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Numerical calculation of the lowest eigenmodes of the Laplacian in compact orientable 3-dimensional hyperbolic spaces.

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Abstract.
A simple method to compute numerically the lowest eigenmodes of the Laplacian in compact orientable hyperbolic spaces of dimension 3 is presented. It is applied to the Thurston manifold, the Weber-Seifert manifold, and to the spaces whose fundamental domain is a regular icosahedron.

1 Introduction.

Compact hyperbolic spaces are chaotic system for classical trajectories, but as quantum systems, there exists well defined solutions. Although the Cosmic Microwave Background anisotropy data suggest a locally flat universe, nothing is really known about its global structure. Compact hyperbolic spaces may still be an attractive possibility, and there is an infinity of them. The calculation of the observed anisotropies requires to know the eigenmodes of the Laplacian. In this note we present a simple method to compute numerically the lowest eigenvalues of the Laplacian (for scalar functions) in compact orientable hyperbolic spaces of dimension 3. It also provides approximate eigenfunctions as expansions on known functions. The method is applied to the Thurston manifold, in order to compare with the results of [4], the Weber-Seifert manifold, and to the spaces whose fundamental domain is a regular icosahedron. This last example allows to compare spaces having the same fundamental domain but completely different structures.

2 Compact hyperbolic manifolds.

The hyperbolic n dimensional space H^n is defined as the « upper part » of the sphere of radius $\sqrt{|K|}$ in the Minkowski space M^{n+1}. More precisely, if $\{x^\alpha\}$ are cartesian coordinates in M^{n+1} with origine O_M , H^n is the surface defined by :

$$\sum_{\alpha=0}^{n-1} x^\alpha x^\alpha - x^n x^n = K$$

with : $K < 0$ and $x^n \geq \sqrt{|K|}$. We set : $R = \sqrt{|K|}$ and use « spherical » coordinates. For H^3 in M^4 :

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$$
\begin{aligned}
x^0 &= R\, sh\chi\, c \qquad ; \qquad c = \cos(\theta) \quad , \quad s = \sin(\theta) \\
x^1 &= R\, sh\chi\, s\, c_\varphi \qquad ; \qquad c_\varphi = \cos(\varphi) \quad , \quad s_\varphi = \sin(\varphi) \\
x^2 &= R\, sh\chi\, s\, s_\varphi \\
x^3 &= R\, ch\chi
\end{aligned} \tag{1}
$$

Then the linear element of H^3 is :

$$
ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2 = R^2\left[d\chi^2 + sh^2\chi\, ((d\theta)^2 + s^2(d\varphi)^2)\right] \tag{2}
$$

The coordinates (χ, θ, φ) are the Riemann normal (spherical) coordinates with origine at $\chi = 0$, which corresponds to the point $(0,0,0,R)$ in M^4 .

In the following $g_{\alpha\beta}$ is the metric tensor and $g^{\alpha\beta}$ its inverse, whatever the coordinate system used. The curvature tensor is : $R_{\alpha\beta\gamma\delta} = -\frac{1}{R^2}(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma})$, the Ricci tensor is : $R_{\alpha\beta} = -\frac{2}{R^2}\, g_{\alpha\beta}$ and the scalar curvature : $R_H = -\frac{6}{R^2}$. From now on, we set $R = 1$.

H^n is simply connected. From theorem 2.4.10 of [1] : a manifold M^n of dimension n and negative constant curvature is complete and connected if and only if it is isometric to a quotient H^n/Γ where Γ is a group of isometries of H^n acting freely and properly discontinuously.

Γ is a subgroup of the component of $O(n,1)$ which preserves H^n. It is called the group of deck transformation. H^n is the universal covering of M^n.

In the following we shall consider 3-dimensional compact and orientable manifolds only. The elements of the group Γ are screw motions. A screw motion is the product of a transvection and a rotation. H^n, which is of constant curvature, is a symetric space. A transvection in a symetric space is an isometry which generalises the notion of translation. It is defined as the product of two successive symetries whith respect to two different points A and B . The geodesic going through these two points is invariant and is called base geodesic. In H^3 one can perform a rotation around this geodesic. The transvection and this rotation commute and the base geodesic is invariant. The product of two screw motions is still a screw motion. In the following we shall call γ^i the generators of Γ and g^i the corresponding base geodesics. The base geodesic of a motion $\gamma \in \Gamma$ is given by the intersection of the invariant plane, associated to the real eigenvalues of the matrix representing γ in M^4 , with H^3 .

We shall call L the « length » of the transvection, which is twice the distance between A and B . A point whose spherical coordinates are ($\chi, \theta, \varphi,\ c = \cos(\theta)$) is transformed by a transvection along Oz into a point of coordinates ($\chi', \theta', \varphi',\ c' = \cos(\theta')$) given by :

$$
\begin{aligned}
ch(\chi') &= ch(\chi)\, ch(L) + sh(\chi)\, sh(L)\, \mathrm{c} \\
c' &= (ch(\chi)\, sh(L) + c\, sh(\chi)\, ch(L)) / sh(\chi') \\
\varphi' &= \varphi
\end{aligned} \tag{3}
$$

We shall also use cylindrical coordinates wich we define as follows :

The z coordinate of a point M is given by the orthogonal projection of that point on the Oz axis. The distance between M and that projection is the radius ρ. The azimuth φ is the same as for the spherical coordinates. The linear element is given by :

$$ds^2 = (d\rho)^2 + ch^2\rho\,(dz)^2 + sh^2\rho\,(d\varphi)^2 \qquad (4)$$

The two sets of coordinates are related by :

$$ch\chi = ch\rho\, chz \quad , \quad thz = c\, th\chi \quad , \quad sh\rho = s\, sh\chi$$

In cylindrical coordinates the action of a transvection along Oz is simple and given by :

$$\rho' = \rho \quad , \quad z' = z + L \quad , \quad \varphi' = \varphi \qquad (5)$$

In the following, we shall call respectively L_i and ω_i the length and the rotation angle of the screw motion γ^i.

3 Eigenmodes of the Laplacian.

We consider only scalar functions (real or complex), not differential forms, and call Φ the functions defined on M^3. The Lapacian is :

$$\Delta\Phi = \frac{1}{\sqrt{g}}\,\partial_\alpha(\, g^{\alpha\beta}\sqrt{g}\;\partial_\beta\Phi\,) \qquad (6)$$

where g is the determinant of $g_{\alpha\beta}$.

