Radiative and isospin-violating decays of D_s-mesons in the hadrogenesis conjecture

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Abstract

We challenge the hadrogenesis conjecture by a study of radiative and isospin-violating strong decays of charmed mesons with strangeness. The scalar $D^*_{s0}(2317)^\pm$ and the axial vector $D^*_{s1}(2460)^\pm$ states are generated by coupled-channel dynamics based on the leading order chiral Lagrangian. The effect of chiral corrections is investigated. We show that taking into account large-N_c relations implies a measurable signal for an exotic axial vector state in the $\eta\, D^*$ invariant mass distribution. The one-loop contribution to the electromagnetic decay amplitudes of scalar and axial-vector states is calculated using the chiral Lagrangian, where the role of light vector meson degrees of freedom is explored. We confront our results with measured branching ratios. Once the light vector mesons are considered a natural explanation of all radiative decay parameters is achieved.

Keywords: Charmed mesons; $D^*_{s0}(2317)$; $D^*_{s1}(2460)$; Dynamical generation of resonances

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1 Introduction

The observation [1,2] of two narrow, with positive parity, strange charmed mesons at mass lower than expected in quark models [3,4] may provide new insight into the way hadrons are generated. The properties of these two mesons, the scalar $\mathrm{D}^*_{s0}(2317)^\pm$ and the axial vector $\mathrm{D}^*_{s1}(2460)^\pm$, appear indeed sensitive to strong interaction symmetries as well as to the degrees of freedom building up hadronic excitations [5,6]. The purpose of this paper is to study the $\mathrm{D}^*_{s0}(2317)$ and $\mathrm{D}^*_{s1}(2460)$ meson decays in the hadrogenesis conjecture. Earlier work [7,8,9] showed that such states exist and can be produced at their observed masses. This approach for heavy-light mesons exploits both heavy-quark and spontaneously broken chiral symmetries and generates open-charm mesons with strangeness through relativistic coupled-channel dynamics. For the $\mathrm{D}^*_{s0}(2317)^+$ and $\mathrm{D}^*_{s1}(2460)^+$ mesons considered in this work, the calculation involves the ηD^+_s, $K^0 D^+$ and $K^+ D^0$ channels coupled further to the $\pi^0 D^+_s$ channel through isospin mixing parameters.

We display in Figure 1 the $\mathrm{D}^\pm_s$-meson spectrum as presently known [10,11]. The spin and parity of the $\mathrm{D}^\pm_s$-mesons are well-established for the ground state and for the $\mathrm{D}^*_{s1}(2460)^\pm$. The spin and parity of the other states need confirmation. We have quoted their most probable values.

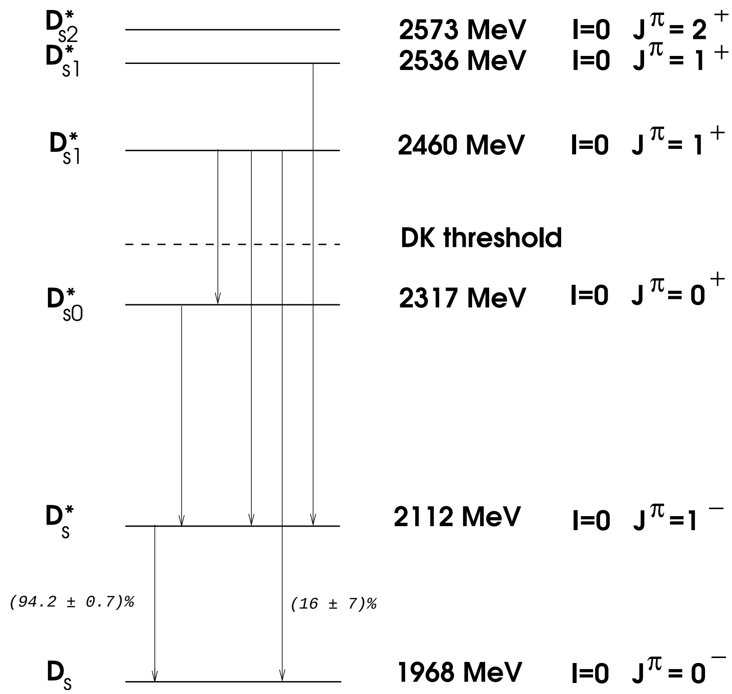

Fig. 1. The $D_s^{\pm}$-meson spectrum with the most probable spin-parity assignments together with the radiative transitions on which there is experimental information [10,11]. The arrows without number indicate that there are experimental constraints on these decays but no absolute branching ratios. The dashed line shows the DK threshold.

The $D_s^*(2112)^{\pm}$ and $D_{s0}^*(2317)^{\pm}$ mesons lie below the DK threshold and are therefore expected to be very narrow states. They can decay either electromagnetically or into the isospin-violating $D_s^{(*)\pm}\,\pi^0$ channel. The closeness of the $D_{s0}^*(2317)^{\pm}$ meson to the DK threshold has lead to suggest that it could be a DK molecule or alternatively a state involving large sea-quark effects [5,6].

The $D_s^*(2112)^{\pm}$ has a width $\Gamma < 1.9$ MeV and decays dominantly by a radiative transition to the ground state with a probability of (94.2 ± 0.7) % [10]. Its decay probability to the $D_s\pi^0$ channel is therefore (5.8 ± 0.7) %, i.e. about 16 times smaller than to the radiative channel.

The most stringent upper limit obtained for the $D_{s0}^*(2317)^{\pm}$ width is $\Gamma <$ 3.8 MeV [12]. The $D_{s0}^*(2317)^{\pm}$ was first observed through its decay into the $D_s(1968)^{\pm}\,\pi^0$ channel [1]. Its radiative decay to the $D_s(1968)^{\pm}$ has never been seen. The present upper limits available on the ratio of the radiative to pionic decay widths of the $D_{s0}^*(2317)^{\pm}$ to the ground state and to the $D_s^*(2112)^{\pm}$ are [10]

$$\frac{\Gamma\,[D_{s0}^{*}(2317)\;\rightarrow\;D_s(1968)\,\gamma]}{\Gamma\,[D_{s0}^{*}(2317)\;\rightarrow\;D_s(1968)\,\pi^0]} <\;0.05 \tag{1}$$

and

$$\frac{\Gamma\,[D_{s0}^{*}(2317)\;\rightarrow\;D_s^{*}(2112)\,\gamma]}{\Gamma\,[D_{s0}^{*}(2317)\;\rightarrow\;D_s(1968)\,\pi^0]} <\;0.059. \tag{2}$$

The $\mathrm{D}_{s1}^{*}(2460)^{\pm}$-meson is located above the DK threshold but appears nevertheless very narrow: its total width was found to be less than 3.5 MeV [12]. Constraints on its radiative decays to the $\mathrm{D}_s(1968)^{\pm}$, to the $\mathrm{D}_s^{*}(2112)^{\pm}$ and to the $\mathrm{D}_{s0}^{*}(2317)^{\pm}$ are as follows [10],

$$\frac{\Gamma\,[D_{s1}^{*}(2460)\;\rightarrow\;D_s(1968)\,\gamma]}{\Gamma\,[D_{s1}^{*}(2460)\;\rightarrow\;D_s^{*}(2112)\,\pi^0]} = 0.31 \pm 0.06, \tag{3}$$

$$\frac{\Gamma\,[D_{s1}^{*}(2460)\;\rightarrow\;D_s^{*}(2112)\,\gamma]}{\Gamma\,[D_{s1}^{*}(2460)\;\rightarrow\;D_s^{*}(2112)\,\pi^0]} <\;0.16, \tag{4}$$

$$\frac{\Gamma\,[D_{s1}^{*}(2460)\;\rightarrow\;D_{s0}^{*}(2317)\,\gamma]}{\Gamma\,[D_{s1}^{*}(2460)\;\rightarrow\;D_s^{*}(2112)\,\pi^0]} <\;0.22. \tag{5}$$

An absolute measurement of the decay probability of the $\mathrm{D}_{s1}^{*}(2460)^{-}$ to the $D_s(1968)^{-}\,\gamma$ channel gave recently $(16 \pm 7)\%$ [12]. The same data also provided the branching fraction $\mathrm{B}(D_{s1}^{*}(2460)^{-}\;\rightarrow\;D_s^{*}(2112)^{-}\pi^0) = (56 \pm 22)\%$. The ratio is in agreement with the value quoted in (3). The $\mathrm{D}_{s1}^{*}(2536)$ is also very narrow ($\Gamma < 2.3$ MeV). The only information available on its radiative decay scheme is the ratio

$$\frac{\Gamma\,[D_{s1}^{*}(2536)\;\rightarrow\;D_s^{*}(2112)\,\gamma]}{\Gamma\,[D_{s1}^{*}(2536)\;\rightarrow\;D^{*0}(2007)\,K]} <\;0.42. \tag{6}$$

Finally the total width of the $\mathrm{D}_{s2}^{*}(2573)$ is measured to be 15+5-4 MeV [10] but nothing is known about its radiative decay widths.

We restrict our calculations to the radiative and isospin-violating π^0 decays of the $\mathrm{D}_{s0}^{*}(2317)^{+}$ and $\mathrm{D}_{s1}^{*}(2460)^{+}$ states on which there are fragmentary but significant data. In view of the uncertainties in the measured widths, our main concern will be to check the ability of the hadrogenesis conjecture to provide a consistent picture of the main features of the decay scheme. We aim at identifying the important contributions to the dynamics of these decays, determining characteristic ranges for the parameters involved and making predictions able to test further the structure of the D_s-mesons and the specific conjecture on which this work relies.

This paper is organized as follows. Section 2 is devoted to the calculation of the strong isospin-violating decay of the $D^*_{s0}(2317)^+$ and $D^*_{s1}(2460)^+$ mesons respectively. Our picture involves the isospin-violating mixing of the $\pi^0 D_s$ channel with both the ηD_s and $\bar{K}D$ channels. We present first a calculation based on the leading order chiral Lagrangian. To obtain gauge-invariant expressions, we describe massive vector fields in terms of antisymmetric tensors and show how the results obtained earlier [8,9] in the vector field representation can be recovered in such a description. We introduce chiral correction terms to the leading order interactions, to take into account s- and u-channel D-meson exchange processes on the one hand and local two-body counter terms on the other hand. Section 3 deals with the coupling of the electromagnetic field to the hadrons. In a first step we gauge the hadronic interactions introduced in Section 2 which can contribute to the electromagnetic decays of the $D^*_{s0}(2317)^+$ and $D^*_{s1}(2460)^+$ mesons. Based on chiral power counting we consider additional gauge-invariant interaction terms which play a significant role in electromagnetic processes. We include interaction vertices probed when considering the light vector mesons as explicit degrees of freedom. We comment on the values of the parameters associated with these terms in relation to QCD symmetries and discuss the renormalization of the ultraviolet singularities. The explicit expressions of the electromagnetic decays are derived in Section 4 for the scalar state $D^*_{s0}(2317)$ and in Section 5 for the axial vector state $D^*_{s1}(2460)$. Our numerical results are presented in Section 6 and compared to the available data. We discuss the role of the different contributions and the constraints expected on the range of values for the coupling constants of specific interaction terms. We conclude in Section 7. We relegate lengthy derivations in seven appendices (A-G).

2 Isospin violation and strong decays

We describe the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ mesons in the coupled-channel framework of Ref. [8,9]. This approach is based on the scattering of Goldstone bosons off heavy-light 0^- and 1^- mesons. The corresponding fields interact as dictated by the chiral SU(3) Lagrangian. The $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ resonances appear as poles in the s-wave scattering amplitudes. This description involves isospin-breaking effects arising from the difference between the up and dawn quark masses which lead to isospin-violating strong decay amplitudes for the processes $D^*_{s0}(2317) \to \pi^0\, D_s$ and $D^*_{s1}(2460) \to \pi^0\, D_s$. To arrive at gauge-invariant expressions, the massive 1^- open charm fields appearing in the Lagrangian are represented in terms of antisymmetric tensor fields and the derivation of Ref. [8,9] is reformulated accordingly. We discuss chiral corrections which were shown to contribute to the masses of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states and play a significant role in the hadronic widths of these mesons.

2.1 Scalar states

We compute first the decay amplitude $D^*_{s0}(2317) \to \pi^0\, D_s$. The open-charm $D^*_{s0}(2317)$ state is dynamically generated as a direct consequence of the leading order chiral interaction [8,9]. We recall the relevant terms of the chiral Lagrangian density at leading order:

$$\begin{aligned}\mathcal{L} = & \frac{1}{4}\,\mathrm{tr}\,(\partial_\mu \Phi)\,(\partial^\mu \Phi) - \frac{1}{4}\,\mathrm{tr}\,\chi_0\,\Phi^2 + \ (\partial_\mu \mathrm{D})\,(\partial^\mu \bar{\mathrm{D}}) - \mathrm{D}\,\mathrm{M}^2_{0^-}\,\bar{\mathrm{D}} \\ & + \frac{1}{8\,f^2}\Big\{(\partial^\mu D)\;[\Phi,(\partial_\mu \Phi)]_-\,\bar{D} - D\;[\Phi,(\partial_\mu \Phi)]_-\,(\partial^\mu \bar{D})\Big\}\,, \end{aligned} \tag{7}$$

where Φ and D are pseudoscalar octet and triplet fields. We use the notation $\bar{D} = D^\dagger$. In the particle representation the Goldstone and ground state open-charm meson fields are

$$\Phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\,\eta \end{pmatrix}, \qquad D = (D^0, -D^+, D_s^+)\,. \tag{8}$$

The Weinberg-Tomozawa term in (7), which is proportional to f^{-2}, is obtained by chirally gauging the kinetic term of the D-mesons. The parameter $f \simeq f_\pi = 92.4$ MeV in (7) is known from the weak decay of the charged pions approximatively. It predicts the leading s-wave interaction of the Goldstone bosons with the open-charm meson fields. A precise determination of f requires a chiral SU(3) extrapolation of some data set. In [13] the value $f \simeq 90$ MeV was obtained from a detailed study of pion- and kaon-nucleon scattering data. We will use $f = 90$ MeV throughout this work.

The ground-state D-meson mass matrix is denoted by M_{0^-}. The mass term of the Goldstone bosons is proportional to the quark-mass matrix

$$\chi_0 = 2\,B_0 \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} = \frac{1}{3}\left(m_\pi^2 + 2\,m_K^2\right)\,1 + \frac{2}{\sqrt{3}}\left(m_\pi^2 - m_K^2\right)\lambda_8\,. \quad (9)$$

At leading order the latter can be expressed in terms of the pion and kaon masses as indicated in (9).

If we admit isospin breaking effects, i.e. $m_u \neq m_d$, there is a term in (7) proportional to $(m_u - m_d)\,\pi^0\,\eta$, inducing $\pi^0\,\eta$ mixing. A unitary transformation is required such that the transformed fields $\tilde{\pi}^0$ and $\tilde{\eta}$ with

$$\pi^0 = \tilde{\pi}^0\,\cos\epsilon - \tilde{\eta}\,\sin\epsilon\,, \qquad \eta = \tilde{\pi}^0\,\sin\epsilon + \tilde{\eta}\,\cos\,\epsilon\,, \quad (10)$$

decouple. The Lagrangian density (7), when written in terms of the new fields, does not show a $\tilde{\pi}^0\,\tilde{\eta}$ term if and only if

$$\frac{\sin(2\,\epsilon)}{\cos(2\,\epsilon)} = \sqrt{3}\,\frac{m_d - m_u}{2\,m_s - m_u - m_d}\,. \quad (11)$$

According to [14] the ratio of quark masses relevant in (11) takes the value

$$\frac{m_d - m_u}{m_s - (m_u + m_d)/2} = \frac{1}{43.7 \pm 2.7}\,, \quad (12)$$

which implies

$$\epsilon = 0.010 \pm 0.001\,, \quad (13)$$

for the mixing angle.

Heavy-light meson resonances with quantum numbers $J^P = 0^+$ manifest themselves as poles in the s-wave scattering amplitude, $T(s)$. We consider the four

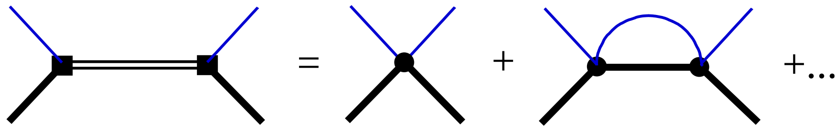

Fig. 2. Diagrams contributing to the s-wave scattering amplitude.

isospin states $\langle K\,D, I|, \langle \pi\,D_s, 1|$ and $\langle \eta\,D_s, 0|$. In the presence of isospin mixing all channels couple. The mixing of the two isospin sectors is of order ϵ. We introduce the four states as follows

$$\begin{aligned}
\langle 1| &= \langle\, \tilde\pi^0\, D_s^+| = \cos\epsilon\, \langle \pi^0\, D_s, 1| + \sin\epsilon\, \langle \eta\, D_s, 0|\,, \\
\langle 2| &= \langle\, \tilde\eta\;\; D_s^+| = \cos\epsilon\, \langle \eta\, D_s, 0| - \sin\epsilon\, \langle \pi^0\, D_s, 1|\,, \\
\langle 3| &= \langle K^0 D^+| = +\tfrac{1}{\sqrt{2}}\,(\langle K\,D, 0| - \langle K\,D, 1|)\,, \\
\langle 4| &= \langle K^+ D^0| = -\tfrac{1}{\sqrt{2}}\,(\langle K\,D, 0| + \langle K\,D, 1|)\,,
\end{aligned} \tag{14}$$

where we apply the phase convention introduced in [8,9]. The Weinberg-Tomozawa interaction (7) implies a scattering amplitude of the simple form [8,9]

$$T(s) = \Big[1 - V(s)\, J(s)\Big]^{-1} V(s)\,. \tag{15}$$

This equation is shown diagrammatically in Figure 2. The matrix of loop functions, $J(s)$, is diagonal and given by [8,9]

$$\begin{aligned}
J(s) &= I(s) - I(\mu_M^2)\,, \\
I(s) &= \frac{1}{16\,\pi^2}\left(\frac{p_{cm}}{\sqrt{s}}\left(\ln\left(1 - \frac{s - 2\,p_{cm}\sqrt{s}}{m^2 + M^2}\right) - \ln\left(1 - \frac{s + 2\,p_{cm}\sqrt{s}}{m^2 + M^2}\right)\right)\right. \\
&\quad \left. + \left(\frac{1}{2}\,\frac{m^2 + M^2}{m^2 - M^2} - \frac{m^2 - M^2}{2\,s}\right)\ln\left(\frac{m^2}{M^2}\right) + 1\right) + I(0)\,,
\end{aligned} \tag{16}$$

where $\sqrt{s} = \sqrt{M^2 + p_{cm}^2} + \sqrt{m^2 + p_{cm}^2}$. Each diagonal element depends on the masses of the light (Goldstone) and heavy (open charm) mesons m and M which constitute the associated coupled-channel state. The matching scale μ_M in (16) should be identified with the ground-state mass of the D_s meson, i.e. $\mu_M \simeq 1968$ MeV. For a properly chosen matching scale s- and u-channel unitarized scattering amplitudes may be smoothly matched around the matching scale as to define a full scattering amplitude that is crossing symmetric by construction [7,8].

The coupled-channel interaction kernel $V_{ij}(s)$ in (15) is determined by the leading order chiral SU(3) Lagrangian (7),

$$V^{(0^+)}_{WT,ij}(s) = \frac{C_{ij}}{8\,f^2}$$
$$\times\Big(3\,s - M^2 - \bar{M}^2 - m^2 - \bar{m}^2 - \frac{M^2 - m^2}{s}\,(\bar{M}^2 - \bar{m}^2)\Big)\,, \tag{17}$$

where (m, M) and $(\bar{m}, \bar{M})$ are the masses of initial and final mesons. The matrix of coefficients C, whose elements characterize the interaction strength in a given channel, is clearly defined with respect to our particular convention for the coupled-channel states.

As a consequence of (14) the 4×4 matrix C_{ij} can be expressed in terms of the mixing angle ϵ and the isospin zero $C^{(0)}_{ij}$ and isospin one $C^{(1)}_{ij}$ coupling matrices of [8]. For the channels of positive strangeness considered in Eq. (14), we have

$$\begin{aligned}
&C_{11} = C^{(0)}_{22}\,\sin^2\epsilon + C^{(1)}_{11}\,\cos^2\epsilon && C_{12} = (C^{(0)}_{22} - C^{(1)}_{11})\,\sin\epsilon\,\cos\epsilon \\
&C_{22} = C^{(0)}_{22}\,\cos^2\epsilon + C^{(1)}_{11}\,\sin^2\epsilon && C_{13} = \tfrac{1}{\sqrt{2}}\left(C^{(0)}_{12}\,\sin\epsilon - \cos\epsilon\,C^{(1)}_{12}\right) \\
&C_{23} = \tfrac{1}{\sqrt{2}}\left(\cos\epsilon\,C^{(0)}_{12} + \sin\epsilon\,C^{(1)}_{12}\right) && C_{33} = \tfrac{1}{2}\left(C^{(0)}_{11} + C^{(1)}_{22}\right) \\
&C_{14} = \tfrac{-1}{\sqrt{2}}\left(C^{(0)}_{12}\,\sin\epsilon + C^{(1)}_{12}\,\cos\epsilon\right) && C_{24} = \tfrac{-1}{\sqrt{2}}\left(C^{(0)}_{12}\,\cos\epsilon - C^{(1)}_{12}\,\sin\epsilon\right) \\
&C_{34} = \tfrac{-1}{2}\left(C^{(0)}_{11} - C^{(1)}_{22}\right) && C_{44} = \tfrac{1}{2}\left(C^{(0)}_{11} + C^{(1)}_{22}\right) \\
&C^{(0)}_{11} = 2 = 2\,C^{(1)}_{12} \qquad C^{(0)}_{12} = \sqrt{3} && C^{(0)}_{22} = C^{(1)}_{11} = C^{(1)}_{22} = 0\,.
\end{aligned} \tag{18}$$

Given the coupled-channel scattering amplitude (15) with the effective interaction (17) it is straightforward to determine the mass and width of possible resonances. The Weinberg-Tomozawa interaction is strongly attractive in the isospin strangeness $(I, S) = (0, 1)$ sector, leading to the formation of a scalar resonance of mass M_{0^+}. The latter manifests itself as a pole in the scattering amplitude which factorizes close to the pole at $s = M^2_{0^+}$

$$T_{ij}(s) \simeq -\frac{2\,g^*_i\,M^2_{0^+}\,g_j}{s - M^2_{0^+} + i\,\Gamma_{0^+}\,M_{0^+}}\,, \tag{19}$$

with the coupling constants g_i and the width parameter Γ_{0^+}.

The numerical results obtained at leading order which we present now should be viewed as qualitative. At the end of the section a more quantitative study where we consider chiral corrections systematically will be given.

Using an effective parameter $f_{eff} = 95.5$ MeV in (17) we can reproduce the empirical mass of 2317.6 MeV at leading order for the 0^+ state. The coupling constants are

$$g_{\eta\, D_s} \simeq 1.95\,, \qquad g_{\pi^0 D_s} \simeq 0.056,$$
$$g_{K^0 D^+} \simeq 2.25\,, \qquad g_{K^+ D^0} \simeq -2.25\,. \tag{20}$$

Isospin breaking effects in the KD coupling constants are found to be negligible, i.e. $g_{K^0 D^+} \simeq -g_{K^+ D^0}$ holds quite accurately. It is therefore meaningful to work with the isospin coupling constants

$$g^{(0+)}_{KD} = \sqrt{2}\, g_{K^0 D^+} \simeq 3.18\,, \qquad g^{(0+)}_{\eta\, D_s} = g_{\eta\, D_s} \simeq 1.95\,. \tag{21}$$

Note that the flavor SU(3) limit suggests $\sqrt{3}\, g^{(0+)}_{\eta D_s} = g^{(0+)}_{KD}$, a result quite compatible with the values given in (20, 21). The total 0^+ width comes at 76 keV using $\epsilon = 0.010$. It is pointed out that our result for the width parameter of the $D^*_{s0}(2317)$ is an order of magnitude larger than the one given in [15] based on the same interaction.

To trace the origin of this difference, it is useful to understand the physics underlying the value obtained for $g_{\pi^0 D_s}$. We have

$$\Gamma_{D^*_{s0}(2317)\to \pi^0 D_s} = |g_{\pi^0 D_s}|^2\, \frac{p_{cm}}{4\,\pi} \simeq |g_{\pi^0 D_s}|^2\, 23.7\,\mathrm{MeV} \simeq 76\,\mathrm{keV}\,. \tag{22}$$

At linear order the relevant coupling constant picks up two terms,

$$g_{\pi^0 D_s} \simeq \tilde{\epsilon}\, g^{(0+)}_{KD} + \epsilon\, g^{(0+)}_{\eta D_s} \simeq 0.056\,. \tag{23}$$

The contribution proportional to $g^{(0+)}_{\eta D_s}$ is unambiguously determined by the angle ϵ. This is a direct consequence of the $\pi^0 - \eta$ mixing as defined in (10). If the coupled-channel dynamics was evaluated with degenerate kaon and D-meson masses ($m_{K^+} = m_{K^0}$ and $M_{D^+} = M_{D^0}$), the coupling constant $g_{\pi^0 D_s}$ would be determined fully by the mixing angle ϵ and the coupling constant $g_{\eta D_s}$. However, when the physical masses are used, there is an additional contribution induced by the mass difference between the neutral and charged kaons ($m_{K^+} \neq m_{K^0}$) and between the neutral and charged D-mesons ($M_{D^+} \neq M_{D^0}$). Like the $\pi^0\eta$ mixing phenomenon, it couples the two isospin sectors. This effect in chiral coupled-channel dynamics is analogous to the mixing phenomenon induced by the exchange of vector K*- and D*-mesons in the molecular picture of Ref. [16,17]. The contribution to $g_{\pi^0 D_s}$ must be proportional to g_{KD} and to the mass differences of neutral and charged mesons, i.e.

$$\tilde{\epsilon} = \tilde{\epsilon}_1 + \tilde{\epsilon}_2\,, \qquad \tilde{\epsilon}_1 \sim (m_{K^+} - m_{K^0})\,, \qquad \tilde{\epsilon}_2 \sim (M_{D^+} - M_{D^0})\,, \tag{24}$$

where the proportionality factors depend on the details of the coupled-channel dynamics. It measures the isospin one transition amplitude $KD \to \pi\, D_s$. Using the leading order chiral interaction we predict that $\tilde{\epsilon} \simeq 0.012$ in (24) is of similar size as $\epsilon \simeq 0.010$, i.e. it is a significant contribution to the width. The value is obtained from the coupled-channel dynamics by computing the coupling constant $g_{\pi^0 D_s}$ and g_{KD} using physical masses for the mesons but imposing $\epsilon = 0$. It is interesting to observe that the effect is somewhat dominated by the isospin breaking induced by $M_{D^+} \neq M_{D^0}$. We derive $\tilde{\epsilon}_1 \simeq 0.004$ and $\tilde{\epsilon}_2 \simeq 0.008$ in (24).

We note that the mass differences $m_{K^+} - m_{K^0}$ and $M_{D^+} - M_{D^0}$ have a contribution that is proportional to ϵ [14]. In our coupled-channel computation we use the known empirical masses, which include contributions from electromagnetic interactions. It is interesting to observe that if we put $\epsilon = 0$ in (18) but keep the isospin violation in the kaon and D-meson masses we arrive at a width of 33 keV, a value significantly larger than the value in [15,19]. If we set $\tilde{\epsilon} = 0$ we would obtain a width of 8.9 keV only, in good agreement with the 8.7 keV obtained for example in [15]. The importance of isospin breaking effects in the kaon and D-meson masses in generating the strong widths of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ mesons was missed in [15,19] but recently pointed out in [16,17]. It should be emphasized that a precise prediction for the width suffers from an uncertainty in the coupling constants $g_{\eta D_s}$ and g_{KD}. In [9] it was demonstrated that chiral correction terms may change these couplings and lead to the values $g_{\eta D_s} \simeq 3.7$ and $g_{KD} \simeq 3.7$. From this result we conclude that the width of 76 keV quoted above defines most likely a lower limit. Inserting the coupling constants of [9] into (22,23) one predicts a width that may yet be significantly larger. We will return to this issue in the final part of this section.

2.2 Axial vector states and tensor fields

We turn now to the strong decay of the axial vector meson $D^*_{s1}(2460) \to \pi^0 \, D_s$.

It is important to study scalar and axial vector mesons in the open-charm sector on equal footing. The properties of spin 0 and spin 1 mesons are indeed closely related by the heavy-quark symmetry of QCD [20,21,22,23,24]. Rather than applying a multiplet formalism, where for instance scalar and vector fields are grouped together in one field, we use the more conventional representation in terms of scalar and vector fields directly. The latter amounts to a partial summation scheme, the fields being defined away from the heavy-quark limit. Due to the semi-heavy character of the charm mass the heavy-quark expansion is not always well-converging in the open-charm sector. How to go beyond the heavy-quark is an important issue.

Since we aim at predicting electromagnetic decay amplitudes, we have to construct gauge invariant expressions. Our Lagrangian involves massive scalar and vector D-meson fields. We are faced with a serious complication, namely, the mixing of scalar and vector modes. It is a quite cumbersome enterprize to arrive at gauge invariant expressions in the presence of such mixing phenomena [25,26,27]. This is a known and non-trivial complication of the standard model where the Higgs boson may mix with the longitudinal component of the Z boson [28]. A solution to this problem is to represent the 1^- open-charm mesons in terms of antisymmetric tensor fields [29,30,31,32]. The massive spin 1 field is proportional to the divergence of the antisymmetric tensor.

To proceed with this particular representation we have to demonstrate first that the results of [8], which were obtained using the conventional vector field representation, can be recovered with the tensor field representation. This will readily be achieved.

We start with the Lagrangian density,

$$\begin{aligned}\mathcal{L} = & -(\partial_\mu D^{\mu\alpha})\,(\partial^\nu \bar{D}_{\nu\alpha}) + \frac{1}{2}\, D^{\mu\alpha}\, M_{1^-}^2\, \bar{D}_{\mu\alpha} \\ & - \frac{1}{8\,f^2}\Big\{(\partial^\nu D_{\nu\alpha})\;[\Phi,(\partial_\mu\Phi)]_-\,\bar{D}^{\mu\alpha} - D_{\nu\alpha}\;[\Phi,(\partial_\nu\Phi)]_-\,(\partial_\mu \bar{D}^{\mu\alpha})\Big\}\,, \qquad (25)\end{aligned}$$

involving the kinetic term and its associated Weinberg-Tomozawa interaction. In (25) we use the antisymmetric triplet fields $D_{\mu\nu} = -D_{\nu\mu}$ and $\bar{D}_{\mu\nu} = D^\dagger_{\mu\nu}$ with $D_{\mu\nu} = (D^0_{\mu\nu}, -D^+_{\mu\nu}, D^+_{s,\mu\nu})$ describing the heavy-quark multiplet partners of the field D introduced in (8). Since the tensor field representation is not frequently applied in the literature we will be more detailed in the presentation. The key objects we will need in this work are the propagator of the tensor field and its associated wave function. The propagator takes the form

$$\begin{aligned}\langle 0|\,T\,\bar{D}_{\mu\nu}(x)\,D_{\alpha\beta}(y)\,|0\rangle = & -\frac{i}{M_{1^-}^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot(x-y)}}{k^2 - M_{1^-}^2 + i\,\epsilon} \\ & \times\Big[(M_{1^-}^2 - k^2)\,g_{\mu\alpha}\,g_{\nu\beta} + g_{\mu\alpha}\,k_\nu\,k_\beta - g_{\mu\beta}\,k_\nu\,k_\alpha - (\mu\leftrightarrow\nu)\Big]\,. \qquad (26)\end{aligned}$$

The wave function

$$\begin{aligned}&\langle 0|D_{\mu\nu}(0)\,|D(p,\lambda)\rangle = \epsilon_{\mu\nu}(p,\lambda) = \frac{i}{M_{1^-}}\Big[p_\mu\,\epsilon_\nu(p,\lambda) - p_\nu\,\epsilon_\mu(p,\lambda)\Big]\,, \\ &\sum_{\lambda=1}^{3}\epsilon^\dagger_\mu(p,\lambda)\,\epsilon_\nu(p,\lambda) = -g_{\mu\nu} + \frac{p_\mu\,p_\nu}{M_{1^-}^2}\,, \qquad (27)\end{aligned}$$

is expressed most economically in terms of the conventional wave function, $\epsilon_\mu(p,\lambda)$, of a vector field in the vector representation.

Given the interaction (25) we need to derive the on-shell scattering amplitude. This is achieved by an appropriate modification of the technique developed in [7]. Assuming the vector representation of spin one fields the scattering process of Goldstone bosons off vector mesons was studied. The on-shell part of the scattering amplitude takes the simple form,

$$T^{\rm on-shell}_{\mu\nu} = \sum_{J,P} M^{(JP)}(s)\,\mathcal{Y}^{(JP)}_{\mu\nu}(\bar q, q; w)\,, \tag{28}$$

where the projectors $\mathcal{Y}^{(JP)}_{\mu\nu}(\bar q, q; w)$ were constructed to carry well-defined total angular momentum J and parity P. The projectors are polynomials in the initial and final 4-momenta of the Goldstone bosons, q_μ and $\bar q_\mu$, as well as the total 4-momentum w_μ with $w^2 = s$. Only the sector with $J^P = 1^+$ is relevant for the present work. We recall the appropriate projector

$$\mathcal{Y}^{(1+)}_{\mu\nu} = \frac{3}{2}\Big(\frac{w^\mu\,w^\nu}{w^2} - g^{\mu\nu}\Big)\,. \tag{29}$$

The merit of the projectors is their property of solving the Bethe-Salpeter coupled-channel equation analytically for the case of quasi-local interactions. The partial-wave amplitudes $M^{(JP)}(s)$ are Lorentz invariant. They can be computed in terms of an effective interaction $V^{(JP)}(s)$ and loop functions $J^{(JP)}(s)$, with

$$\begin{aligned} &M^{(JP)}(s) = \Big[1 - V^{(JP)}(s)\,J^{(JP)}(s)\Big]^{-1}\,V^{(JP)}(s)\,,\\ &J^{(JP)}(s) = N^{(JP)}(s)\,\Big\{I(s) - I(\mu_M^2)\Big\}\,, \end{aligned} \tag{30}$$

where the factors $N^{(JP)}(s)$ reflect the presence of spin and angular momentum. The latter, if multiplied with an appropriate factor s^n, are polynomials in s and the masses of the intermediate states. The universal integral $I(s)$ was introduced already in (16). The specification of the matching scale μ_M was discussed in [8]. In the strangeness sector we identify $\mu_M = 2012$ MeV with the mass of the vector-meson ground state.

We seek a set of tensor projectors, $\mathcal{Y}^{(JP)}_{\mu\nu,\alpha\beta}(\bar q, q; w)$, with properties analogous to those of $\mathcal{Y}^{(JP)}_{\mu\nu}(\bar q, q; w)$. They are readily defined in terms of the previously established ones

$$\begin{aligned} \mathcal{Y}^{(JP)}_{\mu\nu,\alpha\beta}(\bar q, q; w) &= \tfrac{1}{4}\,\bar p_\nu\,\mathcal{Y}^{(JP)}_{\mu\alpha}(\bar q, q; w)\,p_\beta - \tfrac{1}{4}\,\bar p_\nu\,\mathcal{Y}^{(JP)}_{\mu\beta}(\bar q, q; w)\,p_\alpha\\ &\quad - \tfrac{1}{4}\,\bar p_\mu\,\mathcal{Y}^{(JP)}_{\nu\alpha}(\bar q, q; w)\,p_\beta + \tfrac{1}{4}\,\bar p_\mu\,\mathcal{Y}^{(JP)}_{\nu\beta}(\bar q, q; w)\,p_\alpha\,, \end{aligned} \tag{31}$$

where $p_\mu = w_\mu - q_\mu$ and $\bar p_\mu = w_\mu - \bar q_\mu$. By construction it holds

$$\epsilon^{\dagger,\mu\nu}(\bar{p})\,\mathcal{Y}^{(JP)}_{\mu\nu,\alpha\beta}(\bar{q},q;w)\,\epsilon^{\alpha\beta}(p) = \sqrt{\bar{p}^2\,p^2}\,\epsilon^{\dagger,\mu}(\bar{p})\,\mathcal{Y}^{(JP)}_{\mu\nu}(\bar{q},q;w)\,\epsilon^{\nu}(p)\,, \tag{32}$$

where we make use of the explicit representation of the wave function (27). From the identity (32) we can read off the relation we are after. The scattering amplitude takes the form

$$T^{\rm on-shell}_{\mu\nu,\alpha\beta} = \sum_{J,P} M^{(JP)}(s)\,\mathcal{Y}^{(JP)}_{\mu\nu,\alpha\beta}(\bar{q},q;w)\,, \tag{33}$$

where the invariant partial-wave amplitudes are given by an equation of the form (30). The only modification compared to the expressions of [7] are a rescaling of the loop functions by the factor M^2. The relevant normalization factor reads

$$N^{(1+)}(s) = \frac{3}{2}\,M^2 + \frac{p_{cm}^2}{2}\,, \qquad \sqrt{s} = \sqrt{M^2+p_{cm}^2} + \sqrt{m^2+p_{cm}^2}\,. \tag{34}$$

It is left to derive the effective interaction $V^{(1+)}_{ij}(s)$ as implied by (25). We use a convention for the coupled-channel states analogous to (14). A straightforward application of [7] leads to the result

$$V^{(1+)}_{WT,ij}(s) = \frac{\bar{M}^2+M^2}{3\,\bar{M}^2\,M^2}\,V^{(0^+)}_{WT,ij}(s) - \frac{(\bar{M}^2-M^2)}{12\,f^2\,\bar{M}^2\,M^2}\,(\bar{m}^2-m^2)\,C_{ij}\,, \tag{35}$$

in terms of the matrix $V^{(0^+)}_{WT,ij}(s)$ and the C_{ij} coefficients already specified in (17,18). In (35) the parameters M and $\bar{M}$ denote the masses of open-charm vector meson of the initial and final state respectively. Like in (17) the masses m and $\bar{m}$ stand for the masses of the initial and final Goldstone bosons.

We point out that in the particular limit $\bar{M}=M$ we recover the expressions obtained before in [8], i.e. the invariant amplitude $M^{(1+)}(s)$ is identical to that of [8] within a factor $2\,M^2/3$. This observation implies in particular that the predictions for the axial-vector spectrum be consistent with the expectation of the heavy-quark symmetry, as was emphasized in [8]. The $D^*_{s1}(2460)$ state is generated dynamically. The scattering amplitude develops a pole at $s=M^2_{1^+}$. Close to the pole it has the form:

$$M^{(1+)}_{ij}(s) \simeq -\frac{2}{3\,\bar{M}\,M}\,\frac{2\,g_i^*\,M^2_{1^+}g_j}{s-M^2_{1^+}+i\,\Gamma_{1^+}\,M_{1^+}}\,, \tag{36}$$

with the coupling constants g_i and the width parameter Γ_{1^+}. The normalization of the coupling constants, g_i, is such that in the heavy-quark limit they are identical to those of (19). The values discussed in the following can be compared directly to those given in [9].

Using an effective parameter $f_{eff} = 97.1$ MeV in (35) we can reproduce the empirical mass of 2459 MeV for the 1^+ state at leading order. The coupling constants are

$$
\begin{aligned}
&g_{\eta D_s^*} \simeq 1.95\,, && g_{\pi^0 D_s^*} \simeq 0.049\,, \\
&g_{K^0 D_+^*} \simeq 2.25\,, && g_{K^+ D_0^*} \simeq -2.25\,.
\end{aligned}
\tag{37}
$$

in this case. Again isospin breaking effects in the coupling constants are found negligible. For a quantitative study that considers chiral correction terms we refer to the end of this section.

It is worth emphasizing that the values (37) are identical (or very similar for $g^{(1+)}_{\pi^0 D_s^*}$) to those given in (20). An approximate degeneracy is expected from heavy-quark symmetry. The total width comes at 55 keV using $\epsilon = 0.010$. We point out that our result for the width parameter is five times larger larger than the one claimed in [33] based on the same interaction. The source of the conflicting results lies again in the neglect of isospin breaking effects in the kaon and D-meson masses.

In this section we sharpen our prediction for the hadronic decay widths of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ mesons. Following [9] we construct chiral correction terms to the leading order interactions (7, 25) adjusting to the notations and conventions used in the present work. We take into account the chiral correction terms to the effective interactions $V^{(0^+)}(s)$ and $V^{(1^+)}(s)$ relevant at chiral order Q_χ^2. There will be two types of contributions for s-wave scattering. On the one hand we include s- and u-channel exchanges of the D-meson ground states based on the leading order vertices involving a Goldstone boson and two D-mesons. On the other hand local 2-body counter terms (breaking chiral symmetry and chiral symmetric respectively) will be constructed. These different contributions are represented diagrammatically in Figure 3.

We identify the leading order 3-point vertices involving the Goldstone bosons

$$\mathcal{L} = i\,\frac{g_P}{f}\,\Big\{D_{\mu\nu}\,(\partial^\mu\Phi)\,(\partial^\nu \bar{D}) - (\partial^\nu D)\,(\partial^\mu\Phi)\,\bar{D}_{\mu\nu}\Big\} + \frac{\tilde{g}_P}{4\,f}\,\epsilon^{\mu\nu\alpha\beta}\,\Big\{D_{\mu\nu}\,(\partial_\alpha\Phi)\,(\partial^\tau\bar{D}_{\tau\beta}) + (\partial^\tau D_{\tau\beta})\,(\partial_\alpha\Phi)\,\bar{D}_{\mu\nu})\Big\}\,, \tag{38}$$

that are responsible for the s- and u-channel exchange contributions. The vertices (38) will play a decisive role when computing the radiative decay widths of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states. As detailed in Appendix A the decay of the charged D^* mesons implies

$$|g_P| = 0.57 \pm 0.07\,, \tag{39}$$

using $f = 90$ MeV. The parameter $\tilde{g}_P$ in (38) can not be extracted from empirical data directly. An accurate evaluation within unquenched QCD lattice simulations would be highly desirable. However, the size of the latter parame-

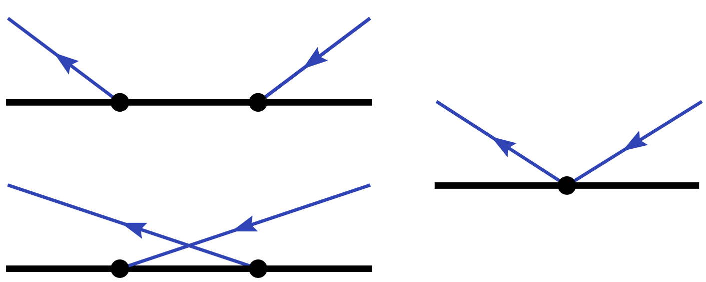

Fig. 3. Chiral correction terms from s- and u-channel exchange processes of the D-meson (left) and local 2-body counter terms (right).

(I,S)	Channel	$C^{(I,S)}$	$C^{(I,S)}_{\pi,0}$	$C^{(I,S)}_{\pi,1}$	$C^{(I,S)}_{K,0}$	$C^{(I,S)}_{K,1}$	$C^{(I,S)}_{2}$	$C^{(I,S)}_{3}$	$C^{(I,S)}_{s-ch}$	$C^{(I,S)}_{u-ch}$
(0,+1)	11	2	0	0	4	0	2	0	4	0
	12	$\sqrt{3}$	0	$-\frac{\sqrt{3}}{2}$	0	$\frac{5}{2\sqrt{3}}$	0	$-\frac{1}{2\sqrt{3}}$	$\frac{4}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$
	22	0	$-\frac{4}{3}$	$-\frac{2}{3}$	$\frac{16}{3}$	0	2	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
(1,+1)	11	0	4	-2	0	0	2	1	0	0
	12	1	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	-2
	22	0	0	0	4	-2	2	1	0	0

Table 1
The coefficients $C^{(I,S)}$ that characterize the interaction of Goldstone bosons with heavy-meson fields as introduced in (14, 18) for given isospin (I) and strangeness (S).

ter can be estimated applying the heavy-quark symmetry of QCD. As detailed in Appendix B one expects

$$\tilde{g}_P = g_P \,, \tag{40}$$

at leading order.

The coupling constant g_P contributes to the effective interaction $V^{(1^+)}(s)$ and $V^{(0^+)}(s)$ via s- and u-channel exchange processes of the D meson ground states. We derive the relevant contributions applying the on-shell reduction scheme of [7]. In the scalar sector only the u-channel exchange of the $J^P = 1^-$ contributes. In contrast to [9] the s-channel process of the 1^- state does not contribute here. This is a consequence of the tensor field representation of the spin-one field. We derive

$$\begin{aligned}
V^{(0^+)}_{u-ch}(s) &= g_P^2\,\frac{C_{u-ch}}{4\,s\,f^2}\left(s-\bar{M}^2+\bar{m}^2\right)\left(s-M^2+m^2\right) \\
&\quad + g_P^2\,\frac{C_{u-ch}}{f^2}\int_{-1}^{1}\frac{dx}{2}\,\frac{(\bar{q}\cdot q)\,\mu_{1^-}^2-(\bar{m}^2-\bar{q}\cdot p)\,(m^2-\bar{p}\cdot q)}{M^2+\bar{M}^2-\mu_{1^-}^2-s+2\,\bar{q}\cdot q}\,, \\
\bar{q}\cdot q &= \sqrt{\bar{m}^2+\bar{p}_{\rm cm}^2}\,\sqrt{m^2+p_{\rm cm}^2}-\bar{p}_{\rm cm}\,p_{\rm cm}\,x\,, \\
\bar{q}\cdot p &= -\bar{q}\cdot q+\frac{s-\bar{M}^2+\bar{m}^2}{2}\,, \qquad \bar{p}\cdot q = -\bar{q}\cdot q+\frac{s-M^2+m^2}{2}\,, \\
\sqrt{s} &= \sqrt{M^2+p_{cm}^2}+\sqrt{m^2+p_{cm}^2} = \sqrt{\bar{M}^2+\bar{p}_{cm}^2}+\sqrt{\bar{m}^2+\bar{p}_{cm}^2}\,,
\end{aligned} \tag{41}$$

where (m, M) and $(\bar{m}, \bar{M})$ are the masses of initial and final mesons. The parameter $\mu_{1^-} \simeq 2059$ MeV is taken to be the average of the vector D-meson masses. The coefficients $C^{(I,S)}_{u-ch}$ are recalled from [9] in Table 1 for the channels relevant for the formation and decay of the $D^*_{s0}(2317)$.

As already noted in [9] the influence of the u-channel process is of very minor importance for the formation of the $D^*_{s0}(2317)$. At $f = 90$ MeV and $g_P = 0$ we obtain a mass of 2304 MeV which is pulled down by 1 MeV only if we switch on $g_P = 0.57$.

The situation is different for the axial-vector state $D^*_{s1}(2460)$. In this case there are three processes contributing. The s-channel exchange of the 1^- state, and the u-channel exchanges of the 0^- and 1^- charmed mesons. Consider first the two u-channel contributions. After some tedious algebra we obtain:

$$
\begin{aligned}
V^{(1^+)}_{u-ch}(s) &= -\frac{C_{u-ch}}{f^2}\int_{-1}^{1}\frac{dx}{4}\,\Big\{A_1\,(1+x^2)+\bar p_{cm}\,p_{cm}\,x\,(1-x^2)\,A_5\Big\}\,,\\
A_1 &= \frac{\tilde g_P^2}{16\,\bar M^2\,M^2\,\mu_{1^-}^2}\Big\{-\Big(M^4+(3\,\mu_{1^-}^2-u)\,M^2+\mu_{1^-}^2\,u\Big)\,\bar M^4\\
&\quad+\Big((u-3\,\mu_{1^-}^2)\,M^4+(\mu_{1^-}^2\,(2\,s-u)-u^2)\,M^2\\
&\qquad+m^2\,(M^2+\mu_{1^-}^2)\,(M^2-u)+2\,\mu_{1^-}^2\,s\,u\Big)\,\bar M^2\\
&\quad+\bar m^2\,(\bar M^2-m^2-u)\,\Big((M^2+\mu_{1^-}^2)\,\bar M^2+\mu_{1^-}^2\,(M^2+u)\Big)\\
&\quad+\mu_{1^-}^2\,(M^2+u)\,\Big((M^2-u)\,m^2+u\,(-M^2+2\,s+u)\Big)\Big\}\,,\\
A_5 &= \frac{g_P^2}{u-\mu_{0^-}^2}+\tilde g_P^2\left(1+\frac{(M^2+\bar M^2+u)\,\mu_{1^-}^2}{M^2\,\bar M^2}\right)\frac{M^2+\bar M^2-s-u}{8\,\mu_{1^-}^2\,(u-\mu_{1^-}^2)}\,,\\
&\qquad u=\bar M^2+M^2-s+2\,\bar q\cdot q\,,
\end{aligned}
\tag{42}
$$

where we apply the kinematics of (41). We use $\mu_{0^-} = 1918$ MeV as the average mass of the 0^- charmed mesons. The influence of the u-channel exchange interaction (42) on the formation of the $D^*_{s1}(2460)$ state is somewhat more important than it is in the scalar sector. For $f = 90$ MeV and $g_P = \tilde g_P = 0$ we obtain 2441 MeV, a mass which is pushed up by 5 MeV upon incorporation of (42) with $g_P = \tilde g_P = 0.57$. The effect is dominated largely by the exchange of the 1^- state.

We turn to the s-channel exchange. In contrast to [9] which was based on the vector-field representation of the spin-one D mesons, there is a contribution from the s-channel exchange of the 1^- state within the tensor-field approach. This reflects the fact that the Green's function (26) contains a non-propagating 1^+ component. It is analogous to the non-propagating 0^+ component of a conventional Green's function for a 1^- particle as implied by the Proca formalism. We derive

$$V^{(1^+)}_{s-ch}(s) = \frac{2\,\tilde g_P^2}{3\,\mu^2_{1^-}}\,\frac{C_{s-ch}}{4\,s\,f^2}\left(s-\bar M^2+\bar m^2\right)\left(s-M^2+m^2\right), \tag{43}$$

where the coefficients $C^{(I,S)}_{s-ch}$ are recalled from [9] in Table 1. The combined effect of (42) and (43) yields a resonance mass of 2433 MeV for $f = 90$ MeV and $g_P = \tilde g_P = 0.57$. Clearly, further correction terms are needed in order to reproduce quantitatively the masses of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states.

