{89}$$

Furthermore, let us choose $C_{0}^{\mathrm{in}}=0$. It follows that

$$C_{0}^{\mathrm{ex}}=-D_{0}^{\mathrm{ex}}\ln R_{\mathrm{y}} \tag{90}$$

Equation (87) gives

$$\begin{cases} D_{\mathrm{n}}^{\mathrm{ex}}{R_{\mathrm{y}}}^{-2n}+\dfrac{1}{\mu_{\mathrm{r}}}C_{\mathrm{n}}^{\mathrm{in}}=-\dfrac{\mu_0 I}{2\pi n}\left(1-\dfrac{1}{\mu_{\mathrm{r}}}\right)\dfrac{1}{a^{n}} & \text{for } n,\ n\geq 1 \\ D_{0}^{\mathrm{ex}}=0 \end{cases} \tag{91}$$

Then, by combining Eqs. (89) and (91), we find

$$\begin{cases} C_{\mathrm{n}}^{\mathrm{in}}=-\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}+1}\right)\left(\dfrac{1}{a}\right)^{n} \\ D_{\mathrm{n}}^{\mathrm{ex}}=-\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}+1}\right)\left(\dfrac{{R_{\mathrm{y}}}^{2}}{a}\right)^{n} \end{cases} \quad \text{for } n,\ n\geq 1 \tag{92}$$

and by using Eq. (91)

$$C_{0}^{\mathrm{in}}=C_{0}^{\mathrm{ex}}=D_{0}^{\mathrm{ex}}=0 \tag{93}$$

By inserting the above expressions into Eqs. (83), (84) and (85), we finally get

$$A_{1\mathrm{z}}=-\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{2\mu_{\mathrm{r}}}{\mu_{\mathrm{r}}+1}\right)\left(\frac{r}{a}\right)^{n}\right]\cos(n\theta)+\frac{\mu_0 I}{2\pi}\ln(a) \qquad \text{for } r,\ 0<r<R_{\mathrm{y}} \tag{94}$$

$$A_{2\mathrm{z}}=-\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^{n}+\left(\frac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}+1}\right)\left(\frac{{R_{\mathrm{y}}}^{2}}{ar}\right)^{n}\right]\cos(n\theta)+\frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r,\ R_{\mathrm{y}}<r<a \tag{95}$$

$$A_{3z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\left(\frac{R_y{}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) \quad \text{for } r,\ r > a \qquad (96)$$

Note that, as expected, these expressions are continuous at $r = R_y$ and a.

5.3 IMAGE CURRENTS

Let us add and subtract

$$-\frac{\mu_0 I}{2\pi}\left(\frac{\mu_r - 1}{\mu_r + 1}\right)\ln(r) \qquad (97)$$

from Eqs. (95) and (96).

Then, it appears that the vector potential A_{2z} outside the yoke corresponds to the vector potential due to the current line, (–*I*), and to the effect of the ferromagnetic medium, represented by two current lines: a current line of intensity, $\left(\frac{\mu_r - 1}{\mu_r + 1}\right)(I)$, localized at the origin, and a current line of intensity, $\left(\frac{\mu_r - 1}{\mu_r + 1}\right)(-I)$, localized at the so called point of inversion of the exterior current line. The point of inversion is situated at a distance $\left(\frac{R_y{}^2}{a}\right)$ from the origin. These current lines are represented in Figure 8.

Indeed, the potential vector created by the current line (–*I*), is given by Eqs. (19) and (20). Hence, the one created by a line $\left(\frac{\mu_r - 1}{\mu_r + 1}\right)(-I)$, located at a distance $\left(\frac{R_y{}^2}{a}\right)$ from O, can be expressed as

$$A_z = -\frac{\mu_0 I}{2\pi}\left(\frac{\mu_r - 1}{\mu_r + 1}\right)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{R_y^2}{ar}\right)^n\cos(n\theta) + \frac{\mu_0 I}{2\pi}\left(\frac{\mu_r - 1}{\mu_r + 1}\right)\ln(r) \qquad \text{for } r,\ \left(\frac{R_y{}^2}{a}\right) < r \qquad (98)$$

Moreover, the vector potential due to the current $\left(\frac{\mu_r - 1}{\mu_r + 1}\right)(I)$ localized at the origin is

$$A_z = -\frac{\mu_0 I}{2\pi}\left(\frac{\mu_r - 1}{\mu_r + 1}\right)\ln(r) \qquad (99)$$

