Systematics of geometric scaling

F. Gelis,* R. Peschanski,† and G. Soyez‡§
Service de physique théorique, CEA/Saclay, 91191 Gif-sur-Yvette cedex, France
URA 2306, unité de recherche associée au CNRS

L. Schoeffel¶
DAPNIA/Service de physique des particules, CEA/Saclay, 91191 Gif-sur-Yvette cedex, France

Using all available data on the deep-inelastic cross-sections at HERA at $x \leq 10^{-2}$, we look for geometric scaling of the form $\sigma^{\gamma^* p}(\tau)$ where the scaling variable τ behaves alternatively like $\log Q^2 - \lambda Y$, as in the original definition, or $\log Q^2 - \lambda\sqrt{Y}$, which is suggested by the asymptotic properties of the Balitsky-Kovchegov (BK) equation with *running* QCD coupling constant. A "Quality Factor" (QF) is defined, quantifying the phenomenological validity of the scaling and the uncertainty on the intercept λ. Both choices have a good QF, showing that the second choice is as valid as the first one, predicted for *fixed* coupling constant. A comparison between the QCD asymptotic predictions and data is made and the QF analysis shows that the agreement can be reached, provided going beyond leading logarithmic accuracy for the BK equation.

1. Geometric scaling [1] is a remarkable empirical property verified by the data on the high-energy deep-inelastic scattering (DIS) cross-sections $\sigma^{\gamma^* p}$. It has been realized that one can represent with reasonable accuracy the cross-section by the formula

$$\sigma^{\gamma^* p}(Y,Q) = \sigma^{\gamma^* p}\left(\frac{Q^2}{Q_s^2(Y)}\right) \ , \tag{1}$$

where Q is the virtuality of the photon, Y the total rapidity in the γ^*-proton system and $Q_s^2 \propto e^{\lambda Y}$ an increasing function of Y. The value found for $\lambda \sim 0.3$ has been confirmed by the well-known Golec-Biernat and Wüsthoff model [2] where geometric scaling has been explicitly assumed in the parametrisation.

The property (1), also observed in DIS on nucleus [3] and diffractive processes [4], has been intimately related [5] to the concept of *saturation* [6], i.e. the behaviour of perturbative QCD amplitudes when the density of partons becomes high enough to exercise the unitarity bound. Indeed, there has been many theoretical arguments to infer that in a Y, Q^2 domain where saturation effects set in, geometric scaling is expected to occur. Within this framework, the function $Q_s(Y)$ can be called the saturation scale, since it delineates the approximate upper bound of the saturation domain.

Following a theoretical approach, it has been possible to derive the property (1) from the nonlinear Balitsky-Kovchegov (BK) equation which represents the "mean-field" approximation of the evolution equation for high energy (high density) QCD amplitudes. This equation is supposed to capture essential features of saturation effects. In the 1-dimensional approximation it reads for the density of gluons with transverse momenta k in some target [7]

$$\partial_Y \mathcal{N}(L,Y) = \bar\alpha\,\chi(-\partial_L)\,\mathcal{N}(L,Y) - \bar\alpha\,\mathcal{N}^2(L,Y), \tag{2}$$

where $L = \log(k^2/k_0^2)$, with k_0^2 being an arbitrary constant. In Eq.(2), the coupling constant will be considered alternatively in the following as fixed (we shall use $\bar\alpha = 0.15$), or running such that $\bar\alpha(L) = 1/bL$, $b = (11N_c - 2N_f)/12N_c$. The kernel will also be either taken with leading logarithm (LL) accuracy [8], as $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$, or at next-to-leading logarithm (NLL) accuracy (see *e.g.* [9, 10, 11]).

The key ingredient to theoretically prove geometric scaling of the asymptotic solutions of the nonlinear equation (2) is the *travelling wave* method [12]. Indeed, the BK equation admits solutions in the form of *travelling waves* $\mathcal{N}(L - v_g\bar\alpha Y)$. L has the interpretation of a space variable while $t = \bar\alpha Y$, interpreted as time, is an increasing function of rapidity Y. v_g is the *critical velocity* of the wave, defined in this case as the minimum of the phase velocity. The travelling-wave solution for the quantity $\mathcal{N}$ can be easily translated to the property (2) (at least assuming negligible quark masses) since it yields $\mathcal{N}(k^2/Q_s^2(Y))$. Hence the asymptotic travelling-wave solutions of the BK equation satisfy geometric scaling (up to subdominant scaling violations which we will not analyse in the present work). Similar results can be obtained [13, 14] from an approximation of the linear equation with absorptive boundary conditions. The main theoretical results for the asymptotic saturation scale are the following:

$$\log Q_s^2 = \bar\alpha\frac{\chi(\gamma_c)}{\gamma_c}(Y - Y_0) - \frac{3}{2\gamma_c}\log(Y - Y_0) - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar\alpha\chi''(\gamma_c)}}\frac{1}{\sqrt{(Y+Y_0)}} + \dots\ , \tag{3}$$