We turn to the local 2-body interaction terms, considering the chiral symmetry breaking and the chiral symmetric terms successively [9]. At chiral order Q^2_χ the following terms break chiral symmetry explicitly,

$$\begin{aligned}
\mathcal{L}_{\chi-SB} &= -2\,c_1\,D\,\chi_0\,\bar D - (4\,c_0 - 2\,c_1)\,(D\,\bar D)\,\mathrm{tr}\,\chi_0 \\
&\quad + \frac{2\,c_0 - c_1}{f^2}\,D\,\bar D\,\mathrm{tr}\left(\Phi\,\chi_0\,\Phi\right) + \frac{c_1}{4\,f^2}\,D\left\{\Phi,\left\{\Phi\,,\chi_0\right\}\right\}\bar D \\
&\quad + \tilde c_1\,D_{\alpha\beta}\,\chi_0\,\bar D^{\alpha\beta} + (2\,\tilde c_0 - \tilde c_1)\,(D_{\alpha\beta}\,\bar D^{\alpha\beta})\,\mathrm{tr}\,\chi_0 \\
&\quad - \frac{2\,\tilde c_0 - \tilde c_1}{2\,f^2}\,D_{\alpha\beta}\,\bar D^{\alpha\beta}\,\mathrm{tr}\left(\Phi\,\chi_0\,\Phi\right) - \frac{\tilde c_1}{8\,f^2}\,D_{\alpha\beta}\left\{\Phi,\left\{\Phi\,,\chi_0\right\}\right\}\bar D^{\alpha\beta}\,,
\end{aligned} \tag{44}$$

where the matrix χ_0 was introduced already in (9). The parameters c_1 and $\tilde c_1$ are determined by the empirical mass differences of the $J^P = 0^-$ and $J^P = 1^-$ charmed mesons. According to [9] we have

$$c_1 \simeq 0.44\,, \qquad \tilde c_1 \simeq 0.47\,. \tag{45}$$

The parameters c_0 and $\tilde c_0$ could in principle be determined by unquenched lattice QCD simulation upon studying the pion- and kaon-mass dependence of the D-meson ground states. So far they are unknown. We construct the effective interaction

$$\begin{aligned}
V^{(0^+)}_{\chi-SB}(s) &= 2\,\frac{m_\pi^2}{f^2}\left(c_0\,C_{\pi,0} + c_1\,C^{(I,S)}_{\pi,1}\right) + 2\,\frac{m_K^2}{f^2}\left(c_0\,C_{K,0} + c_1\,C_{K,1}\right), \\
V^{(1^+)}_{\chi-SB}(s) &= \frac{2}{3\,\bar M\,M}\,V^{(0^+)}_{\chi-SB}(s)\Big|_{c_i\to\tilde c_i} + \cdots\,,
\end{aligned} \tag{46}$$

where the dots represent higher order terms that we neglect. The coefficients $C^{(I,S)}_{\pi,0}, C^{(I,S)}_{\pi,1}, C^{(I,S)}_{K,0}$ and $C^{(I,S)}_{K,1}$ are recalled in Table 1.

The order-Q^2_χ terms that are chiral symmetric and contribute to s-wave scattering are

$$\mathcal{L}_{CT} = \frac{2\,c_2 + c_3}{f^2}\,D\,\bar D\,\mathrm{tr}\left((\partial_\mu\Phi)\,(\partial^\mu\Phi)\right) - \frac{c_3}{f^2}\,D\,(\partial_\mu\Phi)\,(\partial^\mu\Phi)\,\bar D$$

$$-\frac{2\,\tilde c_2+\tilde c_3}{2\,f^2}\,D_{\alpha\beta}\,\bar D^{\alpha\beta}\,\mathrm{tr}\,\Big((\partial_\mu\Phi)\,(\partial^\mu\Phi)\Big)+\frac{\tilde c_3}{2\,f^2}\,D_{\alpha\beta}\,(\partial_\mu\Phi)\,(\partial^\mu\Phi)\,\bar D^{\alpha\beta}\,. \tag{47}$$

They imply the effective s-wave interactions

$$\begin{aligned}
V^{(0^+)}_{CT}(s) &= \Big(c_2\,\frac{C_2}{s\,f^2}+c_3\,\frac{C_3}{s\,f^2}\Big)\,\Big(s-\bar M^2+\bar m^2\Big)\,\Big(s-M^2+m^2\Big)\,,\\
V^{(1^+)}_{CT}(s) &= \frac{2}{3\,\bar M\,M}\,V^{(0^+)}_{CT}(s)+\cdots\,,
\end{aligned} \tag{48}$$

where the dots represent higher order terms that we neglect. The coefficients $C_2^{(I,S)}$ and $C_3^{(I,S)}$ are given in Table 1. All together the effective interactions $V^{(0^+)}(s)$ and $V^{(1^+)}(s)$ receive the following contributions

$$V(s)=V_{WT}(s)+V_{u-ch}(s)+V_{s-ch}(s)+V_{\chi-SB}(s)+V_{CT}(s)\,, \tag{49}$$

where the various terms are detailed in (17, 35, 41, 42, 43, 46, 48). Clearly, the number of unknown parameters, $c_{0,2,3}$ and $\tilde c_{0,2,3}$ appears large at first. A free fit to the masses of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states only would not be significant.

Additional constraints from QCD should be used. According to [9] the parameters c_i and $\tilde c_i$ are degenerate in the heavy-quark mass limit, i.e. we expect

$$c_i\simeq\tilde c_i\,. \tag{50}$$

It is reassuring that the values for c_1 and $\tilde c_1$ given in (45) are quite compatible with the expectation of the heavy-quark symmetry relations (50). We consider further constraints from QCD as they arise in the limit of large number of colors N_c [34]. For that purpose the representation (47) is advantageous over the one used in [9]. Since at leading order in a $1/N_c$ expansion single-flavour trace interactions are dominant, we anticipate the relations

$$c_0\simeq\frac{c_1}{2}\,,\qquad c_2\simeq-\frac{c_3}{2}\,,\qquad \tilde c_0\simeq\frac{\tilde c_1}{2}\,,\qquad \tilde c_2\simeq-\frac{\tilde c_3}{2}\,. \tag{51}$$

In the combined heavy-quark and large-N_c limit with (51, 50) we are left with one free parameter only. We may vary $c_3=\tilde c_3$. The optimal value $c_3=\tilde c_3=1.2$ together with $c_1,\tilde c_1$ as dictated by (45), $g_P=\tilde g_P=0.57$ and $f=90$ MeV predicts 2330 MeV and 2449 MeV for the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states respectively.

Since for the determination of the widths parameters it is important to reproduce the masses accurately we allow for small variations of the parameters

$J^P_{(I,S)}$	$0^+_{(0,1)}$	$0^+_{(0,1)}$	$0^+_{(\frac{1}{2},0)}$	$1^+_{(0,1)}$	$1^+_{(0,1)}$	$1^+_{(\frac{1}{2},0)}$
M_R [MeV]	2317.6	2317.6	2410.5	2459.2	2459.2	2568
Γ [MeV]	0.14	0.25	2.18	0.14	0.25	18
$\lvert g_1 \rvert$	3.27	3.27	0.24	2.97	2.97	<0.05
$\lvert g_2 \rvert$	2.50	2.50	1.37	2.42	2.42	1.4
$\lvert g_3 \rvert$	-	-	2.12	-	-	2.5
ϵ	0.01	0.02	0	0.01	0.02	0

Table 2
Masses, widths and coupling constants for dynamically generated 0^+ and 1^+ states. The coupling constants g_i are given in the isospin basis (see [9]). We use $f = 90$ MeV, $g_P = \tilde{g}_P = 0.57$, $c_3 = 1.0$, $\tilde{c}_3 = 1.4$. The remaining parameters are implied by the relations (45, 51, 50).

around the heavy-quark scenario but leaving the large-N_c relations (51) untouched. A precise reproduction of the scalar and axial-vector states is achieved with $c_3 = 1.0$ and $\tilde{c}_3 = 1.4$. Detailed results are collected in Table 2. For given mixing angle $\epsilon = 0.01$ the decay widths of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ come at 140 keV, values significantly larger than the leading order estimates discussed above. The effect is, in part, a consequence of somewhat larger coupling constants $g^{(0^+)}_{\eta D_s} = 2.50$ and $g^{(1^+)}_{\eta D^*_s} = 2.42$ than given in Table 2. The sensitivity to the mixing angle ϵ is illustrated by the 3rd and 6th row where the results with $\epsilon = 0.02$ are given. Though the isospin coupling constants are basically unchanged the decay width is almost doubled with 250 keV.

We briefly comment on the previous results of [9]. In that work a different scenario was investigated. Applying the conventional vector-field representation of the 1^- charmed mesons, it was assumed that the axial-vector resonance $D^*_1(2420)$ was a member of the exotic sextet, predicted at leading order by chiral coupled-channel dynamics [8,9]. The crucial question is whether chiral correction terms reduce the weak attraction predicted at leading order or possibly enhance it. In [9] chiral correction terms were tuned in such a way as to pull down the exotic axial state with $(I,S) = (\frac{1}{2}, 0)$ to match the properties of the $D^*_1(2420)$. The invariant πD and πD^* mass distributions as measured by the BELLE collaboration [35] were used as an additional constraint. It was argued that the scalar heavy-quark partner of the exotic axial state decouples from the πD channel, and therefore is not seen in the data. Based on the large-N_c relations, that were not considered in [9], we would deem this scenario unlikely. In order to compare to the large-N_c scenario advocated in the present work we included in Table 2 the rows 4 and 7, which give the characteristics of the exotic $(\frac{1}{2}, 0)$ states. Like in [9] the scalar state is quite narrow with a mass below the ηD threshold. We obtain a mass and width of

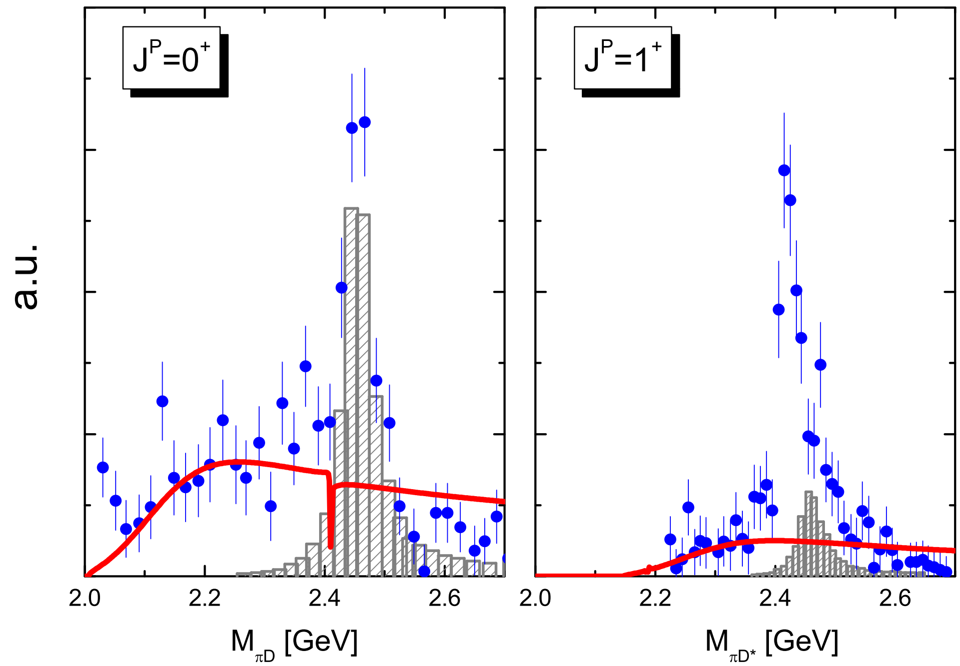

Fig. 4. Mass spectra of the $(\frac{1}{2}, 0)$-resonances as seen in the $\pi\, D(1867)$-channel (l.h. panel $J^P = 0^+$) and $\pi\, D^*(2008)$-channel (r.h. panel $J^P = 1^+$). The solid lines show the theoretical mass distributions. The data are taken from [35] as obtained from the $B \to \pi\, D(1867)$ and $B \to \pi\, D^*(2008)$ decays. The histograms indicate the contribution from the $J = 2$ resonances $D_2^*(2460)$ as given in [35].

2411 MeV and 2.2 MeV respectively. As seen from the coupling constants in Table 2 the state couples most strongly to the ηD and $K\, D_s$ channels.

In Figure 4 we confront the imaginary part of the $\pi D \to \pi D$ amplitude with the invariant πD mass distribution measured by the BELLE collaboration [35]. The empirical distribution is dominated by the broad $(\frac{1}{2}, 0)$ state, a member of the triplet to which the $D_{s0}^*(2317)$ belongs, as well as the tensor state $D_2^*(2460)$, the contribution of which is illustrated by the histograms. The possible presence of a narrow $(\frac{1}{2}, 0)$ state is not excluded by the present data. It is interesting to observe that the exotic state leads to a dip in the mass distribution rather than a peak. This is a consequence of the nearby ηD channel that couples strongly to that state. We note that, with the exception of a strong cusp effect at the $\bar{K}D$ threshold in the $(0, -1)$ sector, there is no further strong signal of any sextet state in this scenario.

A striking prediction of the large-N_c scenario is a clear measurable signal of the exotic axial state in the ηD^* invariant mass distribution. The exotic axial state at mass 2568 MeV lies above the ηD^* threshold giving it a width of about 18 MeV. In Figure 4 we confront the imaginary part of the $\pi D^* \to \pi D^*$

amplitude with the invariant πD^* mass distribution measured by the BELLE collaboration [35]. The empirical distributions shows two axial and one tensor states. The contribution of the tensor state state is illustrated by the histograms. In conventional approaches the tensor state $D_2^*(2460)$ is grouped together with the $D_1^*(2420)$ state to form a heavy-quark multiplet. Within the hadrogenesis conjecture we would expect to generate that multiplet dynamically via coupled-channel effects once the light vector mesons are considered as additional and explicit degrees of freedom. The theoretical amplitude of the present work describes only the broad state, which has a width of about 300 MeV. In contrast to the πD distribution shown in the left panel of Figure 4 we do not predict any significant signal in the πD^* distribution that one may use to discover the exotic axial state. This reflects a coupling constant of that state to the πD^* channel that is almost compatible with zero. Nonetheless, we deem the exotic axial state to be easily discovered by ongoing experiments once the invariant ηD^* mass distribution is analyzed. The discovery of the scalar state, in contrast, would require a measurement of the πD invariant mass with an energy resolution of a few MeV as may be possible with the PANDA experiment at FAIR.

We close this section with the remark that very similar results are obtained applying the vector-field representation once the parameters are chosen in accordance with the expectation of large-N_c QCD [36].

3 Electromagnetic interactions

In order to compute the radiative decay of the scalar and axial-vector states generated in the last section we need to couple the photon field to hadronic fields. In doing so we have to ensure consistency with the U(1) gauge symmetry of electromagnetic interactions. We consider first electromagnetic interactions in the framework defined by the hadronic interactions of the previous section. We complete this scheme by additional hadronic and electromagnetic interaction terms which are relevant when considering the light vector mesons as additional explicit degrees of freedom. They are expected to be responsible for the formation of tensor states within the hadrogenesis approach. Even though we do not consider the effect of light vector mesons in the coupled-channel part of this work, we do explore it in the radiative decay properties of the scalar and axial vector D_s^* states.

Eqs. (7, 25, 38) are restricted to hadronic interactions and do not include terms linear in the photon field which are implied by the minimal gauge principle. They are easily constructed using the gauge covariant derivatives

$$\partial_\mu \Phi + i\, e\, [Q, \Phi]\, A_\mu \,, \qquad\qquad \partial_\mu D + i\, e\, D\, Q'\, A_\mu \,, \tag{52}$$

with $e = |e|$ and the charge matrices are defined as

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \qquad Q' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{53}$$

The kinetic terms in (7) and (25) imply the couplings

$$\begin{aligned} \mathcal{L}_{\text{e.m.}} = i\,\frac{e}{2}\, A^\mu\, \text{tr}\, \Big((\partial_\mu \Phi)\, [Q,\, \Phi]_- \Big) + \, i\, e\, A^\mu \Big(D\, Q' (\partial_\mu\, \bar{D}) - (\partial_\mu\, D)\, Q'\, \bar{D} \Big) \\ - i\, e\, A_\mu\, D_{\mu\nu}\, Q'\, (\partial^\beta \bar{D}_{\beta\nu}) + i\, e\, A_\mu\, (\partial_\alpha D^{\alpha\nu})\, Q'\, \bar{D}_{\mu\nu} \,. \end{aligned} \tag{54}$$

The gauging of 4-point vertices, such as the Weinberg-Tomozawa terms in (7) and (25) or the chiral correction terms (47), does not lead to any vertex that will be of relevance in this work. It would not contribute to the radiative decay processes.

We point out that, given the interactions (7, 25, 54) only, the electromagnetic decay amplitudes $0^+ \to \gamma 1^-$ and $1^+ \to \gamma\, 0^-, \gamma\, 1^-$ are zero identically. These decay processes probe 3-point hadronic vertices introduced in (38). The importance of these terms in determining decays contrasts with the minor role they play for the formation of the $D_{s0}^*(2317)$ and $D_{s1}^*(2460)$ states. The gauging of the interaction (38) yields the terms

$$
\begin{aligned}
\mathcal{L}_{e.m.} =& - \frac{e\, g_P}{f} \Big\{ D_{\mu\nu}\, [Q,\Phi]\, A_\mu\, (\partial^\nu \bar D) - (\partial^\nu D)\, [Q,\Phi]\, A^\mu\, \bar D_{\mu\nu} \Big\} \\
&+ \frac{e\, g_P}{f} \Big\{ D_{\mu\nu}\, (\partial^\mu \Phi)\, A^\nu\, Q'\, \bar D + A^\nu D\, Q'\, (\partial^\mu \Phi)\, \bar D_{\mu\nu} \Big\} \\
&+ i\, \frac{e\, \tilde g_P}{4\, f}\, \epsilon^{\mu\nu\alpha\beta} \Big\{ D_{\mu\nu}\, [Q,\Phi]\, A_\alpha\, (\partial^\tau \bar D_{\tau\beta}) + (\partial^\tau D_{\tau\beta})\, [Q,\Phi]\, A_\alpha\, \bar D_{\mu\nu} \Big\} \\
&- i\, \frac{\tilde g_P}{4\, f}\, \epsilon^{\mu\nu\alpha\beta} \Big\{ D_{\mu\nu}\, (\partial_\alpha \Phi)\, A^\tau\, Q' \bar D_{\tau\beta} - A^\tau\, D_{\tau\beta}\, Q'\, (\partial_\alpha \Phi)\, \bar D_{\mu\nu} \Big\} \,. \qquad (55)
\end{aligned}
$$

It is important to realize that further interactions, that are gauge invariant separately, are possible and, in fact, will play an important role in this work. Note that the terms (55) are part of the gauge invariant vertex that is associated a leading chiral power Q_χ. This illustrates the fact that the standard counting scheme require the photon field A_μ to be counted as order Q_χ. Like in section 2 we incorporate chiral correction terms.

We construct the leading order correction terms that involve the electromagnetic field strength tensor $F_{\mu\nu} \sim Q_\chi^2$. There are two relevant terms involving one Goldstone boson field. According to the heavy-quark symmetry such terms must involve the charge matrix Q_χ of the light quarks. Terms that are proportional to the charge of the charm quark are suppressed in the heavy-quark mass limit. We write

$$
\begin{aligned}
\mathcal{L}_{\rm e.m.} =& \frac{e_P}{f\, m_V^2}\, F^{\mu\nu} \Big\{ D_{\mu\alpha}\, [(\partial_\nu \Phi), Q]\, (\partial^\alpha \bar D) - (\partial^\alpha D)\, [(\partial_\nu \Phi), Q]\, \bar D_{\mu\alpha} \Big\} \\
&- i\, \frac{\tilde e_P}{4\, f\, m_V^2}\, F_{\mu\nu}\, \epsilon^{\sigma\tau\mu\beta} \Big\{ D_{\sigma\tau}\, [(\partial^\nu \Phi), Q]\, (\partial^\alpha \bar D_{\alpha\beta}) \\
&\qquad\qquad + (\partial^\alpha D_{\alpha\beta})\, [(\partial^\nu \Phi), Q]\ \bar D_{\sigma\tau}) \Big\} \,, \qquad (56)
\end{aligned}
$$

where heavy-quark symmetry predicts the relation (see Appendix B)

$$
e_P = \tilde e_P \,. \qquad (57)
$$

According to standard counting rules the vertices (56) carry the leading chiral power Q_χ^3. It should be emphasized, however, that the counting rules depend on a naturalness assumption, i.e. that the dimension full parameters of the effective Lagrangian density scale with appropriate power of the chiral symmetry breaking scale $4\pi\, f \simeq 1131$ MeV, or more pragmatically with the mass m_V of the lightest degree of freedom that is integrated out. This is the rational behind the particular representation of the vertex (38) in terms of the dimension less parameters e_P and $\tilde e_P$. Based on this assumption one would expect

$|e_P| \sim e \simeq 0.303$. Unfortunately, at present there is no empirical estimate available for the size of the parameters e_P and $\tilde{e}_P$. The parameter e_P is determined by the three-body decay processes $D^*_+ \to \gamma\,\pi_+\,D_0$ or $D^*_0 \to \gamma\,\pi_-\,D_+$.

We anticipate from the numerical result section that the magnitude of the parameter e_P is somewhat larger than expected from the naive naturalness assumption if we omit the light vector mesons as explicit degrees of freedom. Such a phenomenon is not unusual in effective field theories, though asking for a physical explanation. Typically one would promote such a counter term to carry a formal counting power that takes this into account. Thus one may assign the vertex (56) the order Q_χ^2. On a formal level that would justify to consider the effect of (56) while neglecting additional hadronic vertices of chiral order Q_χ^3, such as the SU(3) flavor breaking effects in the coupling of the Goldstone bosons to the D mesons (see (38)). A more physical resolution would be to incorporate additional degrees of freedom, the most prominent ones the light vector mesons. Once they are incorporated one would expect the required magnitude for e_P to reduce significantly. We take this as additional motivation to explore the role played by the light vector mesons in the radiative decays of scalar and axial-vector molecules. We return to this issue once we completed the collection of chiral correction terms of order Q_χ^2.

Additional chiral correction terms of chiral order Q_χ^2 describe anomalous processes. A photon hitting a pseudo-scalar charmed meson may convert the latter into its heavy-quark partner, a charmed vector meson. In the absence of such processes the decay amplitude $D^*_{s1}(2460) \to \gamma\,D^*_{s0}(2317)$ would vanish identically, given the hadronic interactions of section 2. The leading order anomalous vertex is readily identified

$$\mathcal{L}_{\rm e.m.} = \frac{1}{2\,M_V^2}\,F^{\mu\nu}\,\epsilon_{\mu\nu\alpha\beta}\,\Big\{(\partial_\tau D^{\tau\beta})\Big(e_C + e_Q\,Q\Big)\,(\partial^\alpha \bar{D}) + (\partial^\alpha D)\Big(e_C + e_Q\,Q\Big)\,(\partial_\tau \bar{D}^{\tau\beta})\Big\}\,. \tag{58}$$

The vertices (58) carry the leading chiral power Q_χ^2 (see [38,39,40]). This should be compared to the leading chiral power Q_χ of the last four terms in (54). As worked out in Appendix A the radiative decay properties of the charmed vector mesons suggest

$$e_Q = 0.91 \pm 0.10\,, \qquad e_C = 0.13 \pm 0.05\,, \tag{59}$$

where we use $M_V = 2000$ MeV. The values (59) together with (39) reproduce the empirical branching ratios of the $D^*_- \to D_-\,\gamma$ and $D^*_0 \to D_0\,\gamma$ decays. As detailed in Appendix A the empirical constraints on the $D^*_s \to D_s\,\gamma$ and $D^*_s \to D_s\,\pi_0$ suggest the somewhat smaller value $e_Q \simeq 0.52$.

It should be emphasized that the anomalous interaction (58) by itself is at odds with the heavy-quark symmetry of QCD. Heavy-quark symmetry relates the interactions of pseudo-scalar and vector D mesons. Additional terms are required that will parameterize the magnetic moments of the charmed vector mesons. We write

$$\mathcal{L}_{\rm e.m.} = \frac{i}{M_V^2}\, F^{\mu\nu}\,(\partial^\alpha D_{\alpha\mu})\,\Big(e\,Q' - \tilde e_C + \tilde e_Q\,Q\Big)\,(\partial^\beta \bar D_{\beta\nu})\,. \tag{60}$$

In order to provide a physical interpretation for the parameters $\tilde e_C$ and $\tilde e_Q$ we derive expressions for the magnetic moments of the charmed vector mesons. Following the original works by Jones and Kyriakopoulos [29,30,31] we obtain

$$\begin{aligned} &\mu_{D^*_0} = \frac{M_{D^*_0}}{2\,M_V^2}\,\Big(\tilde e_C - \frac{2}{3}\,\tilde e_Q\Big)\,, \qquad \mu_{D^*_+} = \frac{M_{D^*_+}}{2\,M_V^2}\,\Big(\tilde e_C + \frac{1}{3}\,\tilde e_Q\Big)\,,\\ &\mu_{D^*_s} = \frac{M_{D^*_s}}{2\,M_V^2}\,\Big(\tilde e_C + \frac{1}{3}\,\tilde e_Q\Big)\,, \end{aligned} \tag{61}$$

where we recognize that at present there is no empirically information on the magnetic moments available. Note that the term proportional to Q' in (60) cancels a corresponding contribution which is implied by minimally gauging the kinetic term (54). This construction proves convenient when working out the consequences of the heavy-quark symmetry. The term defines a finite renormalization of the parameters $\tilde e_C$ and $\tilde e_Q$. It is instructive to interpret the result (61) in terms of the constituent quark model. The contribution from e_C reflects the magnetic moment of the charm quark. Therefore it is SU(3) flavor blind. This reflects the fact that in the heavy-quark mass limit the parameter e_C approaches a constant. In contrast the term proportional to e_Q models the contribution of the magnetic moment of the light quark: it is flavor dependent, being proportional to the charge matrix of the light quarks, Q_χ. In the heavy-quark mass limit the parameter the parameter e_Q scales linear in the charm quark mass. As recalled in Appendix B one expects the relations

$$e_Q = \tilde e_Q\,, \qquad e_C = \tilde e_C\,, \tag{62}$$

at leading order.

We turn to the interaction vertices that are probed when considering the light vector mesons as explicit degrees of freedom. While doing so we one may be worried whether one leaves the path of rigorous effective field theory: a well defined chiral power counting scheme for the light vector mesons is not established. We advocate a pragmatic argument as to justify the consideration of light vector mesons as relevant physical degrees of freedom in effective field theories. Rather than relying on the strict chiral counting we apply large-N_c arguments as a justification that the effect of multiple loop effects involving light vector mesons must be suppressed as compared to the leading one-loop diagrams.

Consider the anomalous process where a pion hit by an energetic photon may be transformed into a ρ meson. Such a process is analogous to (58) and it is an important contribution to the three-body decay process $D^*_+ \to \rho_+ D_0 \to \gamma\,\pi_+ D_0$, but will also affect the radiative decays of the scalar and axial vector molecules. The relevant anomaly vertex derived for the flavor SU(3) sector of QCD reads

$$\mathcal{L}_{\rm e.m.} = \frac{e_A}{8\,f\,m_V}\,F^{\mu\nu}\,\epsilon_{\mu\nu\alpha\beta}\,{\rm tr}\Big((\partial^\alpha\Phi)\,[Q,\,\partial_\tau V^{\tau\beta}]_+\Big)\,, \tag{63}$$

where we use the representation

$$V_{\mu\nu} = \begin{pmatrix} \rho^0_{\mu\nu}+\omega_{\mu\nu} & \sqrt{2}\,\rho^+_{\mu\nu} & \sqrt{2}\,K^+_{\mu\nu} \\ \sqrt{2}\,\rho^-_{\mu\nu} & -\rho^0_{\mu\nu}+\omega_{\mu\nu} & \sqrt{2}\,K^0_{\mu\nu} \\ \sqrt{2}\,K^-_{\mu\nu} & \sqrt{2}\,\bar K^0_{\mu\nu} & \sqrt{2}\,\phi_{\mu\nu} \end{pmatrix}\,. \tag{64}$$

The particular form (63) follows at leading order in a chiral expansion applying the external field method. From the radiative decays of the light vector mesons considered in Appendix A one derives conflicting values for the parameter e_A of (63). A quantitative description requires the inclusion of SU(3) flavor breaking effects. However, since for this work the three processes $K^*_0 \to K_0\gamma, K^*_\pm \to K_\pm\gamma$ and $\phi \to \eta\gamma$ are relevant only, we may use coupling constants as dictated by the latter processes.

$$\begin{aligned} &|e_A^{(K^*_0\to K_0\gamma)}| = 0.119 \pm 0.006\,, \qquad &|e_A^{(K^*_\pm\to K_\pm\gamma)}| = 0.090 \pm 0.004\,, \\ &|e_A^{(\,\phi\,\to\,\eta\,\gamma\,)}| = 0.053 \pm 0.001\,, \end{aligned} \tag{65}$$

where we use $f = 90$ MeV and $m_V = 776$ MeV.

It is observed that the anomaly vertex (63) can contribute to the radiative decay processes of the D mesons only in the presence of additional hadronic 3-point vertices. The light vector meson that is created by the photon must be absorbed by the heavy meson. An appropriate effective interaction is readily constructed:

$$
\begin{aligned}
\mathcal{L} = & \, i \, \frac{g_V}{2\,f} \Big\{ D\,(\partial_\alpha V^{\alpha\mu})\,(\partial_\mu \bar{D}) - (\partial_\mu D)\,(\partial_\alpha V^{\alpha\mu})\,\bar{D} \Big\} \\
& - i \, \frac{\tilde{g}_V}{2\,f} \Big\{ D^{\mu\nu}\,(\partial^\alpha V_{\alpha\mu})\,(\partial^\beta \bar{D}_{\beta\nu}) - (\partial^\beta D_{\beta\nu})\,(\partial^\alpha V_{\alpha\mu}) \bar{D}^{\mu\nu} \Big\} \\
& + \frac{\tilde{g}_T}{4\,f}\,\epsilon^{\mu\nu\alpha\beta} \Big\{ (\partial_\alpha D)\,V_{\mu\nu}\,(\partial^\tau \bar{D}_{\tau\beta}) + (\partial^\tau D_{\tau\beta})\,V_{\mu\nu}\,(\partial_\alpha \bar{D}) \Big\} \\
& + i \, \frac{g_T}{2\,f}\,(\partial^\alpha D_{\alpha\mu})\,V^{\mu\nu}\,(\partial^\beta \bar{D}_{\beta\nu}) \\
& + \frac{g_E}{4\,f}\,\epsilon_{\mu\nu\alpha\beta} \Big\{ D^{\mu\nu}\,(\partial_\tau V^{\tau\beta})\,(\partial^\alpha \bar{D}) + (\partial^\alpha D)\,(\partial_\tau V^{\tau\beta})\,\bar{D}^{\mu\nu} \Big\} \,,
\end{aligned}
\tag{66}
$$

where we assume additional terms that are implied by the covariant derivatives (52). The particular form of (66) was guided by the requirement that (66) together with (38) has a well defined flavor SU(4) limit. We complement (66) by further electromagnetic interactions that are analogous to (56). We write

$$
\begin{aligned}
\mathcal{L} = & \, \frac{e_V}{2\,f\,m_V^2}\,F^{\mu\nu} \Big\{ D\,[(\partial_\alpha V^{\alpha\nu}),\,Q]\,(\partial_\mu \bar{D}) - (\partial_\mu D)\,[(\partial_\alpha V^{\alpha\nu}),\,Q]\,\bar{D} \Big\} \\
& - \frac{\tilde{e}_V}{2\,f\,m_V^2}\,F^{\mu\nu} \Big\{ D_{\mu\tau}\,[(\partial_\alpha V^{\alpha\nu}),\,Q]\,(\partial_\beta \bar{D}^{\beta\tau}) - (\partial_\beta D^{\beta\tau})\,[(\partial_\alpha V^{\alpha\nu}),\,Q]\,\bar{D}_{\mu\tau} \Big\} \\
& - i\,\frac{\tilde{e}_T}{4\,f\,m_V^2}\,F^{\mu\nu}\,\epsilon_{\mu\sigma\alpha\beta} \Big\{ (\partial^\alpha D)\,[V_\nu^{\ \sigma},\,Q]\,(\partial_\tau \bar{D}^{\tau\beta}) \\
& \qquad\qquad + (\partial_\tau D^{\tau\beta})\,[V_\nu^{\ \sigma},\,Q]\,(\partial^\alpha \bar{D}) \Big\} \\
& + \frac{e_T}{2\,f\,m_V^2}\,F^{\mu\nu}\,(\partial^\alpha D_{\alpha\mu})\,[V^{\nu\tau},\,Q]\,(\partial^\beta \bar{D}_{\beta\tau}) \\
& - \frac{e_E}{f\,m_V^2}\,F^{\mu\nu}\,(\partial_\alpha D)\,\{V_{\mu\nu},\,Q\}\,(\partial^\alpha \bar{D}) \\
& + \frac{\tilde{e}_E}{f\,m_V^2}\,F^{\mu\nu}\,(\partial_\alpha D^{\alpha\tau})\,\{V_{\mu\nu},\,Q\}\,(\partial^\beta \bar{D}_{\beta\tau}) \,.
\end{aligned}
\tag{67}
$$

We remark that the hadronic vertices (38, 66) should also have some impact on the formation of the molecules as discussed in section 2 based on the leading orders coupled-channel interaction. Note that the influence of the parameter g_P and $\tilde{g}_P$ was studied already in [9] and also in section 2.2. of this work. Its effects was found to be of rather minor importance. This results confirms the expectation that the latter interactions are of subleading order in a chiral expansion. Nevertheless, this issue deserves further detailed studies, in particular the role played by additional inelastic channels that involve the light vector mesons should be worked out. We anticipate that the latter channels are responsible for the formation of tensor molecules with angular momentum $J = 2$. We will return to this issue in the subsequent section when discussing the radiative decay of scalar and axial-vector molecules.

The various parameters in (66, 67) are correlated by the heavy-quark symmetry of QCD. As detailed in Appendix B one expects

$$\begin{aligned} &\tilde g_V = g_V\,, \qquad &&\tilde g_T = g_T\,, \\ &e_V = \tilde e_V\,, \qquad &&e_T = \tilde e_T\,, \qquad &&e_E = \tilde e_E\,. \end{aligned} \tag{68}$$

at leading order. The size of g_V and $\tilde g_V$ can be estimated by the phenomenological assumption of universally coupled light vector mesons. If supplemented by the KSFR relation we identify

$$g_V = \tilde g_V = \frac{g\,f}{m_V} = \frac{\pm 1}{\sqrt{2}} \simeq \pm 0.71\,, \tag{69}$$

with the universal vector-coupling constant $|g| \simeq 6$ [37]. Note that the phase of the coupling constant g_V is not determined by this assumption.

The estimate of the size of the coupling constant g_T and g_E is more difficult. While lacking QCD lattice simulations we may use a flavor SU(4) ansatz, admittedly a rather rough and questionable tool. In order to do so we introduce flavor SU(4) multiplet fields

$$\begin{aligned} V^{\mu\nu}_{[16]} &= \begin{pmatrix} V^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2}\,\bar D^{\mu\nu} \\ \sqrt{2}\,D^{\mu\nu} & J/\psi \end{pmatrix}\,, \\ \Phi_{[16]} &= \begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2}\,\bar D \\ \sqrt{2}\,D & \eta_c \end{pmatrix}\,, \end{aligned} \tag{70}$$

in terms of the SU(3) multiplet fields Φ, D and $V_{\mu\nu}, D_{\mu\nu}$. We construct a minimal SU(4) invariant Lagrangian density that comprises the interaction terms introduced in (38, 66):

$$\begin{aligned} \mathcal{L}_{SU(4)} &= \frac{g_1}{4\,f}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\,(\partial^\alpha \Phi_{[16]})\,\Big\{ \mathrm{V}^{\mu\nu}_{[16]}\,(\partial_\tau \mathrm{V}^{\tau\beta}_{[16]}) + (\partial_\tau \mathrm{V}^{\tau\beta}_{[16]})\,\mathrm{V}^{\mu\nu}_{[16]} \Big\} \\ &\quad + \frac{i\,g_2}{4\,f}\,\mathrm{tr}\,(\partial_\alpha V^{\alpha\mu}_{[16]})\,V^{[16]}_{\mu\nu}\,(\partial_\beta V^{\beta\nu}_{[16]}) - \frac{i\,g_3}{2\,f}\,\mathrm{tr}\,(\partial_\mu \Phi_{[16]})\,\mathrm{V}^{\mu\nu}_{[16]}\,(\partial_\nu \Phi_{[16]}) \end{aligned} \tag{71}$$

The ansatz (71) suggests the identification

$$\tilde{g}_T = g_E = \tilde{g}_P = g_1 \,, \qquad \tilde{g}_V = g_T = g_2 \,, \qquad g_V = g_P = g_3 \,. \quad (72)$$

The result (72) deserves and requires some discussion. We observe that according to (40, 68) the coupling constants g_i with $i = 1, 2, 3$ in (71) should approach a universal value in the heavy-quark mass limit. This would imply, for instance, that the parameters g_P and g_V should approach a common value. If confronted with the values (39) and(69) this appears consistent with an uncertainty of less than 20%. However, the result (72) would predict also a common value for g_P and g_T. We point out that this implication is troublesome: whereas g_P approaches a finite value, the parameter g_T must vanish in the heavy-quark mass limit. This is an immediate consequence of the fact that the QCD action is linear in the charm quark mass. Note that, while the term proportional to g_P involves one derivative on a heavy field, the term proportional to g_T involves two. From this discussion it should be evident that we must not use the relations (72) as they stand. From (72) we take serious the predicted phase relations for the coupling constants, i.e. we assume all coupling constants to take positive values. In addition, for the unknown parameter g_E we would anticipate the range

$$g_P \leq g_E \leq g_V \,, \qquad (73)$$

a conjecture consistent with the expected scaling behavior at large charm-quark masses. The estimate of the remaining parameters $g_T \simeq \tilde{g}_T$ is most uncertain. From (72) we would expect the range

$$0 \leq g_T \leq \frac{m_V}{M_V}\, g_P \,, \qquad (74)$$

with $m_V = 776$ MeV and $M_V = 2000$ MeV the typical masses of a light and heavy vector mesons.

In Figure 5 we analyze the various decay processes of a given molecule graphically. At first we consider only contributions that are linear in the hadronic three-point vertices. In this case the central building block is the sum of the bubble diagrams defining the effective propagator of the resonance state. It is drawn with a double line. The corresponding formal expression at leading order are (15) or (30) for the scalar and axial-vector molecule respectively. Solid lines represent the propagation of the pseudo-scalar or vector mesons, where the thin lines stand for the light mesons and the thick lines for the heavy mesons. The wavy line is the photon.

In Figure 6 we depict additional diagrams that, at first sight, should be relevant for the radiative decay width of scalar and axial-vector molecules. The figure shows contributions where the photon couples to the resonance line directly. Here we do not resolve the structure of the latter vertex, since it will not be relevant. The diagrams vanish identically. This is an immediate consequence of using the tensor-field representation of the spin-one particles in this work. Consider for instance the decay of a scalar molecule, where the final state carries $J^P = 1^-$ quantum numbers. All diagrams in Figure 6 factorize into two contributions. Since the left part is contracted with the anti-symmetric wave function of the 1^- state it vanishes identically. This follows, since this contribution carries the two indices α and β but depends on one 4-momentum only, the 4-momentum of the final state. Thus it must by symmetric in the indices α and β and therefore vanishes if contracted with the wave function. Analogous arguments hold for the decay modes of axial-vector molecules.

As a striking consequence of our discussion we conclude that the radiative decay process of a scalar and axial-vector molecule is determined fully by an effective hadronic resonance vertex. The detailed structure of the resonance

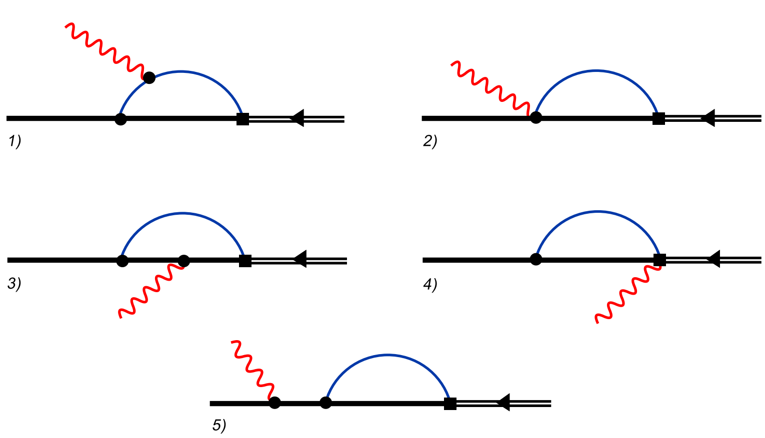

Fig. 5. Diagrams contributing to the decay amplitude of a scalar or axial-vector molecule.

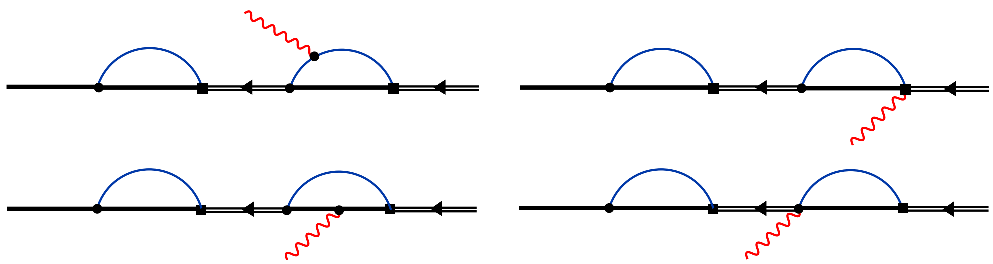

Fig. 6. Additional diagrams that are irrelevant for the decay width of scalar and axial-vector molecules.

propagator is not probed. It is emphasized that this is a consequence of using a tensor-field representation of the spin-one particles in this work. The application of the more traditional vector-field representation would require a detailed study of the resonance propagator. In general this would be quite cumbersome. In particular, enforcing gauge invariance is highly non-trivial. Our observation implies that even in the presence of the additional coupled-channel interactions (38, 66), the formal evaluation of the diagrams in Figure 5 is unchanged.

Thus it is useful to introduce flavor triplet fields, R and $R_{\mu\nu}$, that interpolate the scalar and axial-vector molecules. All what is needed are the coupling strengths of the molecules to the hadronic final states. We introduce the effective coupling constants, g_R and $\tilde{g}_R$, by

$$\begin{aligned} \mathcal{L}_R = & -\frac{g_R}{2\,f}\left\{(\partial_\mu D)\,(\partial^\mu \Phi)\,\bar{R} + R\,(\partial^\mu \Phi)\,(\partial_\mu \bar{D})\right\} \\ & - i\,\frac{\tilde{g}_R}{2\,f}\,\left\{D^{\mu\nu}\,(\partial_\mu \Phi)\,(\partial^\tau \bar{R}_{\tau\nu}) - (\partial^\tau R_{\tau\nu})\,(\partial^\mu \Phi)\,\bar{D}^{\mu\nu}\right\}, \end{aligned} \tag{75}$$

where we suppress additional terms linear in the photon field that are required by gauge invariance. It should be emphasized that we may write down further terms in (75) involving the electromagnetic field strength tensor, $F_{\mu\nu}$, that are gauge invariant separately. Like the parameters g_R and $\tilde{g}_R$ the structure and size of such terms had to be extracted form a coupled-channel computation. However, as will become clear in the subsequent section the role of such terms is very minor. Within the given renormalization scheme such contributions vanish identically for the particular choice where the matching scale, μ_M, and the mass of the hadronic final state degenerate. This is almost the case.

For on-shell conditions the first vertex of (75) is equivalent to a vertex of the generic form $D\,\Phi\,\bar{R}$ as is implied by the projector technique of section 2.1. This follows from the rewrite

$$2\,(\partial_\mu D)\,(\partial^\mu \Phi)\,\bar{R} \to D\,\Phi\,(\partial^\mu \partial_\mu \bar{R}) - (\partial^\mu \partial_\mu D)\,\Phi\,\bar{R} - D\,(\partial^\mu \partial_\mu \Phi)\,\bar{R}\,. \tag{76}$$

Though in general the two vertices would give different results in one loop diagrams, within our renormalization scheme to be discussed in the next subsection the vertices are equivalent. The decay amplitudes implied coincide up to terms that may be generated by effective molecule vertices that involve the field strength tensor $F_{\mu\nu}$. As argued above such terms vanish identically for the particular choice where the matching scale μ_M is identified with the mass of the hadronic final state. An analogous argument shows the equivalence of the second vertex of (75) with the generic form $(\partial_\alpha D^{\alpha\tau})\,\Phi\,(\partial^\beta \bar{R}_{\beta\tau})$ implied in section 2.2.

At leading order the magnitude of the parameters g_R and $\tilde{g}_R$ are determined by the coupling constants as extracted from the pole structure of the scattering amplitude (see (19, 36)). We identify

$$
\begin{aligned}
g_R &= \frac{4\sqrt{2}\,f\,M_{0^+}}{M_{0+}^2 - M_D^2 - m_K^2}\, g_{KD}^{(0+)} = \frac{4\sqrt{6}\,f\,M_{0^+}}{M_{0+}^2 - M_{D_s}^2 - m_\eta^2}\, g_{\eta D_s}^{(0+)}\,, \\
\tilde{g}_R &= \frac{4\sqrt{2}\,f\,M_{D^*}}{M_{1+}^2 - M_{D^*}^2 - m_K^2}\, g_{KD^*}^{(1+)} = \frac{4\sqrt{6}\,f\,M_{D_s^*}}{M_{1+}^2 - M_{D_s^*}^2 - m_\eta^2}\, g_{\eta D_s^*}^{(1+)}\,.
\end{aligned}
\tag{77}
$$

As detailed in Appendix B heavy-quark symmetry predicts

$$
g_R = \tilde{g}_R\,, \tag{78}
$$

at leading order. While (78) is realized quite accurately, the SU(3) flavor breaking effects in the coupling constant g_R and $\tilde{g}_R$ are sizeable with the values given in Table 2. The relation (78) is satisfied at the 10 % level

$$
\frac{\tilde{g}_R}{g_R} \simeq 0.986\, \frac{g_{KD^*}^{(1+)}}{g_{KD}^{(0+)}} \simeq 0.989\, \frac{g_{\eta D_s^*}^{(1^+)}}{g_{\eta D_s}^{(0+)}}\,. \tag{79}
$$

In contrast, the SU(3) relation

$$
g_R \simeq 0.720\, g_{KD}^{(0+)} \simeq 1.710\, g_{\eta D_s}^{(0+)}\,, \qquad \tilde{g}_R \simeq 0.710\, g_{KD^*}^{(1+)} \simeq 1.691\, g_{\eta D_s^*}^{(1+)}\,, \tag{80}
$$

are violated at the 95% level. The values for $g_R \simeq 4.28$ as derived from ηD_s channel with $f = 90$ MeV is almost a factor two larger than the one obtained from KD channel.

Before turning to the renormalization issue we discuss further resonance vertices involving light vector mesons. The latter arise necessarily in a coupled-channel computation once any of the parameters $g_V, \tilde{g}_V, g_T, \tilde{g}_T$ or g_E is non-vanishing. Though we are not presenting results of such a computation we study here the relevance of light vector meson in radiative decay processes of scalar and axial-vector molecules. We introduce the effective resonance vertices

$$
\begin{aligned}
\mathcal{L}_R = & \frac{g_H}{2\,f} \Big\{ R\,(\partial^\tau V_{\tau\nu})\,(\partial_\mu \bar{D}^{\mu\nu}) + (\partial_\mu D^{\mu\nu})\,(\partial^\tau V_{\tau\nu})\,\bar{R} \Big\} \\
& + i\,\frac{\tilde{g}_H}{2\,f}\, \Big\{ (\partial^\mu R_{\mu\nu})\,(\partial_\tau V^{\tau\nu})\,\bar{D} - D\,(\partial_\tau V^{\tau\nu})\,(\partial^\mu \bar{R}_{\mu\nu}) \Big\} \\
& - \frac{\hat{g}_H}{4\,f}\,\epsilon^{\alpha\beta\mu\nu} \Big\{ (\partial^\sigma R_{\sigma\mu})\,(\partial^\tau V_{\tau\nu})\,\bar{D}_{\alpha\beta} + D_{\alpha\beta}\,(\partial^\tau V_{\tau\nu})\,(\partial^\sigma \bar{R}_{\sigma\mu}) \Big\}\,,
\end{aligned}
\tag{81}
$$

where the parameters $g_H, \tilde{g}_H$ and $\hat{g}_H$ are unknown at this stage. Again additional terms linear in the photon field that are required by gauge invariance are assumed. We emphasize that the three parameters degenerate in the heavy-quark limit. As worked out in Appendix B it holds

$$
g_H = \tilde{g}_H = \hat{g}_H\,, \tag{82}
$$

in this case. Note that for the axial-vector molecules additional vertices that are not on-shell equivalent to those of (81) may be constructed. Such terms are suppressed by their d-wave phase-space behavior. The presence of the vertices (81) leads to additional diagrams contributing to the radiative decay amplitudes of scalar and axial-vector molecules. Note that such diagrams are part of Figure 5 given the association of the solid lines as representing the propagation of either pseudo-scalar or vector mesons.

3.3 Renormalization

We finally turn to the renormalization issue. It is noted that the decay diagrams are ultraviolet diverging. Applying the Passareno-Veltman reduction [41], which is rigourously justified within dimensional regularization, the integrals of Figure 5 may be expressed in terms of a set of scalar integrals of the form

$$
\begin{aligned}
I_a &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\,, & S_a(l) &= \frac{1}{l^2 - m_a^2 + i\,\epsilon}\,,\\
I_{ab} &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+p)\,, & \bar{I}_{ab} &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+\bar{p})\,,\\
J_{abc} &= +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+q)\, S_c(l+p)\,, \\
\bar{J}_{abc} &= +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+\bar{p})\, S_c(l+p)\,,
\end{aligned}
\tag{83}
$$

where we focus on the physical limit with space-time dimension four. In our convention the 4-momentum p_μ characterizes the decaying molecule, q_μ is the 4-momentum of the photon and $\bar{p}_\mu = p_\mu - q_\mu$ is the 4-momentum of the hadronic final state.