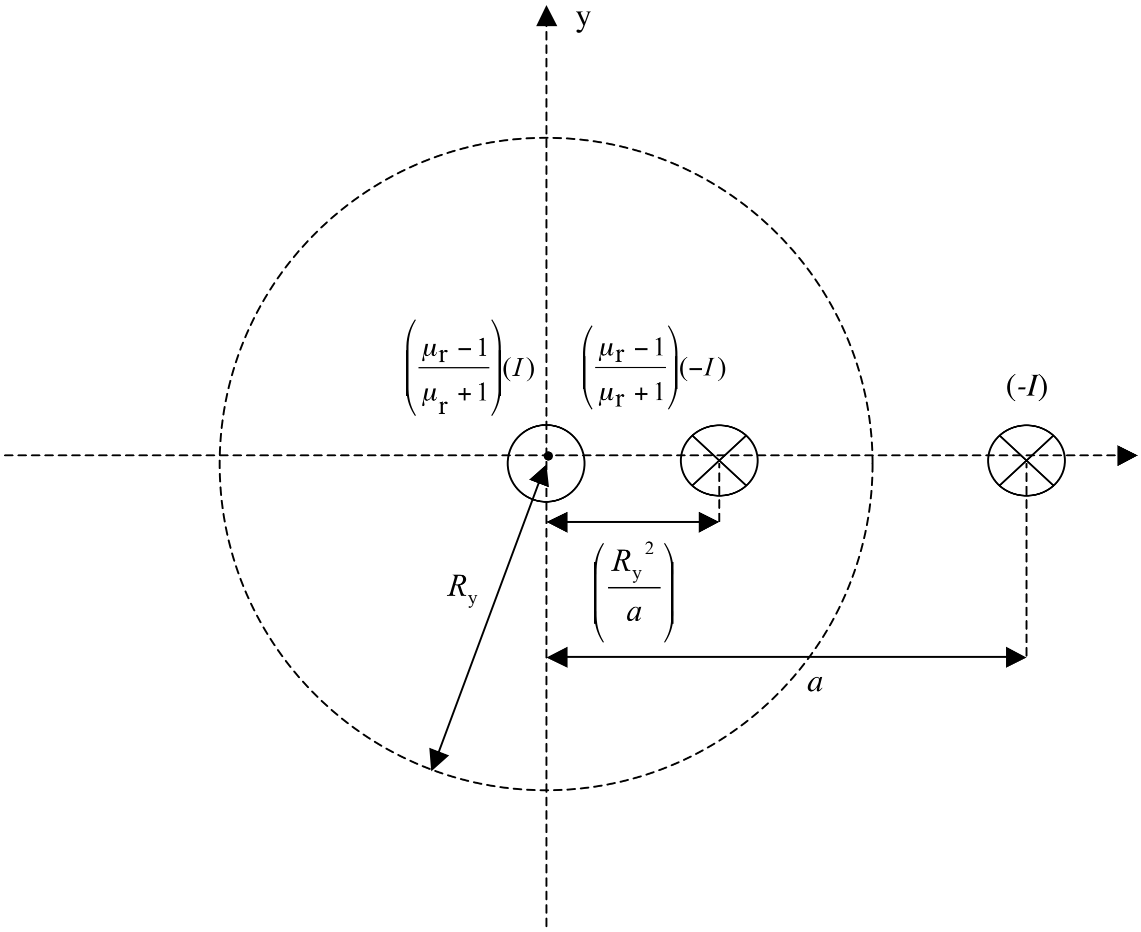

Figure 8. Image currents for the problem of Figure 7 seen outside the ferromagnetic cylinder.

By superimposing Eqs. (98) and (99), we eliminate the terms in (ln *r*) and we get the 'image' vector potential, A_z^{im}

$$A_z^{\mathrm{im}} = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\mu_r - 1}{\mu_r + 1}\right) \left(\frac{R_y^2}{ar}\right)^n \cos(n\theta) \tag{100}$$

This term is present in Eqs. (95) and (96), where it is added to the contribution of the current line (–*I*) located at **a**.

The result is the same as the one presented in Ref. [2] for the case of electrostatic.

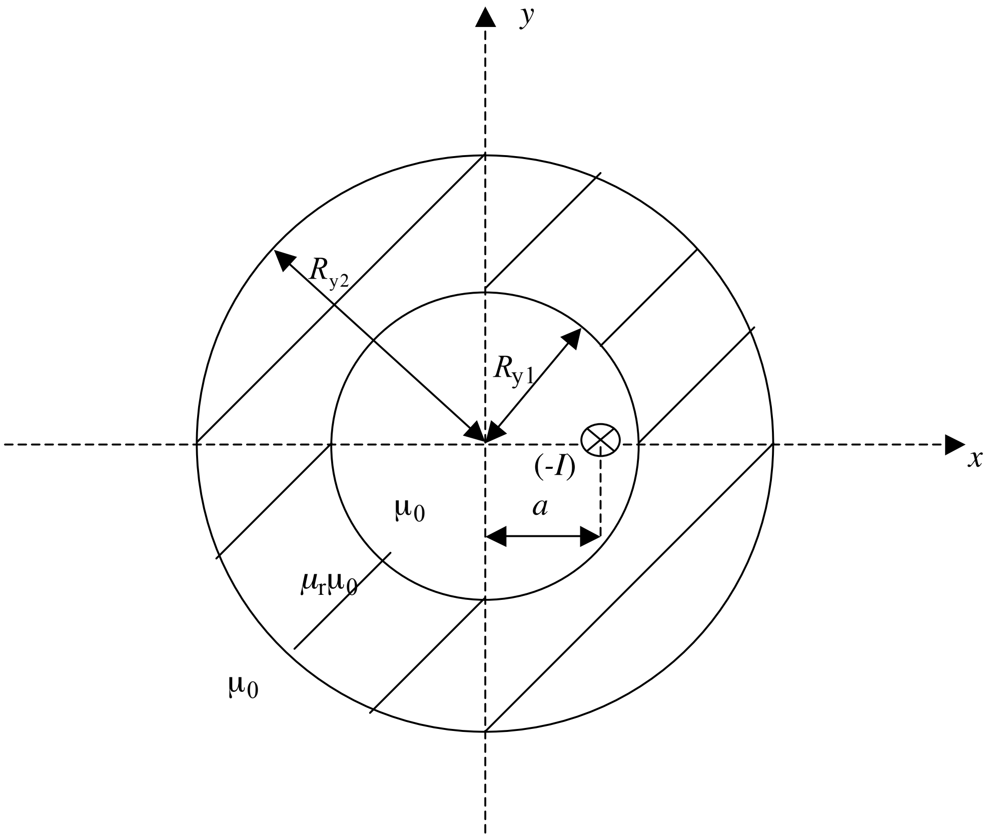

Figure 9. Ferromagnetic tube surrounding a current line.

6 CASE OF A FERROMAGNETIC TUBE OF FINITE INTERNAL AND EXTERNAL RADII SURROUNDING A CURRENT LINE

6.1 GEOMETRY OF THE PROBLEM

Let us consider a ferromagnetic tube, infinite along the *z*-axis, of internal radius R_{y1} and external radius R_{y2}, of permeability $\mu_r\mu_0$, and surrounding a current line, (–*I*), localized at a distance *a* from the *z*-axis. Furthermore, les us assume that the inside and outside of the tube are vacuum of permeability μ_0, as shown in Figure 9.

The potential vector will be noted $\overrightarrow{A_i}$ with $i = 1$ in the area 1 where $r < a$, $i = 2$ in the area 2 where $a < r < R_{y1}$, $i = 3$ in the area 3 where $R_{y1} < r < R_{y2}$ and $i = 4$ in the area 4 where $r > R_{y2}$.

Once again, we will estimate the vector potential by superimposition of the vector potentials of the current line [as given by Eqs. (19) and (20)] and of the ferromagnetic tube. The potential vector of the tube can be estimated from Eq. (56) in the area $0 < r < R_{y1}$, Eq. (53) in the area $R_{y1} < r < R_{y2}$ and Eq. (59) in the area $r > R_{y2}$.

