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ENVELOPE MODES OF A MISMATCHED BUNCHED BEAM

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Abstract

Frequencies of the 3 mismatched envelope modes of a bunched beam are calculated as a function of :

- *the beam aspect ratio* $a = \frac{\gamma Z_0}{R_0}$, *$R_0$ and Z_0 being the matched transverse and longitudinal envelope sizes of the beam and γ being the Lorentz factor of the beam centroid in the lab ref.,*
- *the transverse space-charge tune depression* $\eta_t = \frac{k_t}{k_{t0}}$, *$k_t$ and k_{t0} being the transverse phase advance per unit length of particle in the homogenous beam respectively with and without space-charge forces,*
- *the external force aspect ratio* $\alpha = \frac{k_{l0}}{k_{t0}}$, *$k_{l0}$ and k_{t0} being the phase advance per unit length of particles respectively in the transverse and the longitudinal direction.*

The calculated wave numbers of the 3 modes are then compared to those obtained from the Fourier analysis of the envelope of mismatched bunched beam computed numerically. A good Agreement is found.

As with a continuous beam, some particles always feel ½ order resonances with this envelope modes, and can moreover be excited by a 1^{st} order resonance.

Contents

1. Introduction

The mismatch is now known as being an important source of halo [1] [2]. In a continuous beam, the mismatch modes excite the ½ order resonance whatever the beam depressed tune. The betatron amplitude of some beam particles, under the effect of one of these resonances, is modulated and can reach a high value. This effect, now well known with continuous beams, has to be explored with bunched beams. The first step consists in the calculation of the envelope modes frequencies of mismatched bunched beam in a continuous focusing channel. This work has been already done by K. Bongardt and M. Pabst but using rough approximations in order to get analytical "easy to write" solutions [3]. We propose an exact calculation of these modes in the case of a small mismatch in a continuous focusing channel. The solutions suit very well with numerical results.

2. Envelope equation of bunched beam

Envelope sizes of a bunched beam are : X, Y and Z (Z being the longitudinal size of the beam). In an accelerator, the longitudinal focusing force is sinusoidal, then non linear. However, this force can be linearised the same way as is the space-charge force in [4]. The linear part of the space-charge force is the one of an uniformly filled ellipsoid whose sizes X, Y and Z are 5 times the RMS sizes of the bunch.

In a continuous focusing channel, envelope equations are :

$$\begin{cases} X'' + k_{x0}^2 \cdot X - I_x(X,Y,Z)\cdot X - \dfrac{\varepsilon_x^2}{X^3} = 0, \\ Y'' + k_{y0}^2 \cdot Y - I_y(X,Y,Z)\cdot Y - \dfrac{\varepsilon_y^2}{Y^3} = 0, \\ Z'' + k_{z0}^2 \cdot Z - I_z(X,Y,Z)\cdot Z - \dfrac{\varepsilon_z^2}{Z^3} = 0, \end{cases} \qquad \textbf{(Eq. 1)}$$

with :

$$I_i = \frac{\gamma\cdot\lambda}{2}\int_0^\infty \frac{dt}{\left(A_i^2+t\right)\left[\left(X^2+t\right)\left(Y^2+t\right)\left(\gamma^2Z^2+t\right)\right]^{1/2}}, \qquad \textbf{(Eq. 2)}$$

where $A_i = X$, Y or γZ for $i = x$, y or z.

and $\lambda = \dfrac{q}{mc^2}\dfrac{3}{4\pi\varepsilon_0}\dfrac{I}{f_p\beta^2\gamma^2}$.

q is the charge, mc^2 the rest energy of the particles, f_p is the bunch frequency rate, I is the mean beam courant, β and γ are the normalised speed and energy of the beam centroid[1].

[1] Which limits the model validity to beam with small velocity or energy spreads.