The function Φ can be considered as a periodic function defined on H^3 (we give the same name) satisfying :

$$\Phi(\gamma\, x) = \Phi(x) \qquad (7)$$

for any point x of H^3 and any element γ of Γ. This condition is equivalent to : $\Phi(\gamma^i\, x) = \Phi(x)$ for any generator γ^i and their inverse. In particular, Φ must be periodic on the base geodesics.

Since the Laplacian commutes with any isometry we can search for functions which are eigenfunctions of the Laplacian and of some commuting subset of Γ. We shall consider commuting subsets of Γ of the form $(\gamma^i)^p$ for some i and $p \in \mathbb{Z}$. We shall choose, for i, the isometry which has the longest transvection, although this is not necessary, and consider its base geodesic as the polar axis for spherical or cylindrical coordinates. For the rest of this note, we rename the generators such that γ^0 becomes the one with the longest transvection and such that its base geodesic g^0 defines the Oz axis of the spherical coordinates (this is always possible by performing a transvection and a rotation around the origin).

In cylindrical coordinates functions of the form :

$$\Phi \sim I(\mu_z, \nu, \rho)\exp(i\,(\mu_z z + \nu\varphi)) \quad , \quad \nu \text{ integer} \qquad (8)$$

are invariant under the action of γ^0 provided that : $\mu_z L_0 + \nu\omega_0 = 2\pi\, m_z$, where m_z is an integer. This last constraint defines μ_z.

At this stage one has two possibilities. Either one expands Φ on the known basis of eigenfunctions of the Laplacian in spherical coordinates, or one keeps on using cylindrical coordinates.

Spherical coordinates.

With equation (2) in mind we naturally try to use a basis of eigenfunctions of Δ in spherical coordinates. The solutions are known and of the form (β real) :

$$\Delta\Phi = -(1+\beta^2)\Phi \quad , \quad \Phi \simeq \phi_\beta^l(\chi)\, Y_l^m(\theta,\varphi) \tag{9}$$

where $Y_l^m(\theta,\varphi)$ are the usual spherical harmonic functions, $\phi_\beta^l \simeq \frac{1}{\sqrt{sh\chi}} B_\lambda^\mu(\chi)$, and $B_\lambda^\mu(\chi)$ are Legendre functions with $\lambda = -\frac{1}{2} + i\,\beta$ and $\mu = -\frac{1}{2} - l$ (see [2] for instance). The solution $\Phi = C$onstant corresponds to the 0 eigenvalue, and will not be considered anymore. These functions are regular and for example : $\phi_\beta^0 = \frac{\sin(\beta\chi)}{\beta sh\chi}$.

An infinitesimal transvection along Oz has the form :

$$\Phi(x') = \Phi(x) + \varepsilon\, T\ \Phi \quad , \quad T = c\,\partial_\chi + \frac{ch\chi}{sh\chi} s^2\ \partial_c \tag{10a}$$

while in cylindrical coordinates it is simply : $T = \partial_z$ (10b)

The expression of the transvection component of γ^0 is the exponentiation of the operator T .

A rotation by an angle ω is simply : $\chi \to \chi \quad , \quad \theta \to \theta \quad , \quad \varphi \to \varphi + \omega$.

Using the recurrence relations for the Legendre functions [2], one obtains :

$$\begin{aligned}(2l+1)\, T(\phi_\beta^l Y_l^m) = {} & (l+m)\sqrt{\frac{2l+1}{2l-1}\frac{l-m}{l+m}}\ \phi_\beta^{l-1} Y_{l-1}^m \\ & - (l+1-m)\,(\beta^2 + (l+1)^2)\sqrt{\frac{2l+1}{2l+3}\frac{l+1+m}{l+1-m}}\ \phi_\beta^{l+1} Y_{l+1}^m\end{aligned} \tag{11}$$

It is then possible to build functions that are eigenfunctions of Δ and γ^0. We write such functions as : $\Phi \simeq I(\mu_z,\nu,\rho)\exp(i\,(\mu_z z + \nu\varphi)) = \sum_{l,m} \bar{a}_{lm}\ \phi_\beta^l(\chi)\, Y_l^m(\theta,\varphi)$. The orthogonality of the $e^{i\nu\varphi}$ on $[0,2\pi]$ implies $m = \nu$, and by applying twice the operator T to both members of the latter equation one has : $-{\mu_z}^2\Phi = \sum_{l,m} \bar{a}_{lm}\ T^2\left(\phi_\beta^l(\chi)\, Y_l^m(\theta,\varphi)\right)$ which, using (11), gives a matrix equation for the calculation of the $\bar{a}_{lm}$ coefficients.

Unfortunately, although these equations are simple, their use in numerical calculations meets important problems as we explain now.

Let us consider the example of the Weber-Seifert manifold which has a high degree of symetry. Its fondamental domain is a regular dodecahedron. The transvection of the generators γ^i have all the same length, and their rotation angles are all equal to $3\pi/5$. Their base geodesics go through the origin, and through the middle of the corresponding faces. The generators can be deduced from one another by simple rotation around the origin : $\gamma^j = R\gamma^i R^{-1}$. Therefore it seems that spherical coordinates are well suited to study that case.

As said above, Φ must be periodic on the base geodesics of the generators γ^i, and, on them, we can expand Φ as : $\Phi = \sum r_j\, e^{i\,k_j\,\chi}$ where : $k_j = 2\pi j/L$ and j is an integer . By analogy with the spherical Bessel functions, one can write :

$$e^{ik\chi} = \sum_l i^l (2l+1) b^l\, \phi^l_\beta(\chi) \tag{12}$$

By derivating twice both members of this equation and using the Legendre function recurrence relations, the b^l cofficients are related by :

$$k\,(2l+1) b^l = (l+1) b^{l+1} + l(\beta^2 + l^2) b^{l-1} \tag{13}$$

$$b^0 = 1 \quad , \quad b^1 = k \quad , \quad b^1 = \tfrac{1}{2}(3k^2 - (1+\beta^2)) \quad , \quad ...$$

When one wants to implement this numerically, because of the shape of the functions $\phi^l_\beta(\chi)$ the number of terms necessary for χ larger than 2, increases very rapidly. But at high l values and for χ smaller than or of the order of 1, the recurrence relations used to compute these functions fail, unless one uses 128 bits calculations. For these reasons we have worked with cylindrical coordinates.