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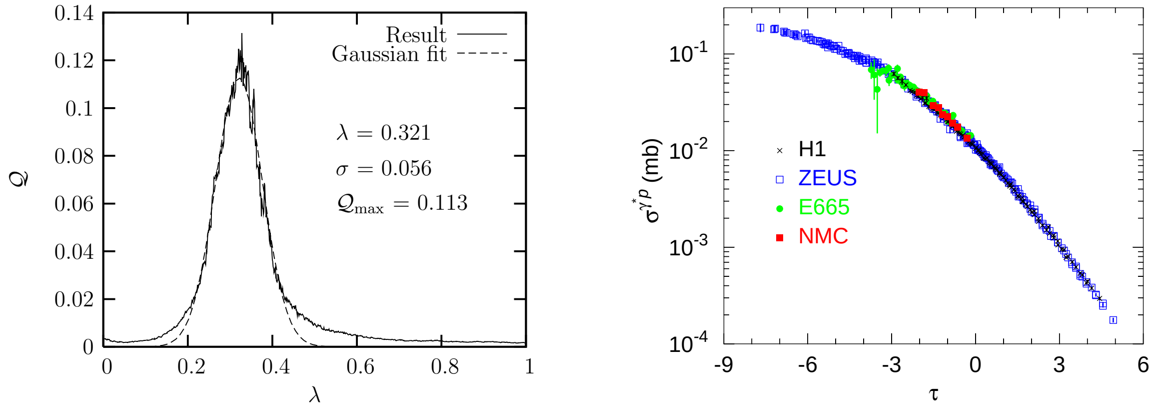


(a) Quality Factor as a function of λ.

(b) Scaling curve.

FIG. 1: Geometric Scaling in Y

for fixed coupling and

$$\log Q_s^2 = \sqrt{\frac{2\chi(\gamma_c)}{b\gamma_c}(Y - Y_0)} + \tfrac{3}{4}\left(\frac{\chi''(\gamma_c)}{\sqrt{2b\gamma_c\chi(\gamma_c)}}\right)^{1/3} \xi_1\ (Y - Y_0)^{1/6} + \dots\ , \qquad (4)$$

for running coupling. γ_c is the solution of the implicit equation $\chi(\gamma_c) = \gamma_c \chi'(\gamma_c)$ while $\xi_1 = -2.338$ is the first zero of the Airy function. Y_0 is an arbitrary constant which may parameterize the unknown "non-universal" preasymptotic contributions, depending on the initial conditions.

The questions we want to ask deal with the confrontation between the empirical formula (2) with the asymptotic predictions for the saturation scale (3,4). They may be expressed as follows

- Is geometric scaling, demonstrated for $\log(Q_s^2) \sim Y$ in agreement with the leading term in (3), remains valid for $\log(Q_s^2) \sim Y^{1/2}$ as suggested by (4)?
- Are the theoretical factors and subasymptotic terms given in Eqs.(3,4) visible in the data?
- Are NLL effects in the theoretical predictions phenomenologically important?
- What is the role of "non-universal" terms?

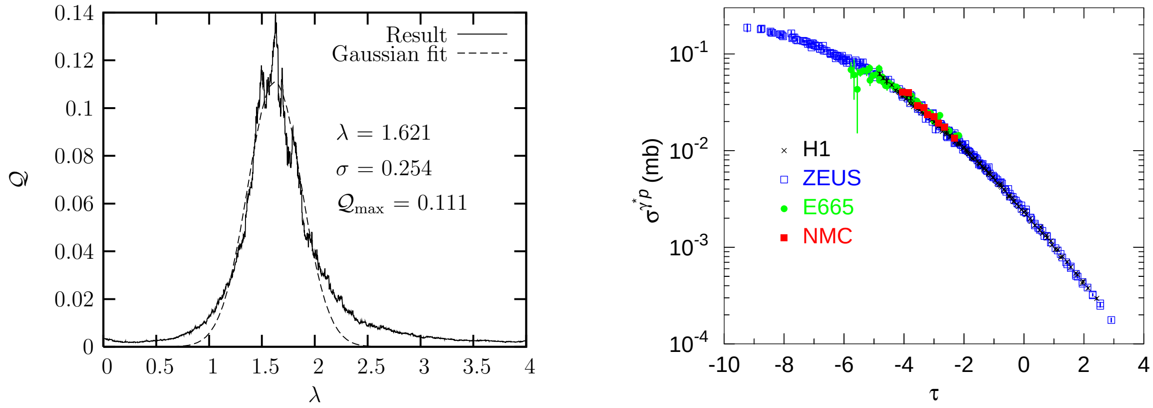
2. We focus our analysis on the measurements of the $\gamma^* p$ cross-section $\sigma^{\gamma^* p} = (4\pi^2\alpha_e/Q^2)F_2$ for which geometric scaling is predicted. The latter being valid at high-energy, we shall restrict ourselves to $x \le 0.01$ for all values of Q^2. Within that range, we shall use all available data (404 points from E665 [15], H1 [16], NMC [17] and ZEUS [18]).