3. Mismatched beam

3.1 Mismatched-beam envelopes

X_0, Y_0 and Z_0 being the envelope sizes of the matched beam. The envelope of a lightly mismatched beam can be written :

$$\begin{cases} X(z) = X_0 + \Delta X(z), \\ Y(z) = Y_0 + \Delta Y(z), \\ Z(z) = Z_0 + \Delta Z(z). \end{cases} \qquad \textbf{(Eq. 3)}$$

Let's study the equation :

$$X'' + k_{x0} \cdot X - I_x(X,Y,Z) \cdot X - \frac{\varepsilon_x}{X^3} = 0 . \qquad \textbf{(Eq. 4)}$$

With a Taylor expansion of I_x at first order :

$$\begin{aligned} I_x(X,Y,Z) = {} & I_x(X_0, Y_0, Z_0) \\ & + \Delta X \cdot \left.\frac{\partial I_x}{\partial X}\right]_{X_0,Y_0,Z_0} + \Delta Y \cdot \left.\frac{\partial I_x}{\partial Y}\right]_{X_0,Y_0,Z_0} + \Delta Z \cdot \left.\frac{\partial I_x}{\partial Z}\right]_{X_0,Y_0,Z_0} \\ & + O(2^{nd} \text{ order}) \end{aligned} \qquad \textbf{(Eq. 5)}$$

Equation **(Eq. 4)** becomes :

$$\Delta X'' + \left(k_{x0}^2 - I_{x0} - \frac{\partial I_x}{\partial X} X_0 + \frac{3 \cdot \varepsilon_x^2}{X_0^4} \right) \cdot \Delta X - \frac{\partial I_x}{\partial Y} X_0 \cdot \Delta Y - \frac{\partial I_x}{\partial Z} X_0 \cdot \Delta Z = 0 , \qquad \textbf{(Eq. 6)}$$

where: $I_{u0} = I_u(X_0, Y_0, Z_0)$ and $\frac{\partial I_u}{\partial V} = \frac{\partial I_u}{\partial V}(X_0, Y_0, Z_0)$, u and v being x, y or z.

Partial derivative of I_x with X, Y and Z are :

$$\begin{cases} \dfrac{\partial I_x}{\partial X} = -3X_0 \dfrac{\lambda \cdot \gamma}{2} \displaystyle\int_0^\infty (X_0^2 + t)^{-5/2} (Y_0^2 + t)^{-1/2} \left((\gamma Z_0)^2 + t\right)^{-1/2} dt = -3X_0 \cdot F_{xx} , \\ \dfrac{\partial I_x}{\partial Y} = -Y_0 \dfrac{\lambda \cdot \gamma}{2} \displaystyle\int_0^\infty (X_0^2 + t)^{-3/2} (Y_0^2 + t)^{-3/2} \left((\gamma Z_0)^2 + t\right)^{-1/2} dt = -Y_0 \cdot F_{yx} , \\ \dfrac{\partial I_x}{\partial Z} = -\gamma^2 Z_0 \dfrac{\lambda \cdot \gamma}{2} \displaystyle\int_0^\infty (X_0^2 + t)^{-3/2} (Y_0^2 + t)^{-1/2} \left((\gamma Z_0)^2 + t\right)^{-3/2} dt = -\gamma^2 Z_0 \cdot F_{zx} . \end{cases} \qquad \textbf{(Eq. 7)}$$

The same calculus can be done with equation containing ΔY" et ΔZ".

Assuming[2] :

$$X_0 = Y_0 = R_0,$$
$$\varepsilon_x = \varepsilon_y = \varepsilon_t, \qquad \varepsilon_z = \varepsilon_l, \qquad \textbf{(Eq. 8)}$$
$$k_{x0} = k_{y0} = k_{t0}, \qquad k_{z0} = k_{l0},$$

We have :

$$\mathbf{I_t} = I_{x0} = I_{y0} = \frac{\lambda\cdot\gamma}{2}\int_0^\infty \left(R_0^2+t\right)^{-2}\left(\left(\gamma Z_0\right)^2+t\right)^{-1/2}dt\,, \qquad \textbf{(Eq. 9)}$$

$$\mathbf{I_l} = I_{z0} = \frac{\lambda\cdot\gamma}{2}\int_0^\infty \left(R_0^2+t\right)^{-1}\left(\left(\gamma Z_0\right)^2+t\right)^{-3/2}dt\,, \qquad \textbf{(Eq. 10)}$$