Note that, in the case $\nu = 0$, the coefficients $\bar{a}_{lm}$ are simply :

$$\bar{a}_{l0}\, \frac{Y^0_l(c=1)}{Y^0_0} = (2l+1)\, i^l\, b^l(k = \frac{2\pi m_z}{L})$$

Cylindrical coordinates.

In cylindrical coordinates the Laplacian takes the form :

$$\frac{1}{sh\rho\, ch\rho}\, \partial_\rho (sh\rho\, ch\rho\, \partial_\rho \Phi) + \frac{1}{ch^2\rho}\, \partial^2_z \Phi + \frac{1}{sh^2\rho}\, \partial^2_\varphi \Phi = -(1+\beta^2)\Phi$$

and inserting (8) :

$$\frac{1}{sh\rho\, ch\rho}\, \partial_\rho (sh\rho\, ch\rho\, \partial_\rho I) - \frac{\mu_z^2}{ch^2\rho}\, I - \frac{\nu^2}{sh^2\rho}\, I = -(1+\beta^2) I \tag{14}$$

The radial function has the following properties :

For $\rho \to 0$ we set : $I = q(u)/ch\rho$ where : $u = sh\rho$. With that, (14) looks like a Bessel function when $\rho \ll 1$, and one has the following expansion :

$$q = u^\alpha\, (1 + a_1 u + a_2 u^2 + ... + a_p u^p + ...) \tag{15}$$

with : $\alpha = \nu$ (in order to have a regular function at $\rho = 0$) , $a_{2p+1} = 0$,

$$4(\nu+1)\, a_2 = \mu_z^2 + 1 - (\beta^2 + \nu^2) \quad ,$$

$$8(\nu+2)\, a_4 = a_2\,(\mu_z^2 + 1 - \beta^2 - \nu^2) + (\beta^2 + \nu^2 - 2(\mu_z^2 + 1)) \;\; , \;\; \dots$$

(4) may suggest also to make the variable change $u = th\rho$ and try an expansion of the form :

$$I = u^\alpha\, (1 + a_1 u + a_2 u^2 + ... + a_p u^p + ...)$$

one obtains coefficients which are very similar to the above ones : $\alpha = \nu$ (in order to have a regular function at $\rho = 0$) , $a_{2p+1} = 0$,

$$4(\nu+1)\, a_2 = \mu_z^2 + \nu^2 - (1+\beta^2) \quad ,$$

$$8(\nu+2)\, a_4 = a_2\,(\mu_z^2 + \nu^2 - (1+\beta^2)) + (\mu_z^2 + \nu^2 - 2(1+\beta^2)) \;\; , \;\; \dots$$

but the former expansion (with $u = sh\rho$) has better convergence properties, in the parameter domain in which we have used them.

For $\rho \to \infty$ we set : $I = q(\rho)\big/\sqrt{sh\rho\, ch\rho}$ (this is suggested by the fact that, assymptotically, we expect an oscillating function and that $\sqrt{g} = sh\rho\, ch\rho$). The radial equation becomes :

$$\partial_{\rho\rho} q + \beta^2 q - \frac{(\mu_z^2 + \frac{1}{4})}{ch^2\rho} q - \frac{(\nu^2 - \frac{1}{4})}{sh^2\rho} q = 0 \qquad (16)$$

We write : $q = A\cos(S)$, the radial function has the following assymptotic expansion :

$$A = A_0\,(1 + \frac{a_2}{ch^2\rho} + \frac{a_4}{ch^4\rho} + ...) \quad , \quad \frac{dS}{d\rho} = \frac{\beta}{(A/A_0)^2} \qquad (17)$$

$$a_2 = \frac{\mu_z^{\,2} + \nu^2}{4(1+\beta^2)} \quad , \quad S = \varphi(\beta,\nu,\mu_z) + \beta\rho + 2\,a_2\,\beta(1 - th\rho) + ...$$

where $\varphi(\beta,\nu,\mu_z)$ is a phase , and A_0 is an overall normalisation constant which must be determined numerically.

The values of the radial function at low values of ρ ($\rho \le 0.25$) are calculated using the expansion (15) up to the twelvth order in $sh\rho$. For higher ρ values, the differental equation (14) is solved numerically. Figure 1 shows some examples of radial functions and comparisons with the $\rho \to \infty$ assymptotic expansion up to terms of order $1\big/ch^8\rho$ (included), where the phase and A_0 are numerically adjusted.

We could also have set : $I = q(u)\big/\sqrt{ch\rho}$ with $u = ch\rho$

In that case the radial equation looks like a Legendre equation when $\rho \to \infty$:

$$(1-u^2)\frac{d^2q}{du^2} - 2u\frac{dq}{du} - (\beta^2 + \frac{1}{4})\,q + (\mu_z^{\,2} + \frac{1}{4})\,\frac{1}{u^2}q + \frac{\nu^2}{u^2-1}q = 0$$

$$I \xrightarrow[\rho\to\infty]{} \frac{1}{\sqrt{ch\rho}}\,B^{\mu}_{-\frac{1}{2}+i\beta}(\rho) \quad , \quad \mu^2 = \mu_z^{\,2} + \nu^2 + \frac{1}{4} \qquad (18)$$

I have not been able to find any reccurrence relations which could link the radial functions at different ν or μ_z values.