6.2 LINEAR SUPERIMPOSITION OF THE VECTOR POTENTIAL DUE TO THE CURRENT LINE AND DUE TO THE FERROMAGNETIC TUBE

6.2.1 EXPRESSION OF THE VECTOR POTENTIAL IN THE AREA WHERE $r < a$

By combining Eqs. (19) and (56), the vector potential can be written under the form

$$A_{1\mathrm{z}} = \sum_{n=1}^{\infty} C_{\mathrm{n}}^{\mathrm{in}} \cos(n\theta) r^n - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n \cos(n\theta) - \ln(a)\right] + C_0^{\mathrm{in}} \quad (101)$$

6.2.2 EXPRESSION OF THE VECTOR POTENTIAL IN THE CASE WHERE $a < r < R_{\mathrm{y1}}$

By combining Eqs. (20) and (56), the vector potential can be written under the form

$$A_{2\mathrm{z}} = \sum_{n=1}^{\infty} C_{\mathrm{n}}^{\mathrm{in}} \cos(n\theta) r^n - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n \cos(n\theta) - \ln(r)\right] + C_0^{\mathrm{in}} \quad (102)$$

6.2.3 EXPRESSION OF THE VECTOR POTENTIAL IN THE CASE WHERE $R_{\mathrm{y1}} < r < R_{\mathrm{y2}}$

By combining Eqs. (20) and (53), the vector potential can be written under the form

$$A_{3\mathrm{z}} = \sum_{n=1}^{\infty}(C_{\mathrm{n}}^{\mathrm{co}} r^n + D_{\mathrm{n}}^{\mathrm{co}} r^{-n})\cos(n\theta) - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n \cos(n\theta) - \ln(r)\right] + C_0^{\mathrm{co}} + D_0^{\mathrm{co}}\ln(r)$$

(103)

6.2.4 EXPRESSION OF THE VECTOR POTENTIAL IN THE CASE WHERE $R_{\mathrm{y2}} < r$

By combing Eqs. (20) and (59), the vector potential can be written under the form

$$A_{4\mathrm{z}} = \sum_{n=1}^{\infty} D_{\mathrm{n}}^{\mathrm{ex}} \cos(n\theta) r^{-n} - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n \cos(n\theta) - \ln(r)\right] + C_0^{\mathrm{ex}} + D_0^{\mathrm{ex}}\ln(r) \quad (104)$$

6.2.5 METHOD OF DETERMINATION OF INTEGRATION CONSTANTS

The coefficients $C_{\mathrm{n}}^{\mathrm{in}}$, $C_{\mathrm{n}}^{\mathrm{co}}$, $D_{\mathrm{n}}^{\mathrm{co}}$, $D_{\mathrm{n}}^{\mathrm{ex}}$, C_0^{in}, C_0^{co}, D_0^{co}, C_0^{ex} and D_0^{ex}, which appear in Eqs. (101), (102), (103) and (104), have now to be determined. These coefficients can be found by writing the boundary conditions at $r = R_{\mathrm{y1}}$ and $r = R_{\mathrm{y2}}$.

6.3 RESOLUTION AND SOLUTION OF THE PROBLEM

6.3.1 GENERAL FORM OF THE SOLUTION

The boundary conditions at $r = R_{y1}$ and $r = R_{y2}$ are

$$\begin{cases} \left(A_{2z} = A_{3z}\right)_{r=R_{y1}} \\ \left(\mu_r \dfrac{\partial A_{2z}}{\partial r} = \dfrac{\partial A_{3z}}{\partial r}\right)_{r=R_{y1}} \end{cases} \tag{105}$$

and

$$\begin{cases} \left(A_{3z} = A_{4z}\right)_{r=R_{y2}} \\ \left(\mu_r \dfrac{\partial A_{4z}}{\partial r} = \dfrac{\partial A_{3z}}{\partial r}\right)_{r=R_{y2}} \end{cases} \tag{106}$$

By combining Eqs. (101), (102), (103) and (104) with Eqs. (105) and (106), we derive the following system

$$\begin{cases} C_n^{in} - C_n^{co} - D_n^{co}\, R_{y1}^{-2n} = 0 \\ C_n^{co} + D_n^{co}\, R_{y2}^{-2n} - D_n^{ex}\, R_{y2}^{-2n} = 0 \\ \mu_r C_n^{in} - C_n^{co} + D_n^{co}\, R_{y1}^{-2n} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{a}{R_{y1}^{2}}\right)^{n}(\mu_r - 1) \\ -C_n^{co} + D_n^{co}\, R_{y2}^{-2n} - \mu_r\, D_n^{ex}\, R_{y2}^{-2n} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{a}{R_{y2}^{2}}\right)^{n}(\mu_r - 1) \end{cases} \quad \text{for } n,\ n \geq 1 \tag{107}$$

and

$$\begin{cases} C_0^{in} - C_0^{co} - D_0^{co} \ln R_{y1} = 0 \\ C_0^{co} - C_0^{ex} + D_0^{co} \ln R_{y2} - D_0^{ex} \ln R_{y2} = 0 \\ \dfrac{\mu_0 I}{2\pi}(\mu_r - 1) - D_0^{co} = 0 \\ \dfrac{\mu_0 I}{2\pi}(\mu_r - 1) + \mu_r D_0^{ex} - D_0^{co} = 0 \end{cases} \tag{108}$$