Divergent structures arise only from the tadpole integral I_a and the two-propagator integrals I_{ab} and $\bar{I}_{ab}$. Since the combinations

$$
I_{ab} - \bar{I}_{ab}\,, \qquad I_{ab} - \frac{I_b - I_a}{m_a^2 - m_b^2}\,,
\tag{84}
$$

are finite as space-time dimension approaches four, one may take the viewpoint that all divergent structures are caused by the tadpole integral I_a (see also [42]).

It is essential to realize that any divergent contribution can not be renormalized by simply adding a counter vertex where the photon coupled directly to

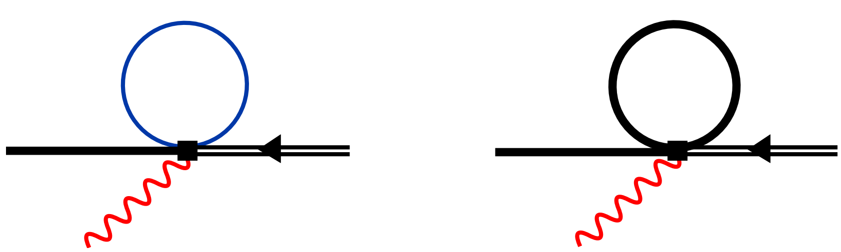

Fig. 7. Examples of counter loop contributions that renormalize the decay amplitude of molecules.

the resonance. Since the resonance is formed by coupled-channel dynamics its radiative decays amplitude must be renormalized by a loop subtraction mechanism similar to the one introduced in [43]. This is illustrated in Figure 7. Typically the counter terms needed are appropriate 5-point vertices as exemplified in Figure 7 by an solid square. A contribution to the radiative decay amplitude is generated by contracting two identical lines giving rise to a tadpole-type contribution. The latter have to cancel the tadpole contributions from Figure 5 as discussed above.

There is yet a further crucial observation we should discuss. In Figure 5 the integral arising from the second and fourth diagrams have a structure similar to the loop functions that build up the resonance state (see e.g. Figure 2). Therefore the renormalization of the coupled-channel dynamics and the one of the radiative-decay amplitude are related necessarily. Since the coupled-channel dynamics involves an infinite number of Feynman diagrams one has to leave the well trotted path of perturbative renormalization, i.e. infinite sets of counter terms present in effective field theories have to be activated at each order. We emphasize that one should carefully discriminate two issues. First, the renormalization scale independence of a scattering amplitude and second, a possible scheme dependence of how the infinite number of diagrams are treated. We point out that the application of the on-shell reduction formalism developed in [43,13,7], as is implied by the unique existence of an algebra of covariant projectors, one may consider as a scheme dependence. The additional ingredient, the requirement of a smooth matching of scattering amplitudes unitarized in different channels, we consider again as a scheme dependence. It is an economical, though not unique, procedure to build crossing symmetry into the scheme. From this point of view the matching parameter μ_M in (15, 30) reflects a scheme dependence. It should not be confused with a renormalization scale dependence.

As a consequence of the on-shell reduction formalism our coupled-channel dynamics is based on, at a leading order computation neither the effective potential $V^{(JP)}(s)$ nor the loop functions $J^{(JP)}(s)$ gain any contribution from a tadpole-type loop integral, I_a. If the leading order two-body interaction would be used as the driving force of a Bethe-Salpeter equation, plenty of tadpole contributions would arise [13,7]. As was discussed in great detail in [13,7] the contribution of reduced tadpoles can be trusted only at a level where one computes one-loop corrections to the Bethe-Salpeter interaction kernel, i.e. two-particle irreducible diagrams. While for the latter conventional power-counting arguments are applicable, they are'nt for the reducible contributions, i.e. the ones summed by the unitarization.

Closing our chain of arguments, we arrive at the result that reduced tadpole contributions in the decay amplitude should be dropped in our leading orders computation. In addition a finite renormalization is applied to the integral,

I_{ab}. It is identified with the expression introduced in (16), with

$$I_{ab} = I(p^2) - I(\mu_M^2)\,, \qquad\qquad m_a = M\,, \qquad m_b = m\,, \tag{85}$$

and the matching scale, μ_M, as used in the coupled-channel computation.

4 Decay amplitude of scalar molecules

We derive expressions for the radiative decay amplitude of the $D_{s0}^*(2317)$. Given the transition amplitude, $M^{\alpha\beta,\mu}_{0^+\to\gamma\,1^-}$, we derive the partial-decay width $\Gamma_{0^+\to\gamma\,1^-}$. Due to gauge invariance the transition tensor is characterized by one scalar number $d_{0^+\to\gamma\,1^-}$. We write:

$$\begin{aligned}
&\epsilon^\dagger_\mu(q,\lambda_q)\,\epsilon^\dagger_{\alpha\beta}(\bar p,\lambda_{\bar p})\,M^{\alpha\beta,\mu}_{0^+\to\gamma\,1^-} \\
&\qquad = d_{0^+\to\gamma\,1^-}\;\epsilon^\dagger_\mu(q,\lambda_q)\,\epsilon^\dagger_{\alpha\beta}(\bar p,\lambda_{\bar p})\,\Big\{g^{\mu\alpha}\,(p\cdot q) - p^\mu\,q^\alpha\Big\}\,\frac{i\,\bar p^\beta}{\sqrt{\bar p^2}}\,, \qquad (86)
\end{aligned}$$

where $\epsilon_\mu(q,\lambda_q)$ and $\epsilon_{\alpha\beta}(\bar p,\lambda_{\bar p})$ denote the wave functions of the outgoing photon and vector meson. The decaying resonance has momentum $p_\mu = \bar p_\mu + q_\mu$. The partial-decay width is readily established

$$\Gamma_{0^+\to\gamma\,1^-} = \frac{|d_{0^+\to\gamma\,1^-}|^2}{4\pi}\left(\frac{M^2_{0^+}-M^2_{1^-}}{2\,M_{0^+}}\right)^3\,, \qquad (87)$$

with $M^2_{1^-} = \bar p^2$ and $M^2_{0^+} = p^2$. It is worth pointing out that the decay parameter can be obtained by the simple projection formula

$$d_{0^+\to\gamma\,1^-} = \frac{-i}{(p\cdot q)\,M_{1^-}}\left\{g_{\mu\alpha} - \frac{p_\mu\,q_\alpha}{(p\cdot q)}\right\}\bar p_\beta\,M^{\alpha\beta,\mu}_{0^+\to\gamma\,1^-}\,. \qquad (88)$$

The result (88) follows since the decay amplitude is transverse with respect to the photon momentum and anti-symmetric in the indices $\alpha\leftrightarrow\beta$. This implies that the off-shell amplitude, $\bar p_\beta\,M^{\alpha\beta,\mu}$ is orthogonal to q_μ and $\bar p_\alpha$, i.e. it is characterized by two parameters only. With (88) we project onto the relevant component.

There are 5 classes of contributions we have to consider. They are depicted in Figure 5. All terms have the topology of a one-loop Feynman diagram. A given contribution is either proportional to the coupling constant g_R or g_H. In the former and latter case the thin and thick lines in Figure 5 that are attached to the resonance vertex stand for the propagation of pseudo-scalar and vector mesons respectively. The outgoing line is a vector D_s^* meson.

It is convenient to group together diagrams that are gauge invariant separately. We discuss the various contributions that are proportional to the coupling constant g_R in some detail. There are two types of gauge invariant combinations that are proportional to $e\,g_P/f$. We form two corresponding transverse tensors $A^{\alpha\beta,\mu}(\bar p,p)$ and $B^{\alpha\beta,\mu}(\bar p,p)$, where the latter and former receive contributions from class 1),2) and 2),3) of Figure 5 respectively. The tensors $A^{\alpha\beta,\mu}(\bar p, p)$ and $B^{\alpha\beta,\mu}(\bar p, p)$ describe the processes where the photon couples to the charge of the light and heavy pseudo-scalar meson. We introduce:

$$
\begin{aligned}
A^{\alpha\beta,\mu}_{ab}(\bar p, p) &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(p+l) \\
&\quad \times \Big\{ S_a(l+q)\,(q+2\,l)^\mu\, \bar p^\alpha\,(p+l)^\beta + g^{\mu\alpha}\,(p+l)^\beta \Big\}\,, \\
B^{\alpha\beta,\mu}_{ab}(\bar p, p) &= +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(p+l) \\
&\quad \times \Big\{ S_b(\bar p+l)\,(\bar p+p+2\,l)^\mu\, \bar p^\alpha\,(\bar p+l)^\beta + g^{\mu\beta}\, l^\alpha \Big\}\,,
\end{aligned} \tag{89}
$$

with

$$
S_a(p) = \frac{1}{p^2 - m_a^2}\,. \tag{90}
$$

We note that for technical simplicity the tensors (89) are constructed with an effective molecule vertex of the generic form $D\,\Phi\,\bar R$ as implied by the on-shell reduction technique applied in the coupled-channel computation of section 2.1. This is the reason why class 4) contributions are not entering the expressions (89).

There remain the contributions that are implied by interactions involving the electromagnetic field strength tensor $F_{\mu\nu}$. They are proportional to the anomalous coupling strengths e_A, e_Q, e_C of (63, 58) or the parameter e_P of (56). The terms proportional to $e_P/(f\,m_V^2)$ probe the class 2) of Figure 5 only. Their effect is encoded into the transverse tensor $\bar A^{\alpha\beta,\mu}(\bar p, p)$, with

$$
\bar A^{\alpha\beta,\mu}_{ab}(\bar p, p) = -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(p+l)\,\Big\{ (q\cdot l)\, g^{\mu\alpha} - l^\mu\, q^\alpha \Big\}\,(p+l)^\beta\,. \tag{91}
$$

The further tensors $C^{\alpha\beta,\mu}(\bar p, p)$ and $\bar C^{\alpha\beta,\mu}(\bar p, p)$ describe the processes where the Goldstone boson, which emits the photon, is converted into a light vector meson as included in class 1). They are proportional to the parameter combinations $e_A\,\tilde g_T/f^2$ and $e_A\,\tilde g_E/f^2$ respectively. We introduce

$$
\begin{aligned}
C^{\alpha\beta,\mu}_{abc}(\bar p, p) &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_c(p+l) \\
&\quad \times 2\, q_\rho\, l_\sigma\, \epsilon^{\rho\mu\sigma}{}_{\tau}\, \bar p^\alpha\,(p+l)^{\bar\rho}\, \epsilon_{\bar\sigma\bar\tau\bar\rho}{}^{\beta}\, S_b^{\bar\sigma\bar\tau,\tau}(l+q)\,,
\end{aligned}
$$

$$\bar{C}^{\alpha\beta,\mu}_{abc}(\bar{p},p) = -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_c(p+l)$$
$$\times 2\, q_\rho\, l_\sigma\, \epsilon^{\rho\mu\sigma}{}_{\tau} (p+l)_{\bar{\rho}}\, \epsilon^{\alpha\beta}{}_{\bar{\rho}\bar{\sigma}}\, S_b^{\bar{\sigma}\tau}(l+q)\,, \tag{92}$$

where

$$S_a^{\mu\nu}(p) = p_\alpha\, S_a^{\alpha\mu,\beta\nu}(p)\, p_\beta\,,$$
$$S_a^{\mu\nu,\alpha\beta}(p) = -\frac{1}{m_a^2}\, \frac{1}{p^2 - m_a^2 + i\,\epsilon} \Big[(m_a^2 - p^2)\, g_{\mu\alpha}\, g_{\nu\beta}$$
$$+ g_{\mu\alpha}\, p_\nu\, p_\beta - g_{\mu\beta}\, p_\nu\, p_\alpha - (\mu \leftrightarrow \nu) \Big]\,,$$
$$S_a^{\mu,\beta\nu}(p) = p_\alpha\, S_a^{\alpha\mu,\beta\nu}(p)\,, \qquad S_a^{\alpha\mu,\nu}(p) = S_a^{\alpha\mu,\beta\nu}(p)\, p_\beta\,. \tag{93}$$

Processes that are analogous to the anomalous ones described by the tensors (92) are driven by the parameters e_Q and e_C. In this case the photon is emitted by a pseudo-scalar D-meson, which is converted into a vector D-meson. The contribution is included in class 3) of Figure 5. We introduce a corresponding loop tensor

$$D^{\alpha\beta,\mu}_{abc}(\bar{p},p) = +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_c(p+l) \tag{94}$$
$$\times 2\, q_\rho\, (l+p)_\sigma\, \epsilon^{\rho\mu\sigma}{}_{\tau} \Big\{ l_{\bar{\rho}}\, \epsilon^{\alpha\beta\bar{\rho}}{}_{\bar{\tau}}\, S_b^{\bar{\tau}\tau}(\bar{p}+l) \; - \bar{p}^\alpha\, l^{\bar{\rho}}\, \epsilon_{\bar{\sigma}\bar{\tau}\bar{\rho}}{}^{\beta}\, S_b^{\bar{\sigma}\bar{\tau},\tau}(\bar{p}+l) \Big\}\,.$$

We emphasize that each of the tensor integrals introduced in (89, 91, 92, 94) is gauge invariant separately, i.e if contracted with q_μ it vanishes identically. This was checked by explicit calculations.

Collecting all contributions from the KD and $\eta\, D_s$ channels we arrive at the following decay amplitude

$$i\, M^{\alpha\beta,\mu}_{0^+ \to \gamma\, 1^-} = -\frac{e\, g_P}{f}\; h^{(0+)}_{KD} \Big\{ A^{\alpha\beta,\mu}_{KD}(\bar{p},p) + B^{\alpha\beta,\mu}_{KD}(\bar{p},p) \Big\}$$
$$-\frac{e_P}{f\, m_V^2}\; h^{(0+)}_{KD}\; \bar{A}^{\alpha\beta,\mu}_{KD}(\bar{p},p) - 2\, \frac{e\, g_P}{\sqrt{3}\, f}\; h^{(0+)}_{\eta D_s}\; B^{\alpha\beta,\mu}_{\eta D_s}(\bar{p},p)$$
$$+\frac{e_A}{48\, f^2\, m_V}\, h^{(0+)}_{KD} \Big\{ \tilde{g}_T\, C^{\alpha\beta,\mu}_{KK^*D}(\bar{p},p) + g_E\, \bar{C}^{\alpha\beta,\mu}_{KK^*D}(\bar{p},p) \Big\}$$
$$+\frac{e_A}{\sqrt{3}\, 12\, f^2\, m_V}\, h^{(0+)}_{\eta D_s} \Big\{ \tilde{g}_T\, C^{\alpha\beta,\mu}_{\eta\phi D_s}(\bar{p},p) + g_E\, \bar{C}^{\alpha\beta,\mu}_{\eta\phi D_s}(\bar{p},p) \Big\}$$
$$+\frac{\tilde{g}_P}{4\, f\, M_V^2} \Big\{ \frac{2}{\sqrt{3}}\, \frac{3\, e_C - e_Q}{6}\, h^{(0+)}_{\eta D_s}\, D^{\alpha\beta,\mu}_{\eta D_s^* D_s}(\bar{p},p)$$
$$+\frac{6\, e_C + e_Q}{6}\, h^{(0+)}_{KD}\, D^{\alpha\beta,\mu}_{KD^*D}(\bar{p},p) \Big\}\,, \tag{95}$$

where exact isospin symmetry is assumed with the coupling constant

$$h^{(0+)}_{KD} = \sqrt{2}\, M_{0^+}\, g^{(0+)}_{KD} \leftrightarrow \frac{M^2_{0^+} - M^2_D - m^2_K}{2\, f}\, g_R \,,$$
$$h^{(0+)}_{\eta D_s} = \sqrt{2}\, M_{0^+}\, g^{(0+)}_{\eta D_s} \leftrightarrow \frac{M^2_{0^+} - M^2_{D_s} - m^2_\eta}{2\,\sqrt{3}\, f}\, g_R \,. \tag{96}$$

The values for the coupling constants $g^{(0+)}_{KD}$ and $g^{(0+)}_{\eta D_s}$ are given in Table 2. Note that the tensors introduced in (89, 91, 92, 94) are not anti-symmetric in the indices $\alpha \leftrightarrow \beta$ as of notational convenience. While deriving the decay parameter $d_{0^+ \to \gamma\, 1^+}$ according to (86) the wave-function projects onto the relevant component, the application of the projection formula (88) requires an explicit anti-symmetrization.

The contributions of the $\phi\, D^*_s$ and $K^* D$ channels to the decay parameter as well as explicit results for the Passareno-Veltman reduction of the tensor integrals of (95) are detailed in Appendix D. Such contributions are proportional to g_H.

5 Decay amplitudes of axial-vector molecules

We derive expressions for the radiative decay amplitudes of the $D^*_{s1}(2640)$. There are three types of processes we have to study. The first two are characterized by a photon and a pseudo-scalar or scalar particle, the third by a photon and a vector particle in the final state.

5.1 The process $1^+ \to \gamma\, 0^-$

Given the transition amplitude, $M^{\mu,\alpha\beta}_{1^+\to\gamma\,0^-}$, we derive the partial-decay width $\Gamma_{1^+\to\gamma\,0^-}$. It is determined by one scalar number, $d_{1^+\to\gamma\,0^-}$. We write:

$$\begin{aligned}
&\epsilon^\dagger_\mu(q,\lambda_q)\, M^{\mu,\alpha\beta}_{1^+\to\gamma\,0^-}\, \epsilon^{\alpha\beta}(p,\lambda_p) \\
&\qquad = d_{1^+\to\gamma\,0^-}\, \epsilon^\dagger_\mu(q,\lambda_q) \Big\{ g^{\mu\alpha}\,(\bar p\cdot q) - \bar p^\mu\, q^\alpha \Big\} \frac{p^\beta}{\sqrt{p^2}}\, \epsilon^{\alpha\beta}(p,\lambda_p)\,, \qquad (97)
\end{aligned}$$

which implies

$$\begin{aligned}
\Gamma_{1^+\to\gamma\,0^-} &= \frac{|d_{1^+\to\gamma\,0^-}|^2}{12\pi} \left(\frac{M^2_{1^+} - M^2_{0^-}}{2\, M_{1^+}} \right)^3 , \\
d_{1^+\to\gamma\,0^-} &= \frac{1}{(p\cdot q)\, M_{1^+}} \left\{ g_{\mu\alpha} - \frac{p_\mu\, q_\alpha}{(p\cdot q)} \right\} p_\beta\, M^{\mu,\alpha\beta}\,, \qquad (98)
\end{aligned}$$

with $\bar p^2 = (p-q)^2 = M^2_{0^-}$ and $p^2 = M^2_{1^+}$. Like in (88) we exploit the fact that the decay amplitude is anti-symmetric in $\alpha \leftrightarrow \beta$.

There are 5 classes of contributions we have to consider. They are depicted in Figure 5. Any given diagram is proportional to either one of the three coupling constants $\tilde g_R, \tilde g_H$ or $\hat g_H$. In the first case the thin and thick lines in Figure 5 that are attached to the resonance vertex stand for the propagation of Goldstone bosons and vector D-mesons. In the second case the role of thin and thick lines is interchanged in the sense that the thin lines correspond to light vector mesons, whereas the thick lines describe heavy pseudo-scalar mesons. In the third case both lines stand for vector mesons. The outgoing line is a pseudo-scalar D_s-meson always.

It is convenient to group together diagrams that are gauge invariant separately. We discuss the various contributions in some detail and start with diagrams that are proportional to the coupling constant $\tilde g_R$. There are two types of gauge invariant combinations that are proportional to $e\, g_P/f$. We form two corresponding transverse tensors $A^{\mu,\alpha\beta}(\bar p, p)$ and $B^{\mu,\alpha\beta}(\bar p, p)$, where the latter and former receive contributions from class 1),2),4) and 2),3),4) of Figure 5 respectively. The tensors $A^{\mu,\alpha\beta}(\bar p,p)$ and $B^{\mu,\alpha\beta}(\bar p,p)$ describe the processes where the photon couples to the charge of the light and heavy meson. We introduce:

$$
\begin{aligned}
A^{\mu,\alpha\beta}_{ab}(\bar p,p) &= +i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\,\Big\{(l+\bar p)_{\bar\beta}\,S_b^{\bar\beta\mu,\beta}(p+l)\,p^\alpha \\
&\quad +\bar p_{\bar\beta}\,S_b^{\bar\beta\beta}(\bar p+l)\,g^{\mu\alpha}+S_a(l+q)\,\bar p_{\bar\beta}\,S_b^{\bar\beta\beta}(p+l)\,(q+2\,l)^\mu\,p^\alpha\Big\}\,,\\
B^{\mu,\alpha\beta}_{ab}(\bar p,p) &= +i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\,\Big\{\bar p^{\bar\alpha}\,S_{b,\bar\alpha\bar\beta}(\bar p+l)\,S_b^{\mu\bar\beta,\beta}(p+l)\,p^\alpha \\
&\quad +\bar p_{\bar\alpha}\,S_b^{\bar\alpha,\mu\bar\beta}(\bar p+l)\,S_{b,\bar\beta}{}^{\beta}(p+l)\,p^\alpha\;-l_{\bar\beta}\,S_b^{\mu\bar\beta,\beta}(p+l)\,p^\alpha \\
&\quad +\bar p_{\bar\beta}\,S_b^{\bar\beta,\mu\beta}(\bar p+l)\,p^\alpha+\bar p_{\bar\beta}\,S_b^{\bar\beta\beta}(\bar p+l)\,g^{\mu\alpha}\Big\}\,,
\end{aligned}
\tag{99}
$$

where we apply the notations (90, 93). For technical simplicity the tensors (99) are constructed with an effective molecule vertex of the generic form $(\partial_\alpha D^{\alpha\tau})\,\Phi\,(\partial^\beta\,\bar R_{\beta\tau})$ as is implied by the projector technique of section 2.2. Within the given framework the latter is equivalent to the one introduced in (75) that is proportional to $\tilde g_R$.

There remain the contributions that are implied by interactions involving the electromagnetic field strength tensor $F_{\mu\nu}$. They are proportional to the anomalous coupling strengths $e_A, \tilde e_Q, \tilde e_C$ of (63, 58) or the parameter e_P of (56). The terms proportional to $e_P/(f\,m_V^2)$ probe the class 2) of Figure 5 only. Their effect is encoded into the transverse tensor $\bar A^{\mu,\alpha\beta,}(\bar p,p)$, with

$$
\bar A^{\mu,\alpha\beta}_{ab}(\bar p,p) = +i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\,\Big\{(l\cdot q)\,g^\mu{}_\sigma-l^\mu\,q_\sigma\Big\}\,\bar p_{\bar\beta}\,S_b^{\bar\beta\sigma,\beta}(p+l)\,p^\alpha \tag{100}
$$

The further tensors $C^{\mu,\alpha\beta}(\bar p,p)$ and $\bar C^{\mu,\alpha\beta}(\bar p,p)$ describe the processes where the Goldstone boson, which emits the photon, is converted into a light vector meson as included in class 1). They are proportional to the parameter combinations $e_A\,\tilde g_T/f^2$ and $e_A\,g_E/f^2$ respectively. We introduce

$$
\begin{aligned}
C^{\mu,\alpha\beta}_{abc}(\bar p,p) &= +2\,i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\,\bar p^{\bar\beta}\,S_b^{\sigma\tau,\tilde\alpha}(q+l)\,S_c^{\bar\alpha\beta}(p+l)\,p^\alpha \\
&\quad\times\,\epsilon_{\sigma\tau\bar\alpha\bar\beta}\,\epsilon^\mu{}_{\nu\tilde\alpha\tilde\beta}\,q^\nu\,l^{\tilde\beta}\,,\\
\bar C^{\mu,\alpha\beta}_{abc}(\bar p,p) &= -2\,i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\,\bar p^{\bar\beta}\,S_b^{\bar\alpha\tilde\alpha}(q+l)\,S_c^{\sigma\tau,\beta}(p+l)\,p^\alpha
\end{aligned}
$$

$$\times\, \epsilon_{\sigma\tau\bar{\alpha}\bar{\beta}}\, \epsilon^{\mu}{}_{\nu\tilde{\alpha}\tilde{\beta}}\, q^{\nu}\, l^{\tilde{\beta}}\,. \tag{101}$$

The terms proportional to $\tilde{e}_C, \tilde{e}_Q$ are described by the tensor $D^{\mu,\alpha\beta}(\bar{p}, p)$. In the convention of Figure 5 they correspond to contributions of class 3), which probe the anomalous magnetic moments of the vector D-mesons. We introduce

$$D^{\mu,\alpha\beta}_{ab}(\bar{p}, p) = +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l) \Big\{ \bar{p}_{\bar{\alpha}}\, S_b^{\bar{\alpha}\mu}(\bar{p}+l)\, q_\tau\, S_b^{\tau\beta}(p+l)\, p^\alpha$$
$$-\bar{p}_{\bar{\alpha}}\, S_b^{\bar{\alpha}\tau}(\bar{p}+l)\, q_\tau\, S_b^{\mu\beta}(p+l)\, p^\alpha \Big\}\,. \tag{102}$$

We emphasize that each of the tensor integrals introduced in (99-102) is gauge invariant separately, i.e if contracted with q_μ it vanishes identically. This was checked by explicit calculations.

All terms implied by the $K D^*$ and $\eta\, D_s^*$ channels are collected. We arrive at the decay amplitude

$$M^{\mu,\alpha\beta}_{1^+\to\gamma\, 0^-} = -\frac{e\, g_P}{f}\, h^{(1+)}_{KD^*} \Big\{ A^{\mu,\alpha\beta}_{KD^*}(\bar{p}, p) + B^{\mu,\alpha\beta}_{KD^*}(\bar{p}, p) \Big\}$$
$$-\frac{e_P}{f\, m_V^2}\, h^{(1+)}_{KD^*}\, \bar{A}^{\mu,\alpha\beta}_{KD^*}(\bar{p}, p) - \frac{2}{\sqrt{3}}\, \frac{e\, g_P}{f}\, h^{(1+)}_{\eta D_s^*}\, B^{\mu,\alpha\beta}_{\eta D_s^*}(\bar{p}, p)$$
$$+\frac{e_A}{48\, f^2\, m_V}\, h^{(1+)}_{KD^*} \Big\{ \tilde{g}_T\, C^{\mu,\alpha\beta}_{KK^*D^*}(\bar{p}, p) + g_E\, \bar{C}^{\mu,\alpha\beta}_{KK^*D^*}(\bar{p}, p) \Big\}$$
$$+\frac{e_A}{\sqrt{3}\, 12\, f^2\, m_V}\, h^{(1+)}_{\eta D_s^*} \Big\{ \tilde{g}_T\, C^{\mu,\alpha\beta}_{\eta\phi D_s^*}(\bar{p}, p) + g_E\, \bar{C}^{\mu,\alpha\beta}_{\eta\phi D_s^*}(\bar{p}, p) \Big\}$$
$$-\frac{g_P}{f\, M_V^2} \Big\{ \frac{2}{\sqrt{3}}\, \frac{3\, \tilde{e}_C + \tilde{e}_Q - 3\, e}{3}\, h^{(1+)}_{\eta D_s^*}\, D^{\mu,\alpha\beta}_{\eta D_s^*}(\bar{p}, p)$$
$$+\frac{6\, \tilde{e}_C - \tilde{e}_Q - 3\, e}{3}\, h^{(1+)}_{KD^*}\, D^{\mu,\alpha\beta}_{KD^*}(\bar{p}, p) \Big\}\,, \tag{103}$$

where

$$h^{(1+)}_{KD^*} = \sqrt{2}\, \frac{g^{(1+)}_{KD^*}}{M_{D^*}} \leftrightarrow \frac{M_{1^+}^2 - M_{D^*}^2 - m_K^2}{2\, f\, M_{1^+}^2}\, \tilde{g}_R\,,$$
$$h^{(1+)}_{\eta D_s^*} = \sqrt{2}\, \frac{g^{(1+)}_{\eta D_s^*}}{M_{D_s^*}} \leftrightarrow \frac{M_{1^+}^2 - M_{D_s^*}^2 - m_\eta^2}{2\, \sqrt{3}\, f\, M_{1^+}^2}\, \tilde{g}_R\,. \tag{104}$$

The values for the coupling constants $g^{(1+)}_{KD}$ and $g^{(1+)}_{\eta D_s}$ are given in Table 2. We repeat that the application of the projection formula (98) requires an anti-symmetrization of the tensors (99-102).

We turn to the contributions implied by the channels involving the K^* or ϕ

meson. Such terms are proportional to the coupling constants $\tilde{g}_H$ or $\hat{g}_H$ introduced in (81). While the terms proportional to $\hat{g}_H$ are deferred to Appendix E we detail the ones proportional to $\tilde{g}_H$ here. The former encode the physics of the $K^* D^*$ and $\phi\, D^*$ channels. Appendix E provides in addition explicit results for the Passareno-Veltman reduction of the tensor integrals (99-102, 105).

The evaluation of terms proportional to $\tilde{g}_H$ is analogous to the ones proportional to g_R. Formally the role of light and heavy intermediate lines in Figure 5 is interchanged. Thus the tensor integrals formed in (99-102) will occur in the desired result. Additional tensors are required to describe the implications of $e_V \neq 0$ or $e_E \neq 0$ as introduced in (67). The latter give rise to contributions of class 2) in Figure 5. They are analogous to the contributions proportional to e_P in (103). We form the gauge invariant tensors $\bar{B}^{\mu,\alpha\beta}(\bar{p},p)$ and $\tilde{B}^{\mu,\alpha\beta}(\bar{p},p)$ with

$$
\begin{aligned}
\bar{B}^{\mu,\alpha\beta}_{ab}(\bar{p},p) &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b^{\tau\beta}(p+l)\, p^\alpha \\
&\qquad \times\Big((l\cdot q)\, g^\mu_{\;\;\tau} - l^\mu\, q_\tau - (\bar{p}\cdot q)\, g^\mu_{\;\;\tau} + \bar{p}^\mu\, q_\tau \Big)\,, \\
\tilde{B}^{\mu,\alpha\beta}_{ab}(\bar{p},p) &= -2\, i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, q_\tau\, S_b^{\tau\mu,\beta}(p+l)\, p^\alpha\, (\bar{p}\cdot l)\,.
\end{aligned}
\tag{105}
$$

We collect the contributions of the $K^* D$ and $\phi\, D_s$ channels to the decay amplitude

$$
\begin{aligned}
M^{\mu,\alpha\beta}_{1^+\to\gamma 0^-} &= -\frac{e\, g_V\, \tilde{g}_H}{f^2} \Big\{ A^{\mu,\alpha\beta}_{DK^*}(\bar{p},p) + B^{\mu,\alpha\beta}_{DK^*}(\bar{p},p) + A^{\mu,\alpha\beta}_{D_s\phi}(\bar{p},p) \Big\} \\
&\quad - \frac{e_V\, \tilde{g}_H}{2\, f^2\, m_V^2}\, \bar{B}^{\mu,\alpha\beta}_{DK^*}(\bar{p},p) + \frac{e_E\, \tilde{g}_H}{3\, f^2\, m_V^2}\, \tilde{B}^{\mu,\alpha\beta}_{DK^*}(\bar{p},p) + \frac{2\, e_E\, \tilde{g}_H}{3\, f^2\, m_V^2}\, \tilde{B}^{\mu,\alpha\beta}_{D_s\phi}(\bar{p},p) \\
&\quad + \frac{(e_C + e_Q/6)\, \tilde{g}_H}{4\, f^2\, M_V^2} \Big\{ g_E\, C^{\mu,\alpha\beta}_{DD^*K^*}(\bar{p},p) + \tilde{g}_T\, \bar{C}^{\mu,\alpha\beta}_{DD^*K^*}(\bar{p},p) \Big\} \\
&\quad + \frac{(e_C - e_Q/3)\, \tilde{g}_H}{8\, f^2\, M_V^2} \Big\{ g_E\, C^{\mu,\alpha\beta}_{D_s D_s^*\phi}(\bar{p},p) + \tilde{g}_T\, \bar{C}^{\mu,\alpha\beta}_{D_s D_s^*\phi}(\bar{p},p) \Big\}\,,
\end{aligned}
\tag{106}
$$

in terms of the tensor integrals of (99-101, 105). Note that there is no term involving the tensor (102). This follows since we neglect the effect of anomalous magnetic moment of the light vector mesons.

Given the transition amplitude, $M^{\mu,\alpha\beta}_{1^+\to\gamma\,0^+}$, we derive the partial-decay width $\Gamma_{1^+\to\gamma\,0^+}$. It is determined by one scalar number, $d_{1^+\to\gamma\,0^+}$. We write:

$$
\begin{aligned}
&\epsilon^\dagger_\mu(q,\lambda_q)\, M^{\mu,\alpha\beta}_{1^+\to\gamma\,0^+}\, \epsilon^{\alpha\beta}(p,\lambda_p) \\
&\qquad = d_{1^+\to\gamma\,0^+}\, \epsilon^\dagger_\mu(q,\lambda_q)\, q_\tau\, \epsilon^{\mu\tau}{}_{\sigma\alpha}\, p^\sigma\, \frac{p^\beta}{\sqrt{p^2}}\, \epsilon^{\alpha\beta}(p,\lambda_p)\,,
\end{aligned}
\tag{107}
$$

which implies

$$
\begin{aligned}
\Gamma_{1^+\to\gamma\,0^+} &= \frac{|d_{1^+\to\gamma\,0^+}|^2}{12\pi}\left(\frac{M^2_{1^+}-M^2_{0^+}}{2\,M_{1^+}}\right)^3, \\
d_{1^+\to\gamma\,0^+} &= \frac{-1}{(p\cdot q)^2\, M_{1^+}}\, q^\tau\, \epsilon_{\mu\tau\sigma\alpha}\, p^\sigma\, p_\beta\, M^{\mu,\alpha\beta}\,,
\end{aligned}
\tag{108}
$$

with $\bar p^2 = (p-q)^2 = M^2_{0^+}$ and $p^2 = M^2_{1^+}$. Like in (88, 98) we exploit the fact that the decay amplitude is anti-symmetric in $\alpha \leftrightarrow \beta$.

There are 4 classes of contributions we have to consider. They are depicted in Figure 8. Like in Figure 5 the solid lines stand for the propagation of pseudo-scalar or vector mesons. The thick ones are used for the heavy mesons, the thin for the light mesons. The solid and dashed double lines describe the molecule of the initial and final state respectively. Any given diagram is proportional to either one of the four products of coupling constants $g_R\,\tilde g_R$, $g_R\,\tilde g_H$, $g_H\,\tilde g_R$, or $g_H\,\tilde g_H$. We discuss the four possibilities case by case.

In the first two cases, with terms that are proportional to $g_R\,\tilde g_R$ or $g_R\,\tilde g_H$, the thin and thick lines in Figure 8 that are attached to the final molecule vertex stand for the propagation of pseudo-scalar light and heavy mesons. There is only one generic tensor, $A^{\mu,\alpha\beta}_{+,abc}(\bar p, p)$, describing these processes. They are of

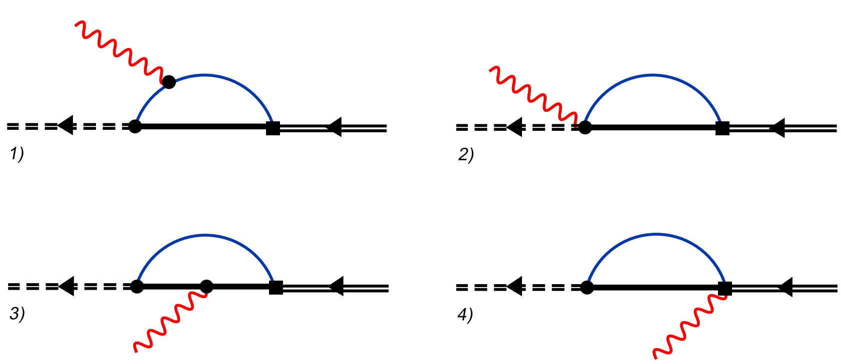

Fig. 8. Diagrams contributing to the process $1^+ \to \gamma\, 0^+$.

class 1) or 3) in the convention of Figure 8. The contributions probe anomalous electromagnetic vertices being proportional to e_A or e_C, e_Q. We introduce

$$A^{\mu,\alpha\beta}_{+,abc}(\bar{p},p) = -2\,i\,q_\tau\,\epsilon^{\mu\tau}{}_{\sigma\rho}\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\,S_b(l+\bar{p})\,(l+\bar{p})^\rho\,S_c^{\sigma\beta}(l+p)\,p^\alpha\,. \tag{109}$$

We turn the terms proportional to $g_H\,\tilde{g}_R$. In this case there is a class 1) contribution only, with thick lines describing the propagation of vector mesons. The thin line changes its character from a pseudo-scalar to vector line at the photon vertex. The contributions are proportional to the anomalous coupling constant e_A. We introduce the corresponding gauge invariant tensor

$$D^{\mu,\alpha\beta}_{+,abc}(\bar{p},p) = -2\,i\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,g_{\bar{\kappa}\kappa}\int \frac{d^4l}{(2\pi)^4}\,l^\tau\,p^\alpha\,S_a(l)\,S_b^{\bar{\kappa}\sigma}(l+q)\,S_c^{\kappa\beta}(p+l)\,. \tag{110}$$

There remain the contributions proportional to $g_H\,\tilde{g}_H$. There are two types of gauge invariant combinations. We form two corresponding transverse tensors $B^{\mu,\alpha\beta}_{+}(\bar{p},p)$ and $C^{\mu,\alpha\beta}_{+}(\bar{p},p)$, where the latter and former receive contributions from class 2),3),4) and 1),2),4) of Figure 8. respectively. The tensors $B^{\mu,\alpha\beta}_{+}(\bar{p},p)$ and $C^{\mu,\alpha\beta}_{+}(\bar{p},p)$ describe the processes where the photon couples to the charge of the heavy and light meson. We introduce the gauge invariant tensors

$$\begin{aligned}
B^{\mu,\alpha\beta}_{+,ab}(\bar{p},p) &= -\,i\,\epsilon_{\tilde{\alpha}\tilde{\beta}\sigma}{}^{\beta}\,g_{\bar{\rho}\rho}\int \frac{d^4l}{(2\pi)^4}\,S_a^{\bar{\rho}\sigma}(l)\,\Big\{g^{\mu\alpha}\,S_b^{\rho,\tilde{\alpha}\tilde{\beta}}(l+\bar{p}) \\
&\quad + p^\alpha\,S_b^{\mu\rho,\tilde{\alpha}\tilde{\beta}}(l+p) + g_{\bar{\kappa}\kappa}\,p^\alpha\,\Big\{S_b^{\rho\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tilde{\alpha}\tilde{\beta}}(p+l) \\
&\qquad + S_b^{\bar{\rho},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\Big\}\Big\}\,,
\end{aligned} \tag{111}$$

$$\begin{aligned}
C^{\mu,\alpha\beta}_{+,ab}(\bar{p},p) &= +\,i\,\epsilon_{\tilde{\alpha}\tilde{\beta}\tau}{}^{\beta}\,g_{\bar{\rho}\rho}\int \frac{d^4l}{(2\pi)^4}\,S_a^{\bar{\rho},\tilde{\alpha}\tilde{\beta}}(l)\,\Big\{p^\alpha S_b^{\rho,\mu\tau}(l+\bar{p}) \\
&\quad + g^{\mu\alpha}\,S_b^{\rho\tau}(l+\bar{p}) + p^\alpha\,S_b^{\mu\rho,\tau}(l+p) \\
&\quad + g_{\bar{\kappa}\kappa}\,p^\alpha\,\Big\{S_b^{\rho\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tau}(p+l) + S_b^{\bar{\rho},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa\tau}(p+l)\Big\}\Big\}\,.
\end{aligned}$$

We emphasize that each of the tensor integrals introduced in (109, 110, 111) is gauge invariant separately, i.e if contracted with q_μ it vanishes identically. This was checked by explicit calculations.

Collecting all terms we arrive at the decay amplitude

$$
\begin{aligned}
M^{\mu,\alpha\beta}_{1^+\to\gamma 0^+} &= \frac{e_C+e_Q/6}{2\,M_V^2}\,h^{(0+)}_{KD}\,h^{(1+)}_{KD^*}\,A^{\mu,\alpha\beta}_{+,KDD^*}(\bar p,p)\\
&+\frac{e_C-e_Q/3}{2\,M_V^2}\,h^{(0+)}_{\eta D_s}\,h^{(1+)}_{\eta D_s^*}\,A^{\mu,\alpha\beta}_{+,\eta D_s D_s^*}(\bar p,p)\\
&+\frac{e_A}{6\sqrt{3}\,f^2\,m_V}\Big\{\tilde g_H\,h^{(0+)}_{\eta D_s}\,A^{\mu,\alpha\beta}_{+,D_s\eta\phi}(\bar p,p)+g_H\,h^{(1+)}_{\eta D_s}\,D^{\mu,\alpha\beta}_{+,\phi\eta D_s^*}(\bar p,p)\Big\}\\
&+\frac{e_A}{24\,f^2\,m_V}\Big\{\tilde g_H\,h^{(0+)}_{KD}\,A^{\mu,\alpha\beta}_{+,DKK^*}(\bar p,p)+g_H\,h^{(1+)}_{KD}\,D^{\mu,\alpha\beta}_{+,K^*KD}(\bar p,p)\Big\}\\
&+e\,\frac{\hat g_H\,g_H}{4\,f^2}\Big\{B^{\mu,\alpha\beta}_{+,\phi D_s^*}(\bar p,p)+B^{\mu,\alpha\beta}_{+,K^*D^*}(\bar p,p)+C^{\mu,\alpha\beta}_{+,D^*K^*}(\bar p,p)\Big\}\\
&+\frac{g_H\,\tilde g_H}{2\,f^2}\Big\{\frac{e_C+e_Q/6}{2\,M_V^2}\,D^{\mu,\alpha\beta}_{+,DD^*K^*}(\bar p,p)+\frac{e_C-e_Q/3}{2\,M_V^2}\,D^{\mu,\alpha\beta}_{+,D_sD_s^*\phi}(\bar p,p)\Big\}
\end{aligned}
\tag{112}
$$

where the coupling constants h_{KD} and $h_{\eta D_s}$ are specified in (96, 104) in terms of the values collected in Table 2. Appendix F provides the results of a Passareno-Veltman reduction of the tensor integrals (109-111).

We turn to the third radiative decay mode of an axial-vector molecule. It is described by a rank-five transition tensor, $M^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{1^+\to\gamma\, 1^-}$. The decay amplitude has a slightly more complicated structure than the ones studied in (86, 107). A detailed analysis reveals that it is characterized by two scalar decay parameters. The latter reflect the fact that the decay may go via a s-wave or a d-wave transition. We write

$$\epsilon^\dagger_\mu(q,\lambda_q)\,\epsilon^\dagger_{\bar\alpha\bar\beta}(\bar p,\lambda_{\bar p})\,M^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{1^+\to\gamma\, 1^-}\,\epsilon_{\alpha\beta}(p,\lambda_p) = \epsilon^\dagger_\mu(q,\lambda_q)\,\frac{-i\,\bar p^{\bar\beta}\,\epsilon^\dagger_{\bar\alpha\bar\beta}(\bar p,\lambda_{\bar p})}{\sqrt{\bar p^2}}$$
$$\times\, q_\tau\,\epsilon^{\mu\tau\bar\alpha\sigma}\left\{d^{(1)}_{1^+\to\gamma\, 1^-}\,\frac{q^\alpha\,p_\sigma}{(q\cdot p)} + d^{(2)}_{1^+\to\gamma\, 1^-}\left(g^\alpha_{\;\;\sigma} - \frac{q^\alpha\,p_\sigma}{(q\cdot p)}\right)\right\}\frac{p^\beta\,\epsilon_{\alpha\beta}(p,\lambda_p)}{\sqrt{p^2}}\,. \tag{113}$$

In order to verify that there are indeed only two independent decay parameters it is useful to recall the identity:

$$g_{\sigma\tau}\epsilon_{\alpha\beta\gamma\delta} = g_{\alpha\tau}\epsilon_{\sigma\beta\gamma\delta} + g_{\beta\tau}\epsilon_{\alpha\sigma\gamma\delta} + g_{\gamma\tau}\epsilon_{\alpha\beta\sigma\delta} + g_{\delta\tau}\epsilon_{\alpha\beta\gamma\sigma}\,. \tag{114}$$

For the decay width we derive

$$\Gamma_{1^+\to\gamma\, 1^-} = \frac{1}{12\,\pi}\left\{\left|\frac{d^{(1)}_{1^+\to\gamma\, 1^-}}{M_{1^+}}\right|^2 + \left|\frac{d^{(2)}_{1^+\to\gamma\, 1^-}}{M_{1^-}}\right|^2\right\}\left(\frac{M^2_{1^+}-M^2_{1^-}}{2\,M_{1^+}}\right)^3, \tag{115}$$

with $\bar p^2 = M^2_{1^-}$ and $p^2 = M^2_{1^+}$. Due to the identity (114) it is quite cumbersome to extract the two decay parameters from a given amplitude. Fortunately, this task can be streamlined considerably by the projection identities:

$$d^{(1)}_{1^+\to\gamma\, 1^-} = \frac{-i\,16\,M_{1^+}}{M_{1^-}\,(M^2_{1^+}-M^2_{1^-})^3}\,p^\sigma\,q^\tau\,\epsilon_{\sigma\tau\bar\alpha\mu}\,\bar p_{\bar\beta}\,M^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{1^+\to\gamma\, 1^-}\,q_\alpha\,p_\beta\,,$$
$$d^{(2)}_{1^+\to\gamma\, 1^-} = \frac{-i\,16\,M_{1^-}}{M_{1^+}\,(M^2_{1^+}-M^2_{1^-})^3}\,p^\sigma\,q^\tau\,\epsilon_{\sigma\tau\mu\alpha}\,q_{\bar\alpha}\,\bar p_{\bar\beta}\,M^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{1^+\to\gamma\, 1^-}\,p_\beta\,, \tag{116}$$

which are a consequence of the anti-symmetry and transversality of the decay amplitude.

There are 5 classes of contributions we have to consider. They are depicted in Figure 5. Any given diagram is proportional to either one of the three coupling constants $\tilde g_R$, $\tilde g_H$ or $\hat g_H$. In the first case the thin and thick lines in Figure 5 that are attached to the resonance vertex stand for the propagation of Goldstone bosons and vector D-mesons. In the second case the role of thin and thick lines is interchanged in the sense that the thin lines correspond to light vector mesons, whereas the thick lines describe heavy pseudo-scalar mesons. In the third case both lines stand for vector mesons. The outgoing line is a vector D_s^*-meson always.

It is convenient to group together diagrams that are gauge invariant separately. We discuss the various contributions in some detail and start with diagrams that are proportional to the coupling constant $\tilde{g}_R$. There are two types of gauge invariant combinations that are proportional to $e\,\tilde{g}_P/f$. We form two corresponding transverse tensors $A_+^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)$ and $B_+^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)$, where the latter and former receive contributions from class 1),2),4),5) and 2),3),4),5) of Figure 5 respectively. The tensors $A_+^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)$ and $B_+^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)$ describe the processes where the photon couples to the charge of the light and heavy meson. We introduce:

$$
\begin{aligned}
A^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{\pm,ab}(\bar{p},p) &= -i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\Big\{\epsilon^{\bar{\alpha}\bar{\beta}}{}_{\sigma\tau}\Big[-g^{\mu\sigma}\,S_b^{\tau\beta}(p+l)\,p^\alpha \\
&+ l^\sigma\,S_b^{\tau\beta}(\bar{p}+l)\,g^{\mu\alpha} + (l+q)^\sigma\,S_a(l+q)\,S_b^{\tau\beta}(p+l)\,(q+2\,l)^\mu\,p^\alpha\Big] \\
&+ \epsilon^{\tilde{\alpha}\tilde{\beta}}{}_{\sigma\tau}\,\bar{p}^{\bar{\alpha}}\,S^{\mu\bar{\beta}}_{\bar{p},\;\tilde{\alpha}\tilde{\beta}}(p)\,l^\sigma\,S_b^{\tau\beta}(p+l)\,p^\alpha \mp \epsilon_{\sigma\tau}{}^{\rho\bar{\beta}}\Big[-g^\mu{}_\rho\,\bar{p}^{\bar{\alpha}}\,S_b^{\sigma\tau,\beta}(p+l)\,p^\alpha \\
&+ l_\rho\,g^{\mu\bar{\alpha}}\,S_b^{\sigma\tau,\beta}(p+l)\,p^\alpha + \bar{p}^{\bar{\alpha}}\,l_\rho\,S_b^{\sigma\tau,\beta}(\bar{p}+l)\;g^{\mu\alpha} \\
&+ \bar{p}^{\bar{\alpha}}\,(l+q)_\rho\,S_a(l+q)\,S_b^{\sigma\tau,\beta}(p+l)\,(q+2\,l)^\mu\,p^\alpha\Big]\Big\}\,,
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
B^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{\pm,ab}(\bar{p},p) &= -i\int \frac{d^4l}{(2\pi)^4}\,S_a(l)\Big\{\epsilon^{\bar{\alpha}\bar{\beta}}{}_{\sigma\tau}\Big[+l_\sigma\,S_b^{\mu\tau,\beta}(p+l)\,p^\alpha \\
&+ l^\sigma\,S_b^{\tau\beta}(\bar{p}+l)\,g^{\mu\alpha} + l^\sigma\,S_b^{\tau,\mu\beta}(\bar{p}+l)\,p^\alpha \\
&+ l^\sigma\Big\{S_b^{\tau,\mu\kappa}(\bar{p}+l)\,S_b^{\kappa\beta}(p+l) + S_b^{\tau\kappa}(\bar{p}+l)\,S_b^{\mu\kappa,\beta}(p+l)\Big\}\,p^\alpha\Big] \\
&+ \epsilon^{\tilde{\alpha}\tilde{\beta}}{}_{\sigma\tau}\,\bar{p}^{\bar{\alpha}}\,S^{\mu\bar{\beta}}_{\bar{p},\;\tilde{\alpha}\tilde{\beta}}(p)\,l^\sigma\,S_b^{\tau\beta}(p+l)\,p^\alpha \mp \epsilon_{\sigma\tau}{}^{\rho\bar{\beta}}\Big[+l_\rho\,g^{\mu\bar{\alpha}}\,S_b^{\sigma\tau,\beta}(p+l)\,p^\alpha \\
&+ l_\rho\,\bar{p}^{\bar{\alpha}}\,S_b^{\sigma\tau,\beta}(\bar{p}+l)\,g^{\mu\alpha} + l_\rho\,\bar{p}^{\bar{\alpha}}\,S_b^{\sigma\tau,\mu\beta}(\bar{p}+l)\,p^\alpha \\
&+ l_\rho\,\bar{p}^{\bar{\alpha}}\Big\{S_b^{\sigma\tau,\kappa}(\bar{p}+l)\,S_b^{\mu\kappa,\beta}(p+l) + S_b^{\sigma\tau,\mu\kappa}(\bar{p}+l)\,S_b^{\kappa\beta}(p+l)\Big\}\,p^\alpha\Big]\Big\}\,.
\end{aligned}
$$

Note that in the tensors $A_{\pm,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}$ and $B_{\pm,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}$ there are contributions where the photon couples to the final vector particle. In the convention of Figure 5 this is a diagram of class 5). They give rise to terms proportional to $S_{\bar{p}}^{\bar{\alpha}\bar{\beta},\alpha\beta}(p)$, where we identify $m_{\bar{p}}^2 = \bar{p}^2$. Such contributions are not at odds with parity conservation. This follows since the tensor field carries spin one quanta with both parities. Analogous terms where the photon couples to the initial vector meson do not arise due to parity conservation. Since the resonance field couples always with $(\partial_\tau R^{\tau\alpha})$, only the parity $P=+1$ component is accessed.