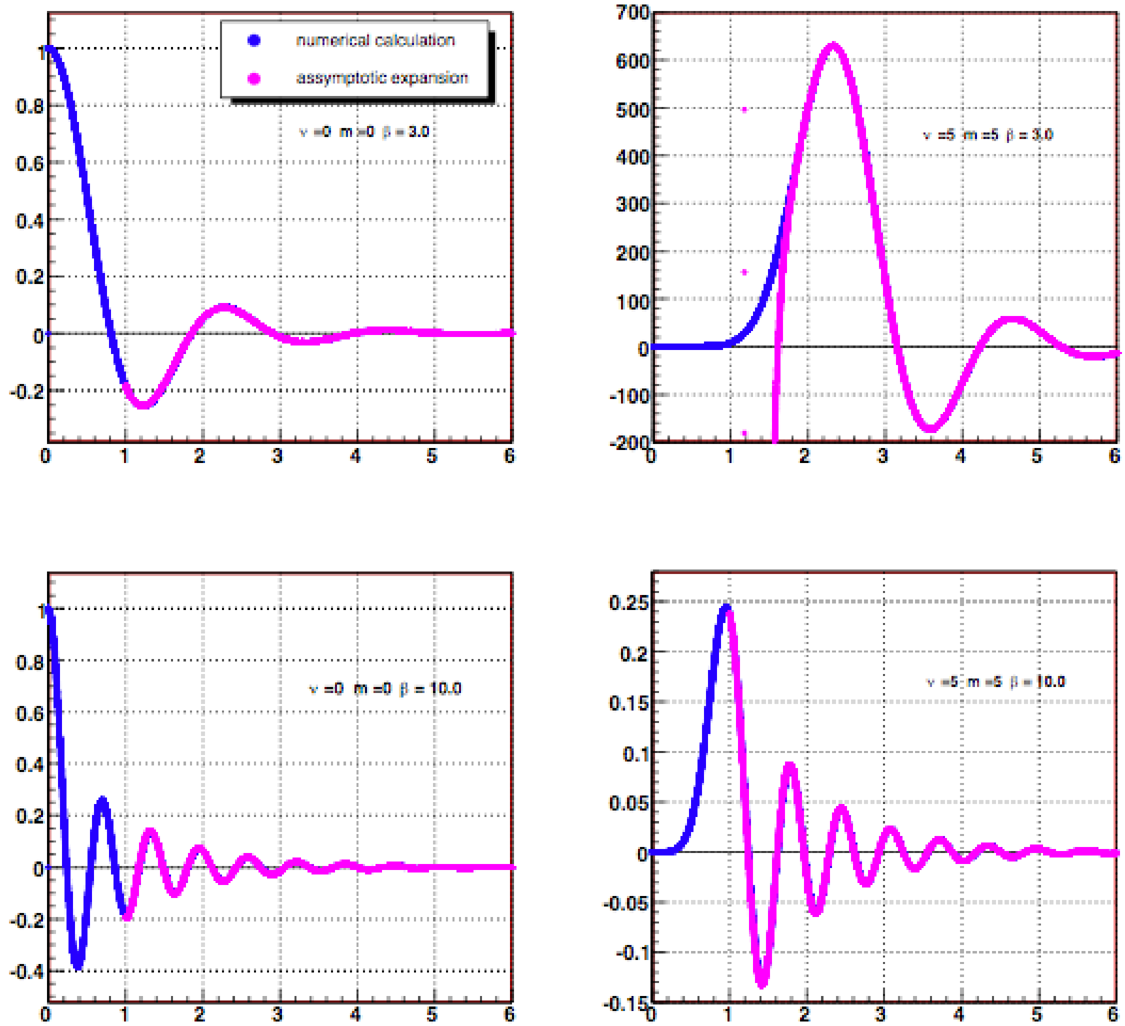

Figure 1. Examples of cylindrical functions.

We shall choose the following basis of real functions :

$$\Phi_{0,\, m,\nu} = I(\beta, \mu_z, \nu, \rho)\cos(\mu_z z + \nu\varphi) \tag{19a}$$

$$\Phi_{1,\, m,\nu} = I(\beta, \mu_z, \nu, \rho)\sin(\mu_z z + \nu\varphi) \tag{19b}$$

where : $\mu_z L_0 + \nu\omega_0 = 2\pi\, m_z$, m_z integer .

These functions are orthogonal on any cylinder of axis Oz and height L. This is important for the least square method used to find solutions satisfying the periodicity relations (7), as described in the next section.

A first consequence of the assymptotic expansion (17) is that Φ can not be simply of the form (19). This can be seen by using again the example of the Weber-Seifert manifold. Φ must be periodic on the base geodesics of the generators γ^i , $i \neq 0$ which go through the origine O , that is to say, Φ must be a periodic function of χ . When $\chi \to \infty$, $z \to z_{\lim}$

such that $thz_{\lim} = 1/\sqrt{5}$ and $\chi \to \rho + Ct$. The expansion (17) shows that functions of the form (19) can not satisfy the periodicity condition since they are decreasing when ρ increases. Analogously to (12), Φ must be a sum :

$$\Phi = \sum_{m,\nu} a_{m,\nu}\, \phi_{m,\nu} \tag{20}$$

with $m \geq 0$, because if both m and ν change sign, $\phi_{m,\nu}$ remains the same up to a sign. We use again the example of the Weber-Seifert manifold to argue that, in (20) the sum can not be reduced to a sum over m or ν only. In this example, Φ , wich must be a periodic function of χ on the base geodesics g_i can be expanded on each of them as : $\Phi = \sum r_k\, e^{i\,2\pi k\,\chi/L}$, $k \in \mathbb{Z}$. The $e^{i\,2\pi k\,\chi/L}$ can be expanded on a set $\{\phi_\beta^\mu\}$ (the functions ϕ_β^μ are linearly independent). The relations (13) shows that the decomposition implies the $\{\phi_\beta^{\mu+j}\}$, $j \in \mathbb{Z}$, But, according to (18), this is impossible if m or ν is fixed, and (20) must be a sum on both indices m and ν.

In practice, one can only calculate a limited number of terms in the sum (20), and the method used will be described in section 5. Since we consider only manifolds without border, we can use Rayleigh’s theorem [3], which puts some constraints on the eigenvalues, by saying that the lowest eigenvalues are associated to the states having the lowest « kinetic energy ».
In the remaining of this section we consider functions f and g defined on a manifold M (without border, by hypothesis) which satisfy (using (6)) :

$$\int_M f^+\, \Delta g\, dV = -\int_M \partial_\alpha f\, g^{\alpha\beta}\, \partial_\beta g\, dV = \int_M (\Delta f)^+\, g\, dV \tag{21}$$

where the second integral is the scalar product of the gradients of the functions.