The solution of Eq. (107) are

$$\begin{cases} C_{\mathrm{n}}^{\mathrm{in}} = -\dfrac{\mu_0 I}{2\pi n} a^n (\mu_{\mathrm{r}}{}^2 - 1) \left[\dfrac{\dfrac{R_{\mathrm{y}2}^{2n} - R_{\mathrm{y}1}^{2n}}{R_{\mathrm{y}1}^{2n}}}{(\mu_{\mathrm{r}}^2 + 1)(R_{y2}^{2n} - R_{y1}^{2n}) + 2\mu_{\mathrm{r}}(R_{y2}^{2n} + R_{y1}^{2n})} \right] \\ C_{\mathrm{n}}^{\mathrm{co}} = -\dfrac{\mu_0 I}{2\pi n} a^n (\mu_{\mathrm{r}} - 1) \left[\dfrac{-2\mu_{\mathrm{r}}}{(\mu_{\mathrm{r}}^2 + 1)(R_{\mathrm{y}2}^{2n} - R_{\mathrm{y}1}^{2n}) + 2\mu_{\mathrm{r}}(R_{\mathrm{y}2}^{2n} + R_{\mathrm{y}1}^{2n})} \right] \\ D_{\mathrm{n}}^{\mathrm{co}} = -\dfrac{\mu_0 I}{2\pi n} a^n (\mu_{\mathrm{r}} - 1) \left[\dfrac{R_{\mathrm{y}2}^{2n}(\mu_{\mathrm{r}} + 1) + R_{\mathrm{y}1}^{2n}(\mu_{\mathrm{r}} - 1)}{(\mu_{\mathrm{r}}^2 + 1)(R_{\mathrm{y}2}^{2n} - R_{\mathrm{y}1}^{2n}) + 2\mu_{\mathrm{r}}(R_{\mathrm{y}2}^{2n} + R_{\mathrm{y}1}^{2n})} \right] \\ D_{\mathrm{n}}^{\mathrm{ex}} = +\dfrac{\mu_0 I}{2\pi n} a^n (\mu_{\mathrm{r}} - 1)^2 \left[\dfrac{(R_{\mathrm{y}2}^{2n} - R_{\mathrm{y}1}^{2n})}{(\mu_{\mathrm{r}}^2 + 1)(R_{\mathrm{y}2}^{2n} - R_{\mathrm{y}1}^{2n}) + 2\mu_{\mathrm{r}}(R_{\mathrm{y}2}^{2n} + R_{\mathrm{y}1}^{2n})} \right] \end{cases} \quad \text{for } n,\ n \ge 1 \qquad (109)$$

Furthermore, let us choose $C_0^{\mathrm{in}} = 0$. It follows from Eq. (108) that

$$\begin{cases} C_0^{\mathrm{co}} = -D_0^{\mathrm{co}} \ln R_{\mathrm{y}1} \\ C_0^{\mathrm{ex}} = C_0^{\mathrm{co}} + D_0^{\mathrm{co}} \ln R_{\mathrm{y}2} \\ D_0^{\mathrm{co}} = \dfrac{\mu_0 I}{2\pi}(\mu_{\mathrm{r}} - 1) \\ D_0^{\mathrm{ex}} = 0 \end{cases} \qquad (110)$$

which can be simplified into

$$\begin{cases} C_0^{\mathrm{co}} = -\dfrac{\mu_0 I}{2\pi}(\mu_{\mathrm{r}} - 1)\ln R_{\mathrm{y}1} \\ C_0^{\mathrm{ex}} = \dfrac{\mu_0 I}{2\pi}(\mu_{\mathrm{r}} - 1)\ln\left(\dfrac{R_{\mathrm{y}2}}{R_{\mathrm{y}1}}\right) \\ D_0^{\mathrm{co}} = \dfrac{\mu_0 I}{2\pi}(\mu_{\mathrm{r}} - 1) \\ C_0^{\mathrm{in}} = D_0^{\mathrm{ex}} = 0 \end{cases} \qquad (111)$$

The expressions of the vector potential in the four areas of space are obtained by inserting Eqs. (109) and (111) into Eqs. (101), (102), (103) and (104).

In area 1, the vector potential has the expression

$$A_{1z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n + \frac{(\mu_r - 1)}{(\mu_r + 1) + \frac{4\mu_r}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{ar}{R_{y1}^2}\right)^n\right]\cos(n\theta) \tag{112}$$

$$+\frac{\mu_0 I}{2\pi}\ln(a) \qquad \text{for } r,\ r < a$$

In area 2, the vector potential has the expression

$$A_{2z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n + \frac{(\mu_r - 1)}{(\mu_r + 1) + \frac{4\mu_r}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{ar}{R_{y1}^2}\right)^n\right]\cos(n\theta) \tag{113}$$

$$+\frac{\mu_0 I}{2\pi}\ln(r) \qquad \text{for } r,\ a < r < R_{y1}$$

In area 3, the vector potential has the expression

$$A_{3z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{2\mu_r\frac{R_{y2}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}{(\mu_r + 1) + \frac{4\mu_r}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{a}{r}\right)^n - \frac{\frac{2\mu_r(\mu_r - 1)}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}{(\mu_r + 1) + \frac{4\mu_r}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{ar}{R_{y1}^2}\right)^n\right]\cos(n\theta)$$

$$+\frac{\mu_0 I}{2\pi}\ln(r) + (\mu_r - 1)\frac{\mu_0 I}{2\pi}\ln\left(\frac{r}{R_{y1}}\right) \qquad \text{for } r,\ R_{y1} < r < R_{y2} \tag{114}$$

In area 4, the vector potential has the expression

$$A_{4z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\frac{4\mu_r}{\mu_r + 1}\frac{R_{y2}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}{(\mu_r + 1) + \frac{4\mu_r}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{a}{r}\right)^n\right]\cos(n\theta) \tag{115}$$

$$+\frac{\mu_0 I}{2\pi}\ln(r) + (\mu_r - 1)\frac{\mu_0 I}{2\pi}\ln\left(\frac{R_{y2}}{R_{y1}}\right) \qquad \text{for } r,\ R_{y2} < r$$

Note that, as expected, these expressions are continuous at $r = a$, R_{y1} and R_{y2}. Also, they are in agreement with results published in the literature (p. 4 of Ref. [5]).