$$\mathbf{F_t} = F_{xx} = F_{yy} = F_{yx} = F_{xy} = \frac{\lambda\cdot\gamma}{2}\int_0^\infty \left(R_0^2+t\right)^{-3}\left(\left(\gamma Z_0\right)^2+t\right)^{-1/2}dt\,, \qquad \textbf{(Eq. 11)}$$

$$\mathbf{F_c} = F_{xz} = F_{yz} = F_{zx} = F_{zy} = \frac{\lambda\cdot\gamma}{2}\int_0^\infty \left(R_0^2+t\right)^{-2}\left(\left(\gamma Z_0\right)^2+t\right)^{-3/2}dt\,, \qquad \textbf{(Eq. 12)}$$

$$\mathbf{F_l} = F_{zz} = \frac{\lambda\cdot\gamma}{2}\int_0^\infty \left(R_0^2+t\right)^{-1}\left(\left(\gamma Z_0\right)^2+t\right)^{-5/2}dt\,. \qquad \textbf{(Eq. 13)}$$

System **(Eq. 1)** becomes:

$$\begin{pmatrix}\Delta X''\\ \Delta Y''\\ \Delta Z''\end{pmatrix} = -\begin{pmatrix}A & B & C\\ B & A & C\\ D & D & E\end{pmatrix}\cdot\begin{pmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{pmatrix}, \qquad \textbf{(Eq. 14)}$$

with :

$$\begin{cases} A = k_{t0}^2 - I_t + 3R_0^2 F_t + \dfrac{3\varepsilon_t^2}{R_0^4},\\ B = F_t R_0^2,\\ C = \gamma^2 R_0 Z_0 F_c = \gamma^2 D,\\ D = R_0 Z_0 F_c = \gamma^{-2} C,\\ E = k_{l0}^2 - I_l + 3\gamma^2 Z_0^2 F_l + \dfrac{3\varepsilon_l^2}{Z_0^4}. \end{cases} \qquad \textbf{(Eq. 15)}$$

The preview system **(Eq. 14)** depends on **8** parameters :

$\Rightarrow$ k_{t0}^2, k_{l0}^2, R_0, γZ_0, γ, λ, ε_t and ε_l.

In a first time, let's add 2 other parameters :

$$k_t^2 = k_{t0}^2 - I_t \quad \text{and} \quad k_l^2 = k_{l0}^2 - I_l\,. \qquad \textbf{(Eq. 16)}$$

[2] Which is generally the case in linac.

They are the transverse and longitudinal phase advances per meter, with phase-space, of particle in a matched uniform beam. Indeed, as shown in **(Eq. 1)**, the movement of a particle in this beam is given by :

$$\begin{cases} x'' + \left(k_{t0}^2 - I_t\right)\cdot x = 0, \\ y'' + \left(k_{t0}^2 - I_t\right)\cdot y = 0, \\ z'' + \left(k_{l0}^2 - I_l\right)\cdot z = 0. \end{cases} \qquad \textbf{(Eq. 17)}$$

When the beam is matched, $R_0'' = 0$ and $Z_0'' = 0$. This allows to suppress the dependence with emittance.

$$\frac{\varepsilon_t^2}{R_0^4} = k_t^2 \qquad \text{and} \qquad \frac{\varepsilon_l^2}{Z_0^4} = k_l^2 . \qquad \textbf{(Eq. 18)}$$

Given **a**, the aspect ratio of the beam :

$$\mathbf{a} = \frac{\gamma Z_0}{R_0}. \qquad \textbf{(Eq. 19)}$$

Equations **(Eq. 9)** to **(Eq. 13)** become:

$$I_t = \frac{\lambda\cdot\gamma}{2\cdot R_0^2\cdot\gamma Z_0}\int_0^\infty (1+\alpha)^{-2}\left(1+a^{-2}\cdot\alpha\right)^{-1/2} d\alpha , \qquad \textbf{(Eq. 20)}$$

$$I_l = \frac{\lambda\cdot\gamma}{2\cdot R_0^2\cdot\gamma Z_0}\int_0^\infty (1+a\cdot\alpha)^{-1}(1+\alpha)^{-3/2} d\alpha , \qquad \textbf{(Eq. 21)}$$