Rayleigh’s theorem states that :
Let $\lambda_1 \leq \lambda_2 \leq ...$ be the eigenvalues repeated the number of times equal to their multplicity. Then for any function $f \neq 0$ satisfying (21), we have : $\lambda_1 \leq (\int \nabla f . \nabla g\, dV) / \|f\|^2$ with equality if and only if f is an eigenfunction of λ_1 . If $\phi_1, \phi_2, ...$ is a complete orthonormal basis such that ϕ_j is an eigenfunction of λ_j , then for $f \neq 0$ satisfying : $(f, \phi_1) = ... = (f, \phi_{k-1}) = 0$ we have : $\lambda_k \leq (\int \nabla f . \nabla g\, dV) / \|f\|^2$, with equality if and only if f is an eigenfunction of λ_k .

In order to use directly this theorem, one needs to construct functions on H^3 which satisfy (7). One possibility woulb be, given any function h on H^3, to write : $f(x) \sim \sum_{\gamma\in\Gamma} h(\gamma x)$, but this would be very time consuming.
In fact we shall use this theorem only qualitatively. The inequalities say that the lowest eigenvalues will be best constrained if we use functions which do not vary rapidly, or, in other words, that the main contribution in (20) should come from the low m and ν values of the basis functions (19). This justify a limited expansion in (20).
Note that, since the first eigenfunction is the constant function associated to the eigenvalue 0, if Φ , in equation (20), is periodic on H^3 , it is orthogonal to the first eigenfunction on M by (21), and the requirements of Rayleigh’s theorem are automatically satisfied for the second eigenfunction.

4 Description of the manifolds used as examples.

Thurston manifold.
Γ has 8 generators, and the fundamental domain domain has 16 faces. It is the second smallest compact hyperbolic manifold known today, with a volume of ~ 0.98. (the smallest known manifold being the Weeks manifold with 9 generators and a volume of ~ 0.94).
The spectrum of the Laplacian eigenvalues has been calculated in [4] using a different method. We shall compare our results with those of this reference.

Weber_Seifert manifold.
It has been briefly described above. The fundamental domain is a regular dodecahedron. Its high degree of symetry, and the fact that the base geodesics of the generators all go through the origine makes it an attractive toy for understanding and tentative analytic calculations.
The Weber-Seifert manifold is not the only one whose fundamental domain is a regular dodecahedron, there are 7 others (see [5]). In the tiling of H^3, each edge is common to 5 dodecahedrons and each summit is common to 20 volumes. The volume of the fondamental domain is ~ 11.2 . The distance h between the center of the fundamental domain and the center of the faces is : $ch^2(h) = (5+2\sqrt{5})/4$. There exists ideal dodecahedron (whose summits are at infinity but having finite volume), but this case will not be considered here.

Manifolds whose fundamental domains is a regular icosahedron.

They have 10 generators and the fundamental domain has 20 faces and 12 summits. There are 7 compact mainfolds. The tables of Appendix A show how the faces and their summits are identified. The volume is ~ 4.69 . The distance h between the center of the fundamental domain and the center of the faces is : $ch^2(h) = 3/2(3-\sqrt{5})$. There is no ideal icosahedron. In the tiling of H^3, each edge is common to 3 icosahedrons and each summit is common to 12 volumes.

5 Numerical calculation of approximate solutions.

For any value of β, functions defined by equation (19) are eigenfunctions of both the Laplacian and γ^0 (wich is such that its based geodesic g_0 defines the Oz axis), but the solutions must also satisfy all the constraints (7). In order to find approximate solutions, Φ is developped, as in (20), on the basis of cylindrical functions (19) using a limited number of terms, and the constraints (7) are enforced using a χ^2 method to determine a set of coefficients $a_{m,\nu}$. Then we look at the behaviour of the function $\chi^2(\beta)$ to find satisfying solutions. The fact that the functions (19) forms an orthogonal basis, on cylinders of axis Oz, allows a good mode decoupling.
The calculations are done as follows :

- points x_j are chosen randomly in a volume containing the fundamental domain, then their images $y_j^k = \gamma^k x_j$ are calculated for all the generators γ^k , $k \neq 0$ and their inverse.

- Since Φ is defined up to a multiplicative constant, we set either : $\Phi(O)=1$, or, for some arbitrary point A on Oz : $\Phi(O)=0$, $\Phi(A)=1$.
- We define : $\chi^2 = \sum_i (\zeta_i - a_{m,\nu}\,\psi_{m,\nu})^2 / \sigma_i^2$, where the index i corresponds to a pair (x_j, y_j^k) . If one wants to enforce the periodicity condition one sets : $\zeta_i = 0$ and $\psi_{m,\nu} = \phi_{m,\nu}(y_i) - \phi_{m,\nu}(x_i)$ where the $\phi_{m,\nu}$ are the basis functions (19) . For the normalisation constraints : $\psi_{m,\nu} = \phi_{m,\nu}(x_i)$ and $\zeta_i = 0$ or 1 . σ_i is an arbitrary error.

The calculations are local, which means that :

- both elements of the couple (x_j, y_j^k) must be inside the fondamental domain or close to it.
- The periodicity conditions are imposed for the generators γ^k (and their inverse) not for all the elements of Γ .

The reason for local calculations is due to the behaviour of the radial function $I(\beta, \mu_z, \nu, \rho)$. At small radii it is given by (15). The position of the fisrst maximum increases if ν increases. In other words, the larger the working volume the larger the number of modes one has to consider.
The covariance matrix is inverted using the SVD decomposition of the SGL library.

6 Results

For the following calculations, the parameter values used are typically $|\nu| \leq 20$, $m \leq 5$. In order to impose the periodicity conditions (7), points were chosen in a sphere of radius 2, and their image should lie in a cylinder of radius ranging from 1.4 to 2. The number of these points is chosen to be ~ 4000. With such representative parameters, the calculation time necessary to explore the range $0 \leq \beta \leq 9$ by steps of 0.025, is around 140 mn on a MacBookPro, without special efforts to optimise the program.

Thurston manifold.