There remain the contributions that are implied by interactions involving the electromagnetic field strength tensor $F_{\mu\nu}$. They are proportional to the anomalous coupling strengths e_A, e_Q, e_C of (63, 58) or the parameter $\tilde{e}_P$ of (56). The terms proportional to $\tilde{e}_P/(f\, m_V^2)$ probe the class 2) of Figure 5 only. Their effect is encoded into the transverse tensor $\bar{A}_+^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta,}(\bar{p},p)$, with

$$
\begin{aligned}
\bar{A}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{\pm,ab}(\bar{p},p) &= -i\int\frac{d^4l}{(2\pi)^4}\,S_a(l)\left\{\epsilon^{\bar{\alpha}\bar{\beta}}{}_{\sigma\tau}\left(l^\mu\,q^\sigma-(l\cdot q)\,g^{\mu\sigma}\right)S_b^{\tau\beta}(p+l)\right.\\
&\quad\left.\mp\,\epsilon_{\sigma\tau}{}^{\rho\bar{\beta}}\left(l^\mu\,q_\rho-(l\cdot q)\,g^\mu_{\;\;\rho}\right)\bar{p}^{\bar{\alpha}}\,S_b^{\sigma\tau,\beta}(p+l)\right\}p^\alpha\,,
\end{aligned}
\tag{118}
$$

$$
\begin{aligned}
\bar{B}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{ab}(\bar{p},p) &= +i\int\frac{d^4l}{(2\pi)^4}\,S_a(l)\left\{\epsilon^{\bar{\alpha}\bar{\beta}}{}_{\sigma\tau}\left[l^\sigma\,q_\kappa\left\{S_b^{\tau\kappa}(\bar{p}+l)\,S_b^{\mu\beta}(p+l)\right.\right.\right.\\
&\quad\left.\left.-\,S_b^{\tau\mu}(\bar{p}+l)\,S_b^{\kappa\beta}(p+l)\right\}p^\alpha\right]-\epsilon_{\sigma\tau}{}^{\rho\bar{\beta}}\left[l_\rho\,q_\kappa\,\bar{p}^{\bar{\alpha}}\left\{S_b^{\sigma\tau,\kappa}(\bar{p}+l)\,S_b^{\mu\beta}(p+l)\right.\right.\\
&\quad\left.\left.\left.-\,S_b^{\sigma\tau,\mu}(\bar{p}+l)\,S_b^{\kappa\beta}(p+l)\right\}p^\alpha\right]\right\},
\end{aligned}
$$

The further tensors $C_+^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)$ and $\bar{C}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)$ describe the processes where the Goldstone boson, which emits the photon, is converted into a light vector meson as included in class 1). They are proportional to the parameter combinations $e_A\,\tilde{g}_V/f^2$ and $e_A\,g_T/f^2$ respectively. We introduce

$$
\begin{aligned}
C^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{\pm,abc}(\bar{p},p) &= +2\,i\int\frac{d^4l}{(2\pi)^4}\,S_a(l)\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,l^\tau\left\{S_b^{\bar{\alpha}\sigma}(l+q)\,S_c^{\bar{\beta}\beta}(l+p)\,p^\alpha\right.\\
&\quad\left.\pm\,\bar{p}^{\bar{\alpha}}\,g_{\bar{\kappa}\kappa}\,S_b^{\bar{\kappa}\sigma}(l+q)\,S_c^{\kappa\bar{\beta},\beta}(l+p)\,p^\alpha\right\},\\
\bar{C}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{abc}(\bar{p},p) &= +2\,i\int\frac{d^4l}{(2\pi)^4}\,S_a(l)\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,l^\tau\,\bar{p}^{\bar{\alpha}}\\
&\quad\times\,g_{\bar{\kappa}\kappa}\,S_b^{\bar{\kappa}\bar{\beta},\sigma}(l+q)\,S_c^{\kappa\beta}(l+p)\,p^\alpha\;.
\end{aligned}
\tag{119}
$$

Processes that are analogous to the anomalous ones described by the tensors (119) are driven by the parameters e_Q and e_C. In this case the photon is emitted by a pseudo-scalar D-meson, which is converted into a vector D-meson. The contribution is included in class 3) of Figure 5. We introduce a corresponding loop tensor

$$D_{abc}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p) = +2\,i\int\frac{d^4l}{(2\pi)^4}\,S_a(l)\,S_b(l+\bar p)$$
$$\times l^{\bar\alpha}\,\bar p^{\bar\beta}\,q_\tau\,\epsilon^{\mu\tau}{}_{\rho\sigma}\,(l+\bar p)^\sigma\,S_c^{\rho\beta}(l+p)\,p^\alpha\,. \tag{120}$$

We emphasize that each of the tensor integrals introduced in (117, 118, 119, 120) is gauge invariant separately, i.e if contracted with q_μ it vanishes identically. This was checked by explicit calculations.

All together we arrive at the decay amplitude

$$\begin{aligned}
-i\,M_{1^+\to\gamma\,1^-}^{\bar\alpha\bar\beta,\mu,\alpha\beta} &= \frac{e\,\tilde g_P}{4\,f}\,h_{KD^*}^{(1+)}\Big\{A_{+,KD^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)+B_{+,KD^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\Big\}\\
&+\frac{\tilde e_P}{4\,f\,m_V^2}\,h_{KD^*}^{(1+)}\,\bar A_{+,KD^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)+\frac{2}{\sqrt3}\,\frac{e\,\tilde g_P}{4\,f}\,h_{\eta D_s^*}^{(1+)}\,B_{+,\eta D_s^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\\
&+\frac{\tilde g_P}{4\,f\,M_V^2}\Big\{\frac{2}{\sqrt3}\,\frac{3\,\tilde e_C+\tilde e_Q-3\,e}{3}\,h_{\eta D_s^*}^{(1+)}\,\bar B_{\eta D_s^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\\
&\qquad+\frac{6\,\tilde e_C-\tilde e_Q-3\,e}{3}\,h_{KD^*}^{(1+)}\,\bar B_{KD^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\Big\}\\
&-\frac{e_A}{24\,f^2\,m_V}\,h_{KD^*}^{(1+)}\Big\{\tilde g_V\,C_{+,KK^*D^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)+g_T\,\bar C_{KK^*D^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\Big\}\\
&-\frac{e_A}{\sqrt3\,6\,f^2\,m_V}\,h_{\eta D_s^*}^{(1+)}\Big\{\tilde g_V\,C_{+,\eta\phi D_s^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)+g_T\,\bar C_{\eta\phi D_s^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\Big\}\\
&-\frac{g_P}{f\,M_V^2}\Big\{\frac{2}{\sqrt3}\,\frac{3\,e_C-e_Q}{6}\,h_{\eta D_s^*}^{(1+)}\,D_{\eta D_sD_s^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\\
&\qquad+\frac{6\,e_C+e_Q}{6}\,h_{KD^*}^{(1+)}\,D_{KDD^*}^{\bar\alpha\bar\beta,\mu,\alpha\beta}(\bar p,p)\Big\}\,.
\end{aligned}\tag{121}$$

We turn to the contributions implied by the coupling constants $\tilde g_H$ and $\hat g_H$ introduced in (81). While the terms proportional to $\hat g_H$ are deferred to Appendix G we detail the ones proportional to $\tilde g_H$ here. The former are expected to be more relevant than the latter due to phase space. Appendix G provides in addition explicit results for the Passareno-Veltman reduction of the tensor integrals (117-120, 122).

The evaluation of terms proportional to $\tilde g_H$ is analogous to the ones proportional to g_R. Formally the role of light and heavy intermediate lines in Figure 5 is interchanged. Thus the tensor integrals formed in (117-120) will occur in the desired result. Additional tensors are required to describe the implications of $\tilde e_T \neq 0$ as introduced in (67). The latter give rise to contributions of class 2) in Figure 5. They are analogous to the contributions proportional to $\tilde e_P$ in (112). We form the gauge-invariant tensor $\tilde B^{\mu,\alpha\beta}(\bar p,p)$ with

$$\tilde{B}_{ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = +\,i \int \frac{d^4l}{(2\pi)^4}\, S_a(l)\, \bar{p}^{\bar{\alpha}}\, l_\rho\, \epsilon^{\rho\bar{\beta}}{}_{\bar{\sigma}\tau}\, S_b^{\sigma\tau,\beta}(p+l) \\ \times\Big(g^{\mu}{}_{\sigma}\, q^{\bar{\sigma}} - g^{\mu\bar{\sigma}}\, q^{\sigma}\Big)\, p^{\alpha}\,. \tag{122}$$

The contributions proportional to $\tilde{g}_H$ are

$$\begin{aligned}
-i\, M_{1^+\to\gamma\, 1^-}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta} &= -\frac{e\,(g_E+\tilde{g}_T)\,\tilde{g}_H}{8\,f^2}\Big\{A_{+,DK^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) + B_{+,DK^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)\Big\} \\
&- \frac{e\,(g_E-\tilde{g}_T)\,\tilde{g}_H}{8\,f^2}\Big\{A_{-,DK^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) + B_{-,DK^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)\Big\} \\
&- \frac{\tilde{e}_T\,\tilde{g}_H}{4\,f^2\,m_V^2}\,\tilde{B}_{DK^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) \\
&- \frac{e\,\tilde{g}_H}{8\,f^2}\Big\{(g_E+\tilde{g}_T)\,A_{+,D_s\phi}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) + (g_E-\tilde{g}_T)\,A_{-,D_s\phi}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)\Big\} \\
&+ \frac{(e_C+e_Q/6)\,\tilde{g}_H}{4\,f^2\,M_V^2}\Big\{(\tilde{g}_V+g_T)\,C_{+,DD^*K^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) + 2\,\tilde{g}_V\,\bar{C}_{DD^*K^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) \\
&\qquad + (\tilde{g}_V-g_T)\,C_{-,DD^*K^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)\Big\} \\
&+ \frac{(e_C-e_Q/3)\,\tilde{g}_H}{8\,f^2\,M_V^2}\Big\{(\tilde{g}_V+g_T)\,C_{+,D_sD_s^*\phi}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) + 2\,\tilde{g}_V\,\bar{C}_{D_sD_s^*\phi}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) \\
&\qquad + (\tilde{g}_V-g_T)\,C_{-,D_sD_s^*\phi}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)\Big\} \\
&+ \frac{e_A\,g_P\,\tilde{g}_H}{12\,f^3\,m_V}\,D_{DKK^*}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) + \frac{e_A\,g_P\,\tilde{g}_H}{9\,f^3\,m_V}\,D_{D_s\eta\phi}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)\,.
\end{aligned} \tag{123}$$

6 Numerical results

In this section we confront the results of the previous four sections with the empirical information on the radiative and strong decays of the scalar and axial-vector D_s mesons.

Our predictions of 140 keV from the isospin violating processes $D^*_{s0}(2317)^\pm \to D_s(1968)^\pm \pi^0$ is compatible with the empirical bound $\Gamma < 3.8$ MeV [12]. The present upper limit on the ratio of the radiative to pionic decay width is [10]

$$\frac{\Gamma\,[D^*_{s0}(2317)^\pm \;\to\; D^*_s(2112)^\pm\,\gamma]}{\Gamma\,[D^*_{s0}(2317)^\pm \;\to\; D_s(1968)^\pm\,\pi^0]} <\; 0.059 \tag{124}$$

It is convenient to translate the bound (124) into one for the decay constant $d_{0^+\to\gamma\,1^-}$ as introduced in (87). Given our prediction for the isospin violating π_0 decay width of 140 keV (see Table 2) we obtain

$$\Big|d_{0^+\to\gamma\,1^-}\Big| \;<\; 0.117\,\mathrm{GeV}^{-1}\,. \tag{125}$$

The total width of the $\mathrm{D}^*_{s1}(2460)^\pm$ meson is less than 3.5 MeV [12]. Our predictions of 140 keV from the isospin violating process $D^*_{s1}(2460) \;\to\; D^*_s(2112)\,\pi^0$ is compatible with the latter bound. Constraints on its radiative decays to the $\mathrm{D}_s(1968)^\pm$, to the $\mathrm{D}^*_s(2112)^\pm$ and to the $\mathrm{D}^*_{s0}(2317)^\pm$ are as follows [10],

$$\frac{\Gamma\,[D^*_{s1}(2460) \;\to\; D_s(1968)\,\gamma]}{\Gamma\,[D^*_{s1}(2460) \;\to\; D^*_s(2112)\,\pi^0]} = 0.31 \pm 0.06, \tag{126}$$

$$\frac{\Gamma\,[D^*_{s1}(2460) \;\to\; D^*_s(2112)\,\gamma]}{\Gamma\,[D^*_{s1}(2460) \;\to\; D^*_s(2112)\,\pi^0]} <\; 0.16, \tag{127}$$

$$\frac{\Gamma\,[D^*_{s1}(2460) \;\to\; D^*_{s0}(2317)\,\gamma]}{\Gamma\,[D^*_{s1}(2460) \;\to\; D^*_s(2112)\,\pi^0]} <\; 0.22. \tag{128}$$

Given our prediction for the isospin-violating decay process we derive the implications of (126, 127, 128) on the decay constants $d_{1^+\to\gamma\,0^-}$, $d_{1^+\to\gamma\,0^+}$ and $d^{(1,2)}_{1^+\to\gamma\,1^-}$ as introduced in (98, 108, 115). It holds

$$\Big|d_{1^+\to\gamma\,0^-}\Big| \;=\; 0.138\,^{+0.012}_{-0.014}\,\mathrm{GeV}^{-1}\,, \qquad \Big|d_{1^+\to\gamma\,0^+}\Big| \;<\; 0.665\,\mathrm{GeV}^{-1}\,,$$

$$\left|d^{(1)}_{1^+\to\gamma\,1^-}\right|^2 + (1.164)^2\left|d^{(2)}_{1^+\to\gamma\,1^-}\right|^2 < \Big(0.39\Big)^2 . \tag{129}$$

An additional constraint follows from the absolute measurement of the decay probability of the $D^*_{s1}(2460)^-$ to the $D_s(1968)^-\,\gamma$ channel with $(16\pm 7)\%$ [12]. We observe that the central value is incompatible with (126, 127, 128). The one sigma lower bound of 9 % suggests a minimal branching

$$\frac{\Gamma\,[D^*_{s1}(2460)\;\to\;D^*_s(2112)\,\gamma]}{\Gamma\,[D^*_{s1}(2460)\;\to\;D^*_s(2112)\,\pi^0]} >\;0.12, \tag{130}$$

which implies

$$\left|d^{(1)}_{1^+\to\gamma\,1^-}\right|^2 + (1.164)^2\left|d^{(2)}_{1^+\to\gamma\,1^-}\right|^2 > \Big(0.34\Big)^2 . \tag{131}$$

It is instructive to confront the constraints on the decay parameters (125, 129) with the prediction of heavy-quark symmetry. We confirm the leading order results of [19], which imply the relations

$$M_c\,d_{0^+\to\gamma\,1^-} = M_c\,d_{1^+\to\gamma\,0^-} = d^{(1)}_{1^+\to\gamma\,1^-} = d^{(2)}_{1^+\to\gamma\,1^-} \equiv d\,, \tag{132}$$

with $M_c \simeq M_V$ a typical mass of a charmed meson. In the heavy-quark mass limit the parameter d scales linear in the charm quark mass. We observe that (129) suggests the range $0.25 < |d| < 0.30$ for $M_c = M_V = 2000$ MeV, which

	$10\,\times\,d_{0^+\to\gamma\,1^-}\,[\mathrm{GeV}^{-1}]$		
KD	$+2.371\,g_P + 3.036\,e_P$	$-2.252\,(e_C + e_Q/6)\,\tilde g_P$	$-(6.095\,\tilde g_T + 0.950\,g_E)\,e_A$
ηD_s	$+6.822\,g_P$	$-1.018\,(e_C - e_Q/3)\,\tilde g_P$	$-(14.93\,\tilde g_T + 3.575\,g_E)\,e_A$
	$10\,\times\,d_{1^+\to\gamma\,0^-}\,[\mathrm{GeV}^{-1}]$		
KD	$+1.641\,g_P + 2.920\,e_P$	$+2.272\,(\tilde e_C - \tilde e_Q/6)\,g_P$	$+(0.500\,\tilde g_T + 0.501\,g_E)\,e_A$
ηD_s	$+3.445\,g_P$	$+1.222\,(\tilde e_C + \tilde e_Q/3)\,g_P$	$+(3.149\,\tilde g_T + 0.883\,g_E)\,e_A$
	$10\,\times\,d_{1^+\to\gamma\,0^+}\,[\mathrm{GeV}^{-1}]$		
KD		$+2.104\,(e_C + e_Q/6)$	
ηD_s		$+0.891\,(e_C - e_Q/3)$	

Table 3
Decay constants that are implied by (95, 103, 112). We use the coupling constants of Table 2 together with $f = 90$ MeV, $m_V = 776$ MeV, $M_V = 2000$ MeV.

is barely compatible with the bound (125). This naive estimate favors a decay parameter $d_{0^+\to\gamma 1^-}$ that is not much smaller than the bound of (125).

We proceed and analyze the decay parameters as they arise in the hadrogenesis conjecture. The various contributions from the KD and ηD_s channels are detailed in Tables 3-4, which we discuss successively from left to right. Large terms, proportional to g_P or $\tilde{g}_P$ are implied by the processes where the photon couples to the charge of either the K^+ or a charm meson. Given the empirical value $g_P \simeq 0.57$ the decay parameters would meet the constraints (125, 129) within about 40% with the exception of $d^{(2)}_{1^+\to\gamma 1^-} \simeq 0.52$, which is twice as large as the heavy-quark relations (132) suggest. We stress that performing a formal expansion of the full expressions for the decay parameter in the inverse charm-quark mass leads results that are compatible with the expectation from the heavy-quark symmetry. It is interesting to observe that the KD channel does produce decay parameters that significantly distort the pattern that arises in the limit of infinitely heavy charm quarks (132). Such breaking pattern are even larger in the ηD_s channel. From Tables 3-4 we conclude that the contribution form the latter channel is in fact much larger than the ones from the KD channel. This is an immediate consequence that for the ηD_s channel there is a single contribution only, which is not balanced by a second contribution of opposite sign as is for the KD channel. In the KD channel there are two contributions of opposite sign, where the photon couples either to the charge of the K^+ or the D^+.

Adding up the two contributions from the KD and $\eta\, D_s$ channels we would arrive at decay parameters that are much too large, for instance we would

	$10\,\times\, d^{(1)}_{1^+\to\gamma 1^-}$		
KD	$+5.147\,\tilde{g}_P + 7.347\,\tilde{e}_P$	$+0.221\,(e_C + e_Q/6)\,g_P$	$+(8.544\,g_T - 1.458\,\tilde{g}_V\,)\,e_A$
		$+3.214\,(\tilde{e}_C - \tilde{e}_Q/6)\,\tilde{g}_P$	
ηD_s	$+11.92\,\tilde{g}_P$	$-0.227\,(e_C - e_Q/3)\,g_P$	$+(26.85\,g_T - 5.914\,\tilde{g}_V\,)\,e_A$
		$+1.756\,(\tilde{e}_C + \tilde{e}_Q/3)\,\tilde{g}_P$	
	$10\,\times\, d^{(2)}_{1^+\to\gamma 1^-}$		
KD	$+2.799\,\tilde{g}_P + 6.305\,\tilde{e}_P$	$+2.153\,(e_C + e_Q/6)\,g_P$	$-(8.802\,g_T - 1.335\,\tilde{g}_V\,)\,e_A$
		$+0.0650\,(\tilde{e}_C - \tilde{e}_Q/6)\,\tilde{g}_P$	
ηD_s	$+8.852\,\tilde{g}_P$	$+0.489\,(e_C - e_Q/3)\,g_P$	$-(21.62\,g_T - 3.469\,\tilde{g}_V\,)\,e_A$
		$+0.080\,(\tilde{e}_C + \tilde{e}_Q/3)\,\tilde{g}_P$	

Table 4
Decay constants that are implied by (121). We use the coupling constants of Table 2 together with $f = 90$ MeV, $m_V = 776$ MeV, $M_V = 2000$ MeV.

	I)	II)	III)	IV)
$d_{0^+\to\gamma\,1^-}\,[\mathrm{GeV}^{-1}]$	+0.069	+0.035	−0.003	+0.073
$d_{1^+\to\gamma\,0^-}\,[\mathrm{GeV}^{-1}]$	−0.148	−0.130	−0.120	−0.139
$d_{1^+\to\gamma\,0^+}\,[\mathrm{GeV}^{-1}]$	0	+0.04	+0.04	+0.04
$d^{(1)}_{1^+\to\gamma\,1^-}$	−0.129	−0.097	−0.105	−0.090
$d^{(2)}_{1^+\to\gamma\,1^-}$	−0.282	−0.251	−0.251	−0.251

Table 5
Decay constants that are implied by $e_P = \tilde{e}_P = -1.50$ and $g_H = \tilde{g}_H = \hat{g}_H = 0$. The values used for e_A, $e_C = \tilde{e}_C$ and $e_Q = \tilde{e}_Q$ are discussed in the text. The four scenarios are characterized by I) $e_A = 0$ and $e_Q = e_C = 0$, II) $e_A = 0$, III) $e_A > 0$ and IV) $e_A < 0$. We use $f = 90$ MeV, $m_V = 776$ MeV, $M_V = 2000$ MeV.

obtain $d^{(2)}_{1^+\to\gamma\,1^-} \simeq 1.74$. We emphasize that the significant role played by the ηD_s is predicted by chiral coupled-channel dynamics. We note that our results differ significantly from the recent computations [16,17,18]. Identical results should not be expected since the latter works use a coupling vertex of the Goldstone bosons to the D mesons that is incompatible with the constraints set by chiral symmetry.

We conclude the decay parameters as they arise in the hadrogenesis conjecture must be subject to a cancellation mechanism between the terms $\sim g_P$ and $\sim e_P$. Indeed, a crucial role is played by the unknown parameters $e_P \simeq \tilde{e}_P$ introduced in (56), the contributions of which to the decay parameters are given by the second terms in Tables 3-4. The latter parameterize gauge invariant 4 point vertices, which describe the process where a D-meson emits a photon and a charged Goldstone boson simultaneously. Such interactions must be taken into account in effective field theories. The parameter e_P can be varied to achieve consistency with (125, 129). We discuss first the scenario for vanishing anomalous processes with $e_{C,Q} = 0 = \tilde{e}_{C,Q}$ and $e_A = 0$ in Tables3-4. While from the process $1^+ \to \gamma\,0^-$ we derive two possible ranges with $-0.57 < e_P < -0.49$ or $-1.50 < e_P < -1.42$ the process $1^+ \to \gamma\,1^-$ suggests $-1.54 < \tilde{e}_P < -0.84$ for $g_P = \tilde{g}_P = 0.57$. Finally from the scalar decay we deduce $-2.08 < e_P < -1.37$. It appears that with $e_P = \tilde{e}_P \simeq -1.50$ we arrive at an acceptable scenario, given the assumption of section 3 that the chiral power assigned to the e_P vertex is promoted from order Q^3_χ to order Q^2_χ. The various decay parameters implied are collected in the second column of Table 5. We observe that this scenario, though compatible with the constraints (125, 129), is characterized by decay parameters in striking disagreement with (132). The decay parameters $d_{0^+\to\gamma\,1^-}$ and $d_{1^+\to\gamma\,0^-}$ would have different magnitudes.

We turn to the anomalous contributions proportional to e_C, e_Q or $\tilde{e}_C, \tilde{e}_Q$, the contributions of which to the decay parameters are given by the second last terms in Tables 3-4. In the absence of the light vector mesons as explicit degrees of freedom, the latter parameters control the process $1^+ \to \gamma\, 0^+$. According to Appendix A, which analyzes the radiative decay pattern of the D meson ground states, we should use the universal value $e_C \simeq 0.13$, but discriminate the value for e_Q depending on the channel. While $e_Q \simeq 0.91$ is requested in the KD channel, the somewhat smaller value $e_Q \simeq 0.52$ is needed in the ηD_s channel. Given the coupling constants of Table 2 we predict $d_{1^+ \to \gamma\, 0^+} = 0.04$ GeV^{-1}, a result compatible with the bound of (129). Again we use $e_P = \tilde{e}_P = -1.50$, the implications of which are displayed in the third column of Table 5. It is interesting to observe that the decay parameter $d_{0^+ \to \gamma\, 1^-}$ is reduced significantly as compared to the one given in the 2nd column.

We continue with a discussion of effects induced by the presence of light vector mesons as explicit degrees of freedom. Even in the case where the latter do not couple directly to the scalar and axial-vector molecules with $g_H = \tilde{g}_H = \hat{g}_H = 0$, they have an influence on their radiative decays through the anomalous vertex introduced in (63) that is proportional to e_A. The various contributions to the decay parameters are included in Tables 3-4. According to Appendix A, which studies radiative decays of light vector mesons, the parameter e_A shows significant flavor SU(3) breaking. In order to take this into account we use

$$
\begin{aligned}
& e_A^{(\eta\, D_s)} = e_A^{(\phi \to \gamma\, \eta)} \simeq \pm 0.053\,, \\
& e_A^{(K\, D)} = 2\, e_A^{(K_0^* \to \gamma\, K_0)} - e_A^{(K_+^* \to \gamma\, K_+)} \simeq \pm 0.148\,,
\end{aligned}
\tag{133}
$$

in the ηD_s and KD channels respectively. In (133) we apply the values for e_A as given in (65), but recapitulate that the phase of the parameter e_A is not determined by experiment. Note that the $K_+\, D_0$ and $K_0\, D_+$ channels contribute to the decay amplitudes with opposite signs. The contribution of the $K_0\, D_0$ channel is twice as large as the one induced by $K_+\, D_0$. As is evident form Tables 3-4 the terms proportional to e_A probe the hadronic parameters g_T, g_E and $\tilde{g}_T, \tilde{g}_V$. The phases and the size of the latter parameters were estimated in section 3 from the assumption of universally coupled light vector mesons, together with the ansatz of a combined heavy-quark and flavor SU(4) symmetry. We take the values $\tilde{g}_V \simeq 0.71$, $g_E = g_P$ and $g_T = \tilde{g}_T = 0.5\, m_V\, g_P / M_V$ with $g_P = 0.57$ in the following discussion. Clearly, these assumptions suffer from some ambiguity.

Like before we use $e_P = \tilde{e}_P = -1.50$. The results for the decay parameter are shown in the fourth column of Table 5. While a for $e_A > 0$ a small and negative decay parameter is predicted for the scalar molecule, a positive and much larger value is predicted for the choice $e_A < 0$.

We continue with the discussion of the role played by light vector mesons. Non-zero coupling strength of the scalar and axial-vector molecules to the $K^* D^*$ and $\phi\, D_s^*$ are implied for finite values of the coupling constants g_H and $\hat{g}_H$. Heavy-quark symmetry anticipates $g_H = \hat{g}_H$ at leading order. Moreover it demands the relevance of the additional channels $K^* D$ and $\phi\, D_s$ for the axial-vector molecules. The strength of the latter is parameterized by $\tilde{g}_H$ with $\tilde{g}_H = g_H = \hat{g}_H$ in the heavy-quark mass limit. We consider first the decay process $1^+ \to \gamma\, 0^+$, for which we obtain the contributions

$$
\begin{aligned}
10\,\times\, d_{1^+\to\gamma\,0^+}[\mathrm{GeV}^{-1}] &= \hat{g}_H\, g_H \Big[1.187 - 0.377\Big] \\
&- \tilde{g}_H\, g_H \Big[5.363\,(e_C + e_Q/6)\, + 5.607\,(e_C - e_Q/3)\Big] \\
&+ e_A^{(KD)} \Big[0.429\, g_H + 9.244\,\tilde{g}_H\Big] + e_A^{(\eta D_s)} \Big[1.704\, g_H + 13.62\,\tilde{g}_H\Big]\,, \quad (134)
\end{aligned}
$$

where we discriminate the contributions from channels involving the D and D_s mesons. While the value 1.187 in (134) determines the contribution form the $\phi\, D_s^*$ channel, the value 0.377 corresponds to the $K^* D^*$ channel. Similarly the contribution proportional to $e_C + e_Q/6$ predicts the influence of the $K^* D \to K^* D^*$ transition and $e_C - e_Q/3$ the one of $\phi\, D_s \to \phi\, D_s^*$. Taking the empirical values for e_A, e_Q and e_C as discussed above we may use (134) to constrain the parameters $g_H, \tilde{g}_H$ and $\hat{g}_H$ using (129). Assuming the heavy-quark mass scenario with degenerate coupling constants $g_H = \tilde{g}_H = \hat{g}_H$ we derive the conditions

$$
-\,10.5 < g_H < 2.18\,, \qquad\qquad -2.18 < g_H < 10.5\,, \qquad (135)
$$

for the positive and negative e_A scenarios respectively. The result (135), unfortunately, does not provide a strong constraint on the size of the parameter g_H. If it were as large as $|g_H| \simeq 10$, we certainly had to question the coupled-channel computation of section 2, which assumed that the role played by the light vector mesons is of very minor importance for the formation of the $\mathrm{D}_{s0}^*(2317)$ and $\mathrm{D}_{s1}^*(2460)$ molecules.

In Tables 6-7 the various contributions from channels involving light vector mesons to the remaining decay parameters are detailed. Besides the hadronic parameters together with e_P and $\tilde{e}_P$ encountered already in Tables3-4 the further parameters $e_T, \tilde{e}_T, e_V, \tilde{e}_V$ and $e_E, \tilde{e}_E$ are involved. The latter are introduced in (67). They describe the gauge invariant process where a D-meson while attached to a light vector meson emits a photon. Clearly, it is not possible to determine the latter by the constraints (125, 129) only.

	$10\,\times\,d_{0^+\to\gamma\,1^-}\,[\mathrm{GeV}^{-1}]$	
K^*D^*	$+10.66\,g_T+2.976\,\tilde g_V$	$+1.758\,e_T+2.343\,\tilde e_E-2.765\,\tilde e_V$
$\sim g_H:$	$-27.78\,e_A\,\tilde g_P$	$+(e_C+e_Q/6)\,(3.408\,\tilde g_T+0.611\,g_E)$
		$+(\tilde e_C-\tilde e_Q/6)\,(2.283\,\tilde g_T+0.653\,\tilde g_V)$
$\phi\;D_s^*$	$+29.69\,g_T+8.369\,\tilde g_V$	$+3.091\,\tilde e_E$
$\sim g_H:$	$-60.24\,e_A\,\tilde g_P$	$+(e_C-e_Q/3)\,(1.383\,\tilde g_T+0.322\,g_E)$
		$+(\tilde e_C+\tilde e_Q/3)\,(1.278\,\tilde g_T+0.592\,\tilde g_V)$
	$10\,\times\,d_{1^+\to\gamma\,0^-}\,[\mathrm{GeV}^{-1}]$	
K^*D	$+0.852\,g_V$	$+6.902\,e_E-6.127\,e_V$
$\sim \tilde g_H:$		$+(e_C+e_Q/6)\,(6.777\,\tilde g_T+1.337\,g_E)$
$\phi\;D_s$	$-2.902\,g_V$	$+6.462\,e_E$
$\sim \tilde g_H:$		$+(e_C-e_Q/3)\,(3.548\,\tilde g_T+0.847\,g_E)$
K^*D^*	$+9.798\,\tilde g_T+1.603\,g_E$	$+3.338\,\tilde e_T$
$\sim \hat g_H:$	$-9.070\,e_A\,g_P$	$+1.568\,(e_C+e_Q/6)\,g_V$
		$-(\tilde e_C-\tilde e_Q/6)\,(2.705\,\tilde g_T+0.547\,g_E)$
$\phi\;D_s^*$	$+16.92\,\tilde g_T+3.807\,g_E$	
$\sim \hat g_H:$	$-15.44\,e_A\,g_P$	$+1.072\,(e_C-e_Q/3)\,g_V$
		$-(\tilde e_C+\tilde e_Q/3)\,(1.478\,\tilde g_T+0.359\,g_E)$

Table 6
Decay constants that are implied by (167, 106, 178). The various contributions have to be multiplied by either $g_H, \tilde g_H$ or $\hat g_H$ as indicated in the first column. We use $f = 90$ MeV, $m_V = 776$ MeV, $M_V = 2000$ MeV.

To achieve a qualitative understanding we proceed with the assumption that all parameters are correlated as dictated by the heavy-quark mass limit, i.e. $e_P = \tilde e_P, e_V = \tilde e_V$ etc. In this case there remain five unknown parameters g_H, e_P, e_V, e_T and e_E. Given the constraints (125, 129) only, one may expect to learn little. However, this is not quite so. For given values of g_H and e_P one can always adjust the values for e_V, e_T and e_E such as to reproduce given values of the decay parameters $d_{0^+\to\gamma\,1^-}$ as well as the partial decay widths of the process $1^+\to\gamma\,0^-$, $1^+\to\gamma\,1^-$ as given in (125, 129). Like before the values $\tilde g_V \simeq 0.71$, $g_E = g_P$ and $g_T = \tilde g_T = 0.5\,m_V\,g_P/M_V$ with $g_P = 0.57$ are assumed in the following.

We point out, that the requirement the parameters e_V, e_T and e_E to be real, as implied by charge symmetry conjugation, defines stringent conditions on the allowed ranges of g_H and e_P. Note that the latter are determined by quadratic equations, the solution of which involves a square root. Only if the argument

	$d^{(1)}_{1^+\to\gamma\,1^-}$	
$K^* D$	$+1.024\,\tilde g_T + 0.152\,g_E$	$+0.618\,\tilde e_T$
$\sim \tilde g_H :$	$+2.049\,e_A\,g_P$	$-(e_C + e_Q/6)\,(2.930\,g_T + 0.483\,\tilde g_V)$
$\phi\, D_s$	$+2.674\,\tilde g_T + 0.597\,g_E$	
$\sim \tilde g_H :$	$+5.714\,e_A\,g_P$	$-(e_C - e_Q/3)\,(1.505\,g_T + 0.147\,\tilde g_V)$
$K^* D^*$	$+1.065\,g_T + 0.569\,\tilde g_V$	$+0.528\,e_T - 0.979\,\tilde e_V + 1.069\,\tilde e_E$
$\sim \hat g_H :$	$-5.914\,e_A\,\tilde g_P$	$-(e_C + e_Q/6)\,(0.209\,\tilde g_T + 0.037\,g_E)$
		$+(\tilde e_C - \tilde e_Q/6)\,(0.439\,g_T + 0.002\,\tilde g_V)$
$\phi\, D_s^*$	$+2.597\,g_T + 0.746\,\tilde g_V$	$+1.328\,\tilde e_E$
$\sim \hat g_H :$	$-10.99\,e_A\,\tilde g_P$	$-(e_C - e_Q/3)\,(0.047\,\tilde g_T + 0.011\,g_E)$
		$+(\tilde e_C + \tilde e_Q/3)\,(0.260\,g_T + 0.058\,\tilde g_V)$
	$d^{(2)}_{1^+\to\gamma\,1^-}$	
$K^* D$	$+0.683\,\tilde g_T + 0.091\,g_E$	$+0.521\,\tilde e_T$
$\sim \tilde g_H :$	$-6.998\,e_A\,g_P$	$+(e_C + e_Q/6)\,(1.567\,g_T - 0.202\,\tilde g_V)$
$\phi\, D_s$	$+2.276\,\tilde g_T + 0.504\,g_E$	
$\sim \tilde g_H :$	$-13.99\,e_A\,g_P$	$+(e_C - e_Q/3)\,(0.793\,g_T - 0.190\,\tilde g_V)$
$K^* D^*$	$+1.512\,g_T + 0.446\,\tilde g_V$	$+0.669\,e_T - 0.907\,\tilde e_V + 0.881\,\tilde e_E$
$\sim \hat g_H :$	$+1.435\,e_A\,\tilde g_P$	$+(e_C + e_Q/6)\,(0.247\,\tilde g_T + 0.044\,g_E)$
		$-(\tilde e_C - \tilde e_Q/6)\,(0.886\,g_T - 0.037\,\tilde g_V)$
$\phi\, D_s^*$	$+2.820\,g_T + 0.587\,\tilde g_V$	$+1.041\,\tilde e_E$
$\sim \hat g_H :$	$+2.858\,e_A\,\tilde g_P$	$+(e_C - e_Q/3)\,(0.157\,\tilde g_T + 0.037\,g_E)$
		$-(\tilde e_C + \tilde e_Q/3)\,(0.443\,g_T - 0.014\,\tilde g_V)$

Table 7
Decay constants that are implied by (123, 191). The various contributions have to be multiplied by either $\tilde g_H$ or $\hat g_H$ as indicated in the first column. We use $f = 90$ MeV, $m_V = 776$ MeV, $M_V = 2000$ MeV.

of that square root is positive the determined parameters e_V, e_T and e_E will be real numbers. We arrive at the condition that for given value of e_P, the parameter g_H has to be confined in a small interval

$$g_H^{crit,-} < g_H < g_H^{crit,+} \,. \tag{136}$$

We discriminate the two scenarios with positive or negative values of the parameter e_A, for which we derive

	I)	II)
$d_{0^+\to\gamma\,1^-}\,[\mathrm{GeV}^{-1}]$	$+0.057$	$+0.104$
$d_{1^+\to\gamma\,0^-}\,[\mathrm{GeV}^{-1}]$	-0.139	-0.140
$d_{1^+\to\gamma\,0^+}\,[\mathrm{GeV}^{-1}]$	-0.043	$+0.091$
$d^{(1)}_{1^+\to\gamma\,1^-}$	$+0.303$	$+0.278$
$d^{(2)}_{1^+\to\gamma\,1^-}$	$+0.202$	$+0.071$

Table 8
Decay constants that are implied by $e_P = e_V = e_T = e_E = 0$ and $\tilde{e}_P = \tilde{e}_V = \tilde{e}_T = \tilde{e}_E = 0$. In set I) we use $e_A > 0$ with $g_H = \tilde{g}_H = \hat{g}_H = -0.46$. Set II) assumes $e_A < 0$ and $g_H = \tilde{g}_H = \hat{g}_H = -0.25$. We use $f = 90$ MeV, $m_V = 776$ MeV, $M_V = 2000$ MeV.

$$
\begin{aligned}
g_H^{crit,\pm} &\simeq -0.390 + 1.332\, d_{0^+\to\gamma\,1^-}[\mathrm{GeV}^{-1}] - 0.242\, e_P \pm 0.131\,,\\
g_H^{crit,\pm} &\simeq -0.285 + 0.751\, d_{0^+\to\gamma\,1^-}[\mathrm{GeV}^{-1}] - 0.137\, e_P \pm 0.074\,,
\end{aligned}
\qquad (137)
$$

in respective order. We checked the stability of this result against reasonable variations of the parameters g_V, g_T and g_E. In (137) we allow the decay parameter $d_{0^+\to\gamma\,1^-}$ to take any value. The result (137) is amazing since for any reasonable range of e_P it requires the coupling constant g_H to be quite small, typically $|g_H| < 0.5$. This justifies in retrospect the coupled-channel computation of section 2, which assumed that the light vector mesons are not relevant for the formation of the scalar and axial-vector D_s molecules.

Nevertheless, the light vector mesons may change the radiative decay parameters significantly. This is not surprising since the latter are subject to subtle cancellations. As illustrated in Table 8 we arrive at a remarkably consistent picture. We obtain values for all decay parameters that are compatible with the empirical constraints using vanishing values for all gauge-invariant counter terms $e_P = e_V = e_T = e_E = 0$ and $\tilde{e}_P = \tilde{e}_V = \tilde{e}_T = \tilde{e}_E = 0$. In such a scenario there is one free parameter only $g_H = \tilde{g}_H = \hat{g}_H$ which can be dialed to recover all empirical constraint. The results for the positive and negative e_A scenarios are collected in Table 8. It is interesting to observe that we predict a negative decay constant for the $1^+ \to \gamma\, 0^-$ decay contradicting the naive expectation of heavy-quark symmetry. We emphasize that this follows even though performing a formal expansion of the full expressions for the decay parameter in the inverse charm-quark mass leads results that are compatible with the expectation from the heavy-quark symmetry. Such an expansion assumes for instance $m_\phi \ll M_c$, which is not realized too well in nature. Our results provide a physical explanation, why we had to promote the counter terms proportional to e_P and $\tilde{e}_P$ to carry chiral order Q_χ^2 rather than the expected power Q_χ^3. Once the light vector mesons are accepted as important physical degrees of freedoms, the naturalness assumption for the residual size of e_P and $\tilde{e}_P$ appears to be justified.

We emphasize that at present it appears not possible to predict precise values for the decay parameters. The results of Table 8 should be viewed as possible and natural scenarios. Precise unquenched lattice QCD simulations for the hadronic coupling constants of the Goldstone bosons and light vector mesons to the D mesons would be highly desirable.

7 Summary

Based on the chiral Lagrangian properties of scalar and axial-vector meson molecules with open-charm content were studied. Chiral correction terms were incorporated systematically in the coupled-channel dynamics, where we relied on constraints from large-N_c QCD and the heavy-quark symmetry. We focused on the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states, for which their isospin violating hadronic and electromagnetic decay widths were computed. In order to establish manifestly gauge invariant results for the electromagnetic decay parameters the spin-one particles were represented in terms of anti-symmetric tensor fields, rather than the more conventional vector fields. The role of explicit light vector mesons in the radiative decays was investigated.

The main findings of this work are

- The hadronic isospin-violating decay widths of the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ states are predicted to be 140 keV. Chiral correction terms are important contributions to the decay widths.
- The invariant $\eta\, D^*$ invariant mass distribution shows a signal of a member of an exotic axial-vector sextet at mass 2568 MeV and width 18 MeV. While that state decouples from the πD^* spectrum its heavy-quark partner defines a narrow dip at mass 2410 MeV and width 2 MeV in the $\pi\, D$ mass distribution.
- The radiative decay parameters of the $D^*_{s0}(2317) \to \gamma\, D^*_s$ and $D^*_{s1}(2460) \to \gamma\, D_s, \gamma\, D^*_s$ and $D^*_{s1}(2460) \to \gamma\, D^*_{s0}(2317)$ were computed in the hadrogenesis conjecture. We find that the ηD_s and $\eta\, D^*_s$ channels contribute significantly. The results are compatible with all empirical constraints once one gauge-invariant contact term is considered to be more important than expected from a naive naturalness assumption. The decay parameters are subject to a cancellation mechanism, which makes a prediction of their precise values impossible at the moment.
- Upon considering light vector mesons as explicit degrees of freedom, radiative decay parameters that are compatible with the empirical constraints, can be obtained without envoking subleading contact terms. We predict that the $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$ have small coupling strength to the $K^* D^*$ and $\phi\, D^*_s$ channels. Nevertheless, such channels play a decisive role in the radiative decay processes.

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Appendix A

The modulus of the coupling constant g_P can be determined from the hadronic D meson decay processes with

$$\Gamma_{D^*_+ \to D_0\,\pi_+} = (0.677 \pm 0.005) \times (96 \pm 22)\,\mathrm{keV}\,,$$
$$\Gamma_{D^*_+ \to D_+\,\pi_0} = (0.307 \pm 0.005) \times (96 \pm 22)\,\mathrm{keV}\,, \tag{138}$$

where we take the latest values from the Particle Data Group [44]. We derive

$$\begin{aligned}
\Gamma_{D^*_+ \to D_+\,\pi_0} &= \frac{|g_P|^2}{24\,\pi}\,\frac{q^3_{cm}}{f^2}\,, \qquad && q_{cm} = 38.30\,\mathrm{MeV}\,,\\
\Gamma_{D^*_+ \to D_0\,\pi_+} &= \frac{|g_P|^2}{12\,\pi}\,\frac{q^3_{cm}}{f^2}\,, \qquad && q_{cm} = 39.60\,\mathrm{MeV}\,,\\
\to \Gamma_{D^*_0 \to D_0\,\pi^0} &= \frac{|g_P|^2}{24\,\pi}\,\frac{q^3_{cm}}{f^2} \simeq 42\,\mathrm{keV}\,, \qquad && q_{cm} = 43.12\,\mathrm{MeV}\,,\\
\Gamma_{D^*_s \to D_s\,\pi^0} &= \frac{|\epsilon\,g_P|^2}{18\,\pi}\,\frac{q^3_{cm}}{f^2} \simeq \epsilon^2\,77\,\mathrm{keV}\,, \qquad && q_{cm} = 47.87\,\mathrm{MeV}\,,
\end{aligned} \tag{139}$$

with the $\pi_0 - \eta$ mixing angle $\epsilon \simeq 0.01$ introduced in (11). It follows

$$|g_P| = 0.57 \pm 0.07\,, \tag{140}$$

where we use $f = 90$ MeV.

The modulus of the coupling constant e_Q and e_C can be determined from the $D^*_- \to D_-\,\gamma$, $D^*_0 \to D_0\,\gamma$ and $D^*_s \to D_s\,\gamma$ decays with

$$\begin{aligned}
\Gamma_{D^*_+ \to D_+\,\gamma} &= (96 \pm 22)\,\mathrm{keV} \times (0.016 \pm 0.004)\\
&= (1.5 \pm 0.8)\,\mathrm{keV}\,,\\
\Gamma_{D^*_0 \to D_0\,\gamma} &= (\Gamma_{D^*_0 \to D_0\,\gamma} + 42\,\mathrm{keV}) \times (0.381 \pm 0.029)\\
&= (26.0 \pm 3)\,\mathrm{keV}\,,\\
\Gamma_{D^*_s \to D_s\,\gamma} &= (\Gamma_{D^*_s \to D_s\,\gamma} + \epsilon^2\,77\,\mathrm{keV}) \times (0.942 \pm 0.007)\\
&= \epsilon^2\,(1328 \pm 180)\,\mathrm{keV}\,,
\end{aligned} \tag{141}$$

where we recalled the latest values from [44]. The total decay width of the D^*_0 and D^*_s are estimated from the known branching of 61.9±2.9% and 5.8±0.7% into the $\pi_0 D_0$ and $\pi_0 D_s$ channels respectively. Note that the sum of the branching fraction of radiative decay and the π_0 decay add up to 61.9+38.1=100 and 94.2+5.8=100 per cent in both cases. In (142) we make use of the result (139).

We derive

$$
\begin{aligned}
\Gamma_{D_+^* \to D_+ \gamma} &= \frac{M_{D_+^*}^2}{4\,\pi}\,\frac{(e_Q - 3\,e_C)^2}{9\,M_V^4}\,q_{cm}^3\,, \qquad q_{cm} = 135.78\,\text{MeV}\,,\\
\Gamma_{D_0^* \to D_0 \gamma} &= \frac{M_{D_0^*}^2}{4\,\pi}\,\frac{(2\,e_Q + 3\,e_C)^2}{9\,M_V^4}\,q_{cm}^3\,, \qquad q_{cm} = 137.16\,\text{MeV}\,,\\
\Gamma_{D_s^* \to D_s \gamma} &= \frac{M_{D_s^*}^2}{4\,\pi}\,\frac{(e_Q - 3\,e_C)^2}{9\,M_V^4}\,q_{cm}^3\,, \qquad q_{cm} = 138.91\,\text{MeV}\,. \qquad (142)
\end{aligned}
$$

The $D_+^* \to D_+\gamma$ and $D_0^* \to D_0\gamma$ decay imply

$$
\begin{aligned}
&|e_Q - 3\,e_C| \simeq 0.537^{+0.128}_{-0.170}\,, \qquad |2\,e_Q + 3\,e_C| \simeq 2.201^{+0.124}_{-0.131}\,,\\
\to \qquad &e_Q = 0.91 \pm 0.10\,, \qquad e_C = 0.13 \pm 0.05\,, \qquad (143)
\end{aligned}
$$

where we use $M_V = 2000$ MeV. Note that a value of $\epsilon \simeq 0.01$ would then overestimate the $D_s^* \to D_s\gamma$ decay rate by an order of magnitude. The central values of (143) would imply a width of 1.9 keV, which should be compared to $\epsilon^2 \times 1328 \pm 180$ keV $\simeq 0.133 \pm 0.018$ keV. One may take this is a signal of SU(3) breaking effects in e_Q or the need of an increased value of ϵ. At $\epsilon = 0.01$ and $e_C = 0.13$ an effective $e_Q \simeq 0.52$ would reproduce the empirical width.