$$F_t = \frac{\lambda\cdot\gamma}{2\cdot R_0^4\cdot\gamma Z_0}\int_0^\infty (1+\alpha)^{-3}\left(1+a^{-2}\alpha\right)^{-1/2} d\alpha , \qquad \textbf{(Eq. 22)}$$

$$F_c = \frac{\lambda\cdot\gamma}{2\cdot R_0^2\cdot\left(\gamma Z_0\right)^3}\int_0^\infty (1+\alpha)^{-2}\left(1+a^{-2}\alpha\right)^{-3/2} d\alpha , \qquad \textbf{(Eq. 23)}$$

$$F_l = \frac{\lambda\cdot\gamma}{2\cdot\left(\gamma Z_0\right)^5}\int_0^\infty (1+\alpha)^{-1}\left(1+a^{-2}\alpha\right)^{-5/2} d\alpha . \qquad \textbf{(Eq. 24)}$$

One has :

$$\int_0^\infty (1+t)^{-2}\left(1+a^{-2}\cdot t\right)^{-1/2} dt = \begin{cases} 1+\dfrac{1}{a^2-1}-\dfrac{a}{(a^2-1)^{3/2}}\cdot L(a) & \text{if } a>1, \\ 1+\dfrac{1}{a^2-1}+\dfrac{a}{(1-a^2)^{3/2}}\cdot AT(a) & \text{if } a<1, \end{cases} \qquad \textbf{(Eq. 25)}$$

and

$$\int_0^{\infty}\left(1+a^2\cdot t\right)^{-1}(1+t)^{-3/2}\,dt=\begin{cases}\dfrac{-2}{a^2-1}+\dfrac{2\cdot a}{(a^2-1)^{3/2}}\cdot L(a) & \text{if } a>1,\\ \dfrac{-2}{a^2-1}-\dfrac{2\cdot a}{(1-a^2)^{3/2}}\cdot AT(a) & \text{if } a<1,\end{cases} \qquad \textbf{(Eq. 26)}$$

with :

$$L(a)=\frac{1}{2}\ln\left(2\cdot a^2+2\cdot a\cdot\sqrt{a^2-1}-1\right), \qquad \textbf{(Eq. 27)}$$

$$AT(a)=\frac{\pi}{2}-\arctan\left(\sqrt{\frac{a^2}{1-a^2}}\right). \qquad \textbf{(Eq. 28)}$$

$$LAT(a)=\begin{cases}L(a) & \text{if } a>1,\\ AT(a) & \text{if } a<1.\end{cases} \qquad \textbf{(Eq. 29)}$$

with $2I_t+I_l$, one gets :

$$\frac{\lambda\cdot\gamma}{\gamma Z_0 R_0^2}=2\cdot\Delta k_t^2+\Delta k_l^2\,. \qquad \textbf{(Eq. 30)}$$

With $2\cdot I_t+a^2 I_l$, one gets:

$$\frac{\lambda\cdot\gamma}{\gamma Z_0 R_0^2}\cdot\frac{a}{\left|1-a^2\right|^{1/2}}LAT(a)=2\cdot\Delta k_t^2+a^2\Delta k_l^2\,. \qquad \textbf{(Eq. 31)}$$

with $\Delta k_t^2=k_{t0}^2-k_t^2$ and $\Delta k_l^2=k_{l0}^2-k_l^2$.

Including equations **(Eq. 30)** and **(Eq. 31)** in F_t, F_c et F_l (**(Eq. 22)** to **(Eq. 24)**), and using :

$$\int_0^{\infty}(1+t)^{-3}\left(1+a^{-2}\cdot t\right)^{-1/2}dt=\frac{1}{2}-\frac{1}{4a^2}-\frac{3}{4(a^2-1)^2}-\frac{1}{4a^2(a^2-1)}+\frac{3\cdot a}{4\left|a^2-1\right|^{5/2}}\cdot LAT(a), \qquad \textbf{(Eq. 32)}$$