In order to get a quantitative answer, we shall introduce an "estimator" or *quality factor* (QF) for determining the scaling quality, whose working definition is the following. Given a set of points $(x_i, y_i, f_i \equiv f(x_i, y_i))$ and a parametric *scaling law* $\tau(x, y; \lambda)$, we want to determine if for some values of λ, $f(x, y)$ can be considered a function of τ only. To achieve this we define a QF which is large when the points $(\tau_i = \tau(x_i, y_i; \lambda), f_i)$ "lies on a unique curve". To quantify this still ill-defined concept, we shall first rescale the set (τ_i, f_i) into (u_i, v_i) such that $0 \le u_i, v_i \le 1$ and assume that the u_i's are ordered. We then introduce

$$\mathcal{Q}(\lambda) = \left[\sum_i \frac{(v_i - v_{i-1})^2}{(u_i - u_{i-1})^2 + \varepsilon^2}\right]^{-1} . \qquad (5)$$

This definition for the quality factor obviously achieves what we want: when two successive points are close in u and far in v, we expect them "not to lie on the same curve" and, indeed, they give a large contribution to the sum in (5), leading to a small quality factor. The constant ε^2 in (5) is a small number (we have taken $\varepsilon = 1/n$ with n being the number of data points) which prevents the sum from becoming infinite when two points have the same value for u. Finally, by studying the dependence on λ of $\mathcal{Q}$ (as we shall see, it usually shows a Gaussian peak) we can determine the best choice for the parameter in the scaling function τ and its uncertainty.

3. In order to answer the first question concerning the compared validity of geometric scaling in Y *vs.* $\sqrt{Y}$, we performed the QF analysis in both cases. In Fig.1, we show the data and QF for the usual geometric scaling definition with Y. In Fig.1-(b), is displayed the data plot with scale redefinition. The quality of scaling on this plot can be read from Fig.1-(a), where the scatter plot of the QF is displayed together with a Gaussian fit of the peak. As obvious from Fig.1-(b), the peak value larger than 0.1



(a) Quality Factor as a function of λ.

(b) Scaling curve.

FIG. 2: Geometric Scaling in $\sqrt{Y}$

showss a good scaling and a Gaussian fit of the bump gives the best value and error for the lambda parameter $\lambda = 0.321 \pm 0.056$. Note that the results displayed on Fig.1 can be considered as an update for the usual geometric scaling tests and a quantitative determination of the scaling parameter and its attached uncertainty. The new point is that geometric scaling is also verified, and with the same level of quality than previously, when the saturation scale is chosen as a function of $\sqrt{Y}$. This is manifest on Fig.2, where comparable QF heights at the peak and width attest of the quality of scaling. The peak value getting larger than 0.11 givess a good scaling. The Gaussian fit of the bump gives $\lambda = 1.621 \pm 0.254$.

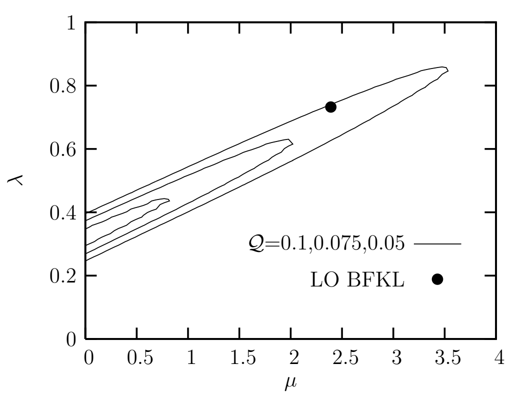


FIG. 3: Quality Factor for the LL prediction.

4. Having verified geometric scaling, as suggested by the asymptotic solutions of the QCD equations with both fixed and running coupling constant, it is tempting to go one step further and to ask whether the theoretical formulae (3,4) with their theoretically predicted parametrisations are valid?