Let us introduce $\eta = R_{y1} / R_{y2}$, where η assumes values between 0 and 1. Then, for $r \leq R_{y1}$, the vector potential in area 1 can be written

$$A_{1z} = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left(\frac{r}{a}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right) \left[\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2} \frac{\eta^{2n}}{1 - \eta^{2n}}} \right] \left(\frac{ar}{R_{y1}^2}\right)^n \right\} \cos(n\theta) \tag{116}$$

$$+ \frac{\mu_0 I}{2\pi} \ln(a) \qquad \text{for } r,\ r < a$$

while the one in area 2 becomes

$$A_{2z} = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left(\frac{a}{r}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right) \left[\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2} \frac{\eta^{2n}}{1 - \eta^{2n}}} \right] \left(\frac{ar}{R_{y1}^2}\right)^n \right\} \cos(n\theta) \tag{117}$$

$$+ \frac{\mu_0 I}{2\pi} \ln(r) \qquad \text{for } r,\ a < r < R_{y1}$$

6.3.2 CASE WHEN THE EXTERNAL RADIUS TENDS TOWARDS INFINITY

Let us verify that when the external radius of the ferromagnetic yoke tends towards infinity, Eqs. (116) and (117) tend toward Eqs. (72) and (73). Indeed, when R_{y2} becomes very large while R_{y1} is kept constant, η tends towards zero. Then, we get

$$\begin{aligned} A_{1z} &= -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(\frac{r}{a}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right) \left(\frac{ar}{R_{y1}}\right)^n \right] \cos(n\theta) + \frac{\mu_0 I}{2\pi} \ln(a) \quad \text{for } r,\ r < a \\ A_{2z} &= -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(\frac{a}{r}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right) \left(\frac{ar}{R_{y1}}\right)^n \right] \cos(n\theta) + \frac{\mu_0 I}{2\pi} \ln(r) \quad \text{for } r,\ a < r < R_{y1} \end{aligned} \tag{118}$$

where we recognize Eqs. (72) and (73). Similarly, it can also be shown that when R_{y2} tends towards infinity, Eq. (114) tends towards Eq. (74).

6.3.3 CASE WHEN R_{y2} TENDS TOWARDS R_{y1}

Another cross check can be carried out by letting R_{y2} tend towards R_{y1} (which corresponds to η tending towards 1). Then, the ferromagnetic tube disappears and we are left with the current line in free space. In this case, it is easy to verify that Eq. (116) tend towards Eq. (19) and that Eqs. (115) and (117) tend towards Eq. (20).

6.4 MULTIPOLE FIELD COEFFICIENTS

Referring to the definition of the complex magnetic flux density given in section 3.3, we get from Eq. (116)

$$\mathbf{B}^{\mathbf{tot}}(\mathbf{s}) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\left\{1+\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left[\frac{1}{1+\dfrac{4\mu_\mathrm{r}}{(\mu_\mathrm{r}+1)^2}\dfrac{\eta^{2n}}{1-\eta^{2n}}}\right]\left(\frac{a}{R_\mathrm{y1}}\right)^{2n}\right\}\frac{\mathbf{s}^{n-1}}{a^n} \quad \text{for } r,\, r<a \tag{119}$$

Hence, the multipole field coefficients $B_\mathrm{n}^\mathrm{tot} + \mathrm{i}\, A_\mathrm{n}^\mathrm{tot}$ are given by

$$B_\mathrm{n}^\mathrm{tot} + \mathrm{i}\, A_\mathrm{n}^\mathrm{tot} = \frac{\mu_0 I}{2\pi R_\mathrm{ref}}\left(\frac{R_\mathrm{ref}}{a}\right)^n\left\{1+\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left[\frac{1}{1+\dfrac{4\mu_\mathrm{r}}{(\mu_\mathrm{r}+1)^2}\dfrac{\eta^{2n}}{1-\eta^{2n}}}\right]\left(\frac{a}{R_\mathrm{y1}}\right)^{2n}\right\} \tag{120}$$

which can be written

$$B_\mathrm{n}^\mathrm{tot} + \mathrm{i}\, A_\mathrm{n}^\mathrm{tot} = \left\{1+\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left[\frac{1}{1+\dfrac{4\mu_\mathrm{r}}{(\mu_\mathrm{r}+1)^2}\dfrac{\eta^{2n}}{1-\eta^{2n}}}\right]\left(\frac{a}{R_\mathrm{y1}}\right)^{2n}\right\}(B_\mathrm{n}^\mathrm{line} + \mathrm{i}\, A_\mathrm{n}^\mathrm{line}) \tag{121}$$

It follows that, in this case, $B_\mathrm{n}^\mathrm{yoke} + \mathrm{i}\, A_\mathrm{n}^\mathrm{yoke}$ are given by

$$B_\mathrm{n}^\mathrm{yoke} + \mathrm{i}\, A_\mathrm{n}^\mathrm{yoke} = \frac{\mu_0 I}{2\pi R_\mathrm{ref}}\left(\frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1}\right)\left[\frac{1}{1+\dfrac{4\mu_\mathrm{r}}{(\mu_\mathrm{r}+1)^2}\dfrac{\eta^{2n}}{1-\eta^{2n}}}\right]\left(\frac{a\, R_\mathrm{ref}}{R_\mathrm{y1}^2}\right)^n \tag{122}$$

where we can verify that when R_y2 tends towards infinity and η tends towards 0, Eq. (122) becomes Eq. (79).