$$\int_0^{\infty}(1+t)^{-2}\left(1+a^{-2}\cdot t\right)^{-3/2}dt=1+\frac{3\cdot a^2}{(a^2-1)^2}+\frac{1}{a^2-1}-\frac{3\cdot a^3}{\left|a^2-1\right|^{5/2}}\cdot LAT(a), \qquad \textbf{(Eq. 33)}$$

$$\int_0^{\infty}(1+t)^{-1}\left(1+a^{-2}\cdot t\right)^{-5/2}dt=\frac{-2\cdot a^4}{(a^2-1)^2}-\frac{2\cdot a^2}{3\cdot\left(a^2-1\right)}+\frac{2\cdot a^5}{\left|a^2-1\right|^{5/2}}\cdot LAT(a), \qquad \textbf{(Eq. 34)}$$

one finally gets :

$$R_0^2 F_t = \frac{\Delta k_t^2 (3 - 2a^2) - \Delta k_l^2 a^2}{4 \cdot (1 - a^2)}, \quad \textbf{(Eq. 35)}$$

$$(\gamma Z_0)^2 F_l = \frac{2 \cdot \Delta k_t^2 + \Delta k_l^2 (1 - 3a^2)}{3 \cdot (1 - a^2)}, \quad \textbf{(Eq. 36)}$$

$$\gamma Z_0 R_0 F_c = \frac{a \cdot (\Delta k_l^2 - \Delta k_t^2)}{(1 - a^2)}. \quad \textbf{(Eq. 37)}$$

Then :

$$\begin{cases} A = 4 \cdot k_t^2 + 3 \cdot B, \\ B = \dfrac{\Delta k_t^2 (3 - 2a^2) - \Delta k_l^2 \cdot a^2}{4 \cdot (1 - a^2)}, \\ C = \gamma \dfrac{a \cdot (\Delta k_l^2 - \Delta k_t^2)}{(1 - a^2)}, \\ D = \gamma^{-2} C, \\ E = 4 \cdot k_l^2 + \dfrac{2 \cdot \Delta k_t^2 + \Delta k_l^2 (1 - 3a^2)}{(1 - a^2)}. \end{cases} \quad \textbf{(Eq. 38)}$$

System defined by **(Eq. 38)** depends on only **6** parameters :

$\Rightarrow \quad k_t^2$, k_t^2, k_{l0}^2, k_l^2, a and γ.

As $I_t = k_{t0}^2 - k_t^2$ (from **(Eq. 16)**), one has :

$$k_{t0}^2 - k_t^2 = \frac{\lambda \cdot \gamma}{2 \cdot R_0^2 \cdot \gamma Z_0} \int_0^\infty (1 + \alpha)^{-2} (1 + a^{-2} \cdot \alpha)^{-1/2} d\alpha . \quad \textbf{(Eq. 39)}$$

This equation gives a relation between R_0 and λ.

As $I = k_0^2 - k^2$ (from **(Eq. 16)**), one has :

$$k_{l0}^2 - k_l^2 = \frac{\lambda \cdot \gamma}{2 \cdot R_0^2 \cdot \gamma Z_0} \int_0^\infty (1 + a^2 \cdot \alpha)^{-1} (1 + \alpha)^{-3/2} d\alpha . \quad \textbf{(Eq. 40)}$$

Using **(Eq. 39)** and **(Eq. 40)** :

$$\frac{k_{t0}^2 - k_t^2}{k_{l0}^2 - k_l^2} = \frac{\int_0^\infty (1 + \alpha)^{-2} (1 + a^{-2} \cdot \alpha)^{-1/2} d\alpha}{\int_0^\infty (1 + a^2 \cdot \alpha)^{-1} (1 + \alpha)^{-3/2} d\alpha} = \frac{1}{P(a)}, \quad \textbf{(Eq. 41)}$$

with :

$$P(a) = \frac{\int_0^{\infty} \left(1 + a^2 \cdot \alpha\right)^{-1} (1+\alpha)^{-3/2} \, d\alpha}{\int_0^{\infty} (1+\alpha)^{-2} \left(1 + a^{-2} \cdot \alpha\right)^{-1/2} \, d\alpha} . \qquad \textbf{(Eq. 42)}$$