The modulus of the coupling constant, e_A, is determined from the radiative decay of the light vector mesons. For this work relevant are the three processes $K_0^* \to K_0\gamma, K_\pm^* \to K_\pm\gamma$ and $\phi \to \eta\gamma$ only. The decay widths are readily derived from the interaction (63)

$$
\begin{aligned}
\Gamma_{K_\pm^* \to K_\pm \gamma} &= \frac{|e_A|^2}{24 \times 144\,\pi}\,\frac{m_{K_\pm^*}^5}{m_V^2\,f^2}\left(1 - \frac{m_{K_\pm}^2}{m_{K_\pm^*}^2}\right)^3\,,\\
\Gamma_{K_0^* \to K_0 \gamma} &= \frac{|e_A|^2}{24 \times \;\; 36\,\pi}\,\frac{m_{K_0^*}^5}{m_V^2\,f^2}\left(1 - \frac{m_{K_0}^2}{m_{K_0^*}^2}\right)^3\,,\\
\Gamma_{\phi \to \eta\,\gamma} &= \frac{|e_A|^2}{24 \times \;\; 54\,\pi}\,\frac{m_{\phi}^5}{m_V^2\,f^2}\left(1 - \frac{m_{\eta}^2}{m_{\phi}^2}\right)^3\,, \qquad (144)
\end{aligned}
$$

where we use $f = 90$ MeV and $m_V = 776$ MeV. The empirical decay widths of the Particle Data Group [44] imply conflicting values for e_A. We obtain

$$
\begin{aligned}
\Gamma_{K_0^* \to K_0 \gamma} &= (116 \pm 10)\,\text{keV}\,, \qquad \to \qquad |e_A| = 0.119 \pm 0.006\,,\\
\Gamma_{K_\pm^* \to K_\pm \gamma} &= (50 \pm 5)\,\text{keV}\,, \qquad \to \qquad |e_A| = 0.090 \pm 0.004\,,
\end{aligned}
$$

$$\Gamma_{\phi \to \eta \gamma} \; = \; (55.38 \, \pm \, 1.68) \, \mathrm{keV} \,, \quad \to \quad |e_A| = 0.053 \pm 0.001 \,. \tag{145}$$

Appendix B

In order to work out the implications of the heavy-quark symmetry of QCD it is useful to introduce auxiliary fields, $P_\pm(x)$ and $P^\mu_\pm(x)$. We write

$$
\begin{aligned}
D(x) \;\; &= e^{-i\,(v\cdot x)\,M_c}\,P_+(x) + e^{+i\,(v\cdot x)\,M_c}\,P_-(x)\,,\\
D^{\mu\nu}(x) &= i\,e^{-i\,(v\cdot x)\,M_c}\Big\{v^\mu\,P^\nu_+(x) - v^\nu\,P^\mu_+(x) + \frac{i}{M_c}\Big(\partial^\mu P^\nu_+ - \partial^\nu P^\mu_+\Big)\Big\}\\
&\quad + i\,e^{+i\,(v\cdot x)\,M_c}\Big\{v^\mu\,P^\nu_-(x) - v^\nu\,P^\mu_-(x) - \frac{i}{M_c}\Big(\partial^\mu P^\nu_- - \partial^\nu P^\mu_-\Big)\Big\}\,,
\end{aligned}
\tag{146}
$$

with a 4-velocity normalized by $v^2 = 1$. The mass parameter, M_c, is an averaged value of the pseudo-scalar and vector charmed-meson ground-state masses. This implies that the fields $P_\pm(x)$ and $P^\mu_\pm(x)$ are varying slowly in space and time. As a consequence time and spatial derivatives of the fields $\partial_\alpha P_\pm$ are small compared to $M_c\,v_\alpha\,P$. In the limit of large mass $M_c \to \infty$ the former terms can be neglected. Note that the fields $P_\pm$ and $P^\mu_\pm$ annihilate quanta with charm content ±1.

We rewrite the interaction terms introduced in (38, 56, 66) in terms of the auxiliary fields, where we focus on the 'plus' components. It holds

$$
\begin{aligned}
\mathcal{L} = &\; i\,\frac{g_P\,M_c}{f}\Big\{P_\mu\,(\partial^\mu\Phi)\,\bar P - P\,(\partial^\mu\Phi)\,\bar P_\mu\Big\} + i\,\frac{\tilde g_P\,M_c}{f}\,v_\mu\,\epsilon^{\mu\nu\alpha\beta}\,P_\nu\,(\partial_\alpha\Phi)\,\bar P_\beta\\
&- \frac{g_V\,M_c}{f}\,v_\mu\,P\,(\partial_\alpha V^{\alpha\mu})\,\bar P + \frac{\tilde g_V\,M_c}{f}\,v_\mu\,P_\nu\,(\partial_\alpha V^{\alpha\mu})\,\bar P^\nu\\
&+ i\,\frac{\tilde g_T\,M_c^2}{4\,f}\,v_\alpha\,\epsilon^{\mu\nu\alpha\beta}\Big\{P_\beta\,V_{\mu\nu}\,\bar P - P\,V_{\mu\nu}\,\bar P_\beta\Big\} + i\,\frac{g_T\,M_c^2}{2\,f}\,P_\mu\,V^{\mu\nu}\,\bar P_\nu\\
&+ \frac{e_P\,M_c}{f}\,F^{\mu\nu}\Big\{P_\mu\,[(\partial_\nu\Phi),\,Q]\,\bar P - P\,[(\partial_\nu\Phi),\,Q]\,\bar P_\mu\Big\}\\
&+ \frac{\tilde e_P\,M_c}{f}\,F_{\mu\nu}\,v_\tau\,\epsilon^{\tau\alpha\mu\beta}\,P_\alpha\,[(\partial_\nu\Phi),\,Q]\,\bar P_\beta\\
&+ \cdots\,,
\end{aligned}
\tag{147}
$$

where the ellipses stand for additional terms involving derivatives of the soft fields $P = P_+$ and $P^\mu = P^\mu_+$. The negative components P_- and P^μ_- are also omitted in (147). In the course of deriving (147) we made use of the equation of motion for the auxiliary field $P^\mu_\pm(x)$. It holds

$$
\begin{aligned}
&\partial^\mu\partial_\alpha D^{\alpha\nu} - \partial^\nu\partial_\alpha D^{\alpha\mu} + M_c^2\,D^{\mu\nu} = 0 \qquad \leftrightarrow\\
&\{(\partial\cdot\partial) \mp 2\,M_c\,i\,(v\cdot\partial)\}\,P^\mu_\pm = 0 \quad \& \quad \mp i\,M_c\,v_\mu\,P^\mu_\pm + \partial_\mu\,P^\mu_\pm = 0\,,
\end{aligned}
\tag{148}
$$

which teaches us that any term $v_\mu P^\mu$ is suppressed in the heavy-quark mass limit. From (147) we conclude that the coupling constants $g_{P,V}$ and $\tilde g_{P,V}$ must approach a finite value in the limit of infinite charm-quark mass. This is a consequence of the linear quark-mass dependence of the QCD action. The same argument requires the coupling constants g_T and $\tilde g_T$ to scale with $1/M_c$. The fact that there is no term proportional to g_E displayed in (147) signals the fact that the corresponding interaction term is at least subleading in the heavy-quark mass expansion. Note that this observation is compatible with a possible finite and non-zero value of the asymptotic value of g_E approached in the heavy-quark mass limit.

In the limit of infinite quark mass, the fields $P_\pm$ and $P_\pm^\mu$ may be combined into an appropriate multiplet field. This reflects the fact that in this limit the 1^- and 0^- fields are related by a spin flip of the charm quark. The later does not cost any energy. Therefore the properties of pseudo-scalar and vector states should be closely related. We follow here the formalism developed in [20,21,22,23,24] and introduce the multiplet field, H, with respect to the P_+ and P_+^μ fields, as follows

$$\begin{aligned}
&H = \frac{1}{2}\left(1+\not v\right)\left(\gamma_\mu\, P_+^\mu + i\,\gamma_5\, P_+\right)\\
&\bar H = \gamma_0\, H^\dagger\,\gamma_0 = \left(P^\dagger_{+,\mu}\,\gamma^\mu + P^\dagger_+\, i\,\gamma_5\right)\frac{1}{2}\left(1+\not v\right),\\
&P_+^\mu\, v_\mu = 0\,, \qquad v^2 = 1\,,
\end{aligned} \tag{149}$$

in terms of which the interaction should be composed. For latter convenience we introduced a resonance doublet-field S in (149). According to [**?**] a transformation under the heavy-quark spin symmetry group $SU_v(2)$, the elements of which being characterized by the 4-vector θ^α with $\theta\cdot v = 0$, the field transforms as follows:

$$\begin{aligned}
&H \to e^{-i\,S_\alpha\,\theta^\alpha}\, H\,, \qquad \bar H \to \gamma_0\,(e^{-i\,S_\alpha\,\theta^\alpha}\, H)^\dagger\,\gamma_0 = \bar H\, e^{+i\,S_\alpha\,\theta^\alpha}\,,\\
&S^\alpha = \frac{1}{2}\,\gamma_5\,[\not v,\gamma^\alpha]\,, \qquad S^\dagger_\alpha\,\gamma_0 = \gamma_0\, S_\alpha\,, \qquad [\not v, S_\alpha]_- = 0\,.
\end{aligned} \tag{150}$$

Under a Lorentz transformation, characterized by the antisymmetric tensor $\omega_{\mu\nu}$, the spinor part of the fields transform as

$$\begin{aligned}
&H \to e^{i\,S_{\mu\nu}\,\omega^{\mu\nu}}\, H\, e^{-i\,S_{\mu\nu}\,\omega^{\mu\nu}}\,, \qquad \bar H \to e^{i\,S_{\mu\nu}\,\omega^{\mu\nu}}\,\bar H\, e^{-i\,S_{\mu\nu}\,\omega^{\mu\nu}}\,,\\
&S_{\mu\nu} = \frac{i}{4}\,[\gamma_\mu,\gamma_\nu]\,.
\end{aligned} \tag{151}$$

It follows that only combinations of the form where Dirac matrices are right of the field H or left to the field $\bar H$ are invariant under the spin group $SU_v(2)$.

It is straight forward to construct those terms that can be matched to the structures detailed in (147). We introduce

$$\begin{aligned}\mathcal{L} = &- \frac{f_P}{2}\,\mathrm{tr}\,\Big(H\,(\partial^\mu \Phi)\,\gamma_5\,\gamma_\mu\,\bar H\Big) + i\,\frac{\bar f_P}{2}\,F^{\mu\nu}\,\mathrm{tr}\,\Big(H\,\gamma_5\,\gamma_\mu\,\Big[(\partial_\nu\,\Phi),\,Q\Big]\,\bar H\Big)\\ &- \frac{f_V}{2}\,\mathrm{tr}\,\Big(H\,(\partial_\alpha V^{\alpha\mu})\,\gamma_\mu\,\bar H\Big) + i\,\frac{\bar f_V}{2}\,F^{\mu\nu}\,\mathrm{tr}\,\Big(H\,\gamma_\mu\,\Big[(\partial^\alpha V_{\alpha\nu}),\,Q\Big]\,\bar H\Big)\\ &+ \frac{f_T}{8}\,\mathrm{tr}\,\Big(H\,V^{\mu\nu}\,\sigma_{\mu\nu}\,\bar H\Big) - i\,\frac{\bar f_T}{8}\,\mathrm{tr}\,F^\mu_{\;\;\nu}\,\Big(H\,[V^{\nu\alpha},\,Q]\,\sigma_{\mu\alpha}\,\bar H\Big)\,,\\ &+ \frac{\bar f_E}{4}\,F^{\mu\nu}\,\mathrm{tr}\,\Big(H\,\{V^{\mu\nu},\,Q\}\,\bar H\Big) \end{aligned} \tag{152}$$

where we note that the field H is a three-dimensional row in flavor space, each of its components consisting of a 4 dimensional Dirac matrix. Note

$$\mathrm{tr}\,\gamma_5\,\gamma_\mu\,\gamma_\nu\,\gamma_\alpha\,\gamma_\beta = -4\,i\,\epsilon_{\mu\nu\alpha\beta}\,, \tag{153}$$

in our convention. Matching the expressions (147,159) we obtain

$$\begin{aligned} &\frac{g_P\,M_c}{f} = \frac{\tilde g_P\,M_c}{f} = f_P\,, &&\frac{e_P\,M_c}{f\,m_V^2} = \frac{\tilde e_P\,M_c}{f\,m_V^2} = \bar f_P\,,\\ &\frac{g_V\,M_c}{f} = \frac{\tilde g_V\,M_c}{f} = f_V\,, &&\frac{e_V\,M_c}{f\,m_V^2} = \frac{\tilde e_V\,M_c}{f\,m_V^2} = \bar f_V\,,\\ &\frac{g_T\,M_c^2}{f} = \frac{\tilde g_T\,M_c^2}{f} = f_T\,, &&\frac{e_T\,M_c^2}{f} = \frac{\tilde e_T\,M_c^2}{f} = \bar f_T\,,\\ &\frac{e_E\,M_c^2}{f} = \frac{\tilde e_E\,M_c^2}{f} = \bar f_E\,. \end{aligned} \tag{154}$$

We turn to the terms (58, 60). Applying the ansatz (146) we derive

$$\begin{aligned}\mathcal{L}_{e.m.} = &\frac{i}{2}\,F^{\mu\nu}\,v^\alpha\,\epsilon_{\mu\nu\alpha\beta}\,\Big\{P^\beta\,\Big(e_C + e_Q\,Q\Big)\,\bar P - P\,\Big(e_C + e_Q\,Q\Big)\,\bar P^\beta\Big\}\\ &+ i\,F^{\mu\nu}\,P_\mu\,\Big(-\tilde e_C + \tilde e_Q\,Q\Big)\,\bar P_\nu + \cdots\,. \end{aligned} \tag{155}$$

where we identify $M_c = M_V$ for convenience. Like in (147), we drop in (155) additional irrelevant terms. The interaction terms displayed in (155) are readily reproduced using the heavy-quark multiplet field H. We follow [39,40] and write

$$\mathcal{L}_{e.m.} = +\frac{e_Q}{4}\,F_{\mu\nu}\,\mathrm{tr}\,H\,Q\,\sigma_{\mu\nu}\,\bar H + \frac{e_C}{4}\,F_{\mu\nu}\,\mathrm{tr}\,\sigma_{\mu\nu}\,H\,\bar H\,, \tag{156}$$

where the term proportional to e_Q respects the heavy-quark symmetry, but the term e_C breaks it. Thus we expect $|e_Q| \gg |e_C|$, since the parameter e_C should vanish in the heavy-quark limit. The requirement that (155) and (156) agree yields the desired relations

$$\tilde{e}_C = e_C \,, \qquad\qquad \tilde{e}_Q = e_Q \,. \tag{157}$$

Note that the empirical values (143) confirm the expectation $|e_Q| \gg |e_C|$.

Finally we provide an analysis of the effective resonance interaction terms (75, 81). Assuming a decomposition for the resonance fields R and $R_{\mu\nu}$ analogous to (146) we introduce a resonance multiplet field

$$\begin{aligned} S &= \frac{1}{2}\left(1+\not{v}\right)\left(\gamma_5\,\gamma_\mu\,R_+^\mu + R_+\right), \\ \bar{S} &= \gamma_0\,S^\dagger\,\gamma_0 = \left(R^\dagger_{+,\mu}\,\gamma_5\,\gamma_\mu + R^\dagger_+\right)\frac{1}{2}\left(1+\not{v}\right), \qquad v^\mu\,R_+^\mu = 0\,, \end{aligned} \tag{158}$$

where the field S transforms like the field H under a spin rotation (see 150). At leading order in a heavy-quark mass expansion the interaction terms (75, 81) are described by

$$\begin{aligned} \mathcal{L} &= \frac{f_H}{4}\,\mathrm{tr}\left(S\,\gamma_\mu\,(\partial_\tau V^{\tau\mu})\,\bar{H}\; + H\,\gamma_\mu\,(\partial_\tau V^{\tau\mu})\,\bar{S}\right) \\ &\quad + \frac{f_R}{4}\,\mathrm{tr}\left(S\,\gamma_5\,\gamma_\mu\,(\partial^\mu\Phi)\,\bar{H} + H\,\gamma_5\,\gamma_\mu\,(\partial^\mu\Phi)\,\bar{S}\right), \end{aligned} \tag{159}$$

which implies the identifications

$$\frac{g_R\,M_c}{f} = \frac{\tilde{g}_R\,M_c}{f} = f_R\,, \qquad \frac{g_H\,M_c^2}{f} = \frac{\tilde{g}_H\,M_c^2}{f} = \frac{\hat{g}_H\,M_c^2}{f} = f_H\,. \tag{160}$$

Appendix C

All tensor integrals will be decomposed into scalar objects of the form

$$
\begin{aligned}
I_{ab} &= -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+p)\,, \qquad \bar{I}_{ab} = -i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+\bar{p})\,, \\
J_{abc} &= +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+q)\, S_c(l+p)\,, \\
\bar{J}_{abc} &= +i \int \frac{d^4 l}{(2\pi)^4}\, S_a(l)\, S_b(l+\bar{p})\, S_c(l+p)\,. \qquad (161)
\end{aligned}
$$

Note that we introduce the two integrals J_{abc} and $\bar{J}_{abc}$ even though they are related to each other by $J_{abc} = \bar{J}_{cba}$. We use this redundancy for a consistency check of the numerical simulation. The integrals I_{ab} and $\bar{I}_{ab}$ are ultraviolet diverging. In fact we encountered those integrals before in (16). We recall the explicit representation

$$
\begin{aligned}
I_{ab}(s) &= \frac{1}{16\,\pi^2} \left(\frac{p_{ab}}{\sqrt{s}} \left(\ln\left(1 - \frac{s - 2\,p_{ab}\sqrt{s}}{m_b^2 + m_a^2}\right) - \ln\left(1 - \frac{s + 2\,p_{ab}\sqrt{s}}{m_b^2 + m_a^2}\right)\right) \right. \\
&\quad \left. + \left(\frac{1}{2}\,\frac{m_b^2 + m_a^2}{m_b^2 - m_a^2} - \frac{m_b^2 - m_a^2}{2\,s} \right) \ln\left(\frac{m_b^2}{m_a^2}\right) + 1 \right) + I_{ab}(0)\,, \\
p_{ab}^2 &= \frac{s - (m_a + m_b)^2}{4\,s} \left(s - (m_a - m_b)^2 \right)\,, \qquad (162)
\end{aligned}
$$

where the logarithmic divergence sits in the subtraction term $I_{ab}(0)$. The difference $I_{ab}(s) - I_{ab}(0)$, as given in (162) is finite. According to the discussion of section 3.3 we specify the renormalized expressions

$$
I_{ab} \to I_{ab}(p^2) - I_{ab}(\mu_M^2)\,, \qquad \bar{I}_{ab} \to I_{ab}(\bar{p}^2) - I_{ab}(\mu_M^2)\,, \qquad (163)
$$

with the matching scale μ_M.

The integrals J_{abc} and $\bar{J}_{abc}$ are finite. They may be evaluated in terms of their dispersion-integral representations

$$
J_{abc} = \int_{(m_a+m_c)^2}^{\infty} \frac{d\,s}{\pi}\, \frac{\rho^{(+)}_{abc}(s)}{s - p^2 - i\,\epsilon}\,, \qquad \bar{J}_{abc} = \int_{(m_a+m_c)^2}^{\infty} \frac{d\,s}{\pi}\, \frac{\rho^{(-)}_{abc}(s)}{s - p^2 - i\,\epsilon}\,, \qquad (164)
$$

with the spectral densities

$$
\begin{aligned}
\rho^{(\pm)}_{abc}(s) &= \frac{\sqrt{p^2_{ac}}}{8\,\pi\,\sqrt{s}}\,\frac{\log\Big(-\mu^2_{\pm,abc}+\sqrt{p^2_{ac}\,k^2}\,\Big)-\log\Big(-\mu^2_{\pm,abc}-\sqrt{p^2_{ac}\,k^2}\,\Big)}{2\,\sqrt{p^2_{ac}\,k^2}}\,,\\
\mu^2_{\pm,abc} &= \left(\frac{m_c^2-m_a^2\mp\bar p^2}{2\,\sqrt{s}}\right)^2-p^2_{ac}-\tfrac{1}{4}\,k^2-m_b^2\,,\\
k^2 &= \left(\frac{\bar p^2}{\sqrt{s}}\right)^2+s-2\,\bar p^2\,,\qquad \sqrt{s}\;=\sqrt{m_a^2+p^2_{ac}}+\sqrt{m_c^2+p^2_{ac}}\,,
\end{aligned}
\tag{165}
$$

Note that we wrote the logarithm in a manner to ensure a smooth spectral density. It is crucial to impose the dispersion-integral representation in terms of the proper variables, i.e. keep $q^2=0$ and $\bar p^2=(p-q)^2$ fixed. This implies that the spectral densities have an implicit dependence on $\bar p^2=p^2-2\,p\cdot q$. It is useful to provide an alternative representation of the integrals J_{abc} and $\bar J_{abc}$, derived via an application of Feynman's parametrization. It holds:

$$
\begin{aligned}
J_{abc} &= \int\frac{\Theta[z_2-z_2^2]\,\Theta[z_1-z_1^2]\,\Theta[z_1-z_2]dz_1\,dz_2/(16\pi^2)}{(m_c^2-z_1\,p^2)\,(1-z_1)+z_1\,m_a^2+2\,(1-z_1)\,z_2\,(p\cdot q)+z_2\,\mu^2_{ba}}\,,\\
\bar J_{abc} &= \int\frac{\Theta[z_2-z_2^2]\,\Theta[z_1-z_1^2]\,\Theta[1-z_1-z_2]\,dz_1\,dz_2/(16\pi^2)}{(m_c^2-z_1\,p^2)\,(1-z_1)+z_1\,m_a^2+2\,z_1\,z_2\,(p\cdot q)\,+z_2\,\mu^2_{bc}}\,,
\end{aligned}
\tag{166}
$$

where we used $q^2=0$ and $\mu^2_{bc}=m_b^2-m_c^2$. Depending on the values of the various mass parameters either (164) or (166) may be more economical in a numerical simulation. For instance, if $p^2>(m_a+m_c)^2$ the representation is advantageous, the integrals being complex.

We performed numerical checks using (164) or (166) that the identity $J_{abc}=\bar J_{cba}$ is indeed satisfied.

Appendix D

We provide explicit expressions for the contributions of the $\phi\, D_s^*$ and $K^* D^*$ channels to the decay amplitude (86). The latter are linear in the resonance coupling constant g_H as introduced in (81). It holds

$$
\begin{aligned}
i\, M^{\alpha\beta,\mu}_{0^+\to\gamma\, 1^-} = & \frac{e\, g_H}{2\, f^2} \Big\{ g_T\, E^{\alpha\beta,\mu}_{+,\phi D_s^*}(\bar p, p) + \tilde g_V \Big(E^{\alpha\beta,\mu}_{\phi D_s^*}(\bar p, p) + E^{\alpha\beta,\mu}_{-,\phi D_s^*}(\bar p, p)\Big)\Big\} \\
& + \frac{g_H}{4\, f^2\, M_V^2} \Big[\tilde e_C - e + \frac{\tilde e_Q}{3}\Big] \Big\{ g_T\, \hat E^{\alpha\beta,\mu}_{+,\phi D_s^*}(\bar p, p) \\
& \qquad + \tilde g_V \Big(\hat E^{\alpha\beta,\mu}_{\phi D_s^*}(\bar p, p) + \hat E^{\alpha\beta,\mu}_{-,\phi D_s^*}(\bar p, p)\Big)\Big\} \\
& + \frac{e\, g_H}{2\, f^2} \Big\{ g_T\, E^{\alpha\beta,\mu}_{+,K^* D^*}(\bar p, p) + \tilde g_V \Big(E^{\alpha\beta,\mu}_{K^* D^*}(\bar p, p) + E^{\alpha\beta,\mu}_{-,K^* D^*}(\bar p, p)\Big)\Big\} \\
& + \frac{g_H}{2\, f^2\, M_V^2} \Big[\tilde e_C - \frac{e}{2} - \frac{\tilde e_Q}{6}\Big] \Big\{ g_T\, \hat E^{\alpha\beta,\mu}_{+,K^* D^*}(\bar p, p) \\
& \qquad + \tilde g_V \Big(\hat E^{\alpha\beta,\mu}_{K^* D^*}(\bar p, p) + \hat E^{\alpha\beta,\mu}_{-,K^* D^*}(\bar p, p)\Big)\Big\} \\
& - \frac{e\, g_H}{2\, f^2} \Big\{ g_T\, E^{\alpha\beta,\mu}_{-,D^* K^*}(\bar p, p) + \tilde g_V \Big(E^{\alpha\beta,\mu}_{D^* K^*}(\bar p, p) + E^{\alpha\beta,\mu}_{+,D^* K^*}(\bar p, p)\Big)\Big\} \\
& - \frac{g_H}{2\, f^2\, m_V^2} \Big\{ e_T\, \bar E^{\alpha\beta,\mu}_{-,D^* K^*}(\bar p, p) + \tilde e_V \Big(\bar E^{\alpha\beta,\mu}_{D^* K^*}(\bar p, p) + \bar E^{\alpha\beta,\mu}_{+,D^* K^*}(\bar p, p)\Big)\Big\} \\
& + \frac{\tilde e_E\, g_H}{3\, f^2\, m_V^2} \Big\{ \tilde E^{\alpha\beta,\mu}_{D^* K^*}(\bar p, p) + 2\, \tilde E^{\alpha\beta,\mu}_{D_s^* \phi}(\bar p, p)\Big\} \\
& + \frac{(e_C + e_Q/6)\, g_H}{4\, f^2\, M_V^2} \Big\{ \tilde g_T\, \bar F^{\alpha\beta,\mu}_{K^* D D^*}(\bar p, p) + g_E\, F^{\alpha\beta,\mu}_{K^* D D^*}(\bar p, p)\Big\} \\
& + \frac{(e_C - e_Q/3)\, g_H}{8\, f^2\, M_V^2} \Big\{ \tilde g_T\, \bar F^{\alpha\beta,\mu}_{\phi D_s D_s^*}(\bar p, p) + g_E\, F^{\alpha\beta,\mu}_{\phi D_s D_s^*}(\bar p, p)\Big\} \\
& - \frac{e_A\, \tilde g_P\, g_H}{48\, f^3\, m_V} \Big\{ F^{\alpha\beta,\mu}_{D^* K K^*}(\bar p, p) + \bar F^{\alpha\beta,\mu}_{D^* K K^*}(\bar p, p)\Big\} \\
& - \frac{e_A\, \tilde g_P\, g_H}{36\, f^3\, m_V} \Big\{ F^{\alpha\beta,\mu}_{D_s^*\, \eta\, \phi}\;(\bar p, p) + \bar F^{\alpha\beta,\mu}_{D_s^*\, \eta\, \phi}\;(\bar p, p)\Big\}\,,
\end{aligned} \tag{167}
$$

with

$$
\begin{aligned}
E^{\alpha\beta,\mu}_{ab}(\bar p, p) = & -i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar\rho\rho}\, S_a^{\alpha\bar\rho}(l) \Big\{ S_b^{\beta,\mu\rho}(\bar p + l) + S_b^{\mu\beta,\rho}(p + l) \\
& + g_{\bar\kappa\kappa} \Big\{ S_b^{\beta\bar\kappa}(\bar p + l)\, S_b^{\mu\kappa,\rho}(p + l) + S_b^{\beta,\mu\bar\kappa}(\bar p + l)\, S_b^{\kappa\rho}(p + l)\Big\}\Big\}\,, \\
E^{\alpha\beta,\mu}_{-,ab}(\bar p, p) = & -i \int \frac{d^4 l}{(2\pi)^4}\, S_{a,\sigma\rho}(l) \Big\{ \bar p^\alpha\, S_b^{\sigma\beta,\mu\rho}(\bar p + l) + g^{\mu\alpha}\, S_b^{\sigma\beta,\rho}(p + l)
\end{aligned}
$$

$$+\bar{p}^{\alpha}\, g_{\bar{\kappa}\kappa} \left\{ S_b^{\sigma\beta,\bar{\kappa}}(\bar{p}+l)\, S_b^{\mu\kappa,\rho}(p+l) + S_b^{\sigma\beta,\mu\bar{\kappa}}(\bar{p}+l)\, S_b^{\kappa\rho}(p+l) \right\} \Bigg\},$$

$$E^{\alpha\beta,\mu}_{+,ab}(\bar{p},p) = -\, i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, g_{\bar{\sigma}\sigma}\, S_a^{\bar{\sigma}\beta,\bar{\rho}}(l) \Bigg\{ \bar{p}^{\alpha}\, S_b^{\sigma,\mu\rho}(\bar{p}+l)$$
$$+\bar{p}^{\alpha}\, S_b^{\mu\sigma,\rho}(p+l) + g^{\alpha\mu}\, S_b^{\sigma\rho}(p+l)$$
$$+\bar{p}^{\alpha}\, g_{\bar{\kappa}\kappa} \left\{ S_b^{\sigma\bar{\kappa}}(\bar{p}+l)\, S_b^{\mu\kappa,\rho}(p+l) + S_b^{\sigma,\mu\bar{\kappa}}(\bar{p}+l)\, S_b^{\kappa\rho}(p+l) \right\} \Bigg\}, \quad (168)$$

and

$$\hat{E}^{\alpha\beta,\mu}_{ab}(\bar{p},p) = -i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, S_a^{\alpha\bar{\rho}}(l) \Bigg\{$$
$$+q_{\kappa} \left\{ S_b^{\beta\kappa}(\bar{p}+l)\, S_b^{\mu\rho}(p+l) - S_b^{\beta\mu}(\bar{p}+l)\, S_b^{\kappa\rho}(p+l) \right\} \Bigg\},$$
$$\hat{E}^{\alpha\beta,\mu}_{-,ab}(\bar{p},p) = -i \int \frac{d^4 l}{(2\pi)^4}\, S_{a,\sigma\rho}(l) \Bigg\{$$
$$+\bar{p}^{\alpha}\, q_{\kappa} \left\{ S_b^{\sigma\beta,\kappa}(\bar{p}+l)\, S_b^{\mu\rho}(p+l) - S_b^{\sigma\beta,\mu}(\bar{p}+l)\, S_b^{\kappa\rho}(p+l) \right\} \Bigg\},$$
$$\hat{E}^{\alpha\beta,\mu}_{+,ab}(\bar{p},p) = -\, i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, g_{\bar{\sigma}\sigma}\, S_a^{\bar{\sigma}\beta,\bar{\rho}}(l) \Bigg\{$$
$$+\bar{p}^{\alpha}\, q_{\kappa} \left\{ S_b^{\sigma\kappa}(\bar{p}+l)\, S_b^{\mu\rho}(p+l) - S_b^{\sigma\mu}(\bar{p}+l)\, S_b^{\kappa\rho}(p+l) \right\} \Bigg\}, \qquad (169)$$

and

$$\bar{E}^{\alpha\beta,\mu}_{ab}(\bar{p},p) = -i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, S_a^{\alpha\bar{\rho}}(l)\, S_b^{\tau\rho}(p+l) \left(g^{\mu}{}_{\tau}\, q^{\beta} - g^{\mu\beta}\, q_{\tau} \right),$$
$$\tilde{E}^{\alpha\beta,\mu}_{ab}(\bar{p},p) = +\, 2\, i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, S_a^{\beta\bar{\rho}}(l)\, \bar{p}^{\alpha}\, S_b^{\tau\mu,\rho}(p+l)\, q_{\tau}\,,$$
$$\bar{E}^{\alpha\beta,\mu}_{+,ab}(\bar{p},p) = -\, i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, S_a^{\sigma\beta,\bar{\rho}}(l)\, \bar{p}^{\alpha}\, S_b^{\tau\rho}(p+l) \left(g^{\mu}{}_{\tau}\, q_{\sigma} - g^{\mu}{}_{\sigma}\, q_{\tau} \right),$$
$$\bar{E}^{\alpha\beta,\mu}_{-,ab}(\bar{p},p) = -\, i \int \frac{d^4 l}{(2\pi)^4}\, g_{\bar{\rho}\rho}\, S_a^{\sigma\bar{\rho}}(l)\, \bar{p}^{\alpha}\, S_b^{\beta\tau,\rho}(p+l) \left(g^{\mu\tau}\, q_{\sigma} - g^{\mu\sigma}\, q_{\tau} \right), \quad (170)$$

and

$$\bar{F}^{\alpha\beta,\mu}_{abc}(\bar{p},p) = -\, 2\, i \int \frac{d^4 l}{(2\pi)^4}\, S_a^{\bar{\tau}\rho}(l)\, q_{\nu}\, \epsilon^{\mu\nu}{}_{\sigma\tau}\, \epsilon^{\alpha\beta}{}_{\bar{\sigma}\bar{\tau}}$$
$$\times (\bar{p}+l)^{\tau}\, (\bar{p}+l)^{\bar{\sigma}}\, S_b(\bar{p}+l)\, S^{\sigma}_{c,\rho}(p+l)\,,$$
$$F^{\alpha\beta,\mu}_{abc}(\bar{p},p) = -\, 2\, i \int \frac{d^4 l}{(2\pi)^4}\, S_a^{\bar{\mu}\bar{\nu},\rho}(l)\, q_{\nu}\, \epsilon^{\mu\nu}{}_{\sigma\tau}\, \epsilon_{\bar{\mu}\bar{\nu}}{}^{\bar{\sigma}\beta}$$

$$\times (\bar p + l)^\tau\, (\bar p + l)_{\bar \sigma}\, \bar p^\alpha\, S_b(\bar p + l)\, S^\sigma_{c,\rho}(p + l)\,. \tag{171}$$

The evaluation of the decay constant $d_{0^+ \to \gamma 1^-}$ as of (88) is determined by the contraction of the gauge invariant tensors (89, 91, 92, 94) and (168-171) with the anti-symmetric tensor

$$P^{(1-)}_{\alpha\beta,\mu} = -\frac{1}{2}\left(\Big\{ g_{\mu\alpha} - \frac{p^\mu\, q^\alpha}{p\cdot q}\Big\}\, \bar p_\beta - \Big\{ g_{\mu\beta} - \frac{p^\mu\, q^\beta}{p\cdot q}\Big\}\, \bar p_\alpha \right). \tag{172}$$

We derive the required contractions in terms of the master loop integrals $I_{ab}, \bar I_{ab}, J_{abc}$ and $\bar J_{abc}$ introduced in (161). We remind the reader that following the arguments of section 3.3 reduced tadpole integrals are dropped systematically. Utilizing the notation $p^2 = M_i^2$ and $(p-q)^2 = M_f^2$ it holds

$$\begin{aligned}
8\,(p\cdot q)\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, A^{\alpha\beta,\mu}_{ab} &= \Big[M_i^6 - \Big(m_a^2 - m_b^2 + 2\,M_f^2\Big)\, M_i^4 \\
&\quad - M_f^2\,\Big(2\,m_a^2 - 2\,m_b^2 + M_f^2\Big)\, M_i^2 + \Big(m_a^2 - m_b^2\Big)\, M_f^4\Big]\, I_{ab} \\
&\quad + 2\,M_f^2\,\Big(m_a^2 - m_b^2 + M_f^2\Big)\, M_i^2\, \bar I_{ab} + 4\,m_a^2\, M_f^2\, M_i^2\, \Big(M_i^2 - M_f^2\Big)\, J_{aab}\,,
\end{aligned}$$

$$\begin{aligned}
48\,(p\cdot q)\, M_i^4\, P^{(1-)}_{\alpha\beta,\mu}\, \bar A^{\alpha\beta,\mu}_{ab} &= -\Big(M_f^2 - M_i^2\Big)^2 \Big[-\Big(3\,\Big(m_a^2 + m_b^2\Big) - M_f^2\Big)\, M_i^4 \\
&\quad + 3\,M_i^6 + \Big(m_a^2 + m_b^2\Big)\, M_f^2\, M_i^2 - 2\,\Big(m_a^2 - m_b^2\Big)^2\, M_f^2\Big]\, I_{ab}\,,
\end{aligned}$$

$$\begin{aligned}
8\,(p\cdot q)\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, B^{\alpha\beta,\mu}_{ab} &= \Big[\Big(M_f^4 - 2\,M_i^2 M_f^2 - M_i^4\Big)\, m_a^2 \\
&\quad + M_i^2\,\Big(M_f^4 + 2\,M_i^2 M_f^2 - M_i^4\Big) + m_b^2\,\Big(-M_f^4 + 2\,M_i^2 M_f^2 + M_i^4\Big)\Big]\, I_{ab} \\
&\quad - 2\,M_f^2\,\Big(-m_a^2 + m_b^2 + M_f^2\Big)\, M_i^2\, \bar I_{ab} + 4\,m_b^2\, M_f^2\, M_i^2\,\Big(M_f^2 - M_i^2\Big)\, \bar J_{abb}\,,
\end{aligned}$$

$$\begin{aligned}
8\,(p\cdot q)\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, C^{\alpha\beta,\mu}_{abc} &= \Big[2\,\Big(m_a^2 - m_c^2\Big)\, M_f^6 + 6\,M_i^4 M_f^4 \\
&\quad -2\,\Big(3\,m_a^2 - 2\,m_b^2 - m_c^2 + M_f^2\Big)\, M_i^2\, M_f^4 - 4\,M_i^6\, M_f^2\Big]\, I_{ac} \\
&\quad + 2\,M_f^2 M_i^2\,\Big[\Big(2\,m_a^2 - 3m_b^2 + m_c^2 - M_f^2\Big)\, M_f^2 + \Big(m_b^2 - m_c^2 + M_f^2\Big)\, M_i^2\Big]\, \bar I_{bc} \\
&\quad - 4\,M_f^2\, M_i^2\,\Big[- m_b^2\, M_f^4 + \Big(m_a^2 - m_b^2\Big)^2\, M_f^2 + 2\,m_b^2\, M_i^2\, M_f^2 - m_b^2\, M_i^4\Big]\, J_{abc}\,,
\end{aligned}$$

$$\begin{aligned}
12\,(p\cdot q)\, M_i^4\, P^{(1-)}_{\alpha\beta,\mu}\, \bar C^{\alpha\beta,\mu}_{abc} &= -\Big[\Big(4\,m_c^2 - 2\,m_a^2\Big)\,\Big(M_f^3 - M_f M_i^2\Big)^2\, m_a^2 \\
&\quad + \Big(\Big(M_f^2 - M_i^2\Big)^2 \Big(M_f^2 + 3\,M_i^2\Big) - 3\,m_b^2\,\Big(M_f^4 - 3\,M_f^2\, M_i^2\Big)\Big)\, M_i^2\, m_a^2 \\
&\quad - \Big(m_c^2 - M_i^2\Big)\,\Big(M_f^2 - M_i^2\Big)^2\,\Big(-3\,M_i^4 + M_f^2\, M_i^2 + 2\,m_c^2\, M_f^2\Big) \\
&\quad + 3\,m_b^2\, M_i^2\,\Big(M_i^2 - M_f^2\Big)\,\Big(2\,M_i^4 - M_f^2\, M_i^2 - m_c^2\, M_f^2\Big) \\
&\quad - 6\,m_b^4\, M_f^2\, M_i^4\Big]\, I_{ac}
\end{aligned}$$

$$
\begin{aligned}
&- 3\, m_b^2\, M_i^4 \Big[M_f^2 \Big(-2\, m_a^2 + 3\, m_b^2 - m_c^2 + M_f^2 \Big) \\
&\qquad - \Big(m_b^2 - m_c^2 + M_f^2 \Big) M_i^2 \Big]\, \bar{I}_{bc} \\
&- 6\, m_b^2\, M_i^4 \Big[- m_b^2\, M_f^4 + \Big(m_a^2 - m_b^2 \Big)^2 M_f^2 + 2\, m_b^2\, M_i^2\, M_f^2 - m_b^2\, M_i^4 \Big]\, J_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
&12\, (p\cdot q)\, M_i^4\, P^{(1-)}_{\alpha\beta,\mu}\, D^{\alpha\beta,\mu}_{abc} = \Big[- 6 \Big(m_b^2 - m_c^2 \Big) \Big(m_b^2 + M_i^2 \Big) M_i^6 \\
&\qquad - 3 \Big(M_i^2 - M_f^2 \Big) \Big(\Big(m_a^2 - 5\, m_b^2 + 3\, m_c^2 \Big) M_i^2 \\
&\qquad + m_b^2 \Big(m_a^2 - 2\, m_b^2 + m_c^2 \Big) \Big) M_i^4 \\
&\qquad + 2 \Big(M_i^2 - M_f^2 \Big)^3 \Big(m_a^4 - 2 \Big(m_c^2 + M_i^2 \Big) m_a^2 + m_c^4 - 2\, M_i^4 + m_c^2 M_i^2 \Big) \\
&\qquad - \Big(M_i^3 - M_f^2\, M_i \Big)^2 \Big(2\, m_a^4 - \Big(3\, m_b^2 + 4\, m_c^2 + 10\, M_i^2 \Big) m_a^2 \\
&\qquad + 2 \Big(m_c^2 - M_i^2 \Big)^2 + 3\, m_b^2 \Big(m_c^2 + M_i^2 \Big) \Big) + 3 \Big(M_i^2 - M_f^2 \Big) M_i^8 \Big]\, I_{ac} \\
&+ 3 \Big(m_b^2 + M_f^2 \Big) M_i^4 \Big[M_f^2 \Big(-m_a^2 + 3\, m_b^2 - 2\, m_c^2 + M_f^2 \Big) \\
&\qquad - \Big(-m_a^2 + m_b^2 + M_f^2 \Big) M_i^2 \Big]\, \bar{I}_{ab} \\
&+ 6 \Big(m_b^2 + M_f^2 \Big) M_i^4 \Big[M_f^2\, m_b^4 - \Big(2\, m_c^2\, M_f^2 + \Big(M_f^2 - M_i^2 \Big)^2 \Big) m_b^2 \\
&\qquad + m_c^4\, M_f^2 \Big]\, \bar{J}_{abc}\,,
\end{aligned} \tag{173}
$$

and

$$
\begin{aligned}
&96\, M_f^2\, M_i^4\, (p\cdot q)\, P^{(1-)}_{\alpha\beta,\mu}\, E^{\alpha\beta,\mu}_{ab} = -3\, M_f^2\, M_i^2 \Big[- 4 \Big(\Big(m_a^2 - M_i^2 \Big)^2 - m_b^4 \Big) M_i^4 \\
&\qquad + \Big(-m_a^2 + m_b^2 + M_i^2 \Big) \Big(M_i^2 - M_f^2 \Big) \Big(-m_a^2 - 3\, m_b^2 + 7\, M_i^2 \Big) M_i^2 \\
&\qquad + \Big(M_f^2 - M_i^2 \Big)^2 \Big(-3\, M_i^4 - 2 \Big(m_a^2 + 3\, m_b^2 \Big) M_i^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big) \Big]\, I_{ab} \\
&+ \Big[- \Big(5\, M_f^4 - \Big(7\, m_a^2 + m_b^2 \Big) M_f^2 + 2 \Big(m_a^2 - m_b^2 \Big)^2 \Big) M_i^8 - 17\, M_f^4\, m_a^4\, M_i^4 \\
&\qquad - M_f^2 \Big(-7\, m_a^4 + 2 \Big(m_b^2 + M_f^2 \Big) m_a^2 + 5\, m_b^4 + 5\, M_f^4 - 22\, m_b^2\, M_f^2 \Big) M_i^6 \\
&\qquad - M_f^4 \Big(\Big(2\, m_b^2 - 19\, M_f^2 \Big) m_a^2 - 19\, m_b^4 + 2\, M_f^4 + 23\, m_b^2\, M_f^2 \Big) M_i^4 \Big]\, \bar{I}_{ab} \\
&- 6\, m_b^2\, M_f^2\, M_i^4 \Big(M_f^2 - M_i^2 \Big) \Big[- M_i^4 + \Big(-5\, m_a^2 + m_b^2 + M_f^2 \Big) M_i^2 \\
&\qquad + M_f^2 \Big(9\, m_a^2 + 3\, m_b^2 - 4\, M_f^2 \Big) \Big]\, \bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
&96\, M_f^2\, M_i^4\, (p\cdot q)\, P^{(1-)}_{\alpha\beta,\mu}\, E^{\alpha\beta,\mu}_{+,ab} = 6\, M_f^2\, M_i^2 \Big[M_i^8 - 2 \Big(m_a^2 - 2 m_b^2 \Big) M_i^6 \\
&\qquad + \Big(m_a^4 + 4\, m_b^2\, m_a^2 - 5\, m_b^4 - 4\, m_b^2\, M_f^2 \Big) M_i^4 \\
&\qquad + M_f^2 \Big(2\, m_a^4 + \Big(8\, m_b^2 - 3\, M_f^2 \Big) m_a^2 - 10\, m_b^4 + M_f^4 - 7\, m_b^2\, M_f^2 \Big) M_i^2 \\
&\qquad + \Big(m_a^2 - m_b^2 \Big) M_f^4 \Big(-m_a^2 - 5 m_b^2 + M_f^2 \Big) \Big]\, I_{ab} \\
&+ 6\, M_f^4\, M_i^4 \Big(-m_a^2 + m_b^2 + M_f^2 \Big) \Big(2\, m_a^2 + 10\, m_b^2 + M_f^2 - 3\, M_i^2 \Big)\, \bar{I}_{ab}
\end{aligned}
$$

$$- 12\, m_b^2\, M_f^4\, M_i^4\, \left(2\, m_a^2 + 10\, m_b^2 + M_f^2 - 3M_i^2\right) \left(M_f^2 - M_i^2\right) \bar{J}_{abb}\,,$$

$$\begin{aligned}
96\, M_f^2\, M_i^4\, m_b^2\, (p\cdot q)\, P^{(1-)}_{\alpha\beta,\mu}\, E^{\alpha\beta,\mu}_{-,ab} &= 3\, M_f^2\, M_i^2 \Big[\\
& - 4\, m_b^2 \left(5\, m_a^2 + m_b^2 - M_i^2\right) \left(-m_a^2 + m_b^2 + M_i^2\right) M_i^4 \\
& + \left(M_f^2 - M_i^2\right)^3 \left(m_a^4 - m_b^4 + M_i^4 - 2\left(m_a^2 - 2\, m_b^2\right) M_i^2\right) \\
& + \left(M_i^2 - M_f^2\right) \Big(m_a^4 + 36\, m_b^2\, m_a^2 + 11\, m_b^4 + M_i^4 \\
& - 2\left(m_a^2 + 10\, m_b^2\right) M_i^2\Big) M_i^4 \\
& + 2\, m_b^2 \left(M_f^2 - M_i^2\right)^2 \Big(- 5\, m_a^4 + 4\, m_b^2 m_a^2 + m_b^4 + 9\, M_i^4 \\
& - 2\left(5\, m_a^2 + 2\, m_b^2\right) M_i^2\Big)\Big]\, I_{ab} \\
& + M_f^2\, M_i^4 \Big[2\, M_f^8 - \left(m_a^2 + 25\, m_b^2\right) M_f^6 + \left(-m_a^4 + 56\, m_b^2\, m_a^2 + 17\, m_b^4\right) M_f^4 \\
& - \left(-5\, m_a^4 + \left(4\, M_f^2 - 8\, m_b^2\right) m_a^2 + 13\, m_b^4 + M_f^4 - 2\, m_b^2\, M_f^2\right) M_i^2\, M_f^2 \\
& + 12\, m_b^2 \left(-5\, m_a^4 + 4\, m_b^2\, m_a^2 + m_b^4\right) M_f^2 \\
& - \left(M_f^4 - \left(5\, m_a^2 + 11\, m_b^2\right) M_f^2 + 4\left(m_a^2 - m_b^2\right)^2\right) M_i^4\Big]\, \bar{I}_{ab} \\
& + 6\, m_b^2\, M_f^4\, M_i^4 \left(M_i^2 - M_f^2\right) \Big[4\left(5m_a^2 + m_b^2 - M_i^2\right) m_b^2 - 2\, \left(M_f^2 - M_i^2\right)^2 \\
& + \left(M_i^2 - M_f^2\right)\left(-3\, m_a^2 + m_b^2 + 3\, M_i^2\right)\Big]\, \bar{J}_{abb}\,, \qquad (174)
\end{aligned}$$

and

$$\begin{aligned}
48\, M_i^4\, M_f^2\, P^{(1-)}_{\alpha\beta,\mu}\, \hat{E}^{\alpha\beta,\mu}_{ab} &= M_f^2 \Big[- 3\, M_i^{10} + 2\left(3\, m_a^2 + 6\, m_b^2 + M_f^2\right) M_i^8 \\
& - \left(3\, m_a^4 + 2\left(12\, m_b^2 + M_f^2\right) m_a^2 + 9\, m_b^4 - M_f^4 + 5\, m_b^2\, M_f^2\right) M_i^6 \\
& - 2\, M_f^2 \left(m_a^4 - 9\, m_b^2\, m_a^2 - 2m_b^4 + 2\left(m_a^2 + m_b^2\right) M_f^2\right) M_i^4 \\
& + M_f^2 \left(7\, m_b^2\, m_a^4 - 8\, m_b^4\, m_a^2 - m_b^6 + \left(5\, m_a^4 - 6\, m_b^2\, m_a^2 + 5\, m_b^4\right) M_f^2\right) M_i^2 \\
& + 2\, M_f^2\, m_a^6\, M_i^2 - 2\left(m_a^2 - m_b^2\right)^2 \left(m_a^2 + m_b^2\right) M_f^4\Big]\, I_{ab} \\
& + M_i^4 \Big[4\left(M_i^2 - M_f^2\right) m_a^6 - 3\left(\left(m_b^2 - 2M_f^2\right) M_f^2 + 2\left(m_b^2 + M_f^2\right) M_i^2\right) m_a^4 \\
& + m_b^2\, M_f^2 \left(6\, m_b^2 + 25\, M_f^2 - 13\, M_i^2\right) m_a^2 + \left(m_b^2 + M_f^2\right) \Big(- 2\, M_f^6 \\
& + 2\, M_i^2\, M_f^4 + m_b^4 \left(M_f^2 + 2\, M_i^2\right) + m_b^2 \left(4\, M_f^4 - 7\, M_f^2\, M_i^2\right)\Big)\Big]\, \bar{I}_{ab} \\
& + 6\, m_b^4\, M_f^2\, M_i^4 \left(3\, m_a^2 + m_b^2 - M_i^2\right) \left(M_i^2 - M_f^2\right) \bar{J}_{abb}\,,
\end{aligned}$$

$$\begin{aligned}
24\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, \hat{E}^{\alpha\beta,\mu}_{+,ab} &= -M_f^2 \Big[3\, m_b^2 \left(-m_a^2 + m_b^2 + M_i^2\right) M_i^2 \\
& + \left(M_i^2 - M_f^2\right) \left(m_a^4 + \left(m_b^2 - 2\, M_i^2\right) m_a^2 - 2\, m_b^4 + M_i^4 + m_b^2\, M_i^2\right)\Big]\, I_{ab} \\
& + \Big[3\left(m_a^4 - 2\left(m_b^2 + M_f^2\right) m_a^2 + \left(m_b^2 - M_f^2\right)^2\right) M_i^4
\end{aligned}$$

$$
\begin{aligned}
&\quad - 3\, M_f^2 \left(m_a^4 - \left(m_b^2 + 2\, M_f^2 \right) m_a^2 + M_f^4 - 3\, m_b^2\, M_f^2 \right) M_i^2 \Big] \, \bar{I}_{ab} \\
&+ 6\, m_b^4\, M_f^2\, M_i^2 \left(M_i^2 - M_f^2 \right) \bar{J}_{abb} \,,
\end{aligned}
$$

$$
\begin{aligned}
48\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, \hat{E}^{\alpha\beta,\mu}_{-,ab} &= \Big[- 6 \left(m_a^2 - 2 m_b^2 \right) M_i^2\, M_f^4 \\
&\quad + 3 \left(m_a^4 - m_b^4 \right) M_f^4 + 3 \left(M_f^2 - 4\, m_b^2 \right) M_i^4\, M_f^2 \Big] \, I_{ab} \\
&+ M_i^2 \Big[\left(-5\, m_a^4 + 4\, m_b^2\, m_a^2 + m_b^4 - 8\, M_f^4 + \left(13\, m_a^2 + m_b^2 \right) M_f^2 \right) M_f^2 \\
&\quad + \left(5\, M_f^4 - \left(7\, m_a^2 + m_b^2 \right) M_f^2 + 2 \left(m_a^2 - m_b^2 \right)^2 \right) M_i^2 \Big] \, \bar{I}_{ab} \\
&- 6\, m_b^2\, M_f^2\, M_i^2 \left(M_f^2 - M_i^2 \right) \left(m_a^2 + m_b^2 - 2 M_f^2 + M_i^2 \right) \bar{J}_{abb} \,, \qquad (175)
\end{aligned}
$$

and

$$
\begin{aligned}
48\, M_i^4\, P^{(1-)}_{\alpha\beta,\mu}\, \bar{E}^{\alpha\beta,\mu}_{ab} &= \left(M_i^2 - M_f^2 \right) \Big[3\, M_i^8 - \left(6 \left(m_a^2 + m_b^2 \right) - M_f^2 \right) M_i^6 \\
&\quad + \left(3 \left(m_a^4 + 10\, m_b^2\, m_a^2 + m_b^4 \right) - 4 \left(m_a^2 + m_b^2 \right) M_f^2 \right) M_i^4 \\
&\quad + \left(5\, m_a^4 - 6\, m_b^2\, m_a^2 + 5\, m_b^4 \right) M_f^2\, M_i^2 \\
&\quad - 2 \left(m_a^2 - m_b^2 \right)^2 \left(m_a^2 + m_b^2 \right) M_f^2 \Big] \, I_{ab} \,,
\end{aligned}
$$

$$
\begin{aligned}
12\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, \tilde{E}^{\alpha\beta,\mu}_{ab} &= M_f^2 \left(M_i^2 - M_f^2 \right) \Big[- 5\, m_a^4 + 4 \left(m_b^2 + M_i^2 \right) m_a^2 \\
&\quad + \left(m_b^2 - M_i^2 \right)^2 \Big] \, I_{ab} \,,
\end{aligned}
$$

$$
\begin{aligned}
24\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, \bar{E}^{\alpha\beta,\mu}_{+,ab} &= M_f^2 \left(M_i^2 - M_f^2 \right) \Big[m_a^4 + \left(4\, m_b^2 - 2\, M_i^2 \right) m_a^2 \\
&\quad - 5\, m_b^4 + M_i^4 + 4\, m_b^2\, M_i^2 \Big] \, I_{ab} \,,
\end{aligned}
$$

$$
\begin{aligned}
24\, M_i^2\, P^{(1-)}_{\alpha\beta,\mu}\, \bar{E}^{\alpha\beta,\mu}_{-,ab} &= - M_f^2 \left(M_i^2 - M_f^2 \right) \Big[- 5\, m_a^4 + 4 \left(m_b^2 + M_i^2 \right) m_a^2 \\
&\quad + \left(m_b^2 - M_i^2 \right)^2 \Big] \, I_{ab} \,, \qquad (176)
\end{aligned}
$$

and

$$
\begin{aligned}
12\, M_i^2\, M_f^2\, (p \cdot q)\, P^{(1-)}_{\alpha\beta,\mu}\, F^{\alpha\beta,\mu}_{abc} &= 3\, m_a^2\, m_c^2\, M_f^2 \Big[2\, M_i^6 - 3\, M_f^2\, M_i^4 \\
&\quad + M_f^2 \left(-m_a^2 - 2 m_b^2 + 3\, m_c^2 + M_f^2 \right) M_i^2 + \left(m_a^2 - m_c^2 \right) M_f^4 \Big] \, I_{ac} \\
&+ m_a^2\, M_i^2 \Big[\left(3\, m_c^2 \left(3 M_f^2 - M_i^2 \right) M_f^2 + 4 \left(m_a^2 + M_f^2 \right) \left(M_f^2 - M_i^2 \right)^2 \right) m_b^2 \\
&\quad - 2 \left(M_f^2 - M_i^2 \right)^2 m_b^4 - 6\, m_c^4\, M_f^4 - 2 \left(m_a^2 - M_f^2 \right)^2 \left(M_f^2 - M_i^2 \right)^2 \\
&\quad + 3\, m_c^2\, M_f^2 \left(M_f^2 - m_a^2 \right) \left(M_f^2 - M_i^2 \right) \Big] \, \bar{I}_{ab} \\
&+ 6\, m_a^2\, m_c^2\, M_f^2\, M_i^2 \Big[M_f^2\, m_b^4 - \left(2\, m_c^2 M_f^2 + \left(M_f^2 - M_i^2 \right)^2 \right) m_b^2
\end{aligned}
$$

$$+\,m_c^4\,M_f^2\Big]\,\bar{J}_{abc}\,,$$

$$12\,M_i^2\,M_f^2\,(p\cdot q)\,P^{(1-)}_{\alpha\beta,\mu}\,\bar{F}^{\alpha\beta,\mu}_{abc} = 3\,m_c^2\,M_f^4\,\Big[2\,M_i^6 - 3\,M_f^2\,M_i^4$$
$$+\,\Big(m_a^2 - m_c^2\Big)\,M_f^4 + M_f^2\,\Big(-m_a^2 - 2\,m_b^2 + 3\,m_c^2 + M_f^2\Big)\,M_i^2\Big]\,I_{ac}$$
$$-\,M_f^2\,M_i^2\,\Big[-\Big(3\,m_c^2\,\Big(3M_f^2 - M_i^2\Big)\,M_f^2 + 4\,\Big(m_a^2 + M_f^2\Big)\,\Big(M_f^2 - M_i^2\Big)^2\Big)\,m_b^2$$
$$+\,2\,\Big(M_f^2 - M_i^2\Big)^2\,m_b^4 + 6\,m_c^4\,M_f^4 + 2\,\Big(m_a^2 - M_f^2\Big)^2\,\Big(M_f^2 - M_i^2\Big)^2$$
$$+\,3\,m_c^2\,M_f^2\,\Big(M_f^2 - m_a^2\Big)\,\Big(M_i^2 - M_f^2\Big)\Big]\,\bar{I}_{ab}$$
$$+\,6\,m_c^2\,M_f^4\,M_i^2\Big[M_f^2\,m_b^4 - \Big(2\,m_c^2\,M_f^2 + \Big(M_f^2 - M_i^2\Big)^2\Big)\,m_b^2$$
$$+\,m_c^4\,M_f^2\Big]\,\bar{J}_{abc}\,. \tag{177}$$