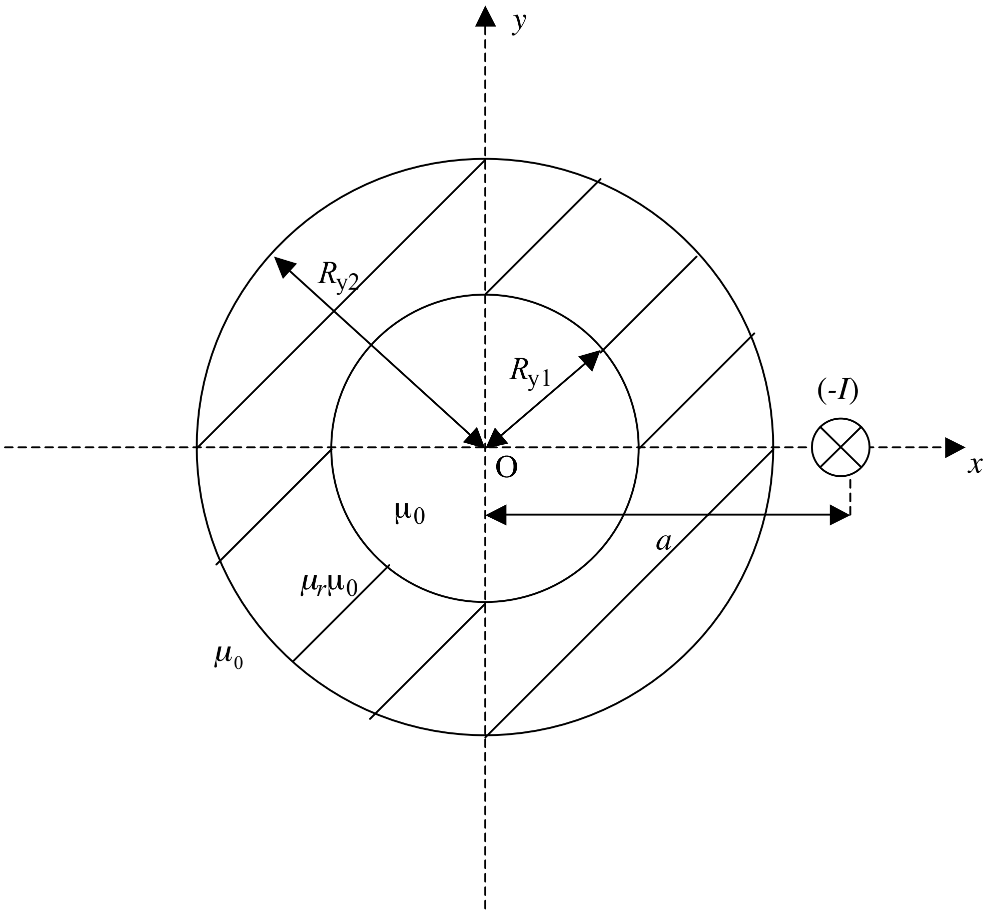

Figure 10. Current line outside a ferromagnetic tube.

7 CASE OF A CURRENT LINE OUTSIDE A FERROMAGNETIC TUBE, WITH FINITE INTERNAL AND EXTERNAL RADII

7.1 GEOMETRY OF THE PROBLEM

Let us consider a current line, –*I*, at a distance, *a*, from the *z*-axis of a ferromagnetic tube, of internal radius R_{y1} and external radius R_{y2}, as shown in Figure 10.

The vector potential will be noted $\overrightarrow{A_i}$, with $i = 1$ in the area 1 where $r < R_{y1}$, $i = 2$ in the area 2 where $R_{y1} < r < R_{y2}$, $i = 3$ in the area 3 where $R_{y2} < r < a$ and $i = 4$ in the area 4 where $r > a$.

Once again, we will estimate the vector potential by superimposition of the vector potentials of the current line and of the tube, as done in previous sections.

7.2 LINEAR SUPERIMPOSITION OF THE VECTOR POTENTIALS DUE TO THE CURRENT AND DUE TO THE FERROMAGNETIC TUBE

7.2.1 Expression of the vector potential in the area where $r < R_{y1}$

By combining Eqs. (19) and (56), the vector potential can be written under the form

$$A_{1z} = \sum_{n=1}^{\infty} C_n^{in} \cos(n\theta) r^n - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \cos(n\theta) - \ln(a)\right] + C_0^{in} \quad (123)$$

7.2.2 Expression of the vector potential in the area where $R_{y1} < r < R_{y2}$

By combining Eqs. (19) and (53), the vector potential can be written under the form

$$A_{2z} = \sum_{n=1}^{\infty} (C_n^{co} r^n + D_n^{co} r^{-n}) \cos(n\theta) - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \cos(n\theta) - \ln(a)\right] + C_0^{co} + D_0^{co} \ln(r)$$

(124)

7.2.3 Expression of the vector potential in the area where $R_{y2} < r < a$

By combining Eqs. (19) and (59), the vector potential can be written under the form

$$A_{3z} = \sum_{n=1}^{\infty} D_n^{ex} \cos(n\theta) r^{-n} - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \cos(n\theta) - \ln(a)\right] + C_0^{ex} + D_0^{ex} \ln(r) \quad (125)$$

7.2.4 Expression of the vector potential in the area where $a < r$

By combining Eqs. (20) and (59), the vector potential can be written under the form

$$A_{4z} = \sum_{n=1}^{\infty} D_n^{ex} \cos(n\theta) r^{-n} - \frac{\mu_0 I}{2\pi}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{a}{r}\right)^n \cos(n\theta) - \ln(r)\right] + C_0^{ex} + D_0^{ex} \ln(r) \quad (126)$$

7.2.5 Method of determination of the constants of integration

Lets now determine the coefficients C_n^{in}, C_n^{co}, D_n^{co}, D_n^{ex}, C_0^{in}, C_0^{co}, D_0^{co}, C_0^{ex} and D_0^{ex}, which appears in Eqs. (123), (124), (125) and (126). These coefficients can be found by writing the boundary conditions at $r = R_{y1}$ and $r = R_{y2}$.

7.3 RESOLUTION AND SOLUTION OF THE PROBLEM

7.3.1 GENERAL FORM OF THE SOLUTION

The boundary conditions at $r = R_{y1}$ and $r = R_{y2}$ are similar to Eqs. (105) and (106), but written at the boundaries between mediums 1 and 2 and mediums 2 and 3. By introducing Eqs. (123), (124) and (125), we derive the following system

$$\begin{cases} C_n^{in} - C_n^{co} - D_n^{co} R_{y1}^{-2n} = 0 \\ C_n^{co} + D_n^{co} R_{y2}^{-2n} - D_n^{ex} R_{y2}^{-2n} = 0 \\ \mu_r C_n^{in} - C_n^{co} + D_n^{co} R_{y1}^{-2n} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{1}{a}\right)^n (1-\mu_r) \\ -C_n^{co} + D_n^{co} R_{y2}^{-2n} - \mu_r D_n^{ex} R_{y2}^{-2n} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{1}{a}\right)^n (1-\mu_r) \end{cases} \quad \text{for } n,\ n \geq 1 \qquad (127)$$

while for the non-harmonic terms, we have

$$\begin{cases} C_0^{in} = C_o^{co} + D_0^{co} \ln(R_{y1}) \\ C_0^{co} + D_0^{co} \ln R_{y2} = C_0^{ex} + D_0^{ex} \ln(R_{y2}) \\ 0 = \dfrac{D_0^{co}}{R_{y1}} \\ \dfrac{D_0^{co}}{R_{y2}} = \mu_r \dfrac{D_0^{ex}}{R_{y2}} \end{cases} \qquad (128)$$

The latter equation can be readily simplified into

$$\begin{cases} C_0^{in} = C_0^{co} = C_0^{ex} \\ D_0^{co} = D_0^{ex} = 0 \end{cases} \qquad (129)$$