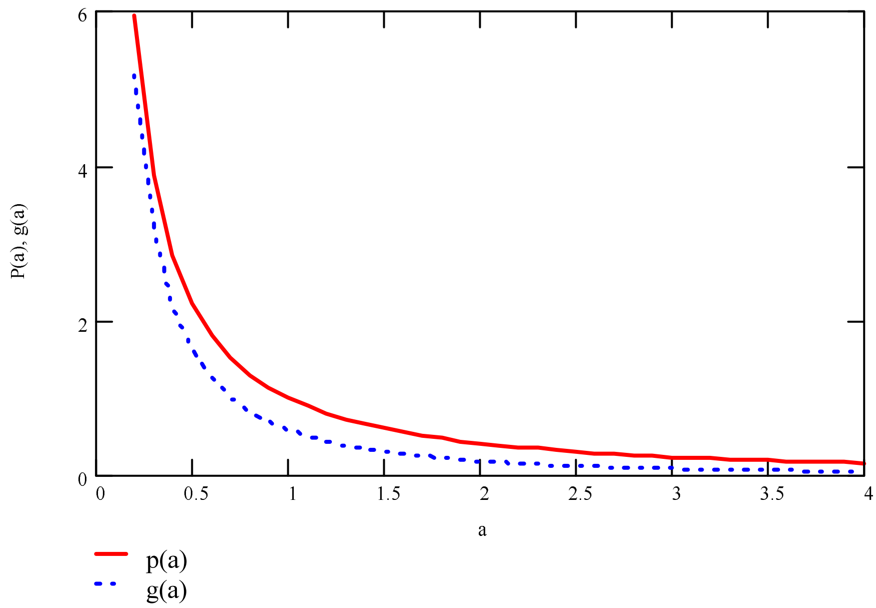

Figure 1 : P(a) and g(a) functions .

This allows to suppress parameter k_l^2 :

$$k_l^2 = k_{l0}^2 - \left(k_{t0}^2 - k_t^2\right) \cdot P(a) . \qquad \textbf{(Eq. 43)}$$

On **Figure 1**, has been represented the P(a) function evolution with **a** between 1/5 and 5. It can be noticed that P(1) = 1.

The number of free parameters is now **5** :

$$\boxed{\Rightarrow \quad k_{t0}^2,\ k_{l0}^2,\ k_t^2,\ a \text{ and } \gamma.}$$

Let's define g(a) as :

$$g(a) = \frac{P(a) - 1}{1 - a^2} \quad \text{and} \quad g(1) = \lim_{a \to 1} \frac{P(a) - 1}{1 - a^2} = 1 . \qquad \textbf{(Eq. 44)}$$

with $\boxed{\eta_t = \frac{k_t}{k_{t0}}}$, the transverse tune depression factor and $\boxed{\alpha = \frac{k_{l0}}{k_{t0}}}$, the ratio between longitudinal and transverse confinement forces, one finally gets :

$$\left\{\begin{array}{l} A = k_{t0}^2\left(4\cdot\eta_t^2 + \frac{3}{4}\cdot(1-\eta_t^2)\left(3-a^2 g(a)\right)\right), \\ B = k_{t0}^2\left(\frac{1}{4}(1-\eta_t^2)\left(3-a^2 g(a)\right)\right), \\ C = k_{t0}^2\left(\gamma\cdot(1-\eta_t^2)\cdot a\cdot g(a)\right), \\ D = k_{t0}^2\left(\gamma^{-1}(1-\eta_t^2)\cdot a\cdot g(a)\right) = \gamma^{-2}C, \\ E = k_{t0}^2\left(4\cdot\alpha^2 - (1-\eta_t^2)\left(1+\left(3-a^2\right)\cdot g(a)\right)\right). \end{array}\right. \quad \textbf{(Eq. 45)}$$

3.2 Mismatch-mode frequencies calculation

In order to calculate the frequencies of the 3 mismatched envelope modes, let's determine the eigenvalues of the matrix defined in **(Eq. 14)** which are the roots of :

$$\text{Det} = \begin{vmatrix} A-k^2 & B & C \\ B & A-k^2 & C \\ D & D & E-k^2 \end{vmatrix}. \quad \textbf{(Eq. 46)}$$

One finds :

$$\text{Det} = \left(A-B-k^2\right)\cdot\left(\left(k^2\right)^2 - (E+A+B)\cdot k^2 + E(A+B) - 2DC\right). \quad \textbf{(Eq. 47)}$$

One remarks that Det does not depend on γ, as depending on the product between D and C which is independant on γ. This reduces to **4** the number of parameters.