For the Thurston manifold, the function $\chi^2(\beta)$ is shown in figure 2 for the case $\Phi(0)=1$. Our results are compared with those of [4] in the following table. The first column reproduces Table 1 of [4], and the second column gives (-) the eigenvalues of the Laplacian, which is $-k^2$ in the notations of [4]. The third column gives the value of β for which we have found a solution. If there is no correspondance, that means we have not found any solution for this value of k. The uncertainties on the values of β are of the order of 0.01, and depends only on the β step used to study the behaviour of $\chi^2(\beta)$. The last column gives the type of solution found according to section 5. $\Phi(0)=0$ does not necessary means that the solution is antisymetric with respect to the origin on the Oz axis.

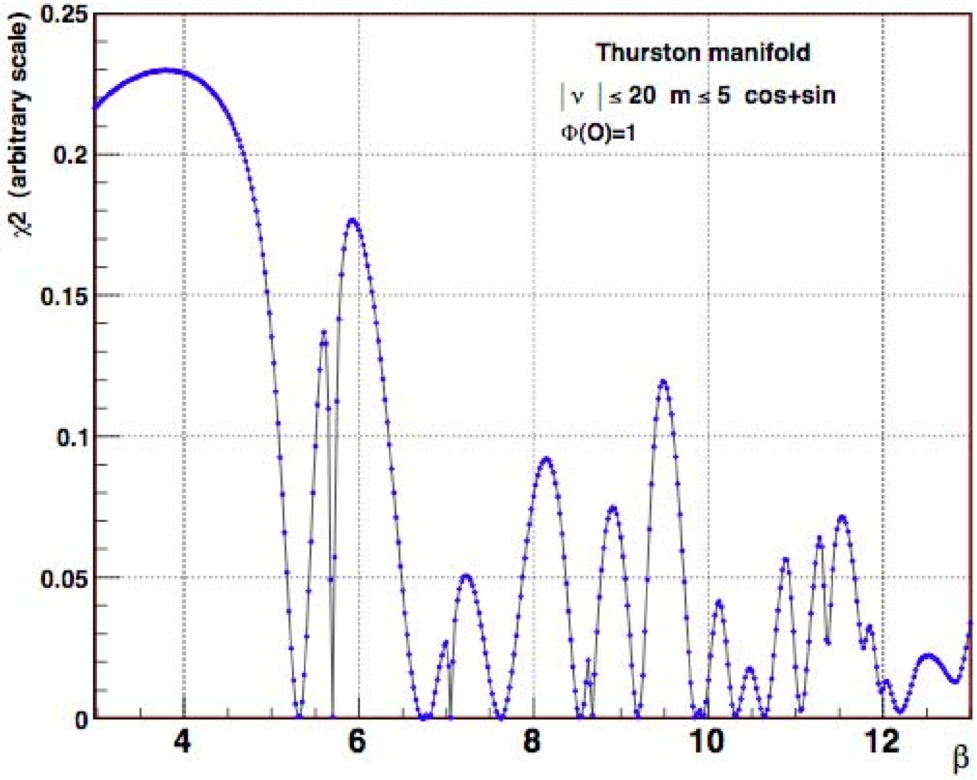

Figure 2.

k [4]	k^2	β	$1+\beta^2$	$\Phi(0)$
5.41	29.27	5.31	29.2	1.
5.79	33.5	5.70	33.5	1.
6.81	46.4	6.72	46.2	1.
6.89	47.5	6.80	47.2	1.
7.12	50.7	7.06	50.8	1.
7.69	59.1	7.62	59.1	0. , 1.
8.30	68.9			
8.60	74.	8.53	73.8	1.
8.73	76.2	8.67	76.2	0. , 1.
9.26	85.7	9.20	85.6	0. , 1.
9.76	95.3			
9.91	98.2	9.85	98.	1.
9.99	99.8	9.94	99.8	1.

In [4] the multiplicity associated to each eigenvalue is one except for $k=9.26$, where it is 2. The shape of the function for the lowest eigenvalue is rather jittery. For the lowest eigenvalue, $\Phi(z)$ on the Oz axis is approximately constant, while for the two next ones, it oscillates with a period equal to the length of the transvection of γ^0 which is $L \simeq 1.04$.

Weber-Seifert manifold.

The function $\chi^2(\beta)$ is shown in figure 3 for the case $\Phi(0)=1$. Except for the lowest β values, the structures of $\chi^2(\beta)$ are less visible than in the case of the Thurston manifold. This is due to the fact that the Weber-Seifert manifold has a large volume. In the expansion (20) the basis functions must be computed at larger radii, but $\phi_{m\,\nu}$ with larger ν becomes non negligible, and should be included in the calculations, which would increases the computation time.

β	$\Phi(O)$
2.925	1
3.79	0
4.275	1
5.58	1
5.65	0
5.775	0
6.57	0
6.925	0
7.225	1

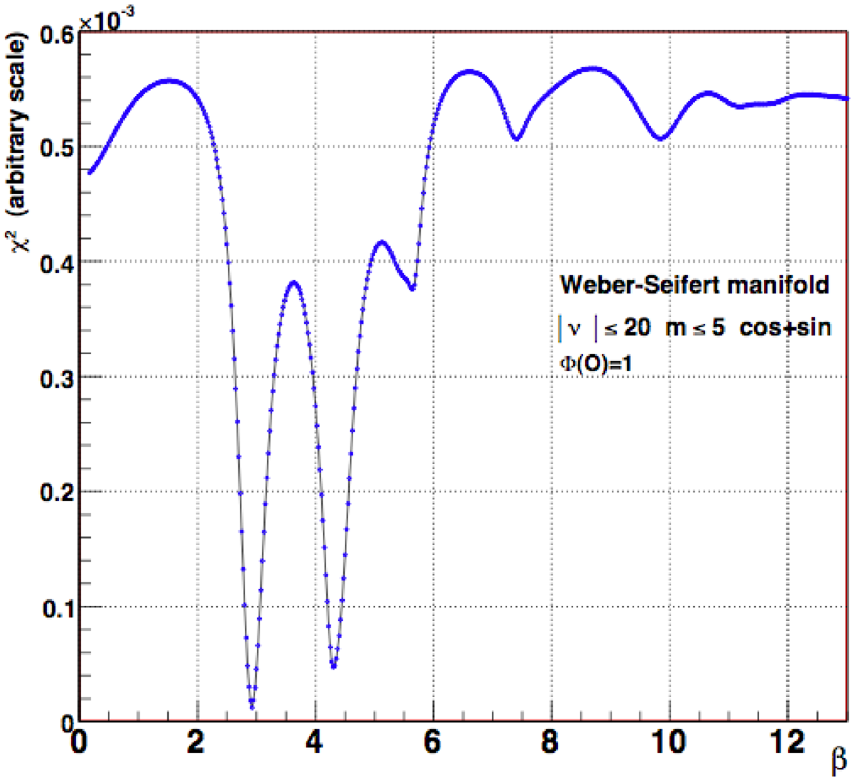

Figure 3.