Appendix E

We detail the contributions of the $\phi\, D_s^*$ and $K^* D^*$ channels to the decay amplitude $M^{\mu,\alpha\beta}_{1^+\to\gamma 0^-}$ as introduced in (97). All terms are linear in the coupling constant $\hat{g}_H$ (see (81)). It is derived

$$
\begin{aligned}
M^{\mu,\alpha\beta}_{1^+\to\gamma 0^-} &= \frac{e\,\tilde{g}_T\,\hat{g}_H}{8\,f^2}\Big\{F^{\mu,\alpha\beta}_{-,\phi D_s^*}(\bar{p},p) + E^{\mu,\alpha\beta}_{+,D^* K^*}(\bar{p},p) + F^{\mu,\alpha\beta}_{-,K^* D^*}(\bar{p},p)\Big\} \\
&+ \frac{e\,g_E\,\hat{g}_H}{8\,f^2}\Big\{F^{\mu,\alpha\beta}_{+,\phi D_s^*}(\bar{p},p) + E^{\mu,\alpha\beta}_{-,D^* K^*}(\bar{p},p) + F^{\mu,\alpha\beta}_{+,K^* D^*}(\bar{p},p)\Big\} \\
&+ \frac{\hat{g}_H}{16\,f^2\,M_V^2}\Big[\tilde{e}_C - e + \frac{\tilde{e}_Q}{3}\Big]\Big\{g_E\,\hat{F}^{\mu,\alpha\beta}_{+,\phi\, D_s^*}(\bar{p},p) + \tilde{g}_T\,\hat{F}^{\mu,\alpha\beta}_{-,\phi\, D_s^*}(\bar{p},p)\Big\} \\
&+ \frac{\hat{g}_H}{8\,f^2\,M_V^2}\Big[\tilde{e}_C - \frac{e}{2} - \frac{\tilde{e}_Q}{6}\Big]\Big\{g_E\,\hat{F}^{\mu,\alpha\beta}_{+,K^* D^*}(\bar{p},p) + \tilde{g}_T\,\hat{F}^{\mu,\alpha\beta}_{-,K^* D^*}(\bar{p},p)\Big\} \\
&+ \frac{\tilde{e}_T\,\hat{g}_H}{8\,f^2\,m_V^2}\,\bar{E}^{\mu,\alpha\beta}_{D^* K^*}(\bar{p},p) + \frac{(2\,e_C + e_Q/3)\,\hat{g}_H}{4\,f^2\,M_V^2}\,g_V\,H^{\mu,\alpha\beta}_{K^* D D^*}(\bar{p},p) \\
&+ \frac{(e_C - e_Q/3)\,\hat{g}_H}{4\,f^2\,M_V^2}\,g_V\,H^{\mu,\alpha\beta}_{\phi D_s D_s^*}(\bar{p},p) \\
&+ \frac{e_A\,g_P\,\hat{g}_H}{24\,f^3\,m_V}\,G^{\mu,\alpha\beta}_{D^* K K^*}(\bar{p},p) + \frac{e_A\,g_P\,\hat{g}_H}{18\,f^3\,m_V}\,G^{\mu,\alpha\beta}_{D_s^*\eta\phi}(\bar{p},p)\,,
\end{aligned}
\tag{178}
$$

with

$$
\begin{aligned}
E^{\mu,\alpha\beta}_{+,ab}(\bar{p},p) &= -i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\tau}{}^{\beta}\,\epsilon_{\bar{\alpha}\bar{\beta}\bar{\sigma}\bar{\tau}}\,S_a^{\bar{\tau},\tilde{\alpha}\tilde{\beta}}(l)\,\Big\{\bar{p}^{\bar{\sigma}}\,p^{\alpha}\,S_b^{\bar{\alpha}\bar{\beta},\mu\tau}(\bar{p}+l) \\
&\quad + \bar{p}^{\bar{\sigma}}\,g^{\mu\alpha}\,S_b^{\bar{\alpha}\bar{\beta},\tau}(\bar{p}+l) + g^{\bar{\sigma}\mu}\,p^{\alpha}\,S_b^{\bar{\alpha}\bar{\beta},\tau}(p+l) \\
&\quad + \bar{p}^{\bar{\sigma}}\,p^{\alpha}\,g_{\bar{\kappa}\kappa}\Big\{S_b^{\bar{\alpha}\bar{\beta},\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tau}(p+l) + S_b^{\bar{\alpha}\bar{\beta},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa\tau}(p+l)\Big\}\Big\}\,, \\
E^{\mu,\alpha\beta}_{-,ab}(\bar{p},p) &= +\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\tau}{}^{\beta}\,\epsilon_{\bar{\alpha}\bar{\beta}\bar{\sigma}\bar{\tau}}\,S_a^{\bar{\alpha}\bar{\beta},\tilde{\alpha}\tilde{\beta}}(l)\,\Big\{\bar{p}^{\bar{\sigma}}\,p^{\alpha}\,S_b^{\bar{\tau},\mu\tau}(\bar{p}+l) \\
&\quad + \bar{p}^{\bar{\sigma}}\,g^{\mu\alpha}\,S_b^{\bar{\tau}\tau}(\bar{p}+l) + g^{\bar{\sigma}\mu}\,p^{\alpha}\,S_b^{\bar{\tau}\tau}(p+l) + \bar{p}^{\bar{\sigma}}\,p^{\alpha}\,S_b^{\mu\bar{\tau},\tau}(p+l) \\
&\quad + \bar{p}^{\bar{\sigma}}\,p^{\alpha}\,g_{\bar{\kappa}\kappa}\Big\{S_b^{\bar{\tau}\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tau}(p+l) + S_b^{\bar{\tau},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa\tau}(p+l)\Big\}\Big\}\,, \\
\bar{E}^{\mu,\alpha\beta}_{ab}(\bar{p},p) &= -\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\tau}{}^{\beta}\,\epsilon_{\bar{\alpha}\bar{\beta}\bar{\sigma}\bar{\tau}}\,S_a^{\bar{\tau},\tilde{\alpha}\tilde{\beta}}(l)\,S_b^{\rho\bar{\beta},\tau}(p+l) \\
&\quad \times p^{\alpha}\,\bar{p}^{\bar{\sigma}}\,\Big(g^{\mu}{}_{\rho}\,q^{\bar{\alpha}} - g^{\mu\bar{\alpha}}\,q_{\rho}\Big)\,,
\end{aligned}
\tag{179}
$$

and

$$
\begin{aligned}
F^{\mu,\alpha\beta}_{+,ab}(\bar p,p) = &+i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde\alpha\tilde\beta\sigma}{}^{\beta}\,\epsilon_{\bar\alpha\bar\beta\bar\sigma\bar\tau}\,S^{\bar\tau\sigma}_a(l)\Big\{\bar p^{\bar\sigma}\,g^{\mu\alpha}\,S^{\bar\alpha\bar\beta,\tilde\alpha\tilde\beta}_b(\bar p+l)\\
&+g^{\bar\sigma\mu}\,p^\alpha\,S^{\bar\alpha\bar\beta,\tilde\alpha\tilde\beta}_b(p+l)+\bar p^{\bar\sigma}\,p^\alpha\,g_{\bar\kappa\kappa}\Big\{S^{\bar\alpha\bar\beta,\bar\kappa}_b(\bar p+l)\,S^{\mu\kappa,\tilde\alpha\tilde\beta}_b(p+l)\\
&\quad+S^{\bar\alpha\bar\beta,\mu\bar\kappa}_b(\bar p+l)\,S^{\kappa,\tilde\alpha\tilde\beta}_b(p+l)\Big\}\Big\}\,,
\end{aligned}
\tag{180}
$$

$$
\begin{aligned}
F^{\mu,\alpha\beta}_{-,ab}(\bar p,p) = &-i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde\alpha\tilde\beta\sigma}{}^{\beta}\,\epsilon_{\bar\alpha\bar\beta\bar\sigma\bar\tau}\,S^{\bar\alpha\bar\beta,\sigma}_a(l)\Big\{\bar p^{\bar\sigma}\,g^{\mu\alpha}\,S^{\bar\tau,\tilde\alpha\tilde\beta}_b(\bar p+l)\\
&+g^{\bar\sigma\mu}\,p^\alpha\,S^{\bar\tau,\tilde\alpha\tilde\beta}_b(p+l)+\bar p^{\bar\sigma}\,p^\alpha\,S^{\mu\bar\tau,\tilde\alpha\tilde\beta}_b(p+l)\\
&+\bar p^{\bar\sigma}\,p^\alpha\,g_{\bar\kappa\kappa}\Big\{S^{\bar\tau\bar\kappa}_b(\bar p+l)\,S^{\mu\kappa,\tilde\alpha\tilde\beta}_b(p+l)+S^{\bar\tau,\mu\bar\kappa}_b(\bar p+l)\,S^{\kappa,\tilde\alpha\tilde\beta}_b(p+l)\Big\}\Big\}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat F^{\mu,\alpha\beta}_{+,ab}(\bar p,p) = &+i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde\alpha\tilde\beta\sigma}{}^{\beta}\,\epsilon_{\bar\alpha\bar\beta\bar\sigma\bar\tau}\,S^{\bar\tau\sigma}_a(l)\Big\{\\
&+\bar p^{\bar\sigma}\,p^\alpha\,q_\kappa\Big\{S^{\bar\alpha\bar\beta,\kappa}_b(\bar p+l)\,S^{\mu,\tilde\alpha\tilde\beta}_b(p+l)-S^{\bar\alpha\bar\beta,\mu}_b(\bar p+l)\,S^{\kappa,\tilde\alpha\tilde\beta}_b(p+l)\Big\}\Big\}\,,
\end{aligned}
\tag{181}
$$

$$
\begin{aligned}
\hat F^{\mu,\alpha\beta}_{-,ab}(\bar p,p) = &-i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde\alpha\tilde\beta\sigma}{}^{\beta}\,\epsilon_{\bar\alpha\bar\beta\bar\sigma\bar\tau}\,S^{\bar\alpha\bar\beta,\sigma}_a(l)\Big\{\\
&+\bar p^{\bar\sigma}\,p^\alpha\,q_\kappa\Big\{S^{\bar\tau\kappa}_b(\bar p+l)\,S^{\mu,\tilde\alpha\tilde\beta}_b(p+l)-S^{\bar\tau\mu}_b(\bar p+l)\,S^{\kappa,\tilde\alpha\tilde\beta}_b(p+l)\Big\}\Big\}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
G^{\mu,\alpha\beta}_{abc}(\bar p,p) = &+2\,i\int\frac{d^4l}{(2\pi)^4}\,q_\nu\,\epsilon^{\mu\nu}{}_{\bar\sigma\bar\tau}\,p^\alpha\,\epsilon_{\tilde\alpha\tilde\beta\tau}{}^{\beta}\,S^{\rho,\tilde\alpha\tilde\beta}_a(l)\\
&\times(l+\bar p)^{\bar\tau}\,\bar p_\rho\,S_b(\bar p+l)\,S^{\bar\sigma\tau}_c(p+l)\,,
\end{aligned}
\tag{182}
$$

$$
\begin{aligned}
H^{\mu,\alpha\beta}_{abc}(\bar p,p) = &-2\,i\int\frac{d^4l}{(2\pi)^4}\,q_\nu\,\epsilon^{\mu\nu}{}_{\bar\sigma\bar\tau}\,p^\alpha\,\epsilon_{\tilde\alpha\tilde\beta\sigma}{}^{\beta}\,S^{\rho\sigma}_a(l)\\
&\times(l+\bar p)^{\bar\tau}\,\bar p_\rho\,S_b(\bar p+l)\,S^{\bar\sigma,\tilde\alpha\tilde\beta}_c(p+l)\,.
\end{aligned}
$$

The decay constant $d_{1^+\to\gamma 0^-}$ as of (98) is computed by contracting of the gauge invariant tensors (99- 102, 105) and (179-182) with the anti-symmetric tensor

$$
P^{(0-)}_{\mu,\alpha\beta}=-\frac{1}{2}\left(\Big\{g_{\mu\alpha}-\frac{p^\mu\,q^\alpha}{p\cdot q}\Big\}\,p_\beta-\Big\{g_{\mu\beta}-\frac{p^\mu\,q^\beta}{p\cdot q}\Big\}\,p_\alpha\right),
\tag{183}
$$

We provide the results in terms of the master loop integrals $I_{ab}, \bar{I}_{ab}, J_{abc}$ and $\bar{J}_{abc}$ introduced in (161). According to the arguments of section 3.3 reduced tadpole integrals are dropped. Using $p^2 = M_i^2$ and $(p-q)^2 = M_f^2$ we derive

$$
\begin{aligned}
16\,(p\cdot q)\,M_f^2\,P^{(0-)}_{\mu,\alpha\beta}\,A^{\mu,\alpha\beta}_{ab} &= 2\,M_f^2\,\Big[M_i^6 - \Big(m_a^2 - m_b^2\Big)^2 M_i^2 \\
&\quad - \Big(m_a^2 + 3\,m_b^2 + M_i^2\Big)\Big(M_i^2 - M_f^2\Big)\,M_i^2 \\
&\quad + \Big(M_f^2 - M_i^2\Big)^2\Big(-m_a^2 + m_b^2 + M_i^2\Big)\Big]\,I_{ab} \\
&+ \Big[M_f^8 - 2\Big(m_a^2 + m_b^2\Big)\,M_f^6 + \Big(m_a^2 - m_b^2\Big)^2 M_f^4 \\
&\quad + 2\Big(\Big(m_a^2 - m_b^2\Big)^2 - M_f^4\Big)\,M_i^2 M_f^2 \\
&\quad - \Big(m_a^4 - 2\Big(m_b^2 + M_f^2\Big)\,m_a^2 + \Big(m_b^2 - M_f^2\Big)^2\Big)\,M_i^4\Big]\,\bar{I}_{ab} \\
&+ 4\,m_a^2\,M_f^2\,M_i^2\Big(m_a^2 - m_b^2 - M_f^2\Big)\Big(M_i^2 - M_f^2\Big)\,J_{aab}\,,
\end{aligned}
$$

$$
\begin{aligned}
24\,M_i^2\,P^{(0-)}_{\mu,\alpha\beta}\,\bar{A}^{\mu,\alpha\beta}_{ab} &= \Big(M_i^2 - M_f^2\Big)\Big[3\,M_i^6 - 3\Big(m_a^2 - m_b^2\Big)^2 M_i^2 \\
&\quad - \Big(M_i^2 - M_f^2\Big)\Big(M_i^4 + \Big(m_a^2 + m_b^2\Big)\,M_i^2 - 2\Big(m_a^2 - m_b^2\Big)^2\Big)\Big]\,I_{ab}\,,
\end{aligned}
$$

$$
\begin{aligned}
16\,(p\cdot q)\,M_f^2\,P^{(0-)}_{\mu,\alpha\beta}\,B^{\mu,\alpha\beta}_{ab} &= M_f^2\Big[-\Big(m_a^2 - m_b^2\Big)\,M_f^4 - \Big(8\,m_b^2 + M_f^2\Big)\,M_i^4 \\
&\quad - \Big(M_f^4 - \Big(5\,m_a^2 + 3\,m_b^2\Big)\,M_f^2 + 2\Big(m_a^2 - m_b^2\Big)^2\Big)M_i^2\Big]\,I_{ab} \\
&+ \Big[M_f^8 - 2\Big(m_a^2 + m_b^2\Big)\,M_f^6 + \Big(m_a^2 - m_b^2\Big)^2 M_f^4 \\
&\quad - \Big(M_f^4 + \Big(m_a^2 - m_b^2\Big)\,M_f^2 - 2\Big(m_a^2 - m_b^2\Big)^2\Big)\,M_i^2\,M_f^2 \\
&\quad - \Big(-2\,M_f^4 + \Big(m_a^2 - 5\,m_b^2\Big)\,M_f^2 + \Big(m_a^2 - m_b^2\Big)^2\Big)\,M_i^4\Big]\,\bar{I}_{ab} \\
&- 2\,m_b^2\,M_f^2\,M_i^2\Big(M_f^2 - M_i^2\Big)\Big(-2\,m_a^2 + 2\,m_b^2 - M_f^2 + 3\,M_i^2\Big)\,\bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
12\,P^{(0-)}_{\mu,\alpha\beta}\,\bar{B}^{\mu,\alpha\beta}_{ab} &= -\Big(M_i^2 - M_f^2\Big)\Big[m_a^4 + \Big(m_b^2 - 2\,M_i^2\Big)\,m_a^2 \\
&\quad - 2\,m_b^4 + M_i^4 + 7\,m_b^2\,M_i^2\Big]\,I_{ab}\,,
\end{aligned}
$$

$$
\begin{aligned}
12\,M_i^2\,P^{(0-)}_{\mu,\alpha\beta}\,\tilde{B}^{\mu,\alpha\beta}_{ab} &= \Big(M_i^2 - M_f^2\Big)\Big[-3\,M_i^6 + 3\Big(m_a^2 - m_b^2\Big)^2 M_i^2 \\
&\quad + \Big(M_i^2 - M_f^2\Big)\Big(M_i^4 + \Big(m_a^2 + m_b^2\Big)\,M_i^2 - 2\Big(m_a^2 - m_b^2\Big)^2\Big)\Big]\,I_{ab}\,,
\end{aligned}
$$

$$
\begin{aligned}
16\,(p\cdot q)\,M_f^2\,P^{(0-)}_{\mu,\alpha\beta}\,C^{\mu,\alpha\beta}_{abc} &= -4\,m_c^2\,M_f^2\Big[-2\,M_i^6 + 3\,M_f^2\,M_i^4 \\
&\quad + M_f^2\Big(-3\,m_a^2 + 2m_b^2 + m_c^2 - M_f^2\Big)\,M_i^2 + \Big(m_a^2 - m_c^2\Big)\,M_f^4\Big]\,I_{ac} \\
&- 4\,m_c^2\,M_f^2\,M_i^2\Big[\Big(2\,m_a^2 - 3\,m_b^2 + m_c^2 - M_f^2\Big)\,M_f^2
\end{aligned}
$$

$$+ \left(m_b^2 - m_c^2 + M_f^2\right) M_i^2\Big]\, \bar{I}_{bc}$$

$$- 8\, m_c^2\, M_f^2\, M_i^2 \left[m_b^2 M_f^4 - \left(m_a^2 - m_b^2\right)^2 M_f^2 - 2 m_b^2\, M_i^2\, M_f^2 + m_b^2\, M_i^4\right] J_{abc}\,,$$

$$12\,(p\cdot q)\, M_f^2\, M_i^2\, P^{(0-)}_{\mu,\alpha\beta}\, \bar{C}^{\mu,\alpha\beta}_{abc} = -M_f^2 \Big[2 \left(M_f^3 - M_f\, M_i^2\right)^2 \left(m_a^2 - 2\, m_c^2\right) m_a^2$$

$$- \Big(\left(M_f^2 - M_i^2\right)^2 \left(M_f^2 + 3\, M_i^2\right) - 3\, m_b^2 \left(M_f^4 - 3\, M_f^2\, M_i^2\right) \Big)\, M_i^2\, m_a^2$$

$$+ \left(m_c^2 - M_i^2\right) \left(M_f^2 - M_i^2\right)^2 \left(-3\, M_i^4 + M_f^2\, M_i^2 + 2\, m_c^2\, M_f^2\right)$$

$$+ 6\, m_b^4\, M_f^2\, M_i^4 - 3\, m_b^2\, M_i^2 \left(M_i^2 - M_f^2\right) \left(2\, M_i^4 - M_f^2\, M_i^2 - m_c^2\, M_f^2\right) \Big]\, I_{ac}$$

$$- 3\, m_b^2\, M_f^2\, M_i^4 \Big[\left(2\, m_a^2 - 3\, m_b^2 + m_c^2 - M_f^2\right) M_f^2 + \left(m_b^2 - m_c^2 + M_f^2\right) M_i^2\Big]\, \bar{I}_{bc}$$

$$- 6\, m_b^2\, M_f^2\, M_i^4 \left[m_b^2\, M_f^4 - \left(m_a^2 - m_b^2\right)^2 M_f^2 - 2\, m_b^2\, M_i^2\, M_f^2 + m_b^2\, M_i^4\right] J_{abc}\,,$$

$$48\, M_f^2\, P^{(0-)}_{\mu,\alpha\beta}\, D^{\mu,\alpha\beta}_{ab} = 2\, M_f^2 \Big[- 3\, m_b^2 \left(-m_a^2 + m_b^2 + M_i^2\right) M_i^2$$

$$- \left(M_i^2 - M_f^2\right) \left(m_a^4 + \left(m_b^2 - 2\, M_i^2\right) m_a^2 - 2\, m_b^4 + M_i^4 + m_b^2\, M_i^2\right) \Big]\, I_{ab}$$

$$+ \Big[6 \left(m_a^4 - 2 \left(m_b^2 + M_f^2\right) m_a^2 + \left(m_b^2 - M_f^2\right)^2 \right) M_i^4$$

$$- 6\, M_f^2 \left(m_a^4 - \left(m_b^2 + 2\, M_f^2\right) m_a^2 + M_f^4 - 3\, m_b^2\, M_f^2\right) M_i^2\Big]\, \bar{I}_{ab}$$

$$+ 12\, m_b^4\, M_f^2\, M_i^2 \left(M_i^2 - M_f^2\right) \bar{J}_{abb}\,, \tag{184}$$

and

$$24\, m_b^2\,(p\cdot q)\, M_f^2\, P^{(0-)}_{\mu,\alpha\beta}\, E^{\mu,\alpha\beta}_{+,ab} = M_f^2 \Big[24\, m_b^2 \Big(\left(m_a^2 - m_b^2\right)^2 - M_i^4\Big)\, M_i^2$$

$$+ 3 \left(M_i^2 - M_f^2\right) \Big(m_a^4 + 20\, m_b^2\, m_a^2 + 11\, m_b^4 + M_i^4$$

$$- 2 \left(m_a^2 + 2\, m_b^2\right) M_i^2\Big)\, M_i^2 - 2 \left(M_f^2 - M_i^2\right)^2 \Big(m_a^4 + \left(7\, m_b^2 - 2\, M_i^2\right) m_a^2$$

$$- 8\, m_b^4 + M_i^4 + m_b^2\, M_i^2\Big)\Big]\, I_{ab}$$

$$-3 \Big[- \left(m_a^2 + 3\, m_b^2\right) M_f^2\, M_i^6 + 4 \left(m_b^3 - m_a^2 m_b\right)^2 \left(2\, M_f^2\, M_i^2 - M_i^4\right)$$

$$- \Big(-M_f^6 + 10\, m_b^2\, M_f^4 + \left(m_a^4 - 8\, m_b^2\, m_a^2 - 9\, m_b^4\right) M_f^2\Big)\, M_i^4$$

$$+ M_f^2 \Big(-M_f^6 + \left(m_a^2 + m_b^2\right) M_f^4 + 2\, m_b^2 \left(m_a^2 - m_b^2\right) M_f^2\Big)\, M_i^2$$

$$+ 4\, m_b^2\, M_f^4 \Big(m_a^4 - 2 \left(m_b^2 + M_f^2\right) m_a^2 + \left(m_b^2 - M_f^2\right)^2 \Big)$$

$$+ \left(m_a^2 - m_b^2\right)^2 M_i^6\Big]\, \bar{I}_{ab}$$

$$+ 6\, m_b^2\, M_f^2\, M_i^2 \left(M_i^2 - M_f^2\right) \Big[- 8\, m_b^4 + 7\, M_f^2\, m_b^2 - 2\, M_f^4 + M_i^4$$

$$+ \left(m_b^2 + M_f^2\right) M_i^2 + m_a^2 \left(8\, m_b^2 + 3\, M_f^2 - 3\, M_i^2\right) \Big]\, \bar{J}_{abb}\,,$$

$$24\, m_a^2\,(p\cdot q)\, M_f^2\, P^{(0-)}_{\mu,\alpha\beta}\, E^{\mu,\alpha\beta}_{-,ab} = M_f^2 \Big[24\, m_b^2 \Big(\left(m_a^2 - m_b^2\right)^2 - M_i^4\Big)\, M_i^2$$

$$
\begin{aligned}
&+3\left(M_i^2-M_f^2\right)\left(m_a^4+20\,m_b^2\,m_a^2+11\,m_b^4+M_i^4\right.\\
&\left.-2\left(m_a^2+2\,m_b^2\right)M_i^2\right)M_i^2+2\left(M_f^2-M_i^2\right)^2\left(m_a^4+\left(m_b^2-2\,M_i^2\right)m_a^2\right.\\
&\left.\left.-2\,m_b^4+M_i^4+7\,m_b^2\,M_i^2\right)\right]I_{ab}\\
&-3\Big[-\left(m_a^2+3\,m_b^2\right)M_f^2\,M_i^6+4\left(m_b^3-m_a^2\,m_b\right)^2\left(2\,M_f^2\,M_i^2-M_i^4\right)\\
&-\left(-M_f^6+10\,m_b^2\,M_f^4+\left(m_a^4-8\,m_b^2\,m_a^2-9\,m_b^4\right)M_f^2\right)M_i^4\\
&+M_f^2\left(-M_f^6+\left(m_a^2+m_b^2\right)M_f^4+2\,m_b^2\left(m_a^2-m_b^2\right)M_f^2\right)M_i^2\\
&+4\,m_b^2\,M_f^4\left(m_a^4-2\left(m_b^2+M_f^2\right)m_a^2+\left(m_b^2-M_f^2\right)^2\right)\\
&+\left(m_a^2-m_b^2\right)^2M_i^6\Big]\,\bar{I}_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(M_i^2-M_f^2\right)\Big[-8\,m_b^4+7\,M_f^2\,m_b^2-2\,M_f^4+M_i^4\\
&+\left(m_b^2+M_f^2\right)M_i^2+m_a^2\left(8\,m_b^2+3\,M_f^2-3\,M_i^2\right)\Big]\,\bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
12\,M_i^2\,P^{(0-)}_{\mu,\alpha\beta}\,\bar{E}^{\mu,\alpha\beta}_{ab}=&\left(M_i^2-M_f^2\right)\Big[-7\,M_i^6+\left(5\left(m_a^2+m_b^2\right)-M_f^2\right)M_i^4\\
&+\left(2\left(m_a^2-m_b^2\right)^2-\left(m_a^2+m_b^2\right)M_f^2\right)M_i^2\\
&+2\left(m_a^2-m_b^2\right)^2M_f^2\Big]\,I_{ab}\,, \qquad (185)
\end{aligned}
$$

and

$$
\begin{aligned}
24\,m_b^2\,(p\cdot q)\,M_f^2\,P^{(0-)}_{\mu,\alpha\beta}\,F^{\mu,\alpha\beta}_{+,ab}=&\,2\,M_f^2\Big[12\,m_a^2\left(-m_a^2+m_b^2+M_i^2\right)^2M_i^2\\
&+3\,m_a^2\left(7\,m_a^2+9\,m_b^2-7\,M_i^2\right)\left(M_i^2-M_f^2\right)M_i^2\\
&+\left(M_f^2-M_i^2\right)^2\Big(-7\,m_a^4+\left(5\,m_b^2+11\,M_i^2\right)m_a^2\\
&+2\left(m_b^2-M_i^2\right)^2\Big)\Big]\,I_{ab}\\
&+6\,m_a^2\Big[-2\,M_f^8+4\left(m_a^2+m_b^2\right)M_f^6-2\left(m_a^2-m_b^2\right)^2M_f^4\\
&+\left(4\,m_a^2-4\,m_b^2+M_f^2\right)\left(-m_a^2+m_b^2+M_f^2\right)M_i^2\,M_f^2\\
&+\Big(-3M_f^4+\left(m_a^2-9\,m_b^2\right)M_f^2+2\left(m_a^2-m_b^2\right)^2\Big)M_i^4\Big]\,\bar{I}_{ab}\\
&-12\,m_a^2\,m_b^2\,M_f^2\,M_i^2\left(4\,m_a^2-4\,m_b^2+M_f^2-5\,M_i^2\right)\left(M_f^2-M_i^2\right)\bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
24\,m_b^2\,(p\cdot q)\,M_f^2\,P^{(0-)}_{\mu,\alpha\beta}\,F^{\mu,\alpha\beta}_{-,ab}=&\,6\,m_b^2\,M_f^2\Big[4\left(-m_a^2+m_b^2+M_i^2\right)^2M_i^2\\
&+\left(M_i^2-M_f^2\right)\left(7\,m_a^2+9\,m_b^2-7\,M_i^2\right)M_i^2\\
&+\left(m_a^2-m_b^2+M_i^2\right)\left(M_i^2-M_f^2\right)^2\Big]\,I_{ab}\\
&-6\,m_b^2\Big[2\left(m_a^4-2\left(m_b^2+M_f^2\right)m_a^2+\left(m_b^2-M_f^2\right)^2\right)M_f^4
\end{aligned}
$$

$$
\begin{aligned}
&- \left(4\, m_a^2 - 4\, m_b^2 + M_f^2\right) \left(-m_a^2 + m_b^2 + M_f^2\right) M_i^2\, M_f^2 \\
&+ \left(3\, M_f^4 - \left(m_a^2 - 9\, m_b^2\right) M_f^2 - 2 \left(m_a^2 - m_b^2\right)^2 \right) M_i^4 \Big] \bar{I}_{ab} \\
&+ 12\, m_b^4\, M_f^2\, M_i^2 \left(M_f^2 - M_i^2\right) \left(-4\, m_a^2 + 4\, m_b^2 - M_f^2 + 5\, M_i^2\right) \bar{J}_{abb}\,, \qquad (186)
\end{aligned}
$$

and

$$
\begin{aligned}
2\, P^{(0-)}_{\mu,\alpha\beta}\, \hat{F}^{\mu,\alpha\beta}_{+,ab} &= m_a^2 \left(-2\, M_i^4 + M_f^2\, M_i^2 + \left(m_a^2 - m_b^2\right) M_f^2\right) I_{ab} \\
&+ m_a^2 \left(-m_a^2 + m_b^2 + M_f^2\right) M_i^2\, \bar{I}_{ab} + 2\, m_a^2\, m_b^2\, M_i^2 \left(M_i^2 - M_f^2\right) \bar{J}_{abb}\,, \\
6\, M_i^2\, P^{(0-)}_{\mu,\alpha\beta}\, \hat{F}^{\mu,\alpha\beta}_{-,ab} &= \Big[3\, M_i^8 - \left(3\, m_a^2 + 9\, m_b^2 + 4\, M_f^2\right) M_i^6 \\
&+ M_f^2 \left(2\, m_a^2 + 5\, m_b^2 + M_f^2\right) M_i^4 - 2 \left(m_a^2 - m_b^2\right)^2 M_f^4 \\
&+ M_f^2 \left(2\, m_a^4 - m_b^2\, m_a^2 - m_b^4 + \left(m_a^2 + m_b^2\right) M_f^2\right) M_i^2\Big] I_{ab} \\
&+ 3\, m_b^2 \left(-m_a^2 + m_b^2 + M_f^2\right) M_i^4\, \bar{I}_{ab} \\
&+ 6\, m_b^4\, M_i^4 \left(M_i^2 - M_f^2\right) \bar{J}_{abb}\,, \qquad (187)
\end{aligned}
$$

and

$$
\begin{aligned}
12\, (p\cdot q)\, M_f^2\, P^{(0-)}_{\mu,\alpha\beta}\, G^{\mu,\alpha\beta}_{abc} &= 3\, m_c^2\, M_f^2 \Big[2\, M_i^6 - 3\, M_f^2\, M_i^4 \\
&+ M_f^2 \left(-m_a^2 - 2m_b^2 + 3\, m_c^2 + M_f^2\right) M_i^2 + \left(m_a^2 - m_c^2\right) M_f^4\Big] I_{ac} \\
&+ M_i^2 \Big[\left(3\, m_c^2 \left(3M_f^2 - M_i^2\right) M_f^2 + 4 \left(m_a^2 + M_f^2\right) \left(M_f^2 - M_i^2\right)^2 \right) m_b^2 \\
&- 2 \left(M_f^2 - M_i^2\right)^2 m_b^4 - 6\, m_c^4\, M_f^4 - 2 \left(m_a^2 - M_f^2\right)^2 \left(M_f^2 - M_i^2\right)^2 \\
&+ 3\, m_c^2\, M_f^2 \left(M_f^2 - m_a^2\right) \left(M_f^2 - M_i^2\right) \Big] \bar{I}_{ab} \\
&+ 6\, m_c^2\, M_f^2\, M_i^2 \Big[M_f^2\, m_b^4 - \left(2\, m_c^2\, M_f^2 + \left(M_f^2 - M_i^2\right)^2 \right) m_b^2 + m_c^4\, M_f^2\Big] \bar{J}_{abc}\,, \\
48\, (p\cdot q)\, M_f^2\, P^{(0-)}_{\mu,\alpha\beta}\, H^{\mu,\alpha\beta}_{abc} &= -2\, M_f^2 \Big[6 \left(m_b^2 - m_c^2\right) \left(m_a^2 + m_b^2 - M_i^2\right) M_i^4 \\
&+ 3 \left(M_i^2 - M_f^2\right) \left(m_a^4 + \left(-3\, m_b^2 + 3\, m_c^2 - 2\, M_i^2\right) m_a^2 \right.\\
&\left. - \left(m_c^2 - M_i^2\right) \left(m_b^2 + M_i^2\right) \right) M_i^2 - \left(M_f^2 - M_i^2\right)^2 \left(4\, m_a^4 \right.\\
&\left. - \left(2\, m_c^2 + 5\, M_i^2\right) m_a^2 - 2\, m_c^4 + M_i^4 + 7\, m_c^2 M_i^2\right)\Big] I_{ac} \\
&- 6\, M_i^2 \Big[\left(2\, M_f^4 - 3\, M_i^2 M_f^2 + M_i^4\right) m_a^4 - 2 \left(m_b^2 \left(3\, M_f^4 - 3\, M_i^2\, M_f^2 + M_i^4\right) \right.\\
&\left. - M_f^2 \left(2\, M_f^2 - M_i^2\right) \left(m_c^2 - M_f^2 + M_i^2\right) \right) m_a^2 - \left(m_b^2 - M_f^2\right) \left(2\, M_f^6 \right.\\
&\left. - \left(-3\, m_b^2 + 2\, m_c^2 + 3\, M_f^2\right) M_i^2\, M_f^2 + \left(M_f^2 - m_b^2\right) M_i^4\right)\Big] \bar{I}_{ab} \\
&- 12\, M_f^2\, M_i^2 \Big[m_c^2 \left(M_i^2 - M_f^2\right)^3 + \left(m_b^2 - m_c^2\right)^2 \left(2\, m_a^2 - M_i^2\right) \left(M_i^2 - M_f^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ m_c^2 \left(m_a^2 + m_b^2 - M_i^2\right) \left(M_f^2 - M_i^2\right)^2 \\
&- \left(m_b^2 - m_c^2\right)^2 M_i^2 \left(m_a^2 + m_b^2 - M_i^2\right) \Big] \bar{J}_{abc} \,. \qquad (188)
\end{aligned}
$$

Appendix F

We derive the contribution of the $\phi\, D_s^*$ and $K^* D^*$ channels to the $1^+ \to \gamma\, 0^+$ process. According to (108) it suffices to evaluate the contractions of the four loop tensors introduced in (109, 110, 111) with the projector

$$P^{(0+)}_{\mu,\alpha\beta} = -\frac{1}{2}\, q^\tau\, p^\sigma \left\{ \epsilon_{\mu\tau\sigma\alpha}\, p_\beta - \epsilon_{\mu\tau\sigma\beta}\, p_\alpha \right\}. \tag{189}$$

It holds

$$\begin{aligned}
&\frac{2\, M_f^2}{(p\cdot q)\, M_i^2}\, P^{(0+)}_{\mu,\alpha\beta}\, A^{\mu,\alpha\beta}_{+,abc} = 2\, m_c^2\, M_f^2\, I_{ac} + 2\, m_c^2 \left(m_c^2 - m_b^2\right) M_f^2\, \bar J_{abc} \\
&- \Big[\left(2\, m_c^2 + \left(M_i^2 - M_f^2\right)\right) M_f^2 + \left(m_a^2 - m_b^2\right) \left(M_f^2 - M_i^2\right) \Big]\, \bar I_{ab}\,, \\[1ex]
&\frac{24\, m_b^2\, M_f^2}{(p\cdot q)}\, P^{(0+)}_{\mu,\alpha\beta}\, B^{\mu,\alpha\beta}_{+,ab} = M_f^2 \Big[-2 M_i^6 + \left(m_a^2 - 11\, m_b^2 - M_f^2\right) M_i^4 \\
&\qquad + \left(m_a^4 + 5\left(8\, m_b^2 + M_f^2\right) m_a^2 + 7\left(m_b^4 - m_b^2\, M_f^2\right)\right) M_i^2 \\
&\qquad - 2\left(2\, m_a^4 - m_b^2\, m_a^2 - m_b^4\right) M_f^2 \Big]\, I_{ab} \\
&+ \Big[3\left(m_a^4 - m_b^4 + M_f^4 - 2\left(m_a^2 - 2\, m_b^2\right) M_f^2\right) M_i^4 \\
&\qquad - 6\, m_b^2\, M_f^2 \left(7\, m_a^2 + m_b^2 - M_f^2\right) M_i^2 \Big]\, \bar I_{ab} \\
&+ 6\, m_b^2\, M_f^2\, M_i^2 \left(3\, m_a^2 + m_b^2 - M_f^2 - 2\, M_i^2\right) \left(M_f^2 - M_i^2\right) \bar J_{abb}\,, \\[1ex]
&\frac{4\, M_f^2}{(p\cdot q)}\, P^{(0+)}_{\mu,\alpha\beta}\, C^{\mu,\alpha\beta}_{+,ab} = M_f^2 \Big[\left(M_f^2 + 7\, M_i^2\right) m_b^2 \\
&\qquad + \left(m_a^2 + M_i^2\right) \left(M_i^2 - M_f^2\right) \Big]\, I_{ab} \\
&+ M_i^2 \Big[\left(m_a^2 + M_f^2\right) \left(M_i^2 - M_f^2\right) - m_b^2 \left(7\, M_f^2 + M_i^2\right) \Big]\, \bar I_{ab} \\
&+ 8\, m_b^2\, M_f^2\, M_i^2 \left(M_f^2 - M_i^2\right) \bar J_{abb}\,, \\[1ex]
&24\, P^{(0+)}_{\mu,\alpha\beta}\, D^{\mu,\alpha\beta}_{+,abc} = \Big[3\, m_b^2 \left(m_a^2 + 3\, m_c^2 + M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^2 \\
&\qquad - \left(M_f^2 - M_i^2\right)^2 \left(m_a^4 + \left(4\, m_c^2 - 2\, M_i^2\right) m_a^2 - 5\, m_c^4 + M_i^4 + 4\, m_c^2\, M_i^2\right) \\
&\qquad + 6\, m_b^2 \left(m_b^2 - m_a^2\right) M_i^4 \Big]\, I_{ac} \\
&+ 3\, m_b^2\, M_i^2 \Big[\left(\left(3\, m_c^2 + M_f^2\right) \left(M_f^2 - M_i^2\right) - m_b^2 \left(M_f^2 + M_i^2\right) \right) \\
&\qquad + 2\, m_a^2\, M_f^2 \Big]\, \bar I_{bc} \\
&+ 6\, m_b^2\, M_i^2 \Big[- \left(m_a^2 - m_b^2\right)^2 M_i^2 - m_a^2 \left(M_f^2 - M_i^2\right)^2
\end{aligned}$$

$$+ \left(m_a^2 - m_b^2\right) \left(m_a^2 + m_c^2 + M_i^2\right) \left(M_i^2 - M_f^2\right) \Big] J_{abc} \,, \tag{190}$$

where we use $p^2 = M_i^2$ and $(p - q)^2 = M_f^2$.