The solution of Eq. (127) is

$$\begin{cases} C_n^{in} = +\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{1}{a}\right)^n (\mu_r - 1)^2 \left[\dfrac{(R_{y2}^{2n} - R_{y1}^{2n})}{(\mu_r^2+1)(R_{y2}^{2n} - R_{y1}^{2n}) + 2\mu_r (R_{y2}^{2n} + R_{y1}^{2n})}\right] \\ C_n^{co} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{1}{a}\right)^n (\mu_r - 1) \left[\dfrac{(\mu_r+1) R_{y2}^{2n} + (\mu_r - 1) R_{y1}^{2n}}{(\mu_r^2+1)(R_{y2}^{2n} - R_{y1}^{2n}) + 2\mu_r (R_{y2}^{2n} + R_{y1}^{2n})}\right] \\ D_n^{co} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{1}{a}\right)^n (\mu_r - 1) \left[\dfrac{-2\mu_r R_{y1}^{2n} R_{y2}^{2n}}{(\mu_r^2+1)(R_{y2}^{2n} - R_{y1}^{2n}) + 2\mu_r (R_{y2}^{2n} + R_{y1}^{2n})}\right] \\ D_n^{ex} = -\dfrac{\mu_0 I}{2\pi n}\left(\dfrac{1}{a}\right)^n ({\mu_r}^2 - 1) \left[\dfrac{(R_{y2}^{2n} - R_{y1}^{2n}) R_{y2}^{2n}}{(\mu_r^2+1)(R_{y2}^{2n} - R_{y1}^{2n}) + 2\mu_r (R_{y2}^{2n} + R_{y1}^{2n})}\right] \end{cases} \quad \text{for } n,\ n \geq 1 \quad (130)$$

Furthermore, let us choose $C_0^{\rm in} = 0$. It follows from Eq. (129) that

$$C_0^{\rm in} = C_0^{\rm co} = C_0^{\rm ex} = D_0^{\rm co} = D_0^{\rm ex} = 0 \tag{131}$$

The expression of the vector potential in the four areas of space is obtained by inserting Eqs (130) and (131) in Eq. (123), (124), (125) and (126).

In area 1, the vector potential has the expression

$$A_{1z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n - \frac{\dfrac{(\mu_r - 1)^2}{\mu_r + 1}}{(\mu_r + 1) + \dfrac{4\mu_r}{\mu_r + 1}\dfrac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{r}{a}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r,\ r < R_{y1} \tag{132}$$

In area 2, the vector potential has the expression

$$A_{2z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{2\mu_r\dfrac{R_{y2}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}{(\mu_r + 1) + \dfrac{4\mu_r}{\mu_r + 1}\dfrac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{r}{a}\right)^n - \frac{\dfrac{2\mu_r(\mu_r - 1)}{\mu_r + 1}\dfrac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}{(\mu_r + 1) + \dfrac{4\mu_r}{\mu_r + 1}\dfrac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r,\ R_{y1} < r < R_{y2} \tag{133}$$

In area 3, the vector potential has the expression

$$A_{3z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n + \frac{(\mu_r - 1)}{(\mu_r + 1) + \dfrac{4\mu_r}{\mu_r + 1}\dfrac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r, R_{y2} < r < a \tag{134}$$

In area 4, the vector potential has the expression

$$A_{4z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n + \frac{(\mu_r - 1)}{(\mu_r + 1) + \frac{4\mu_r}{\mu_r + 1}\frac{R_{y1}^{2n}}{R_{y2}^{2n} - R_{y1}^{2n}}}\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) \quad \text{for } r,\ a < r \tag{135}$$

Note that, as expected, these expressions are continuous at $r = R_{y1},\ R_{y2}$ and a.

Let us introduce $\eta = R_{y1} / R_{y2}$, where η assumes values between 0 and 1. Then, the vector potential in area 1 can be written

$$A_{1z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n - \left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2 \frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^2}{1-\eta^2}}\left(\frac{r}{a}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r,\ r < R_{y1} \tag{136}$$

while the one in area 3 becomes

$$A_{3z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^{2n}}{1-\eta^{2n}}}\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } r,\ R_{y2} < r < a \tag{137}$$

and we have in area 4

$$A_{4z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^{2n}}{(1-\eta^{2n})}}\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) \quad \text{for } r,\ a < r \tag{138}$$

7.3.2 CASE WHEN THE INTERNAL RADIUS TENDS TOWARDS ZERO

Let us verify that when the internal radius of the ferromagnetic yoke tends towards zero, Eqs. (137) and (138) tends towards Eqs. (95) and (96). Indeed, when R_{y1} becomes very little while R_{y2} is kept constant, η tends towards zero. Then, we get

$$A_{3z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(a) \quad \text{for } R_{y2} < r < a$$

$$A_{4z} = -\frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\left(\frac{R_{y2}^2}{ar}\right)^n\right]\cos(n\theta) + \frac{\mu_0 I}{2\pi}\ln(r) \quad \text{for } a < r \qquad (139)$$

where we recognize Eqs. (95) and (96). Similarly, it can also be shown that when R_{y1} tends towards zero, Eq. (133) tends towards Eq. (94).

7.3.3 CASE WHEN R_{y2} TENDS TOWARDS R_{y1}

Another cross check can be carried out by letting R_{y2} tend towards R_{y1} (which corresponds to η tending towards 1). Then, the ferromagnetic tube disappears and we are left with the current line in free space. In this case, it is easy to verify that Eqs. (136) and (137) tend towards Eq. (19) and that Eq. (138) tends towards Eq. (20).