$\Rightarrow$ k^2 , α, η_t et a.

Modes frequencies are solutions of Det=0. There are 3 solutions :

① Transverse mode (quadripolar) :

$$k_Q^2 = A - B. \quad \textbf{(Eq. 48)}$$

With $\eta_Q^2 = \frac{k_Q^2}{k_{t0}^2}$ and A and B given by **(Eq. 45)**, equation **(Eq. 48)** becomes :

$$\eta_Q^2 = 4\cdot\eta_t^2 + \frac{1}{2}\left(1-\eta_t^2\right)\left(3-a^2 g(a)\right). \quad \textbf{(Eq. 49)}$$

This wave number depends on only **2** parameters :

$\Rightarrow$ η_t et a.

② The two other solutions :

$$\eta_{H/L}^2 = \frac{1}{2k_{t0}^2}\left(A + B + E \pm \sqrt{(A + B - E)^2 + 8DC}\right), \quad \textbf{(Eq. 50)}$$

are difficult to calculate analytically.

They depends on **3** parameters :

$\Rightarrow$ α, η_t et a.

3.2.1 Case of continuous beam

The bunched beam becomes continuous for a $\rightarrow \infty$, which gives an asymptotic limit for g(a) $\rightarrow 1/a^2$.

Then, the modes waves number becomes :

$$\eta_Q^2 \rightarrow 1 + 3\eta_t^2$$
$$\eta_H^2 \rightarrow 2 + 2\eta_t^2$$
$$\eta_L^2 \rightarrow 4\alpha^2$$

If $\alpha = 0$ (no longitudinal focusing force), η_Q and η_H are the two well-known modes of continuous beams [].

3.3 Numerical validation

Comparison have been done between frequencies given by a Fourier analysis of the step by step numerical solution of the envelope equations **(Eq. 1)** and the frequencies given by equations **(Eq. 49)** and **(Eq. 50)**. They have been done in the APT linac conditions at 6.7 MeV (transition energy between RFQ and DTL) for η_t **= 0.45**, α **= 0.604** and **a = 2.15**. The initial mismatches were M_x = 1%, M_y = -5‰ and M_z = 8‰

The step by step numerical solution of the envelope equations are presented on **Figure 2**. With a FFT (Fast Fourier Transform), the envelopes spectrum have been calculated and presented on **Figure 3**.