The structures of the function $\chi^2(\beta)$ are not very clear, however, if we require the solutions to be symetric with respect to rotations around the Oz axis with angles multiple of $2\pi/5$ (that we can impose by requiring ν to be a multiple of 5 in (20)), they all appear very neatly. In this case we have used $|\nu| \le 30$, $m \le 10$. The five first eigenmodes are such that the periodicity on the Oz axis is L . On the Oz axis, the first eigenfunction is well described by $\Phi \sim 0.89 + 0.12 * \cos(2\pi z / L) - 0.012 * \cos(4\pi z / L)$. The shape of this function on the other base geodesics is of the form : $\Phi \sim a + b\cos(2\pi\chi / L + phase)$ which shows that it is a « low energy » solution.

Although not detailed here, one can use the expansion (9) in the neighbourhood of the origin to show that there is no solution having the full symetry of the Weber_Seifert manifold. In fact, there is no solution with the same $\Phi(\chi)$ on all the base geodesics associated to the generators.

The fundamental domain is an icosahedron.

The results are presented in the following table, and an example is given in figure 4.
The first line is the number of the space described in appendix A. The second line gives the length of the longest transvection. Each column gives the β values for which there is a solution. A question mark means that the situation is not clear, at least for the limited expansion used. The letter a means a solution of type $\Phi(0) = 1$, and the letter b $\Phi(0) = 0$. Here again a question mark means that the type is uncertain.

	1	2	3	4	5	6	7
L	1.629	1.465	1.629	1.736	1.008	1.736	1.736
β	2.525 a	2.675 a	2.925 a	3.40 b	2.47 a	3.40 b	3.40 a
	3.275 a	2.925 a	3.27 a	4.125 a	4.33 a	4.125 a	3.525 a?
	4.11 a	4.125 a	4.05 b	5.45 a	5.50 a ?	5.45 a	4.125 a
	4.39 b	4.48 a	4.125 a	5.75 b	6.75 a	5.75 b	4.29 b
	4.60 a	5.25 a	4.25 b	5.90 ? b		5.925 ? b	5.45 a
	5.075 a	5.50 b?	4.84 b	6.60 ? b		6.50 ? b	5.75 a?
	5.43 a	5.80 a?	5.10 ? a	7.02 b		6.60 b	
	5.60 b		5.175 b	7.82 b		7.025 b	
			5.45 a	8.11 a		7.79 b	
			6.02 a			8.10 a	

In all the cases, for the lowest eienvalues, the field Φ, on the base geodesic used as coordinate axis, is well approximated by functions of the form : $a + b\cos(2\pi z/L + phase)$, where a and b are constants, and L is the length of the transvection on this geodesic, which means that these functions have « low energy ».

There does not seem to be any correlation between the lowest values of β and the length of the longest transvection, the whole structure of Γ is involved although all these spaces have the same fundamental dommain. The structures of the functions $\chi^2(\beta)$ are not always as clear as in the case of the Thurston manifold. This is the case for spaces number 2, 3, 5 and 7. As already said, this is mainly due to the limitation in the expansion (20), which could be overcome by increasing the computation time.
We have repeated some of the calculations by changing the generator whose base geodesic defines the Oz axis, even if it has a shorter transvection, with identical results.

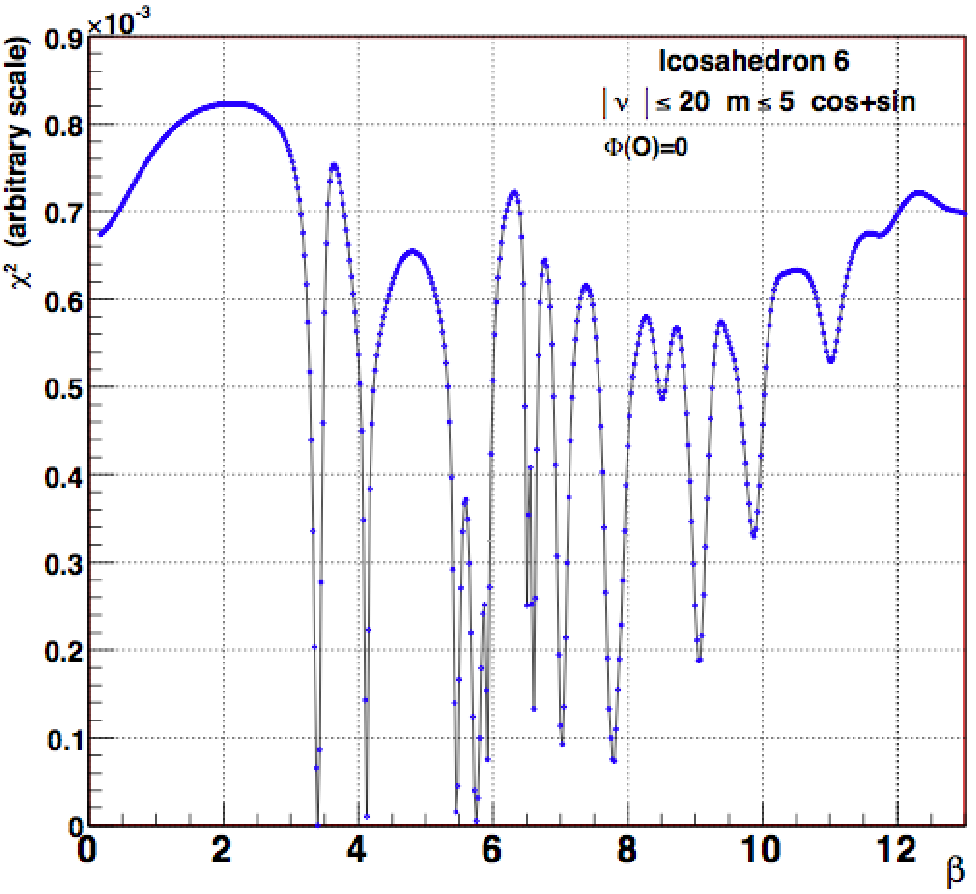

Figure 4.