7.4 MULTIPOLE FIELD COEFFICIENTS

Referring to the definition of the complex magnetic flux density given in section 3.3, we get from Eq. (136)

$$\mathbf{B}^{\mathbf{tot}}(\mathbf{s}) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\left\{1 - \left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2\left[\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^{2n}}{1 - \eta^{2n}}}\right]\right\}\frac{\mathbf{s}^{n-1}}{a^n} \quad \text{for } r, r < a \qquad (140)$$

Hence, the multipole field coefficients B_n^{tot} and A_n^{tot} are given by

$$B_n^{tot} + i\,A_n^{tot} = \frac{\mu_0 I}{2\pi R_{ref}}\left\{1 - \left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2\left[\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^{2n}}{1 - \eta^{2n}}}\right]\right\}\left(\frac{R_{ref}}{a}\right)^n \qquad (141)$$

which can be written

$$B_n^{tot} + i\,A_n^{tot} = \left\{1 - \left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2 \left[\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^{2n}}{1-\eta^{2n}}}\right]\right\}(B_n^{line} + i\,A_n^{line}) \tag{142}$$

It follows that, in this case, B_n^{yoke} and A_n^{yoke} are given by

$$B_n^{yoke} + i\,A_n^{yoke} = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2 \left[\frac{1}{1 + \frac{4\mu_r}{(\mu_r + 1)^2}\frac{\eta^{2n}}{1-\eta^{2n}}}\right]\left(\frac{R_{ref}}{a}\right)^n \tag{143}$$

Note that in Eq. (142), the total multipole coefficients generated inside the tube appears somewhat shielded with respect to the coefficients generated by the current line alone in free space [multiplication by a factor (1–α), where α is positive].

APPENDIX: BOUNDARY EQUATIONS AT THE INTERFACE BETWEEN TWO MEDIUMS OF DIFFERENT MAGNETIC PERMEABILITIES

The boundary equations at the interface between two mediums are derived from Maxwell's equations, judiciously integrated over contour and surface elements on both sides of the interface. In general, they can be written under the form

$$B_{n1} = B_{n2} \tag{144}$$

and

$$\overrightarrow{n_{12}} \wedge (\overrightarrow{H_2} - \overrightarrow{H_1}) = \overrightarrow{j_s} \tag{145}$$

where $\overrightarrow{j_s}$ is the density of surface currents, $\overrightarrow{n_{12}}$ is a unit vector perpendicular to the interface and directed from medium 1 to medium 2, B_n is the component of the magnetic flux density parallel to $\overrightarrow{n_{12}}$, $\overrightarrow{H}$ is the magnetic field, and the indices 1 and 2 relate to the two mediums.

Let us now translate the above boundary conditions on the vector potential.

Starting from $\overrightarrow{B} = \overrightarrow{\mathrm{rot} A}$, we have, by definition of the rotational in cylindrical coordinates

$$\overrightarrow{B} = \left| \begin{array}{l} \left[\dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta} - \dfrac{\partial A_\theta}{\partial z}\right]\overrightarrow{u_r} \\ \left[\dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}\right]\overrightarrow{u_\theta} \\ \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{1}{r}\dfrac{\partial A_r}{\partial \theta}\right]\overrightarrow{u_z} \end{array} \right. \tag{146}$$

As we have seen in section 3.1, the vector potential is parallel to the *z*-axis and is independent of *z*. Then, Eq. (146) reduces to

$$\overrightarrow{B} = \left| \begin{array}{l} \left[\dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta}\right]\overrightarrow{u_r} \\ \left[-\dfrac{\partial A_z}{\partial r}\right]\overrightarrow{u_\theta} \\ 0.\overrightarrow{u_z} \end{array} \right. \tag{147}$$

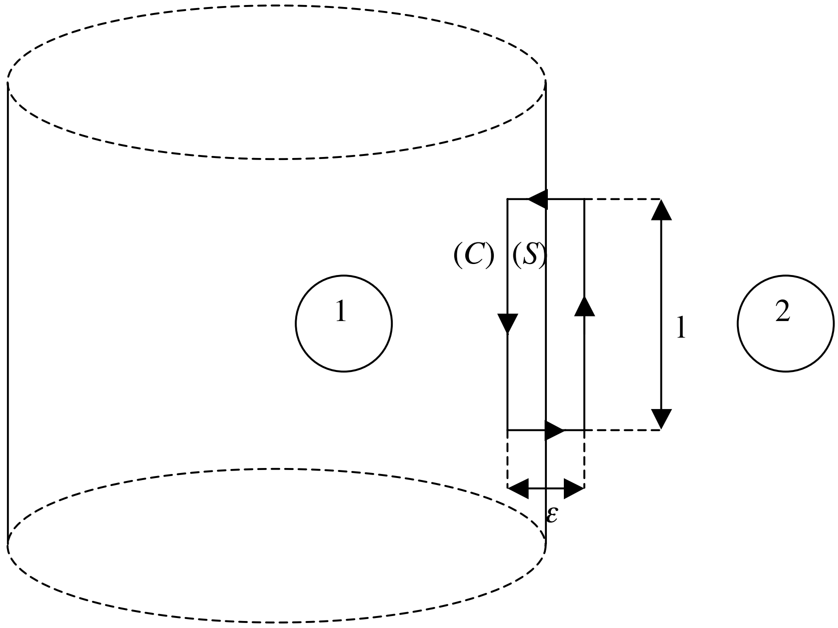

Figure 11. Stokes-Ampere's outline.

In the various problem considered in this paper, the tangential component at the interface between the two mediums corresponds to the azimuthal component. By considering that $\vec{B} = \mu_0 \vec{H}$ in medium 1 (vacuum) and $\vec{B} = \mu_r \mu_0 \vec{H}$ in medium 2 (ferromagnetic), and knowing that there is no real surface current at the interface between the two mediums, Eq. (145) can be written

$$H_{t1} = H_{t2} \tag{148}$$

and, with Eq. (147), gives

$$\left(\frac{\partial A_{1z}}{\partial r}\right) = \frac{1}{\mu_r}\left(\frac{\partial A_{2z}}{\partial r}\right) \tag{149}$$

The second boundary condition is obtained by writing Stokes-Ampere's theorem on the contour (C) around the surface (S) represented in Figure 11. We get

$$\oint_{(C)} \vec{A}.\vec{dl} = A_{2z} l - A_{1z} l + 0 + 0 = \iint_{(S)} \overrightarrow{rot\,A}.\vec{dS} = \iint_{(S)} \vec{B}.\vec{dS} = B_{n1}\frac{\varepsilon}{2} l - B_{n2}\frac{\varepsilon}{2} l \tag{150}$$

which yields

$$A_{2z} - A_{1z} = B_{n1}\frac{\varepsilon}{2} + B_{n2}\frac{\varepsilon}{2} \tag{151}$$

Then, when ε tends toward 0, and if one considers that the magnetic flux density is finite at the interface where there are no surface currents, we get

$$A_{1z} = A_{2z} \tag{152}$$

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