Appendix G

We collect explicit expressions for the contributions to the decay amplitude (113) linear in coupling constant $\hat g_H$ (see (81)). It is derived

$$
\begin{aligned}
-i\, M^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{1^+\to\gamma\, 1^-} &= \frac{e\, g_T\, \hat g_H}{4\, f^2}\Big\{F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\phi D_s^*}(\bar p,p) - E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,D^*K^*}(\bar p,p) + F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,K^*D^*}(\bar p,p)\Big\}\\
&+ \frac{e\, \tilde g_V\, \hat g_H}{4\, f^2}\Big\{F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\phi D_s^*}(\bar p,p) - E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,D^*K^*}(\bar p,p) + F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,K^*D^*}(\bar p,p)\Big\}\\
&+ \frac{e\, \tilde g_V\, \hat g_H}{4\, f^2}\Big\{F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{\phi D_s^*}(\bar p,p) - E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D^*K^*}(\bar p,p) + F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{K^*D^*}(\bar p,p)\Big\}\\
&+ \frac{\hat g_H}{8\, f^2\, M_V^2}\Big[\tilde e_C - e + \frac{\tilde e_Q}{3}\Big]\Big\{g_T\, \hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\phi D_s^*}(\bar p,p)\\
&\qquad\qquad + \tilde g_V\,(\hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{\phi D_s^*}(\bar p,p) + \hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\phi D_s^*}(\bar p,p))\Big\}\\
&+ \frac{\hat g_H}{4\, f^2\, M_V^2}\Big[\tilde e_C - \frac{e}{2} - \frac{\tilde e_Q}{6}\Big]\Big\{g_T\, \hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,K^*D^*}(\bar p,p)\\
&\qquad\qquad + \tilde g_V\,(\hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{K^*D^*}(\bar p,p) + \hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,K^*D^*}(\bar p,p))\Big\}\\
&- \frac{\hat g_H}{4\, f^2\, m_V^2}\Big\{e_T\, \bar E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,D^*K^*}(\bar p,p) + \tilde e_V\Big(\bar E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,D^*K^*}(\bar p,p) + \bar E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D^*K^*}(\bar p,p)\Big)\Big\}\\
&+ \frac{\tilde e_E\, \hat g_H}{6\, f^2\, m_V^2}\Big\{\tilde E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D^*K^*}(\bar p,p) + 2\, \tilde E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D_s^*\phi}(\bar p,p)\Big\}\\
&+ \frac{(e_C + e_Q/6)\, \hat g_H}{8\, f^2\, M_V^2}\Big\{\tilde g_T\, H^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{K^*DD^*}(\bar p,p) + g_E\, \bar H^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{K^*DD^*}(\bar p,p)\Big\}\\
&+ \frac{(e_C - e_Q/3)\, \hat g_H}{16\, f^2\, M_V^2}\Big\{\tilde g_T\, H^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{\phi D_s D_s^*}(\bar p,p) + g_E\, \bar H^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{\phi D_s D_s^*}(\bar p,p)\Big\}\\
&- \frac{e_A\, \tilde g_P\, \hat g_H}{96\, f^3\, m_V}\Big\{G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D^*KK^*}(\bar p,p) + \bar G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D^*KK^*}(\bar p,p)\Big\}\\
&- \frac{e_A\, \tilde g_P\, \hat g_H}{64\, f^3\, m_V}\Big\{G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D_s^*\eta\phi}(\bar p,p) + \bar G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{D_s^*\eta\phi}(\bar p,p)\Big\}\,, \qquad (191)
\end{aligned}
$$

with

$$
\begin{aligned}
E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab}(\bar p,p) &= +\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde\alpha\tilde\beta\rho}{}^{\beta}\, S_a^{\bar\sigma,\tilde\alpha\tilde\beta}(l)\,\Big\{\bar p^{\bar\alpha}\, p^\alpha\, S^{\mu\bar\beta}_{\bar p,\ \bar\sigma\tau}(p)\, S_b^{\tau\rho}(p+l)\\
&\quad + g^{\bar\alpha}_{\ \bar\sigma}\, p^\alpha\, S_b^{\bar\beta,\mu\rho}(\bar p+l) + g^{\bar\alpha}_{\ \bar\sigma}\, g^{\mu\alpha}\, S_b^{\bar\beta\rho}(\bar p+l) + g^{\bar\alpha}_{\ \bar\sigma}\, p^\alpha\, S_b^{\mu\bar\beta,\rho}(p+l)\\
&\quad + g^{\bar\alpha}_{\ \bar\sigma}\, g_{\bar\kappa\kappa}\, p^\alpha\Big\{S_b^{\bar\beta\bar\kappa}(\bar p+l)\, S_b^{\mu\kappa,\rho}(p+l) + S_b^{\bar\beta,\mu\bar\kappa}(\bar p+l)\, S_b^{\kappa\rho}(p+l)\Big\}\Big\}\,,\\
E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,ab}(\bar p,p) &= +\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde\alpha\tilde\beta\rho}{}^{\beta}\, g_{\bar\sigma\sigma}\, S_a^{\bar\sigma\bar\beta,\tilde\alpha\tilde\beta}(l)\,\Big\{\bar p^{\bar\alpha}\, p^\alpha\, S_b^{\sigma,\mu\rho}(\bar p+l)
\end{aligned}
$$

$$+\,\bar{p}^{\bar{\alpha}}\,g^{\mu\alpha}\,S_b^{\sigma\rho}(\bar{p}+l)+\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,S_b^{\mu\sigma,\rho}(p+l)+g^{\bar{\alpha}\mu}\,p^{\alpha}\,S_b^{\sigma\rho}(p+l)$$

$$+\,\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,g_{\bar{\kappa}\kappa}\left\{S_b^{\sigma\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\rho}(p+l)+S_b^{\sigma,\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa\rho}(p+l)\right\}\Bigg\}\,,$$

$$E_{-,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=+\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,g_{\bar{\sigma}\sigma}\,S_a^{\bar{\sigma},\tilde{\alpha}\tilde{\beta}}(l)\Bigg\{g^{\mu\bar{\alpha}}\,p^{\alpha}\,S_b^{\sigma\bar{\beta},\rho}(p+l)$$

$$+\,\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,g_{\bar{\kappa}\kappa}\left\{S_b^{\sigma\bar{\beta},\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\rho}(p+l)+S_b^{\sigma\bar{\beta},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa\rho}(p+l)\right\}$$

$$+\,\bar{p}^{\bar{\alpha}}\,g^{\mu\alpha}\,S_b^{\sigma\bar{\beta},\rho}(\bar{p}+l)+\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,S_b^{\sigma\bar{\beta},\mu\rho}(\bar{p}+l)\Bigg\}\,, \tag{192}$$

and

$$\bar{E}_{ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=\ i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,p^{\alpha}\,S_a^{\bar{\alpha},\tilde{\alpha}\tilde{\beta}}(l)\,S_b^{\tau\rho}(p+l)\left(g^{\mu}{}_{\tau}\,q^{\bar{\beta}}-g^{\mu\bar{\beta}}\,q_{\tau}\right),$$

$$\tilde{E}_{ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=\ 2\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,p^{\alpha}\,S_a^{\bar{\beta},\tilde{\alpha}\tilde{\beta}}(l)\,\bar{p}^{\bar{\alpha}}\,S_b^{\mu\tau,\rho}(p+l)\,q_{\tau}\,,$$

$$\bar{E}_{+,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=\ i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,p^{\alpha}\,S_a^{\sigma\bar{\beta},\tilde{\alpha}\tilde{\beta}}(l)\,\bar{p}^{\bar{\alpha}}\,S_b^{\tau\rho}(p+l)$$

$$\times\left(g^{\mu}{}_{\tau}\,q_{\sigma}-g^{\mu}{}_{\sigma}\,q_{\tau}\right),$$

$$\bar{E}_{-,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=\ i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,p^{\alpha}\,S_a^{\sigma,\tilde{\alpha}\tilde{\beta}}(l)\,\bar{p}^{\bar{\alpha}}\,S_b^{\tau\bar{\beta},\rho}(p+l)$$

$$\times\left(g^{\mu\tau}\,q_{\sigma}-g^{\mu\sigma}\,q_{\tau}\right), \tag{193}$$

and

$$F_{ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=-i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,S_a^{\bar{\sigma}\bar{\rho}}(l)\Bigg\{\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,S^{\mu\bar{\beta}}_{\bar{p},\,\bar{\sigma}\tau}(p)\,S_b^{\tau,\tilde{\alpha}\tilde{\beta}}(p+l)$$

$$+\,g^{\bar{\alpha}}_{\ \bar{\sigma}}\,g^{\mu\alpha}\,S_b^{\bar{\beta},\tilde{\alpha}\tilde{\beta}}(\bar{p}+l)+g^{\bar{\alpha}}_{\ \bar{\sigma}}\,p^{\alpha}\,S_b^{\mu\bar{\beta},\tilde{\alpha}\tilde{\beta}}(p+l)$$

$$+\,g^{\bar{\alpha}}_{\ \bar{\sigma}}\,g_{\bar{\kappa}\kappa}\,p^{\alpha}\left\{S_b^{\bar{\beta}\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)+S_b^{\bar{\beta},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\right\}\Bigg\}\,,$$

$$F_{+,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=-\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,g_{\bar{\sigma}\sigma}\,S_a^{\bar{\sigma}\bar{\beta},\bar{\rho}}(l)\Bigg\{\bar{p}^{\bar{\alpha}}\,g^{\mu\alpha}S_b^{\sigma,\tilde{\alpha}\tilde{\beta}}(\bar{p}+l)$$

$$+\,\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,S_b^{\mu\sigma,\tilde{\alpha}\tilde{\beta}}(p+l)+g^{\bar{\alpha}\mu}\,p^{\alpha}\,S_b^{\sigma,\tilde{\alpha}\tilde{\beta}}(p+l)$$

$$+\,\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,g_{\bar{\kappa}\kappa}\left\{S_b^{\sigma\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)+S_b^{\sigma,\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\right\}\Bigg\}\,,$$

$$F_{-,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p)=-i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,g_{\bar{\sigma}\sigma}\,S_a^{\bar{\sigma}\bar{\rho}}(l)\Bigg\{\bar{p}^{\bar{\alpha}}\,g^{\mu\alpha}\,S_b^{\sigma\bar{\beta},\tilde{\alpha}\tilde{\beta}}(\bar{p}+l)$$

$$+\,g^{\mu\bar{\alpha}}\,p^{\alpha}\,S_b^{\sigma\bar{\beta},\tilde{\alpha}\tilde{\beta}}(p+l)+\,\bar{p}^{\bar{\alpha}}\,p^{\alpha}\,g_{\bar{\kappa}\kappa}\Big\{S_b^{\sigma\bar{\beta},\bar{\kappa}}(\bar{p}+l)\,S_b^{\mu\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)$$

$$+S_b^{\sigma\bar{\beta},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\Big\}\Bigg\}\,, \tag{194}$$

and

$$\hat{F}_{ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = -i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,S_a^{\bar{\sigma}\bar{\rho}}(l)\Bigg\{$$
$$+\,g^{\bar{\alpha}}_{\;\bar{\sigma}}\,q_\kappa\,p^\alpha\Big\{S_b^{\bar{\beta}\kappa}(\bar{p}+l)\,S_b^{\mu,\tilde{\alpha}\tilde{\beta}}(p+l) - S_b^{\bar{\beta},\mu\bar{\kappa}}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\Big\}\Bigg\}\,,$$
$$\hat{F}_{-,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = -i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,g_{\bar{\sigma}\sigma}\,S_a^{\bar{\sigma}\bar{\rho}}(l)\Bigg\{$$
$$+\,\bar{p}^{\bar{\alpha}}\,p^\alpha\,q_\kappa\Big\{S_b^{\sigma\bar{\beta},\kappa}(\bar{p}+l)\,S_b^{\mu,\tilde{\alpha}\tilde{\beta}}(p+l) - S_b^{\sigma\bar{\beta},\mu}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\Big\}\Bigg\}\,,$$
$$\hat{F}_{+,ab}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = -\,i\int\frac{d^4l}{(2\pi)^4}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,g_{\bar{\sigma}\sigma}\,S_a^{\bar{\sigma}\bar{\beta},\bar{\rho}}(l)\Bigg\{$$
$$+\,\bar{p}^{\bar{\alpha}}\,p^\alpha\,q_\kappa\Big\{S_b^{\sigma\kappa}(\bar{p}+l)\,S_b^{\mu,\tilde{\alpha}\tilde{\beta}}(p+l) - S_b^{\sigma\mu}(\bar{p}+l)\,S_b^{\kappa,\tilde{\alpha}\tilde{\beta}}(p+l)\Big\}\Bigg\}\,, \tag{195}$$

and

$$\bar{G}_{abc}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = +\,2\,i\int\frac{d^4l}{(2\pi)^4}\,p^\alpha\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,S_a^{\bar{\tau},\tilde{\alpha}\tilde{\beta}}(l)\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,\epsilon^{\bar{\alpha}\bar{\beta}}{}_{\bar{\sigma}\bar{\tau}}$$
$$\times(\bar{p}+l)^\tau\,(\bar{p}+l)^{\bar{\sigma}}\,S_b(\bar{p}+l)\,S_c^{\sigma\rho}(p+l)\,,$$
$$G_{abc}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = +\,2\,i\int\frac{d^4l}{(2\pi)^4}\,p^\alpha\,\epsilon_{\tilde{\alpha}\tilde{\beta}\rho}{}^{\beta}\,S_a^{\bar{\mu}\bar{\nu},\tilde{\alpha}\tilde{\beta}}(l)\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,\epsilon_{\bar{\mu}\bar{\nu}}{}^{\bar{\sigma}\bar{\beta}}$$
$$\times(\bar{p}+l)^\tau\,(\bar{p}+l)_{\bar{\sigma}}\,\bar{p}^{\bar{\alpha}}\,S_b(\bar{p}+l)\,S_c^{\sigma\rho}(p+l)\,,$$
$$\bar{H}_{abc}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = -\,2\,i\int\frac{d^4l}{(2\pi)^4}\,p^\alpha\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,S_a^{\bar{\tau}\bar{\rho}}(l)\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,\epsilon^{\bar{\alpha}\bar{\beta}}{}_{\bar{\sigma}\bar{\tau}}$$
$$\times(\bar{p}+l)^\tau\,(\bar{p}+l)^{\bar{\sigma}}\,S_b(\bar{p}+l)\,S_c^{\sigma,\tilde{\alpha}\tilde{\beta}}(p+l)\,,$$
$$H_{abc}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}(\bar{p},p) = -\,2\,i\int\frac{d^4l}{(2\pi)^4}\,p^\alpha\,\epsilon_{\tilde{\alpha}\tilde{\beta}\bar{\rho}}{}^{\beta}\,S_a^{\bar{\mu}\bar{\nu},\bar{\rho}}(l)\,q_\nu\,\epsilon^{\mu\nu}{}_{\sigma\tau}\,\epsilon_{\bar{\mu}\bar{\nu}}{}^{\bar{\sigma}\bar{\beta}}$$
$$\times(\bar{p}+l)^\tau\,(\bar{p}+l)_{\bar{\sigma}}\,\bar{p}^{\bar{\alpha}}\,S_b(\bar{p}+l)\,S_c^{\sigma,\tilde{\alpha}\tilde{\beta}}(p+l)\,. \tag{196}$$

According to (116) it suffices to compute the projections of the tensor integrals (117- 120, 122, 192 - 196) with two tensors

$$P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta} = -\frac{1}{4}\,p^\sigma\,q^\tau\Big\{\epsilon_{\sigma\tau\bar{\alpha}\mu}\,\bar{p}_{\bar{\beta}}\,q_\alpha\,p_\beta - \epsilon_{\sigma\tau\bar{\alpha}\mu}\,\bar{p}_{\bar{\beta}}\,q_\beta\,p_\alpha$$

$$- \epsilon_{\sigma\tau\bar{\beta}\mu}\, \bar{p}_{\bar{\alpha}}\, q_\alpha\, p_\beta + \epsilon_{\sigma\tau\bar{\beta}\mu}\, \bar{p}_{\bar{\alpha}}\, q_\beta\, p_\alpha \Big\} ,$$

$$P^{(2)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta} = -\frac{1}{4}\, p^\sigma\, q^\tau \Big\{ \epsilon_{\sigma\tau\mu\alpha}\, q_{\bar{\alpha}}\, \bar{p}_{\bar{\beta}}\, p_\beta - \epsilon_{\sigma\tau\mu\beta}\, q_{\bar{\alpha}}\, \bar{p}_{\bar{\beta}}\, p_\alpha$$
$$- \epsilon_{\sigma\tau\mu\alpha}\, q_{\bar{\beta}}\, \bar{p}_{\bar{\alpha}}\, p_\beta + \epsilon_{\sigma\tau\mu\beta}\, q_{\bar{\beta}}\, \bar{p}_{\bar{\alpha}}\, p_\alpha \Big\} . \qquad (197)$$

We establish the results

$$48\, \frac{M_i^2}{(p\cdot q)}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, A^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{+,\,ab} = \Big[-12 \left(m_a^2 - m_b^2 + M_i^2\right) \left(m_b^2 + M_i^2\right) M_i^4$$
$$+ 6 \left(M_i^2 - M_f^2\right) \left(m_b^4 + 2\, M_i^2\, m_b^2 + 5\, M_i^4 + m_a^2 \left(M_i^2 - m_b^2\right)\right) M_i^2$$
$$- \left(M_f^2 - M_i^2\right)^2 \left(19\, M_i^4 - 8 \left(m_a^2 - 2\, m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2 \right)\Big] I_{ab}$$
$$+ 6 \left(m_a^2 - m_b^2 + M_f^2\right) \left(m_b^2 + M_f^2\right) M_i^2 \left(M_f^2 + M_i^2\right) \bar{I}_{ab}$$
$$- 12\, m_a^2 \left(m_b^2 + M_f^2\right) M_i^2 \left(M_f^2 - M_i^2\right) \left(M_f^2 + M_i^2\right) J_{aab} ,$$

$$48\, \frac{M_i^2}{(p\cdot q)}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, A^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{-,\,ab} = \Big[12 \left(M_i^2 - m_b^2\right) \left(m_a^2 - m_b^2 + M_i^2\right) M_i^4$$
$$- 6 \left(M_i^2 - M_f^2\right) \left(-m_b^4 - 4\, M_i^2\, m_b^2 + 5\, M_i^4 + m_a^2 \left(m_b^2 + M_i^2\right)\right) M_i^2$$
$$- \left(M_f^2 - M_i^2\right)^2 \left(-17\, M_i^4 + 4 \left(m_a^2 + m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2 \right)\Big] I_{ab}$$
$$-6 \left(m_b^2 - M_f^2\right) \left(-m_a^2 + m_b^2 - M_f^2\right) M_i^2 \left(M_f^2 + M_i^2\right) \bar{I}_{ab}$$
$$- 12\, m_a^2 \left(m_b^2 - M_f^2\right) M_i^2 \left(M_f^2 - M_i^2\right) \left(M_f^2 + M_i^2\right) J_{aab} ,$$

$$24\, \frac{M_i^4}{(p\cdot q)^3}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, \bar{A}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{+,\,ab} = \Big[- M_i^8 + 2 \left(m_a^2 - 2 \left(m_b^2 + 2 M_f^2\right)\right) M_i^6$$
$$- \left(m_a^4 + \left(4\, m_b^2 - 3\, M_f^2\right) m_a^2 - 5\, m_b^4 + m_b^2\, M_f^2\right) M_i^4$$
$$+ 2 \left(3\, m_a^4 - 7\, m_b^2\, m_a^2 + 4\, m_b^4\right) M_f^2\, M_i^2 - \left(m_a^2 - m_b^2\right)^3 M_f^2 \Big] I_{ab} ,$$

$$96\, \frac{\pm m_b^2}{M_f^2 \pm m_b^2}\, \frac{M_f^2\, M_i^2}{(p\cdot q)}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, B^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{\pm,\,ab} = M_f^2 \Big[24\, m_b^2 \left(M_i^2 - m_a^2 + m_b^2\right) M_i^4$$
$$- \left(M_i^2 - M_f^2\right) \left(m_a^4 + 10\, m_b^2\, m_a^2 - 11\, m_b^4 + M_i^4 - 2 \left(m_a^2 + m_b^2\right) M_i^2\right) M_i^2$$
$$- \left(M_f^2 - M_i^2\right)^2 \left(M_i^4 - 2 \left(m_a^2 - 5\, m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2\right) \Big] I_{ab}$$
$$- M_i^2 \Big[- \left(m_a^2 + 19\, m_b^2\right) M_f^6 - \left(m_a^4 + 10\, m_b^2\, m_a^2 - 11\, m_b^4\right) M_f^4$$
$$+ 2\, M_f^8 - 3 \left(-m_a^4 + 6\, m_b^2\, m_a^2 - 5\, m_b^4 + M_f^4 - 12\, m_b^2\, M_f^2\right) M_i^2 M_f^2$$
$$+ \left(M_f^4 + \left(m_a^2 + 7\, m_b^2\right) M_f^2 - 2 \left(m_a^2 - m_b^2\right)^2 \right) M_i^4 \Big] \bar{I}_{ab}$$
$$+ 6\, m_b^2\, M_f^2\, M_i^2 \left(M_f^2 - M_i^2\right) \Big[-2\, M_f^4 + M_i^2\, M_f^2 + M_i^4$$

$$+3\,m_a^2\left(M_f^2-M_i^2\right)+m_b^2\left(M_f^2+7\,M_i^2\right)\Big]\,\bar{J}_{abb}\,,$$

$$\begin{aligned}
48\,\frac{M_f^2\,M_i^4}{(p\cdot q)^2}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\bar{B}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} &= M_f^2\,\Big[-M_i^{10}+\left(2\,m_a^2-4\,m_b^2+M_f^2\right)M_i^8\\
&\quad-\left(m_a^4+\left(4\,m_b^2+M_f^2\right)m_a^2-5\,m_b^4-M_f^4-10\,m_b^2\,M_f^2\right)M_i^6\\
&\quad-M_f^2\left(m_a^4+\left(3\,M_f^2-8\,m_b^2\right)m_a^2-m_b^2\left(m_b^2+5\,M_f^2\right)\right)M_i^4\\
&\quad+\left(m_a^2-m_b^2\right)M_f^2\left(m_a^4-\left(m_b^2-3\,M_f^2\right)m_a^2-5\,m_b^2\,M_f^2\right)M_i^2\\
&\quad-\left(m_a^2-m_b^2\right)^3M_f^4\Big]\,I_{ab}\\
&+\left(m_b^2+M_f^2\right)M_i^4\,\Big[-2\,M_f^6+\left(m_a^2-17\,m_b^2\right)M_f^4+\left(m_a^2-m_b^2\right)^2M_f^2\\
&\quad-\Big(-M_f^4-\left(m_a^2+7\,m_b^2\right)M_f^2+2\left(m_a^2-m_b^2\right)^2\Big)M_i^2\Big]\,\bar{I}_{ab}\\
&+6\,m_b^2\,M_f^2\left(m_b^2+M_f^2\right)M_i^4\left(M_f^2-M_i^2\right)\Big[-m_a^2+m_b^2+2\,M_f^2-M_i^2\Big]\,\bar{J}_{abb}\,,
\end{aligned}$$

$$\frac{12\,M_i^2}{(p\cdot q)^3}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\tilde{B}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab}=M_f^2\Big[2\,M_i^4-\left(m_a^2+m_b^2\right)M_i^2-\left(m_a^2-m_b^2\right)^2\Big]\,I_{ab}\,,$$

$$\begin{aligned}
192\,M_f^2\,M_i^4\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,C^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,abc} &= M_f^2\,\Big[12\left(m_b^3-m_a^2\,m_b\right)^2M_i^8\\
&\quad-12\,m_b^2\left(m_b^2-m_a^2\right)\left(-m_a^2+m_b^2-M_i^2\right)\left(M_i^2-M_f^2\right)M_i^6\\
&\quad+m_b^2\left(M_f^2-M_i^2\right)^2\Big(-5\,M_i^4+4\left(m_a^2+m_c^2\right)M_i^2+\left(m_a^2-m_c^2\right)^2\Big)M_i^4\\
&\quad+\left(M_i^2-M_f^2\right)^3\Big(3\,M_i^6-\left(7\,m_a^2-5\,m_b^2+5\,m_c^2\right)M_i^4\\
&\quad+\left(5\,m_a^4+8\,m_b^2\,m_a^2+m_c^4-2\left(3\,m_a^2+8\,m_b^2\right)m_c^2\right)M_i^2\\
&\quad-\left(m_a^2-m_c^2\right)^2\left(m_a^2+m_b^2-m_c^2\right)\Big)\,M_i^2\\
&\quad+\left(M_f^2-M_i^2\right)^4\Big(-2\,M_i^6+\left(5\,m_a^2+9\,m_c^2\right)M_i^4\\
&\quad-2\left(2\,m_a^4-5\,m_c^2\,m_a^2+3\,m_c^4\right)M_i^2+\left(m_a^2-m_c^2\right)^3\Big)\Big]\,I_{ac}\\
&+m_b^2\,M_i^4\,\Big[-2\left(M_f^6+8\,M_i^2\,M_f^4-4\,M_i^4\,M_f^2+M_i^6\right)m_b^4\\
&\quad-\left(M_f^2-M_i^2\right)\Big(8\,M_f^6+5\,m_c^2\,M_f^4-3\left(3\,m_c^2+7\,M_f^2\right)M_i^2M_f^2\\
&\quad+\left(4\,m_c^2+M_f^2\right)M_i^4\Big)\,m_b^2-6\,m_a^4\,M_f^4\left(M_f^2+M_i^2\right)\\
&\quad+3\,m_a^2\,M_f^2\Big(\left(3\,m_b^2+m_c^2+M_f^2\right)M_f^4-2\left(-3\,m_b^2+m_c^2+3\,M_f^2\right)M_i^2\,M_f^2\\
&\quad+\left(-m_b^2+m_c^2+5\,M_f^2\right)M_i^4\Big)+\left(m_c^2-M_f^2\right)\left(M_f^2-M_i^2\right)^2\Big(M_f^2\,m_c^2\\
&\quad-2\,M_i^2\,m_c^2+5\,M_f^2\left(M_i^2-2\,M_f^2\right)\Big)\Big]\,\bar{I}_{bc}\\
&+6\,m_b^2\,M_f^2\,M_i^4\,\Big[2\left(m_a^2-m_b^2\right)^3M_i^4-2\,m_a^2\left(M_f^2-M_i^2\right)^4\\
&\quad+\left(m_a^2-m_b^2\right)^2\left(-3\,m_a^2+2\,m_b^2+m_c^2-3\,M_i^2\right)\left(M_i^2-M_f^2\right)M_i^2
\end{aligned}$$

$$
\begin{aligned}
&- \left(M_i^2 - M_f^2\right)^3 \left(\left(m_b^2 + 2\, m_c^2 - 2\, M_i^2\right) m_a^2 + m_b^2 \left(M_i^2 - 3\, m_c^2\right)\right) \\
&+ \left(m_a^2 - m_b^2\right) \left(M_f^2 - M_i^2\right)^2 \left(\left(3\, m_a^2 - m_b^2\right) M_i^2 \right. \\
&\left. + \left(m_a^2 - m_b^2\right) \left(m_a^2 - m_c^2\right) \right) \Big] J_{abc} \,,
\end{aligned}
$$

$$
\begin{aligned}
192\, M_f^2\, M_i^4\, P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, C^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,abc} = {} & M_f^2 \Big[12 \left(m_b^3 - m_a^2\, m_b\right)^2 M_i^8 \\
&- 12\, m_b^2 \left(m_b^2 - m_a^2\right) \left(-m_a^2 + m_b^2 - 3\, M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^6 \\
&+ m_b^2 \left(M_f^2 - M_i^2\right)^2 \left(7\, M_i^4 + 8 \left(5\, m_a^2 - 3\, m_b^2 - m_c^2\right) M_i^2 \right. \\
&\left. + \left(m_a^2 - m_c^2\right)^2 \right) M_i^4 - \left(M_i^2 - M_f^2\right)^3 \left(M_i^6 - \left(m_a^2 - 7\, m_b^2 + 3\, m_c^2\right) M_i^4 \right. \\
&- \left(m_a^4 - 4\, m_b^2\, m_a^2 - 3\, m_c^4 + 2 \left(m_a^2 - 2\, m_b^2\right) m_c^2\right) M_i^2 \\
&\left. + \left(m_a^2 - m_c^2\right)^2 \left(m_a^2 + m_b^2 - m_c^2\right) \right) M_i^2 \\
&+ \left(m_a^2 - m_c^2 - M_i^2\right) \left(M_f^2 - M_i^2\right)^4 \Big(- 2\, M_i^4 \\
&+ \left(m_a^2 - m_c^2\right)^2 + \left(m_a^2 + m_c^2\right) M_i^2\Big) \Big] I_{ac} \\
+ m_b^2\, M_i^4 \Big[& - 2 \left(M_f^6 + 8\, M_i^2 M_f^4 - 4\, M_i^4\, M_f^2 + M_i^6\right) m_b^4 \\
&- 6\, m_a^4\, M_f^4 \left(M_f^2 + M_i^2\right) + \left(M_f^2 - M_i^2\right) \left(4\, M_f^6 + 33\, M_i^2\, M_f^4 - M_i^4\, M_f^2 \right. \\
&\left. + m_c^2 \left(-5\, M_f^4 + 9\, M_i^2\, M_f^2 - 4\, M_i^4\right) \right) m_b^2 \\
&+ \left(m_c^2 - M_f^2\right) \left(M_f^2 - M_i^2\right)^2 \left(2\, M_f^4 + 5\, M_i^2\, M_f^2 + m_c^2 \left(M_f^2 - 2 M_i^2\right)\right) \\
&- 3\, m_a^2 M_f^2 \Big(\left(-3\, M_f^4 - 6\, M_i^2\, M_f^2 + M_i^4\right) m_b^2 \\
&+ \left(M_f^2 - M_i^2\right) \left(7\, M_f^4 + 5\, M_i^2\, M_f^2 + m_c^2 \left(M_i^2 - M_f^2\right)\right) \Big) \Big] \bar I_{bc} \\
+ 6\, m_b^2\, M_f^2\, M_i^4 \Big[& 2 \left(m_a^2 - m_b^2\right)^3 M_i^4 + 2\, m_a^2 \left(M_f^2 - M_i^2\right)^4 \\
&+ \left(m_a^2 - m_b^2\right)^2 \left(-3\, m_a^2 + 2\, m_b^2 + m_c^2 - 7\, M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^2 \\
&+ \left(m_a^2 - m_b^2\right) \left(M_f^2 - M_i^2\right)^2 \left(4\, M_i^4 + \left(11\, m_a^2 - 5\, m_b^2 - 4\, m_c^2\right) M_i^2 \right. \\
&\left. + \left(m_a^2 - m_b^2\right) \left(m_a^2 - m_c^2\right) \right) - \left(M_i^2 - M_f^2\right)^3 \left(4\, m_a^4 - 3\, m_b^2\, m_a^2 \right. \\
&\left. + \left(-2\, m_c^2 + 6\, M_i^2\right) m_a^2 + m_b^2 \left(m_c^2 - 3\, M_i^2\right) \right) \Big] J_{abc} \,,
\end{aligned}
$$

$$
\begin{aligned}
192\, M_i^2\, P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar C^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = {} & -M_f^2 \Big[12 \left(m_a^2 - m_b^2\right)^2 M_i^6 \\
&+ 12 \left(m_b^2 - m_a^2\right) \left(m_a^2 + m_b^2 - m_c^2\right) \left(M_i^2 - M_f^2\right) M_i^4 \\
&- \left(M_f^2 - M_i^2\right)^3 \left(M_i^4 - 2 \left(m_a^2 - 5\, m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^2 \right) \\
&+ \left(M_i^3 - M_f^2\, M_i\right)^2 \left(m_a^4 + \left(4\, m_c^2 - 2\, M_i^2\right) m_a^2 - 5\, m_c^4 + M_i^4 \right. \\
&\left. + 4 \left(3\, m_b^2 + m_c^2\right) M_i^2\right) \Big] I_{ac}
\end{aligned}
$$

$$
\begin{aligned}
&+ M_i^2\Big[-2\left(2\,M_f^6 + M_i^2\,M_f^4 - 11\,M_i^4 M_f^2 + 2\,M_i^6\right) m_b^4 \\
&\quad - \left(M_f^2 - M_i^2\right)\Big(-2\,M_f^6 - 9\,M_i^2\,M_f^4 + 11\,M_i^4\,M_f^2 \\
&\quad + m_c^2\left(M_f^4 - 21\,M_i^2 M_f^2 + 8\,M_i^4\right)\Big) m_b^2 + 6\,m_a^4\,M_f^4\left(M_f^2 + M_i^2\right) \\
&\quad - \left(m_c^2 - M_f^2\right)\left(M_f^2 - M_i^2\right)^2\left(2\,M_f^4 - M_i^2\,M_f^2 + m_c^2\left(M_f^2 + 4\,M_i^2\right)\right) \\
&\quad - 3\,m_a^2\,M_f^2\Big(\left(-M_f^4 + 6\,M_i^2\,M_f^2 + 3\,M_i^4\right) m_b^2 \\
&\quad + \left(M_f^2 - M_i^2\right)\left(M_f^4 - M_i^2\,M_f^2 + m_c^2\left(M_f^2 + 3\,M_i^2\right)\right)\Big)\Big]\,\bar{I}_{bc} \\
&+ 6\,M_f^2\,M_i^2\Big[\left(m_a^2 - m_b^2\right)^2\left(3\,m_a^2 + 2\,m_b^2 - 3\,m_c^2 + M_i^2\right)\left(M_i^2 - M_f^2\right) M_i^2 \\
&\quad + m_b^2\left(m_a^2 + m_c^2 + M_i^2\right)\left(M_i^2 - M_f^2\right)^3 - 2\left(m_a^2 - m_b^2\right)^3 M_i^4 \\
&\quad - \left(m_a^2 - m_b^2\right)\left(M_f^2 - M_i^2\right)^2\Big(\left(m_a^2 + 3\,m_b^2\right) M_i^2 \\
&\quad + \left(m_a^2 + m_b^2\right)\left(m_a^2 - m_c^2\right)\Big)\Big]\,J_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{192}{(p\cdot q)}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,D^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = 12\,m_c^2\,M_f^2\Big[M_i^4 - \left(m_a^2 - 2\,m_b^2 + m_c^2 + M_f^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right) M_f^2\Big]\,I_{ac} \\
&+ 4\Big[-\left(3\,m_c^2\,M_f^2\left(M_f^2 + M_i^2\right) - 2\left(m_a^2 + M_f^2\right)\left(M_f^2 - M_i^2\right)^2\right) m_b^2 \\
&\quad - \left(M_f^2 - M_i^2\right)^2 m_b^4 + 6\,m_c^4\,M_f^4 - \left(m_a^2 - M_f^2\right)^2\left(M_f^2 - M_i^2\right)^2 \\
&\quad + 3\,m_c^2\,M_f^2\left(M_f^2 - m_a^2\right)\left(M_f^2 - M_i^2\right)\Big]\,\bar{I}_{ab} \\
&- 24\,m_c^2\,M_f^2\Big[\left(m_b^2 - m_c^2\right)\left(M_f^2 - M_i^2\right) m_a^2 \\
&\quad + \left(m_c^2\,M_f^2 - m_b^2\,M_i^2\right)\left(-m_b^2 + m_c^2 + M_f^2 - M_i^2\right)\Big]\,\bar{J}_{abc}\,,
\end{aligned}
\tag{198}
$$

and

$$
\begin{aligned}
&96\,\frac{M_f^2\,M_i^2}{(p\cdot q)}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} = M_f^2\Big[\left(17\,m_a^2 - 25\,m_b^2 + 15\,M_f^2\right) M_i^6 \\
&\quad - 2\left(7\,M_f^4 - 3\left(m_a^2 + 7\,m_b^2\right) M_f^2 + 4\left(2\,m_a^4 + 5\,m_b^2\,m_a^2 - 7\,m_b^4\right)\right) M_i^4 \\
&\quad + \left(m_a^2 + 7\,m_b^2\right) M_f^2\left(3\,m_a^2 - 3\,m_b^2 + M_f^2\right) M_i^2 \\
&\quad - 13\,M_i^8 + \left(m_a^2 - m_b^2\right)^2 M_f^4\Big]\,I_{ab} \\
&+ \Big[-2\left(-5\,M_f^4 + 4\left(m_a^2 - 2m_b^2\right) M_f^2 + \left(m_a^2 - m_b^2\right)^2\right) M_i^6 \\
&\quad + 3\,M_f^2\left(3\,m_a^4 + 2\,m_b^2\,m_a^2 - 5\,m_b^4 + M_f^4 - 4\left(m_a^2 + 3\,m_b^2\right) M_f^2\right) M_i^4 \\
&\quad - M_f^4\left(-5\,m_a^4 - 14\,m_b^2\,m_a^2 + 19\,m_b^4 + M_f^4 + 4\left(m_a^2 + m_b^2\right) M_f^2\right) M_i^2\Big]\,\bar{I}_{ab} \\
&+ 6\,m_b^2\,M_f^2\,M_i^2\left(M_f^2 - M_i^2\right)\Big[-4\,M_i^4 + \left(m_a^2 + 7\,m_b^2 + M_f^2\right) M_i^2
\end{aligned}
$$

$$+ M_f^2 \left(3\, m_a^2 + 5\, m_b^2 - M_f^2\right) \Big]\, \bar{J}_{abb}\,,$$

$$96\, \frac{m_a^2\, M_i^2}{(p\cdot q)}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, E^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{+,\,ab} = -M_f^2 \Big[12\Big(\left(m_a^2 - M_i^2\right)^2 - m_b^4\Big)\, M_i^4$$
$$+ 3\left(M_i^2 - M_f^2\right)\left(-5M_i^4 + 4\left(m_a^2 + 3m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2\right) M_i^2$$
$$+ \left(M_f^2 - M_i^2\right)^2 \left(4M_i^4 + \left(7m_a^2 + m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2\right) \Big]\, I_{ab}$$
$$+ \Big[-2\Big(-5\, M_f^4 + 4\left(m_a^2 - 2\, m_b^2\right) M_f^2 + \left(m_a^2 - m_b^2\right)^2 \Big)\, M_i^6$$
$$+ M_f^2 \Big(M_f^4 - 8\left(m_a^2 + m_b^2\right) M_f^2 + 7\, \left(m_a^2 - m_b^2\right)^2 \Big)\, M_i^4$$
$$+ M_f^4 \left(7\, m_a^4 + 10\, m_b^2\, m_a^2 - 17\, m_b^4 + M_f^4 - 8\left(m_a^2 + m_b^2\right) M_f^2\right) M_i^2 \Big]\, \bar{I}_{ab}$$
$$+ 6\, m_b^2\, M_f^2\, M_i^2 \left(M_f^2 - M_i^2\right) \Big[-4\, M_i^4 + \left(m_a^2 - m_b^2 + M_f^2\right) M_i^2$$
$$+ M_f^2 \left(3\, m_a^2 + 5\, m_b^2 - M_f^2\right) \Big]\, \bar{J}_{abb}\,,$$

$$96\, \frac{m_b^2\, M_i^2}{(p\cdot q)}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, E^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{-,\,ab} = -2\, M_f^2\, M_i^2 \Big[-12\, m_b^2 \left(-m_a^2 + m_b^2 + M_i^2\right) M_i^2$$
$$- \left(M_f^2 - M_i^2\right)\left(M_i^4 - 2\, \left(m_a^2 - 11\, m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2\right) \Big]\, I_{ab}$$
$$+ \Big[2\left(m_a^4 - 2\left(m_b^2 + M_f^2\right) m_a^2 + \left(m_b^2 - M_f^2\right)^2\right) M_i^6$$
$$- 2\, M_f^2 \left(m_a^4 - 2\left(m_b^2 + M_f^2\right) m_a^2 + \left(m_b^2 - M_f^2\right)^2\right) M_i^4$$
$$- 24\, m_b^2\, M_f^4 \left(-m_a^2 + m_b^2 + M_f^2\right) M_i^2 \Big]\, \bar{I}_{ab}$$
$$+ 48\, m_b^4\, M_f^4\, M_i^2 \left(M_f^2 - M_i^2\right) \bar{J}_{abb}\,, \qquad (199)$$

and

$$\frac{12}{(p\cdot q)^3}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, \bar{E}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{ab} = \Big[-m_a^4 - 4\, m_b^2\, m_a^2 + 5\, m_b^4 - M_i^4$$
$$+ 2\left(m_a^2 - 2m_b^2\right) M_i^2 \Big]\, I_{ab}\,,$$

$$\frac{3\, M_i^2}{(p\cdot q)^3}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, \tilde{E}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{ab} = M_f^2 \Big[-2\, M_i^4 + \left(m_a^2 + m_b^2\right) M_i^2$$
$$+ \left(m_a^2 - m_b^2\right)^2 \Big]\, I_{ab}\,,$$

$$24\, \frac{m_a^2\, M_i^2}{(p\cdot q)^3}\, P^{(1)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, \bar{E}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{+,\,ab} = M_f^2 \Big[-3\, M_i^6 + \left(7\, m_a^2 + 5\, m_b^2\right) M_i^4$$
$$- \left(5\, m_a^4 + 18\, m_b^2\, m_a^2 + m_b^4\right) M_i^2 + \left(m_a^2 - m_b^2\right)^3 \Big]\, I_{ab}\,,$$

$$\frac{2}{(p\cdot q)^3}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\bar E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab} = -M_f^2\left[m_a^2+m_b^2-M_i^2\right]I_{ab}\,, \tag{200}$$

and

$$\begin{aligned}
96\,\frac{m_b^2\,M_f^2\,M_i^2}{(p\cdot q)}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} &= M_f^2\left[\,\left(M_f^2-M_i^2\right)^2\left(m_a^6\right.\right.\\
&\quad-\left(2\,m_b^2+3\,M_i^2\right)m_a^4+\left(m_b^4+9\,M_i^2\,m_b^2\right)m_a^2+2\left(M_i^3-m_b^2\,M_i\right)^2\Big)\\
&\quad-\left(M_i^2-M_f^2\right)\Big(m_a^6+18\,m_b^2\,m_a^4-\left(15\,m_b^4+4\,M_i^2\,m_b^2+3\,M_i^4\right)m_a^2\\
&\quad+2\left(-2\,m_b^6+2\,M_i^2\,m_b^4-7\,M_i^4\,m_b^2+M_i^6\right)\Big)\,M_i^2\\
&\quad+12\,m_b^2\left(3\,m_a^2+m_b^2-M_i^2\right)\left(-m_a^2+m_b^2+M_i^2\right)M_i^4\Big]\,I_{ab}\\
&+M_i^2\Big[\left(m_a^2-m_b^2+M_f^2\right)\Big(m_a^4+\left(19\,m_b^2+M_f^2\right)m_a^2+4\,m_b^4-2\,M_f^4\\
&\quad-2\,m_b^2\,M_f^2\Big)\,M_f^4+\Big(2\,M_f^6-4\left(m_a^4+10\,m_b^2\,m_a^2+m_b^4\right)M_f^2\\
&\quad+\left(5\,m_a^2+2\,m_b^2\right)M_f^4-3\left(m_a^6-6\,m_b^2\,m_a^4+m_b^4\,m_a^2+4\,m_b^6\right)\Big)\,M_i^2\,M_f^2\\
&\quad+2\,\Big(m_a^6-3\,m_b^4\,m_a^2+2\,m_b^6-\left(2\,m_a^2-5\,m_b^2\right)M_f^4\\
&\quad+\left(m_a^4-12\,m_b^2\,m_a^2-m_b^4\right)M_f^2\Big)\,M_i^4\Big]\,\bar I_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(M_i^2-M_f^2\right)\Big[-3\left(M_f^2-M_i^2\right)m_a^4\\
&\quad+\left(M_f^4+M_i^2\,M_f^2-2\,M_i^4-m_b^2\left(5\,M_f^2+7\,M_i^2\right)\right)m_a^2\\
&\quad+2\,m_b^2\,M_i^2\left(-2\,m_b^2+M_f^2+M_i^2\right)\Big]\,\bar J_{abb}\,,
\end{aligned}$$

$$\begin{aligned}
192\,m_b^2\,M_i^2\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab} &= -M_f^2\left(M_i^2-M_f^2\right)\Big[-24\,m_b^2\Big(-m_a^2+m_b^2\\
&\quad+M_i^2\Big)\,M_i^4+3\left(-m_a^2+m_b^2+M_i^2\right)^2\left(M_i^2-M_f^2\right)M_i^2\\
&\quad+\left(M_f^2-M_i^2\right)^2\Big(-2\,M_i^4+\left(m_a^2+7\,m_b^2\right)M_i^2+\left(m_a^2-m_b^2\right)^2\Big)\Big]\,I_{ab}\\
&-M_i^2\left(M_i^2-M_f^2\right)\Big[-4\left(m_a^2+m_b^2\right)M_f^6-\left(m_a^4+22\,m_b^2\,m_a^2-23\,m_b^4\right)M_f^4\\
&\quad+5\,M_f^8-\left(7\,M_f^4-2\left(m_a^2+7\,m_b^2\right)M_f^2-5\left(m_a^2-m_b^2\right)^2\right)M_i^2\,M_f^2\\
&\quad+2\left(M_f^4+\left(m_a^2+7\,m_b^2\right)M_f^2-2\left(m_a^2-m_b^2\right)^2\right)M_i^4\Big]\,\bar I_{ab}\\
&-6\,m_b^2\,M_f^2\,M_i^2\left(M_f^2-M_i^2\right)^2\Big[\left(5\,M_f^2+3\,M_i^2\right)m_b^2\\
&\quad+\left(M_i^2-M_f^2\right)\left(-3\,m_a^2+M_f^2+2\,M_i^2\right)\Big]\,\bar J_{abb}\,,
\end{aligned}$$

$$\begin{aligned}
96\,\frac{m_b^2\,M_i^2}{p\cdot q}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab} &= -2M_f^2\,M_i^2\Big[6\left(\left(m_a^2-M_i^2\right)^2-m_b^4\right)M_i^2\\
&\quad+\left(M_i^2-M_f^2\right)\left(-8\,M_i^4+\left(7\,m_a^2-11\,m_b^2\right)M_i^2+\left(m_a^2-m_b^2\right)^2\right)\Big]\,I_{ab}
\end{aligned}$$

$$
\begin{aligned}
&+\Big[2\left(M_f^4-\left(5\,m_a^2+11\,m_b^2\right)M_f^2+4\left(m_a^2-m_b^2\right)^2\right)M_i^6\\
&\quad+2\,M_f^2\left(8\,M_f^4-\left(m_a^2-5\,m_b^2\right)M_f^2-7\left(m_a^2-m_b^2\right)^2\right)M_i^4\\
&\quad-6\,M_f^4\left(-3\,m_a^4+2\left(m_b^2+M_f^2\right)m_a^2+\left(m_b^2-M_f^2\right)^2\right)M_i^2\Big]\,\bar I_{ab}\\
&+12\,m_b^2\,M_f^2\,M_i^2\left(M_f^2-M_i^2\right)\Big[M_i^4-\left(2\,m_a^2-2\,m_b^2+3\,M_f^2\right)M_i^2\\
&\quad+4\,m_a^2\,M_f^2\Big]\,\bar J_{abb}\,, \qquad (201)
\end{aligned}
$$

and

$$
\begin{aligned}
&48\,\frac{M_f^2\,M_i^2}{(p\cdot q)^2}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab}=M_f^2\Big[-2\,M_i^8-5\left(m_a^2-2m_b^2\right)M_i^6\\
&\quad+\left(7\,m_a^4+\left(8\,M_f^2-11\,m_b^2\right)m_a^2+4\,m_b^4\right)M_i^4\\
&\quad-m_a^2\left(7\,m_a^2+m_b^2\right)M_f^2\,M_i^2-m_a^2\left(m_a^2-m_b^2\right)^2M_f^2\Big]\,I_{ab}\\
&+\Big[2\Big(M_f^6+\left(2\,m_a^2-5\,m_b^2\right)M_f^4\\
&\quad-2\left(2\,m_a^4-m_b^2\,m_a^2+m_b^4\right)M_f^2+\left(m_a^3-m_a\,m_b^2\right)^2\Big)M_i^4\\
&\quad-m_a^2\,M_f^2\Big(7\,M_f^4-8\left(m_a^2+m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\Big)M_i^2\Big]\,\bar I_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(M_i^2-M_f^2\right)\Big[-m_a^4+\left(m_b^2-M_f^2+2\,M_i^2\right)m_a^2\\
&\quad-2\,m_b^2\,M_i^2\Big]\,\bar J_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
&48\,\frac{M_i^2}{(p\cdot q)^2}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab}=-M_f^2\Big[\left(M_i^2-M_f^2\right)\Big(10\,M_i^4\\
&\quad-\left(11\,m_a^2+5\,m_b^2\right)M_i^2+\left(m_a^2-m_b^2\right)^2\Big)\\
&\quad+3\left(M_i^2-m_a^2-3\,m_b^2\right)\left(-m_a^2+m_b^2+M_i^2\right)M_i^2\Big]\,I_{ab}\\
&+\Big[2\left(4\,M_f^4-\left(5\,m_a^2-m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\right)M_i^4\\
&\quad+M_f^2\Big(m_a^4+10\,m_b^2\,m_a^2-11\,m_b^4-5\,M_f^4\\
&\quad+4\left(m_a^2-2\,m_b^2\right)M_f^2\Big)M_i^2\Big]\,\bar I_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(m_a^2+3\,m_b^2+M_f^2-2\,M_i^2\right)\left(M_f^2-M_i^2\right)\bar J_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
&24\,\frac{1}{(p\cdot q)^2}\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab}=-M_f^2\Big[5\,m_a^4-\left(4\,m_b^2+7\,M_i^2\right)m_a^2\\
&\quad-m_b^4+2\,M_i^4+5\,m_b^2\,M_i^2\Big]\,I_{ab}\\
&+\Big[-6\,m_a^2\left(-m_a^2+m_b^2+M_f^2\right)M_f^2\\
&\quad-\Big(-2\,M_f^4+\left(m_a^2-5\,m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\Big)M_i^2\Big]\,\bar I_{ab}
\end{aligned}
$$

$$- 6\, m_b^2\, M_f^2 \left(M_f^2 - M_i^2\right) \left(M_i^2 - 2\, m_a^2\right) \bar{J}_{abb}\,, \tag{202}$$

and

$$\begin{aligned}
&96\, m_a^2\, M_f^2\, M_i^2\, P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = m_c^2\, M_f^4 \Big[24 \left(m_b^2 - m_c^2\right)^2 M_i^6 \\
&\quad - 24 \left(m_b^2 - m_c^2\right) \left(m_b^2 - m_c^2 - 2\, M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^4 \\
&\quad + 2 \left(M_f^2 - M_i^2\right)^3 \left(4\, M_i^4 + \left(7\, m_a^2 + m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^2\right) \\
&\quad + 2 \left(M_i^3 - M_f^2\, M_i\right)^2 \Big(M_i^4 - 2 \left(m_a^2 + 9\, m_b^2 - 14\, m_c^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right)^2\Big)\Big]\, I_{ac} \\
&- M_i^2 \Big[\left(M_f^2 - M_i^2\right)^3 \left(m_b^4\, M_f^2 \left(M_f^2 + 5\, M_i^2\right) - \left(M_f^2 - 3\, M_i^2\right) \left(m_a^6 - m_b^6\right)\right) \\
&\quad + \left(M_f^2 - M_i^2\right)^2 \Big(\left(3\, m_b^2 - 2\, m_c^2 + 5\, M_f^2\right) M_f^4 + \left(9\, m_b^2 + 7\, M_f^2\right) M_i^4 \\
&\quad - 4 \left(3\, m_b^2 - m_c^2 + 3\, M_f^2\right) M_i^2\, M_f^2\Big)\, m_a^4 \\
&\quad - \left(M_f^2 - M_i^2\right)^2 \Big(3 \left(M_f^4 - 4\, M_i^2\, M_f^2 + 3\, M_i^4\right) m_b^4 \\
&\quad + 2\, M_f^2 \left(3\, M_f^4 - 4\, M_i^2\, M_f^2 + M_i^4 + m_c^2 \left(4\, M_i^2 - 5\, M_f^2\right)\right) m_b^2 \\
&\quad + M_f^4 \left(6\, m_c^4 - 4 \left(M_f^2 - 2\, M_i^2\right) m_c^2 + 7\, M_f^4 + 5\, M_i^4 - 12\, M_f^2\, M_i^2\right) \Big)\, m_a^2 \\
&\quad + M_f^6 \Big(12 \left(M_f^2 + M_i^2\right) m_c^6 + 6 \left(5\, M_f^4 - 2\, M_i^2\, M_f^2 - 3\, M_i^4\right) m_c^4 \\
&\quad - 2 \left(M_f^2 - 2\, M_i^2\right) \left(M_f^2 - M_i^2\right)^2 m_c^2 + \left(M_f^2 - M_i^2\right)^3 \left(3\, M_f^2 - M_i^2\right)\Big) \\
&\quad + 4\, m_c^2\, m_b^4\, M_f^2 \left(M_f^6 + 8\, M_i^2\, M_f^4 - 4\, M_i^4\, M_f^2 + M_i^6\right) \\
&\quad + m_b^2\, M_f^4 \Big(- 2 \left(M_f^6 + 26\, M_i^2\, M_f^4 - 31\, M_i^4\, M_f^2 + 4\, M_i^6\right) m_c^2 \\
&\quad - 6 \left(3\, M_f^4 + 6\, M_i^2\, M_f^2 - M_i^4\right) m_c^4 - \left(M_f^2 - M_i^2\right)^3 \left(5\, M_f^2 + M_i^2\right)\Big)\Big]\, \bar{I}_{ab} \\
&+ 12\, m_c^2\, M_f^4\, M_i^2 \Big[- 2 \left(m_b^2 - m_c^2\right)^3 M_i^4 + m_c^2 \left(M_f^2 - M_i^2\right)^4 \\
&\quad + \left(m_b^2 - m_c^2\right)^2 \left(m_a^2 + 2\, m_b^2 - 3\, m_c^2 - 5\, M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^2 \\
&\quad + \left(m_b^2 - m_c^2\right) \left(M_f^2 - M_i^2\right)^2 \Big(- 2\, M_i^4 + 2 \left(m_a^2 + 2\, m_b^2 - 4\, m_c^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right) \left(m_c^2 - m_b^2\right)\Big) \\
&\quad + \left(M_i^2 - M_f^2\right)^3 \left(m_a^2\, m_c^2 + \left(2\, m_b^2 - 3\, m_c^2\right) \left(m_c^2 + M_i^2\right)\right) \Big]\, \bar{J}_{abc}\,,
\end{aligned}$$