Conclusion.

We have shown that a basis of functions which are eigenfunctions of the Laplacian and of one generator of the group of deck transformations defining a hyperbolic 3-d compact space, can be used to compute numerically solutions of the Laplacian eigenvalue problem in that space. The method is well suited to compute the lowest eigenmodes.

References.

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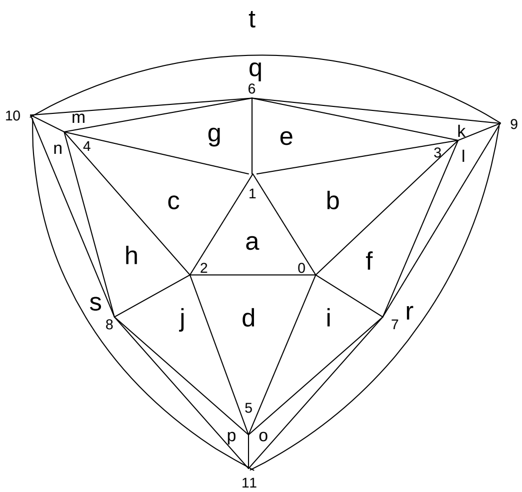
Appendix A.

The following figure represents an icosahedron as if it were a stereographic projection but without respecting the lengths. Faces are named with letters and summits with numbers.



Each of the following tables shows how the faces of the icosahedron are associated to build a manifold. Each line gives the corresponding faces, and how the summits are associated. The first 3 summits belong to the face in the first column, and the next 3 to the face in the second column. For instance in the next table, the correspondance between the faces a and d is done according to : $0 \to 5, \quad 1 \to 2, \quad 2 \to 0$. The next 6 tables correspond to the solutions 14, 12, 13, 10, 9, 11 of [5].

faces		summits				summits		
a	d	0	1	2		5	2	0
b	h	0	1	3		4	8	2
c	i	4	1	2		7	0	5
e	s	6	1	3		11	10	8
f	o	0	7	3		7	11	5
g	r	4	1	6		11	7	9
j	n	8	5	2		10	4	8
k	m	6	9	3		10	4	6
l	t	9	7	3		11	10	9
p	q	8	5	11		6	9	10

faces		summits				summits		
a	d	0	1	2		2	0	5
b	j	0	1	3		5	2	8
c	k	4	1	2		9	6	3
e	i	6	1	3		0	7	5
f	t	0	7	3		10	11	9
g	s	4	1	6		8	11	10
h	o	4	8	2		11	5	7
l	r	9	7	3		11	9	7
m	n	4	10	6		10	8	4
p	q	8	5	11		9	10	6

faces		summits				summits		
a	d	0	1	2		2	0	5
b	j	0	1	3		5	2	8
c	l	4	1	2		3	7	9
e	n	6	1	3		4	8	10
f	q	0	7	3		6	10	9
g	t	4	1	6		9	10	11
h	s	4	8	2		11	10	8
i	r	0	5	7		7	9	11
k	p	6	9	3		5	8	11
m	o	4	10	6		5	11	7

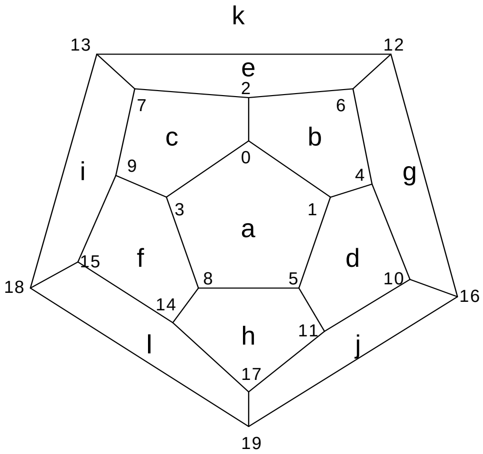
faces		summits				summits		
a	d	0	1	2		2	0	5
b	j	0	1	3		5	2	8
c	q	4	1	2		10	6	9
e	r	6	1	3		11	7	9
f	t	0	7	3		11	9	10
g	m	4	1	6		6	4	10
h	l	4	8	2		3	9	7
i	k	0	5	7		6	9	3
n	o	4	8	10		7	5	11
p	s	8	5	11		10	11	8

faces		summits				summits		
a	j	0	1	2		2	5	8
b	i	0	1	3		5	7	0
c	n	4	1	2		10	8	4
d	o	0	5	2		11	7	5
e	l	6	1	3		3	9	7
f	r	0	7	3		7	9	11
g	q	4	1	6		6	10	9
h	p	4	8	2		8	5	11
k	t	6	9	3		11	10	9
m	s	4	10	6		11	8	10

faces		summits				summits		
a	j	0	1	2		2	5	8
b	l	0	1	3		9	3	7
c	o	4	1	2		5	11	7
d	k	0	5	2		6	3	9
e	n	6	1	3		4	8	10
f	s	0	7	3		11	10	8
g	t	4	1	6		11	9	10
h	q	4	8	2		10	9	6
i	m	0	5	7		4	10	6
p	r	8	5	11		7	11	9

faces		summits				summits		
a	j	0	1	2		2	5	8
b	q	0	1	3		9	10	6
c	k	4	1	2		9	6	3
d	t	0	5	2		11	10	9
e	p	6	1	3		5	11	8
f	g	0	7	3		4	6	1
h	r	4	8	2		7	9	11
i	n	0	5	7		10	4	8
l	o	9	7	3		7	5	11
m	s	4	10	6		10	11	8

The Weber Seifert is described by the next table. The associated faces are diametrically opposed.



faces		summits						summits				
a	k	0	1	5	8	3		16	19	18	13	12
b	l	0	1	4	6	2		18	15	14	17	19
c	j	0	3	9	7	2		17	19	16	10	11
d	i	1	4	10	11	5		13	18	15	9	7
e	h	6	12	13	7	2		8	5	11	17	14
f	g	3	8	14	15	9		10	16	12	6	4