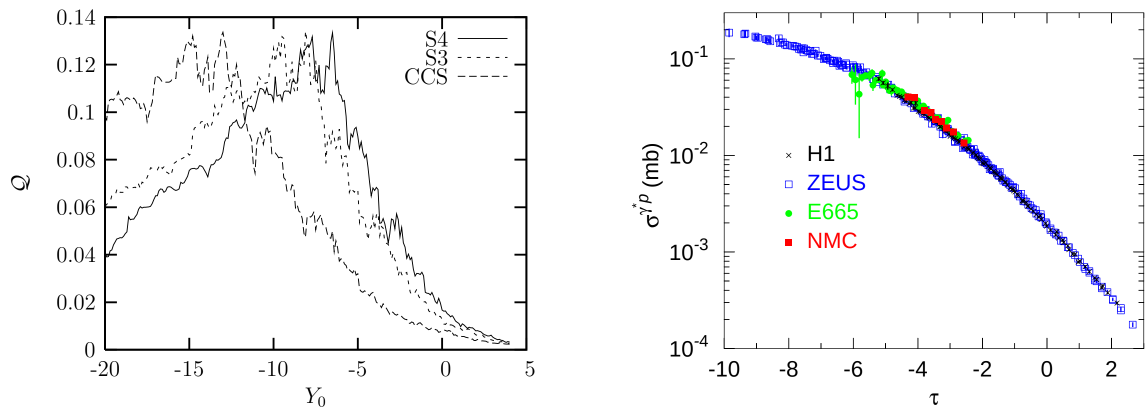
For that sake, we have considered first the LL prediction with fixed coupling (3) with the first subleading correction, *i.e.* of the form $\log(Q_s^2) \propto \lambda Y - \mu \log(Y)$ (see Fig. 3). The predicted values from (3) are represented by a point in the λ, μ plane. It happens that a QF of ~ 0.055 can be attributed to this parametrisation, as shown on the same figure by the equi-QF curves. In this case changing Y_0 or adding the third term of (3) does not improve the QF's. Hence the asymptotic LL theoretical prediction remains only marginally verified at present energies.

Let us now turn to the analysis of the 3 different theoretical NLL schemes that we consider. As now well known, NLL corrections to the LL kernel [9] have to be embedded in a resummation scheme in order to cancel spurious singularities. We consider asymptotic predictions (4) for the so-called S3, S4 schemes [10] and CCS scheme [11] which were recently derived [19].

On Fig.4 we show the resulting analysis. The Quality Factor for the 3 schemes is presented as a function of Y_0, which parameterizes in (4) the non-universal contributions. There always exists a range of Y_0 for which the QF goes beyond 0.1 and thus leads to an acceptable scaling. This has to be contrasted with the situation with LL kernel. However, if Y_0 is too negative, the relevance of geometrical scaling in $\sqrt{Y}$ is questionable. Indeed $t = \sqrt{Y - Y_0} \sim \sqrt{|Y_0|} + Y/2\sqrt{|Y_0|}$ and thus the scaling variable is Y with a non universal $\lambda = 1/2\sqrt{|Y_0|}$ parameter. For that reason the S4 scheme seems favoured by the analysis and the figure displays the resulting scaling curve for $Y_0 = -5.5$.

5. Using the preceding results based on the estimate of Quality Factors for scaling properties, it is now possible to answer the questions asked in the introduction.

- Geometric scaling is definitely as valid for the



(a) Quality Factor as a function of Y_0

(b) Scaling curve for the S4 scheme.

FIG. 4: Geometric Scaling at NLL accuracy

choice $\log(Q_s^2) \sim Y^{1/2}$ as it is for the original $\log(Q_s^2) \sim Y$ suggestion

- The theoretical predictions (3) based on the BK equation at leading log accuracy, *i.e.* with fixed coupling and LL kernel are only marginally verified with a QF of ~ 0.055.

- At next-to-leading accuracy, the theoretical predictions (4) on scaling may be satisfied. There exists preferable NLL schemes *e.g.* the S4 scheme [10].

- Parametrising "non-universal" terms, *i.e.* depending on initial conditions by a constant Y_0 in the rapidity evolution, we find that they do not play an essential role at LL level, while they contribute to the quality of the scaling at NLL level.

The generality of the Quality Factor method indicates its interest for a quantitative evaluation of good scaling properties beyond the specific applications we focussed on in the present work. It can be used to check empirical scaling-law proposals as well as to quantitatively evaluate the "distance" between given theoretical predictions with data. In the present application, it leads to the conclusion that geometrical scaling is a well verified empirical property of deep-inelastic cross-sections on the proton. It also shows that the theoretical predictions based on the QCD saturation mechanism are consistent with the observation, but requires to go beyond leading logarithmic accuracy and the asymptotic terms.

Finally, let us quote that recent high-energy predictions [20] of *diffusive scaling*, namely $\sigma^{\gamma^* p}/\sqrt{Y} = \sigma(\tau)$ with $\tau = \log(Q^2/Q_s^2)/\sqrt{Y}$, have a QF of 0.05, indicating that it might require higher energies.

* Electronic address: gelis@spht.saclay.cea.fr
† Electronic address: pesch@spht.saclay.cea.fr
§ Electronic address: g.soyez@ulg.ac.be
¶ Electronic address: schoeffel@hep.saclay.cea.fr

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