$$\begin{aligned}
&96\, M_f^2\, M_f^2\, M_i^2\, P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar{G}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = 2\, m_c^2\, M_f^4 \Big[12 \left(m_b^2 - m_c^2\right)^2 M_i^6 \\
&\quad - 12 \left(m_c^2 - m_b^2\right) \left(M_i^2 - M_f^2\right) \left(-m_b^2 + m_c^2 + 2\, M_i^2\right) M_i^4 \\
&\quad + \left(M_i^3 - M_f^2\, M_i\right)^2 \left(M_i^4 - 2 \left(m_a^2 + 9\, m_b^2 - 14\, m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(M_f^2-M_i^2\right)^3\left(4\,M_i^4+\left(7\,m_a^2+m_c^2\right)M_i^2+\left(m_a^2-m_c^2\right)^2\right)\Big]\,I_{ac}\\
&-M_i^2\Big[-\left(M_f^2-3\,M_i^2\right)\left(M_f^2-M_i^2\right)^3 m_a^6+m_b^6\left(M_f^2-3\,M_i^2\right)\left(M_f^2-M_i^2\right)^3\\
&\quad+\left(M_f^2-M_i^2\right)^2\Big(\left(3\,m_b^2-2\,m_c^2+5\,M_f^2\right)M_f^4\\
&\quad-4\left(3\,m_b^2-m_c^2+3\,M_f^2\right)M_i^2\,M_f^2+\left(9\,m_b^2+7\,M_f^2\right)M_i^4\Big)\,m_a^4\\
&\quad-\left(M_f^2-M_i^2\right)^2\Big(3\left(M_f^4-4\,M_i^2\,M_f^2+3\,M_i^4\right)m_b^4\\
&\quad+2\,M_f^2\left(3\,M_f^4-4\,M_i^2\,M_f^2+M_i^4+m_c^2\left(4\,M_i^2-5\,M_f^2\right)\right)m_b^2\\
&\quad+M_f^4\left(6\,m_c^4-4\left(M_f^2-2M_i^2\right)m_c^2+7\,M_f^4+5\,M_i^4-12\,M_f^2\,M_i^2\right)\Big)\,m_a^2\\
&\quad+M_f^6\Big(12\left(M_f^2+M_i^2\right)m_c^6+6\left(5\,M_f^4-2\,M_i^2\,M_f^2-3\,M_i^4\right)m_c^4\\
&\quad-2\left(M_f^2-2\,M_i^2\right)\left(M_f^2-M_i^2\right)^2 m_c^2+\left(M_f^2-M_i^2\right)^3\left(3\,M_f^2-M_i^2\right)\Big)\\
&\quad+m_b^4\,M_f^2\Big(\left(M_f^2+5\,M_i^2\right)\left(M_f^2-M_i^2\right)^3\\
&\quad+4\,m_c^2\left(M_f^6+8\,M_i^2\,M_f^4-4\,M_i^4\,M_f^2+M_i^6\right)\Big)\\
&\quad+m_b^2\,M_f^4\Big(-2\left(M_f^6+26\,M_i^2\,M_f^4-31\,M_i^4\,M_f^2+4\,M_i^6\right)m_c^2\\
&\quad-6\left(3\,M_f^4+6\,M_i^2\,M_f^2-M_i^4\right)m_c^4-\left(M_f^2-M_i^2\right)^3\left(5\,M_f^2+M_i^2\right)\Big)\Big]\,\bar{I}_{ab}\\
&+12\,m_c^2\,M_f^4\,M_i^2\Big[m_c^2\left(M_f^2-M_i^2\right)^4-2\left(m_b^2-m_c^2\right)^3 M_i^4\\
&\quad+\left(m_b^2-m_c^2\right)^2\left(m_a^2+2\,m_b^2-3\,m_c^2-5\,M_i^2\right)\left(M_i^2-M_f^2\right)M_i^2\\
&\quad+\left(m_b^2-m_c^2\right)\left(M_f^2-M_i^2\right)^2\Big(-2\,M_i^4+2\left(m_a^2+2\,m_b^2-4\,m_c^2\right)M_i^2\\
&\quad+\left(m_a^2-m_c^2\right)\left(m_c^2-m_b^2\right)\Big)\\
&\quad+\left(M_i^2-M_f^2\right)^3\left(m_a^2\,m_c^2+\left(2\,m_b^2-3\,m_c^2\right)\left(m_c^2+M_i^2\right)\right)\Big]\,\bar{J}_{abc}\,,
\end{aligned}
\tag{203}
$$

and

$$
\begin{aligned}
48\,M_i^2\,P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,H^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc}&=M_f^2\Big[12\left(m_b^2-m_c^2\right)^2 M_i^4\,M_f^2\\
&\quad+\left(M_i^3-M_f^2\,M_i\right)^2\left(M_i^4-2\left(m_a^2+3\,m_b^2-8\,m_c^2\right)M_i^2+\left(m_a^2-m_c^2\right)^2\right)\\
&\quad-\left(M_i^2-M_f^2\right)^3\left(-2\,M_i^4+\left(m_a^2+7\,m_c^2\right)M_i^2+\left(m_a^2-m_c^2\right)^2\right)\Big]\,I_{ac}\\
&+M_i^2\Big[-2\left(M_f^6+8\,M_i^2\,M_f^4-4\,M_i^4\,M_f^2+M_i^6\right)m_b^4\\
&\quad+\Big(M_f^2\Big(3\left(3\,M_f^4+6\,M_i^2\,M_f^2-M_i^4\right)m_c^2+\left(M_f^2-M_i^2\right)^2\left(M_f^2+4\,M_i^2\right)\Big)\\
&\quad-m_a^2\left(5\,M_f^2-4\,M_i^2\right)\left(M_f^2-M_i^2\right)^2\Big)\,m_b^2-6\,m_c^4\,M_f^4\left(M_f^2+M_i^2\right)\\
&\quad-3\,m_c^2\,M_f^2\left(5\,M_f^2-m_a^2\right)\left(M_f^2-M_i^2\right)^2\\
&\quad+\left(m_a^2-M_f^2\right)^2\left(M_f^2-2\,M_i^2\right)\left(M_f^2-M_i^2\right)^2\Big]\,\bar{I}_{ab}
\end{aligned}
$$

$$
\begin{aligned}
&+6\, M_f^2\, M_i^2 \Big[m_c^2 \left(M_f^2 - M_i^2\right)^4 - 2 \left(m_b^2 - m_c^2\right)^3 M_i^4 \\
&\quad + \left(m_b^2 - m_c^2\right)^2 \left(m_a^2 + 2\, m_b^2 - 3\, m_c^2 - M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^2 \\
&\quad + \left(m_b^2 - m_c^2\right) \left(M_f^2 - M_i^2\right)^2 \Big(2 \left(m_b^2 - 2\, m_c^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right) \left(m_c^2 - m_b^2\right) \Big) \\
&\quad + m_c^2 \left(m_a^2 + 2\, m_b^2 - 3\, m_c^2 - M_i^2\right) \left(M_i^2 - M_f^2\right)^3 \Big]\, \bar{J}_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
&48\, M_f^2\, M_i^2\, P^{(1)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar{H}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = m_a^2\, M_f^2 \Big[12 \left(m_b^2 - m_c^2\right)^2 M_i^6 \\
&\quad - 12 \left(m_b^2 - m_c^2\right)^2 \left(M_i^2 - M_f^2\right) M_i^4 + \left(M_i^3 - M_f^2\, M_i\right)^2 \Big(M_i^4 \\
&\quad - 2 \left(m_a^2 + 3\, m_b^2 - 8\, m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^2 \Big) \\
&\quad - \left(M_i^2 - M_f^2\right)^3 \Big(- 2\, M_i^4 + \left(m_a^2 + 7\, m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^2 \Big)\Big]\, I_{ac} \\
&+ m_a^2\, M_i^2 \Big[- 2 \left(M_f^6 + 8\, M_i^2\, M_f^4 - 4\, M_i^4\, M_f^2 + M_i^6\right) m_b^4 \\
&\quad + \Big(M_f^2 \Big(3 \left(6\, M_i^2\, M_f^2 - M_i^4\right) m_c^2 + \left(M_f^2 - M_i^2\right)^2 \left(M_f^2 + 4\, M_i^2\right)\Big) \\
&\quad + 9\, M_f^6\, m_c^2 - m_a^2 \left(5\, M_f^2 - 4\, M_i^2\right) \left(M_f^2 - M_i^2\right)^2 \Big)\, m_b^2 \\
&\quad - 3\, m_c^2\, M_f^2 \left(5\, M_f^2 - m_a^2\right) \left(M_f^2 - M_i^2\right)^2 - 6\, m_c^4\, M_f^4 \left(M_f^2 + M_i^2\right) \\
&\quad + \left(m_a^2 - M_f^2\right)^2 \left(M_f^2 - 2M_i^2\right) \left(M_f^2 - M_i^2\right)^2 \Big]\, \bar{I}_{ab} \\
&+ 6\, m_a^2\, M_f^2\, M_i^2 \Big[- 2 \left(m_b^2 - m_c^2\right)^3 M_i^4 + m_c^2 \left(M_f^2 - M_i^2\right)^4 \\
&\quad + \left(m_b^2 - m_c^2\right)^2 \left(m_a^2 + 2\, m_b^2 - 3\, m_c^2 - M_i^2\right) \left(M_i^2 - M_f^2\right) M_i^2 \\
&\quad + \left(m_b^2 - m_c^2\right) \left(M_f^2 - M_i^2\right)^2 \Big(2 \left(m_b^2 - 2\, m_c^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right) \left(m_c^2 - m_b^2\right) \Big) \\
&\quad + m_c^2 \left(m_a^2 + 2\, m_b^2 - 3\, m_c^2 - M_i^2\right) \left(M_i^2 - M_f^2\right)^3 \Big]\, \bar{J}_{abc}\,. \qquad (204)
\end{aligned}
$$

We turn to the projections with the tensor $P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}$ of (183). It holds

$$
\begin{aligned}
&48\, \frac{M_f^2\, M_i^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, A^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab} = M_i^2 \Big[- 12 \left(m_a^2 - m_b^2 + M_i^2\right) \left(m_b^2 + M_i^2\right) M_i^4 \\
&\quad + 12 \left(M_i^2 - M_f^2\right) \left(m_a^2 + m_b^2 + 3\, M_i^2\right) M_i^4 \\
&\quad + 6 \left(-m_a^2 + m_b^2 + M_i^2\right) \left(M_i^2 - M_f^2\right)^3 \\
&\quad - \left(M_f^2 - M_i^2\right)^2 \Big(m_a^4 - 8 \left(m_b^2 + M_i^2\right) m_a^2 + 7\, m_b^4 \\
&\quad + 31\, M_i^4 + 22\, m_b^2\, M_i^2\Big)\Big]\, I_{ab}
\end{aligned}
$$

$$+ 12\, M_f^2 \left(m_a^2 - m_b^2 + M_f^2\right)\left(m_b^2 + M_f^2\right) M_i^4\, \bar{I}_{ab}$$
$$+ 24\, m_a^2\, M_f^2 \left(m_b^2 + M_f^2\right) M_i^4 \left(M_i^2 - M_f^2\right) J_{aab}\,,$$

$$48\, \frac{M_f^2\, M_i^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, A^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab} = M_i^2 \Big[12 \left(M_i^2 - m_b^2\right)\left(m_a^2 - m_b^2 + M_i^2\right) M_i^4$$
$$- 12 \left(M_i^2 - M_f^2\right)\left(m_a^2 - 3\, m_b^2 + 3\, M_i^2\right) M_i^4$$
$$+ 6 \left(m_a^2 - m_b^2 - M_i^2\right)\left(M_i^2 - M_f^2\right)^3$$
$$- \left(M_f^2 - M_i^2\right)^2 \Big(m_a^4 + \left(4\, M_i^2 - 8\, m_b^2\right) m_a^2 + 7\, m_b^4$$
$$- 29\, M_i^4 + 10\, m_b^2 M_i^2\Big)\Big]\, I_{ab}$$
$$- 12\, M_f^2 \left(M_f^2 - m_b^2\right)\left(m_a^2 - m_b^2 + M_f^2\right) M_i^4\, \bar{I}_{ab}$$
$$+ 24\, m_a^2\, M_f^2 \left(M_f^2 - m_b^2\right) M_i^4 \left(M_f^2 - M_i^2\right) J_{aab}\,,$$

$$24\, \frac{M_i^2}{(p\cdot q)^3}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar{A}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab} = \Big[- 7 M_i^6 + \left(7\, m_a^2 - 3\, m_b^2 - 2\, M_f^2\right) M_i^4$$
$$+ \left(m_a^4 - 10\, m_b^2\, m_a^2 + 9\, m_b^4 - 2\left(m_a^2 + m_b^2\right) M_f^2\right) M_i^2$$
$$- \left(m_a^2 - m_b^2\right)^2 \left(m_a^2 - m_b^2 - 4\, M_f^2\right)\Big]\, I_{ab}\,,$$

$$48\, \frac{m_b^2}{m_b^2 \pm M_f^2}\, \frac{M_f^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, B^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{\pm,\,ab} = \Big[12\, m_b^2 \left(-m_a^2 + m_b^2 + M_i^2\right) M_i^4$$
$$- \left(M_i^2 - M_f^2\right)\left(M_i^4 - 2\left(m_a^2 - 11 m_b^2\right) M_i^2 + \left(m_a^2 - m_b^2\right)^2\right) M_i^2$$
$$+ 6\, m_b^2 \left(M_f^2 - M_i^2\right)^2 \left(m_a^2 - m_b^2 + M_i^2\right)\Big]\, I_{ab}$$
$$+ \Big[\left(m_a^4 - 2\left(m_b^2 + M_f^2\right) m_a^2 + \left(m_b^2 - M_f^2\right)^2\right) M_i^2$$
$$- M_f^2 \left(m_a^4 - 2\left(7\, m_b^2 + M_f^2\right) m_a^2 + 13\, m_b^4 + M_f^4 + 10\, m_b^2\, M_f^2\right)\Big] M_i^2\, \bar{I}_{ab}$$
$$+ 24\, m_b^4\, M_f^2\, M_i^2 \left(M_f^2 - M_i^2\right) \bar{J}_{abb}\,,$$

$$48\, \frac{M_f^2\, M_i^2}{(p\cdot q)^2}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar{B}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} = M_f^2 \Big(- M_i^4 + \left(m_a^2 + m_b^2 + 3\, M_f^2\right) M_i^2$$
$$- \left(m_a^2 - m_b^2\right) M_f^2\Big)\Big[m_a^4 - 2\left(m_b^2 + M_i^2\right) m_a^2 + \left(m_b^2 - M_i^2\right)^2\Big]\, I_{ab}$$
$$+ \Big[- 2\left((m_a - m_b)^2 - M_f^2\right)\left((m_a + m_b)^2 - M_f^2\right)\left(m_b^2 + M_f^2\right) M_i^4\Big]\, \bar{I}_{ab}\,,$$

$$24\, \frac{M_i^2}{(p\cdot q)^3}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \tilde{B}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} = \Big[3\, M_i^6 - \left(3\left(m_a^2 + m_b^2\right) - M_f^2\right) M_i^4$$
$$+ \left(m_a^2 + m_b^2\right) M_f^2\, M_i^2 - 2\left(m_a^2 - m_b^2\right)^2 M_f^2\Big]\, I_{ab}\,,$$

$$
\begin{aligned}
192\, M_f^2\, M_i^2\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, C^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,abc} &= M_f^2 \Big[-12\left(m_b^3 - m_a^2\, m_b\right)^2 M_i^6 \\
&\quad + 6\left(m_a^2 - m_b^2\right) m_b^2 \left(m_a^2 - m_c^2 - M_i^2\right)\left(M_i^2 - M_f^2\right) M_i^4 \\
&\quad + 2\, m_b^2 \left(M_f^2 - M_i^2\right)^2 \Big(-2\, M_i^4 + \left(m_a^2 - 6\, m_b^2 + m_c^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right)^2 \Big) M_i^2 \\
&\quad - 2\left(M_f^2 - M_i^2\right)^4 \Big(- M_i^4 - \left(m_a^2 + m_c^2\right) M_i^2 + 2\left(m_a^2 - m_c^2\right)^2 \Big) \\
&\quad - \left(M_i^2 - M_f^2\right)^3 \Big(-3\, M_i^6 + \left(7\, m_a^2 + 6\, m_b^2 + 5\, m_c^2\right) M_i^4 \\
&\quad - \left(m_a^2 - m_c^2\right)\left(5\, m_a^2 + 6\, m_b^2 - m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^3 \Big)\Big] I_{ac} \\
&+ 2\, m_b^2\, M_i^4 \Big[\left(2\, M_f^4 + 5\, M_i^2\, M_f^2 - M_i^4\right) m_b^4 \\
&\quad + \left(M_f^2 - M_i^2\right)\left(8\, M_f^4 - 5\, M_i^2\, M_f^2 + m_c^2\left(5\, M_f^2 - 2\, M_i^2\right)\right) m_b^2 \\
&\quad + 6\, m_a^4\, M_f^4 - \left(m_c^4 + M_f^2\, m_c^2 - 2\, M_f^4\right)\left(M_f^2 - M_i^2\right)^2 \\
&\quad - 3\, m_a^2\, M_f^2 \Big(\left(3\, M_f^2 + M_i^2\right) m_b^2 + \left(m_c^2 + M_f^2\right)\left(M_f^2 - M_i^2\right)\Big) \Big] \bar I_{bc} \\
&+ 12\, M_f^2\, M_i^4 \Big[\left(M_i^2 - M_f^2\right)^3 m_b^4 + \left(m_b^2 - m_a^2\right)\left(M_f^2 - M_i^2\right)^2 m_b^4 \\
&\quad + \left(m_b^3 - m_a^2 m_b\right)^2 \left(m_a^2 - m_c^2\right)\left(M_i^2 - M_f^2\right) \\
&\quad + \left(m_b^2 - m_a^2\right)^3 M_i^2\, m_b^2 \Big] J_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
192\, M_f^2\, M_i^2\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, C^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,abc} &= M_f^2 \Big[-12\left(m_b^3 - m_a^2\, m_b\right)^2 M_i^6 \\
&\quad + 6\left(m_a^2 - m_b^2\right) m_b^2 \left(M_i^2 - M_f^2\right)\left(m_a^2 - m_c^2 + 7\, M_i^2\right) M_i^4 \\
&\quad + 2\, m_b^2 \left(M_f^2 - M_i^2\right)^2 \Big(-2\, M_i^4 - \left(11\, m_a^2 - 6\, m_b^2 - m_c^2\right) M_i^2 \\
&\quad + \left(m_a^2 - m_c^2\right)^2 \Big) M_i^2 \\
&\quad + 2\left(M_f^2 - M_i^2\right)^4 \Big(-M_i^4 - \left(m_a^2 + m_c^2\right) M_i^2 + 2\left(m_a^2 - m_c^2\right)^2 \Big) \\
&\quad - \left(M_i^2 - M_f^2\right)^3 \Big(M_i^6 - \left(m_a^2 + 6\, m_b^2 + 3\, m_c^2\right) M_i^4 \\
&\quad - \left(m_a^2 - m_c^2\right)\left(m_a^2 - 6\, m_b^2 + 3\, m_c^2\right) M_i^2 + \left(m_a^2 - m_c^2\right)^3 \Big)\Big] I_{ac} \\
&+ 2\, m_b^2\, M_i^4 \Big[\left(2\, M_f^4 + 5\, M_i^2\, M_f^2 - M_i^4\right) m_b^4 \\
&\quad - \left(M_f^2 - M_i^2\right)\left(16\, M_f^4 + 5\, M_i^2\, M_f^2 + m_c^2\left(2\, M_i^2 - 5\, M_f^2\right)\right) m_b^2 \\
&\quad + 6\, m_a^4\, M_f^4 - \left(m_c^4 + M_f^2\, m_c^2 - 2\, M_f^4\right)\left(M_f^2 - M_i^2\right)^2 \\
&\quad - 3\, m_a^2\, M_f^2 \Big(\left(3\, M_f^2 + M_i^2\right) m_b^2 + \left(m_c^2 - 7\, M_f^2\right)\left(M_f^2 - M_i^2\right)\Big) \Big] \bar I_{bc} \\
&+ 12\, M_f^2\, M_i^4 \Big[m_b^2\, M_i^2 \left(m_b^2 - m_a^2\right)^3 \\
&\quad - m_b^2 \left(-4\, m_a^2 + m_b^2 + 2\, m_c^2 - 2\, M_i^2\right)\left(M_f^2 - M_i^2\right)^2 \left(m_b^2 - m_a^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ m_b^2 \left(m_b^2 - 2m_a^2\right) \left(M_f^2 - M_i^2\right)^3 \\
&+ \left(m_b^3 - m_a^2\, m_b\right)^2 \left(M_i^2 - M_f^2\right) \left(m_a^2 - m_c^2 + 4\, M_i^2\right) \Big]\, J_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
96\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar C^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = &\Big[6 \left(m_a^2 - m_b^2\right)^2 M_i^6 \\
&- 3\left(m_a^2 - m_b^2\right)\left(3\, m_a^2 + m_c^2 + M_i^2\right)\left(M_i^2 - M_f^2\right) M_i^4 \\
&- 3\, m_c^2 \left(-m_a^2 + m_c^2 + M_i^2\right)\left(M_i^2 - M_f^2\right)^3 \\
&- \left(M_i^3 - M_f^2\, M_i\right)^2 \Big(M_i^4 - \left(5\, m_a^2 - m_c^2\right) M_i^2 \\
&- 2\left(m_a^4 + m_c^4 + \left(m_a^2 - 3\, m_b^2\right) m_c^2\right)\Big)\Big]\, I_{ac} \\
+ M_i^2 &\Big[\left(M_f^4 - 5\, M_i^2\, M_f^2 - 2\, M_i^4\right) m_b^4 - 6\, m_a^4\, M_f^4 \\
&+ \left(M_f^2 - M_i^2\right)\left(-2\, M_f^4 + 5\, M_i^2 M_f^2 + m_c^2 \left(7\, M_f^2 - 4\, M_i^2\right)\right) m_b^2 \\
&- \left(2\, m_c^4 - M_f^2\, m_c^2 - M_f^4\right)\left(M_f^2 - M_i^2\right)^2 \\
&+ 3\, m_a^2\, M_f^2 \left(m_b^2 \left(M_f^2 + 3\, M_i^2\right) - \left(m_c^2 + M_f^2\right)\left(M_f^2 - M_i^2\right)\right)\Big]\, \bar I_{bc} \\
+ 6\, M_i^2 &\left(m_a^2 - m_b^2 + M_f^2 - M_i^2\right)\left(\left(m_a^2 - m_b\, m_c\right) M_f^2 - m_b\,(m_b - m_c)\, M_i^2\right) \\
&\times\Big[\left(m_a^2 + m_b\, m_c\right) M_f^2 - m_b\,(m_b + m_c)\, M_i^2\Big]\, J_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
48\, \frac{M_f^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, D^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} = &\;3\, m_c^2\, M_f^2 \Big[\left(-M_f^4 + 4\, M_i^2\, M_f^2 + M_i^4\right) m_c^2 \\
&+ \left(M_f^2 - M_i^2\right)^2 \left(m_a^2 + M_i^2\right) - 2\, m_b^2\, M_i^2 \left(M_f^2 + M_i^2\right)\Big]\, I_{ac} \\
- M_i^2 &\Big[\left(M_f^2 - M_i^2\right)^2 m_b^4 - \left(12\, m_c^2\, M_f^4 + 2\left(m_a^2 + M_f^2\right)\left(M_f^2 - M_i^2\right)^2\right) m_b^2 \\
&+ 12\, m_c^4\, M_f^4 + \left(m_a^2 - M_f^2\right)^2 \left(M_f^2 - M_i^2\right)^2\Big]\, \bar I_{ab} \\
+ 6\, m_c^2 &\left(m_c^2 - m_b^2\right) M_f^2\, M_i^2 \Big[M_f^4 - \left(m_a^2 + m_b^2 - 2\, m_c^2\right) M_f^2 \\
&- \left(-m_a^2 + m_b^2 + M_f^2\right) M_i^2\Big]\, \bar J_{abc}\,,
\end{aligned}
\tag{205}
$$

and

$$
\begin{aligned}
48\, \frac{M_f^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} = &\Big[-2\, M_i^8 - \left(-4m_a^2 + 8m_b^2 + M_f^2\right) M_i^6 \\
&+ 2\left(-m_a^4 - 4\, m_b^2\, m_a^2 + 5\, m_b^4 + \left(m_a^2 + m_b^2\right) M_f^2\right) M_i^4 \\
&- M_f^2 \left(7\, m_a^4 + 10\, m_b^2\, m_a^2 - 17\, m_b^4 + 3\, M_f^4 - 3\left(3\, m_a^2 + 5\, m_b^2\right) M_f^2\right) M_i^2 \\
&+ 3\left(m_a^2 - m_b^2\right) M_f^4 \left(m_a^2 + 3\, m_b^2 - M_f^2\right)\Big]\, I_{ab} \\
+ &\Big[-\left(-8M_f^4 + \left(7m_a^2 - 11m_b^2\right) M_f^2 + \left(m_a^2 - m_b^2\right)^2\right) M_i^4
\end{aligned}
$$

$$
\begin{aligned}
&\quad - M_f^2 \Big(- 7\, m_a^4 + 5 \Big(M_f^2 - 2\, m_b^2 \Big)\, m_a^2 + 17\, m_b^4 + 2\, M_f^4 \\
&\quad + 23\, m_b^2\, M_f^2 \Big)\, M_i^2 \Big]\, \bar{I}_{ab} \\
&+ 6\, m_b^2\, M_f^2\, M_i^2 \Big(2\, m_a^2 + 6\, m_b^2 + M_f^2 - 3 M_i^2 \Big) \Big(M_f^2 - M_i^2 \Big)\, \bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
96\, m_a^2\, \frac{M_f^2}{(p \cdot q)}\, P^{(2)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, E^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{+,\,ab} &= M_f^2 \Big[- 12 \Big(\Big(m_a^2 - M_i^2 \Big)^2 - m_b^4 \Big)\, M_i^4 \\
&\quad + 3 \Big(- m_a^2 + m_b^2 + M_i^2 \Big) \Big(M_i^2 - M_f^2 \Big) \Big(- m_a^2 + m_b^2 + 7\, M_i^2 \Big)\, M_i^2 \\
&\quad + 2\, \Big(M_f^2 - M_i^2 \Big)^2 \Big(- 5\, M_i^4 - 2 \Big(m_a^2 + 4\, m_b^2 \Big)\, M_i^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big) \Big]\, I_{ab} \\
&+ \Big[\Big(4\, M_f^4 - \Big(5\, m_a^2 - m_b^2 \Big)\, M_f^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big)\, M_i^6 \\
&\quad + M_f^2 \Big(7\, M_f^4 - 2 \Big(m_a^2 + 19\, m_b^2 \Big)\, M_f^2 - 5\, \Big(m_a^2 - m_b^2 \Big)^2 \Big)\, M_i^4 \\
&\quad + M_f^4 \Big(16\, m_a^4 - \Big(8\, m_b^2 + 17\, M_f^2 \Big)\, m_a^2 - 8\, m_b^4 + M_f^4 \\
&\quad + 37\, m_b^2\, M_f^2 \Big)\, M_i^2 \Big]\, \bar{I}_{ab} \\
&- 6\, m_b^2\, M_f^2\, M_i^2 \Big(M_f^2 - M_i^2 \Big) \Big[M_i^4 - \Big(- 5\, m_a^2 + 5\, m_b^2 + M_f^2 \Big)\, M_i^2 \\
&\quad + M_f^2 \Big(- 9\, m_a^2 + m_b^2 + 4\, M_f^2 \Big) \Big]\, \bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
96\, m_b^2\, \frac{M_f^2}{(p \cdot q)}\, P^{(2)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, E^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{-,\,ab} &= M_f^2 \Big[24\, m_b^2 \Big(- m_a^2 + m_b^2 + M_i^2 \Big)\, M_i^4 \\
&\quad - \Big(M_i^2 - M_f^2 \Big) \Big(M_i^4 - 2 \Big(m_a^2 - 5\, m_b^2 \Big)\, M_i^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big)\, M_i^2 \\
&\quad + 2 \Big(M_f^2 - M_i^2 \Big)^2 \Big(M_i^4 - 2 \Big(m_a^2 + 4\, m_b^2 \Big)\, M_i^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big) \Big]\, I_{ab} \\
&+ \Big[\Big(- 2\, M_f^4 + \Big(m_a^2 - 5\, m_b^2 \Big)\, M_f^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big)\, M_i^6 \\
&\quad - M_f^2 \Big(- 5\, M_f^4 + 4 \Big(m_a^2 + 7\, m_b^2 \Big)\, M_f^2 + \Big(m_a^2 - m_b^2 \Big)^2 \Big)\, M_i^4 \\
&\quad + 3\, M_f^4 \Big(- 8\, m_b^4 + 3\, M_f^2\, m_b^2 - M_f^4 + m_a^2 \Big(8\, m_b^2 + M_f^2 \Big) \Big)\, M_i^2 \Big]\, \bar{I}_{ab} \\
&+ 6\, m_b^2\, M_f^2\, M_i^2 \Big(M_f^2 - M_i^2 \Big) \Big[M_i^4 + \Big(- 3\, m_a^2 + 3\, m_b^2 + M_f^2 \Big)\, M_i^2 \\
&\quad + M_f^2 \Big(3\, m_a^2 + 5\, m_b^2 - 2\, M_f^2 \Big) \Big]\, \bar{J}_{abb}\,, \qquad (206)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{12}{(p \cdot q)^3}\, P^{(2)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, \bar{E}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{ab} &= \Big[- m_a^4 - 4\, m_b^2\, m_a^2 + 5\, m_b^4 - M_i^4 \\
&\quad + 2 \Big(m_a^2 - 2 m_b^2 \Big)\, M_i^2 \Big]\, I_{ab}\,,
\end{aligned}
$$

$$
\frac{6\, M_i^2}{(p \cdot q)^3}\, P^{(2)}_{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}\, \tilde{E}^{\bar{\alpha}\bar{\beta},\mu,\alpha\beta}_{ab} = \Big[- 3 M_i^6 + \Big(3 \Big(m_a^2 + m_b^2 \Big) - M_f^2 \Big)\, M_i^4
$$

$$- \left(m_a^2 + m_b^2\right) M_f^2\, M_i^2 + 2\left(m_a^2 - m_b^2\right)^2 M_f^2\Big]\, I_{ab}\,,$$

$$48\,\frac{m_a^2\, M_i^2}{(p\cdot q)^3}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab} = \Big[\left(3\,M_f^2 - M_i^2\right) m_a^6$$
$$- \left(3\,M_i^4 + 7\,M_f^2\,M_i^2 + m_b^2\left(M_f^2 + 5\,M_i^2\right)\right) m_a^4 + \Big(9\,M_i^6 + 5\,M_f^2\,M_i^4$$
$$+ m_b^4\left(13\,M_i^2 - 7\,M_f^2\right) + 2\,m_b^2\left(5\,M_f^2\,M_i^2 - 23\,M_i^4\right)\Big) m_a^2$$
$$- \left(m_b^2 - M_i^2\right)^2\left(5\,M_i^4 + \left(7\,m_b^2 + M_f^2\right) M_i^2 - 5\,m_b^2\,M_f^2\right)\Big]\, I_{ab}\,,$$

$$\frac{12\,M_i^2}{(p\cdot q)^3}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, \bar E^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab} = \Big[\left(M_i^2 - M_f^2\right)\Big(M_i^4 + \left(m_a^2 + m_b^2\right) M_i^2$$
$$- 2\left(m_a^2 - m_b^2\right)^2\Big) - 6\,M_i^4\left(m_a^2 + m_b^2 - M_i^2\right)\Big]\, I_{ab}\,, \qquad (207)$$

and

$$96\,m_b^2\,\frac{M_f^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab} = \Big[- \left(M_i^2 - M_f^2\right)\Big(M_i^6 - 58\,m_b^2\,M_i^4$$
$$- \left(3\,m_a^4 - 58\,m_b^2\,m_a^2 - 17\,m_b^4\right) M_i^2 + 2\left(m_a^6 - 3\,m_b^4\,m_a^2 + 2\,m_b^6\right)\Big) M_i^2$$
$$+ \left(M_i^2 - M_f^2\right)^3\left(4\,m_a^4 - \left(2\,m_b^2 + 5\,M_i^2\right) m_a^2 - 2\,m_b^4 + M_i^4 + 7\,m_b^2\,M_i^2\right)$$
$$+ \left(M_f^2 - M_i^2\right)^2\Big(2\,m_a^6 + \left(16\,m_b^2 - 7\,M_i^2\right) m_a^4 + 11\,m_b^4\,M_i^2$$
$$+ \left(-14\,m_b^4 + 28\,M_i^2\,m_b^2 + 5\,M_i^4\right) m_a^2 - 4\,m_b^6 - 49\,m_b^2\,M_i^4\Big)$$
$$+ 12\,m_b^2\left(3\,m_a^2 + m_b^2 - M_i^2\right)\left(-m_a^2 + m_b^2 + M_i^2\right) M_i^4\Big]\, I_{ab}$$
$$+ M_i^2\Big[- 2\,M_f^8 - 2\left(2\,m_a^2 - 9\,m_b^2\right) M_f^6 + 8\,m_a^2\left(m_a^2 - 4\,m_b^2\right) M_f^4$$
$$- 2\left(m_a^6 - 18\,m_b^2\,m_a^4 + 9\,m_b^4\,m_a^2 + 8\,m_b^6\right) M_f^2$$
$$- \Big(M_f^4 + 4\left(m_a^2 + 4\,m_b^2\right) M_f^2 - 5\left(m_a^2 - m_b^2\right)^2\Big) M_i^4$$
$$+ \Big(3\,M_f^6 + 2\left(4\,m_a^2 + 5\,m_b^2\right) M_f^4 - \left(13\,m_a^4 + 6\,m_b^2\,m_a^2 + 5\,m_b^4\right) M_f^2$$
$$+ 2\left(m_a^6 - 3\,m_b^4\,m_a^2 + 2\,m_b^6\right)\Big) M_i^2\Big]\, \bar I_{ab}$$
$$+ 6\,m_b^2\,M_f^2\,M_i^2\left(M_i^2 - M_f^2\right)\Big[- 2\,M_i^4 + \left(3\,m_a^2 + m_b^2 + M_f^2\right) M_i^2$$
$$- \left(3\,m_a^2 + m_b^2 - M_f^2\right)\left(4\,m_b^2 + M_f^2\right)\Big]\, \bar J_{abb}\,,$$

$$96\,m_b^2\,\frac{M_f^2}{(p\cdot q)}\, P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\, F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab} = M_f^2\Big[24\,m_b^2\left(-m_a^2 + m_b^2 + M_i^2\right) M_i^4$$
$$- 3\left(-m_a^2 + m_b^2 + M_i^2\right)^2\left(M_i^2 - M_f^2\right) M_i^2$$
$$+ \left(M_f^2 - M_i^2\right)^2\Big(5\,M_i^4 - \left(7\,m_a^2 + 13\,m_b^2\right) M_i^2 + 2\left(m_a^2 - m_b^2\right)^2\Big)\Big]\, I_{ab}$$

$$
\begin{aligned}
&+\Big[\Big(-5\,M_f^4+4\left(m_a^2-2\,m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\Big)M_i^6\\
&\qquad+M_f^2\left(13\,M_f^4-2\left(7\,m_a^2+13\,m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\right)M_i^4\\
&\qquad-2\,M_f^4\left(m_a^4-14\,m_b^2\,m_a^2+13\,m_b^4+4\,M_f^4-5\left(m_a^2+m_b^2\right)M_f^2\right)M_i^2\Big]\,\bar{I}_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(M_i^2-M_f^2\right)\Big[\left(3\,m_a^2-M_f^2-2\,M_i^2\right)\left(M_i^2-M_f^2\right)\\
&\qquad-m_b^2\left(5\,M_f^2+3\,M_i^2\right)\Big]\,\bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
96\,m_b^2\,\frac{M_f^2}{(p\cdot q)}\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab}&=M_f^2\Big[-12\left(\left(m_a^2-M_i^2\right)^2-m_b^4\right)M_i^4\\
&\qquad+\left(M_i^2-M_f^2\right)\left(17\,M_i^4+2\left(m_b^2-5\,m_a^2\right)M_i^2-7\left(m_a^2-m_b^2\right)^2\right)M_i^2\\
&\qquad-\left(M_f^2-M_i^2\right)^2\left(7\,M_i^4+\left(7\,m_a^2+13\,m_b^2\right)M_i^2-2\left(m_a^2-m_b^2\right)^2\right)\Big]\,I_{ab}\\
&+\Big[\Big(7\,M_f^4-4\left(2\,m_a^2-m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\Big)M_i^6\\
&\qquad+M_f^2\left(5\,M_f^4-2\left(5\,m_a^2+11\,m_b^2\right)M_f^2+5\left(m_a^2-m_b^2\right)^2\right)M_i^4\\
&\qquad+6\,M_f^4\left(m_a^4+\left(2\,m_b^2-M_f^2\right)m_a^2-3\,m_b^4+3\,m_b^2\,M_f^2\right)M_i^2\Big]\,\bar{I}_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(M_f^2-M_i^2\right)\Big[-2\,M_i^4+\left(-m_a^2+m_b^2+M_f^2\right)M_i^2\\
&\qquad+M_f^2\left(5\,m_a^2+3\,m_b^2-3\,M_f^2\right)\Big]\,\bar{J}_{abb}\,,
\end{aligned}
\tag{208}
$$

and

$$
\begin{aligned}
48\,\frac{M_f^2}{(p\cdot q)^2}\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\hat{F}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{ab}&=M_f^2\Big[-4\,M_i^6-\left(3\,m_a^2+13\,m_b^2-3\,M_f^2\right)M_i^4\\
&\qquad+\left(9\,m_a^4-m_b^4+3\left(m_a^2+m_b^2\right)M_f^2\right)M_i^2\\
&\qquad-2\,m_a^2\left(m_a^2-m_b^2\right)\left(m_a^2-m_b^2+3\,M_f^2\right)\Big]\,I_{ab}\\
&+M_i^2\Big[2\Big(-\left(2m_a^2+3m_b^2\right)M_f^4+\left(m_a^4-8\,m_b^2\,m_a^2+3\,m_b^4\right)M_f^2\Big)\\
&\qquad+2\left(m_a^3-m_a\,m_b^2\right)^2+\Big(M_f^4+4\left(m_a^2+4\,m_b^2\right)M_f^2\\
&\qquad-5\left(m_a^2-m_b^2\right)^2\Big)M_i^2\Big]\,\bar{I}_{ab}\\
&+6\,m_b^2\,M_f^2\,M_i^2\left(M_i^2-M_f^2\right)\left(-m_a^2+m_b^2-M_f^2+2\,M_i^2\right)\bar{J}_{abb}\,,
\end{aligned}
$$

$$
\begin{aligned}
48\,\frac{M_f^2\,M_i^2}{(p\cdot q)^2}\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\hat{F}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{+,\,ab}&=M_f^2\Big[-3\left(-m_a^2+m_b^2+M_i^2\right)^2M_i^4\\
&\qquad+2\left(M_f^2-M_i^2\right)^2\Big(-M_i^4-\left(m_a^2+m_b^2\right)M_i^2+2\left(m_a^2-m_b^2\right)^2\Big)\\
&\qquad+\left(M_i^2-M_f^2\right)\Big(2\,m_a^4+\left(8\,m_b^2+5\,M_i^2\right)m_a^2-10\,m_b^4-7\,M_i^4\\
&\qquad+11\,m_b^2\,M_i^2\Big)M_i^2\Big]\,I_{ab}
\end{aligned}
$$

$$+\Big[-\Big(-5\,M_f^4+4\left(m_a^2-2m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\Big)\,M_i^6$$
$$-2\,M_f^2\left(M_f^4+\left(m_a^2+m_b^2\right)M_f^2-2\left(m_a^2-m_b^2\right)^2\right)M_i^4\Big]\,\bar I_{ab}$$
$$+6\,m_b^2\,M_f^2\,M_i^4\left(M_i^2-M_f^2\right)\left(-m_a^2+m_b^2-M_f^2+2\,M_i^2\right)\bar J_{abb}\,,$$

$$48\,\frac{M_f^2}{(p\cdot q)^2}\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\hat F^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{-,\,ab}=M_f^2\Big[2\,M_i^6-\left(m_a^2-17\,m_b^2+M_f^2\right)M_i^4$$
$$-\left(-11\,m_a^4+\left(10\,m_b^2+M_f^2\right)m_a^2+m_b^4+7\,m_b^2\,M_f^2\right)M_i^2$$
$$-2\left(5\,m_a^4-4\,m_b^2\,m_a^2-m_b^4\right)M_f^2\Big]\,I_{ab}$$
$$+\Big[-6\left(m_a^2+m_b^2-M_f^2\right)M_i^2\,M_f^4$$
$$-\left(7\,M_f^4-4\left(2\,m_a^2-m_b^2\right)M_f^2+\left(m_a^2-m_b^2\right)^2\right)M_i^4\Big]\,\bar I_{ab}$$
$$+6\,m_b^2\,M_f^2\,M_i^2\left(M_f^2-M_i^2\right)\left(-m_a^2+m_b^2-M_f^2+2\,M_i^2\right)\bar J_{abb}\,,\qquad(209)$$

and

$$48\,m_a^2\,M_f^2\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc}=2\,m_c^2\,M_f^4\Big[-6\left(m_b^2-m_c^2\right)^2M_i^4$$
$$-3\left(m_b^2-m_c^2\right)\left(M_i^2-M_f^2\right)\left(-m_a^2+m_c^2+3\,M_i^2\right)M_i^2$$
$$+\left(M_f^2-M_i^2\right)^2\Big(-2\,M_i^4+\left(m_a^2-5\,m_c^2\right)M_i^2+\left(m_a^2-m_c^2\right)^2\Big)\Big]\,I_{ac}$$
$$+M_i^2\Big[2\,m_b^4\,m_c^2\,M_f^2\left(2\,M_f^4+5\,M_i^2\,M_f^2-M_i^4\right)$$
$$-2\,m_b^2\,m_c^2\left(M_f^2-M_i^2\right)\Big(4\,M_f^4+5\,M_i^2\,M_f^2+m_a^2\left(2\,M_i^2-5\,M_f^2\right)\Big)M_f^2$$
$$+m_b^2\left(3\,m_a^4+2\,M_f^2\,m_a^2+3\,M_f^4\right)\left(M_f^2-M_i^2\right)^3$$
$$-6\,m_b^2\,m_c^4\left(3\,M_f^2+M_i^2\right)M_f^4+3\,m_b^4\left(m_a^2+M_f^2\right)\left(M_i^2-M_f^2\right)^3$$
$$+12\,m_c^6\,M_f^6+\left(m_a^2-M_f^2\right)^2\left(m_a^2+M_f^2\right)\left(M_i^2-M_f^2\right)^3$$
$$+2\,m_c^2\,M_f^2\left(M_f^2-m_a^2\right)\left(m_a^2+2\,M_f^2\right)\left(M_f^2-M_i^2\right)^2$$
$$+\left(M_f^2-M_i^2\right)^3m_b^6+6\,m_c^4\,M_f^4\left(3\,M_f^2-m_a^2\right)\left(M_f^2-M_i^2\right)\Big]\,\bar I_{ab}$$
$$-12\,m_c^2\,M_f^4\,M_i^2\left(-m_b^2+m_c^2+M_f^2-M_i^2\right)\Big[\left(m_b^2-m_c^2\right)\left(M_f^2-M_i^2\right)m_a^2$$
$$+\left(m_c^2\,M_f^2-m_b^2\,M_i^2\right)\left(-m_b^2+m_c^2+M_f^2-M_i^2\right)\Big]\,\bar J_{abc}\,,$$

$$48\,M_f^2\,M_f^2\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\bar G^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc}=2\,m_c^2\,M_f^4\Big[-6\left(m_b^2-m_c^2\right)^2M_i^4$$
$$+3\left(m_c^2-m_b^2\right)\left(M_i^2-M_f^2\right)\left(-m_a^2+m_c^2+3\,M_i^2\right)M_i^2$$
$$+\left(M_f^2-M_i^2\right)^2\Big(-2\,M_i^4+\left(m_a^2-5\,m_c^2\right)M_i^2+\left(m_a^2-m_c^2\right)^2\Big)\Big]\,I_{ac}$$
$$+M_i^2\Big[\left(M_f^2-M_i^2\right)^3m_b^6+3\,m_b^4\left(m_a^2+M_f^2\right)\left(M_i^2-M_f^2\right)^3$$

$$
\begin{aligned}
&+2\,m_c^2\,M_f^2\left(2M_f^4+5M_i^2M_f^2-M_i^4\right)m_b^4-6\,m_c^4\,m_b^2\left(3\,M_f^2+M_i^2\right)M_f^4\\
&-2\,m_c^2\,m_b^2\left(M_f^2-M_i^2\right)\left(4\,M_f^4+5\,M_i^2\,M_f^2+m_a^2\left(2\,M_i^2-5\,M_f^2\right)\right)M_f^2\\
&+\left(3\,m_a^4+2\,M_f^2\,m_a^2+3\,M_f^4\right)\left(M_f^2-M_i^2\right)^3m_b^2\\
&+12\,m_c^6\,M_f^6+\left(m_a^2-M_f^2\right)^2\left(m_a^2+M_f^2\right)\left(M_i^2-M_f^2\right)^3\\
&+2\,m_c^2\,M_f^2\left(M_f^2-m_a^2\right)\left(m_a^2+2M_f^2\right)\left(M_f^2-M_i^2\right)^2\\
&+6\,m_c^4\,M_f^4\left(3\,M_f^2-m_a^2\right)\left(M_f^2-M_i^2\right)\Big]\,\bar{I}_{ab}\\
&+12\,m_c^2\,M_f^4\,M_i^2\left(m_b^2-m_c^2-M_f^2+M_i^2\right)\Big[\left(m_b^2-m_c^2\right)\left(M_f^2-M_i^2\right)m_a^2\\
&+\left(m_c^2\,M_f^2-m_b^2\,M_i^2\right)\left(-m_b^2+m_c^2+M_f^2-M_i^2\right)\Big]\,\bar{J}_{abc}\,, \qquad (210)
\end{aligned}
$$

and

$$
\begin{aligned}
24\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,H^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} &= M_f^2\Big[-6\left(m_b^2-m_c^2\right)^2M_i^4\\
&-3\left(m_b^2-m_c^2\right)\left(-m_a^2+m_c^2+M_i^2\right)\left(M_i^2-M_f^2\right)M_i^2\\
&+\left(M_f^2-M_i^2\right)^2\left(m_a^4-2\left(m_c^2+M_i^2\right)m_a^2+\left(m_c^2-M_i^2\right)^2\right)\Big]\,I_{ac}\\
&+M_i^2\Big[\left(2\,M_f^4+5\,M_i^2\,M_f^2-M_i^4\right)m_b^4-\left(-5\,m_a^2+9\,m_c^2+M_f^2\right)M_f^4\,m_b^2\\
&+\left(-\left(7\,m_a^2+3\,m_c^2+M_f^2\right)M_i^2\,M_f^2+2\left(m_a^2+M_f^2\right)M_i^4\right)m_b^2\\
&+6\,m_c^4\,M_f^4-\left(m_a^2-M_f^2\right)^2\left(M_f^2-M_i^2\right)^2\\
&+3\,m_c^2\,M_f^2\left(M_f^2-m_a^2\right)\left(M_f^2-M_i^2\right)\Big]\,\bar{I}_{ab}\\
&+6\left(m_b^2-m_c^2\right)M_f^2\,M_i^2\Big[\left(m_b^2-m_c^2\right)\left(M_f^2-M_i^2\right)m_a^2\\
&+\left(m_c^2\,M_f^2-m_b^2\,M_i^2\right)\left(-m_b^2+m_c^2+M_f^2-M_i^2\right)\Big]\,\bar{J}_{abc}\,,
\end{aligned}
$$

$$
\begin{aligned}
24\,M_f^2\,P^{(2)}_{\bar\alpha\bar\beta,\mu,\alpha\beta}\,\bar{H}^{\bar\alpha\bar\beta,\mu,\alpha\beta}_{abc} &= m_a^2\,M_f^2\Big[-6\left(m_b^2-m_c^2\right)^2M_i^4\\
&-3\left(m_b^2-m_c^2\right)\left(-m_a^2+m_c^2+M_i^2\right)\left(M_i^2-M_f^2\right)M_i^2\\
&+\left(M_f^2-M_i^2\right)^2\left(m_a^4-2\left(m_c^2+M_i^2\right)m_a^2+\left(m_c^2-M_i^2\right)^2\right)\Big]\,I_{ac}\\
&+m_a^2\,M_i^2\Big[\left(2\,M_f^4+5\,M_i^2M_f^2-M_i^4\right)m_b^4-\left(-5\,m_a^2+9\,m_c^2+M_f^2\right)M_f^4\,m_b^2\\
&+\left(-\left(7\,m_a^2+3\,m_c^2+M_f^2\right)M_i^2\,M_f^2+2\left(m_a^2+M_f^2\right)M_i^4\right)m_b^2\\
&+6\,m_c^4\,M_f^4-\left(m_a^2-M_f^2\right)^2\left(M_f^2-M_i^2\right)^2\\
&+3\,m_c^2\,M_f^2\left(M_f^2-m_a^2\right)\left(M_f^2-M_i^2\right)\Big]\,\bar{I}_{ab}\\
&+6\,m_a^2\,M_f^2\,M_i^2\Big[\left(-m_a^2+m_c^2+M_i^2\right)\left(M_i^2-M_f^2\right)\left(m_b^2-m_c^2\right)^2\\
&+M_i^2\left(m_b^2-m_c^2\right)^3+\left(m_b^2-m_c^2\right)m_c^2\left(M_f^2-M_i^2\right)^2\Big]\,\bar{J}_{abc}\,. \qquad (211)
\end{aligned}
$$

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