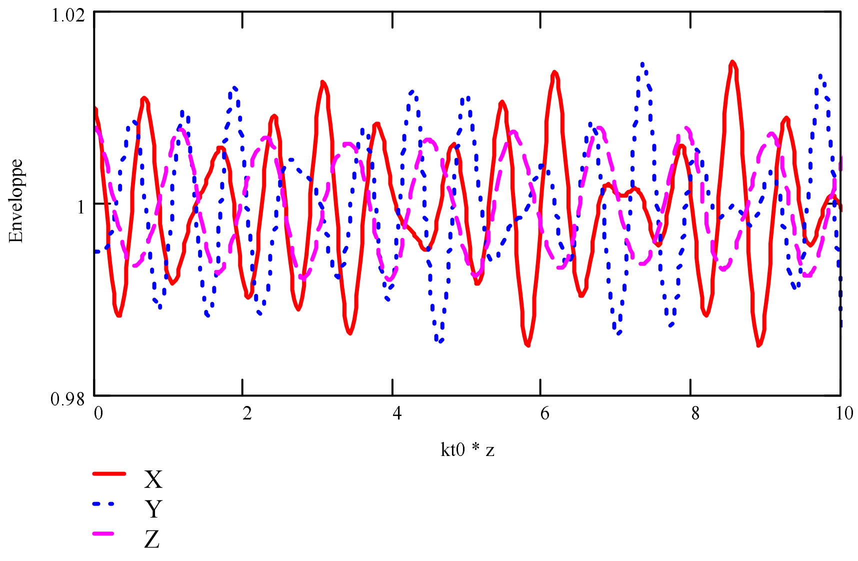

Figure 2 : Mismatched beam envelope

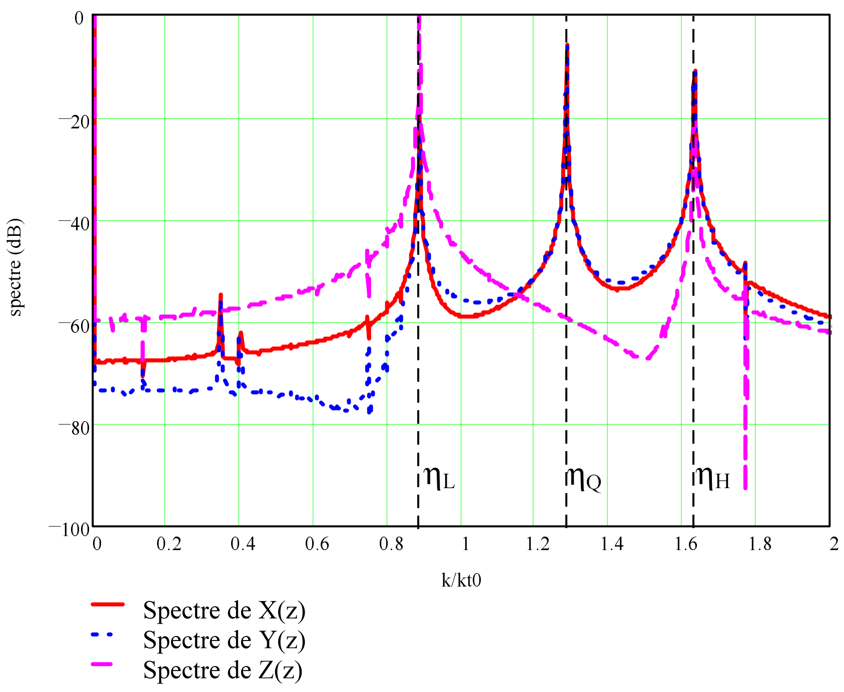

Figure 3 : Mismatched beam envelope spectrum.

From the spectrum analysis, one gets (with an initial mismatch of 1%) :

$\eta_Q = 1.298$; $\eta_L = 0.886$; $\eta_H = 1.634$.

One notes that the quadripolar mode is not present in the longitudinal envelope oscillation.

Equations **(Eq. 49)** and **(Eq. 50)** gives :

$\eta_Q = 1.301$; $\eta_L = 0.887$; $\eta_H = 1.634$.

There is a very good agreement between numerical and analytical solutions.

With an initial mismatch of 10%, Fourier analysis gives :

$\eta_Q = 1.328$; $\eta_L = 0.890$; $\eta_H = 1.637$.

It shows a small frequency shift toward highest frequencies when the mismatch increases.

3.5 Calculation of the excited resonances

Calculations in the APT linac conditions (at 6.7 MeV) have been done. The parameters used are :

- $\alpha = 0.6$,
- $a = 2.187$,
- η_t moving from 0 to 1,

Which gives :

- $\Delta\varepsilon = \frac{\varepsilon_l}{\varepsilon_t} = 2.5$,
- $\eta_l = 0.32$ for $\eta_t = 0.384$.

In these conditions, the evolution of the ratio between particles betatron wave number and the mismatch-mode wave number has been represented on **Figure 4** as a function of the transverse tune depression. It shows that ½ order resonance with quadripolar and high frequency modes is always excited, and that 1 order resonance with low frequency mode can be excited for $\eta_t < 0.65$. The evolution of the ratio between particles synchrotron wave number and the mode wave number has been represented on **Figure 5** as a function of the longitudinal tune depression. It shows that ½ order resonance with low frequency mode is always excited, and only resonance with order lower than ½ can be excited by high frequency mode ($\alpha<1$).

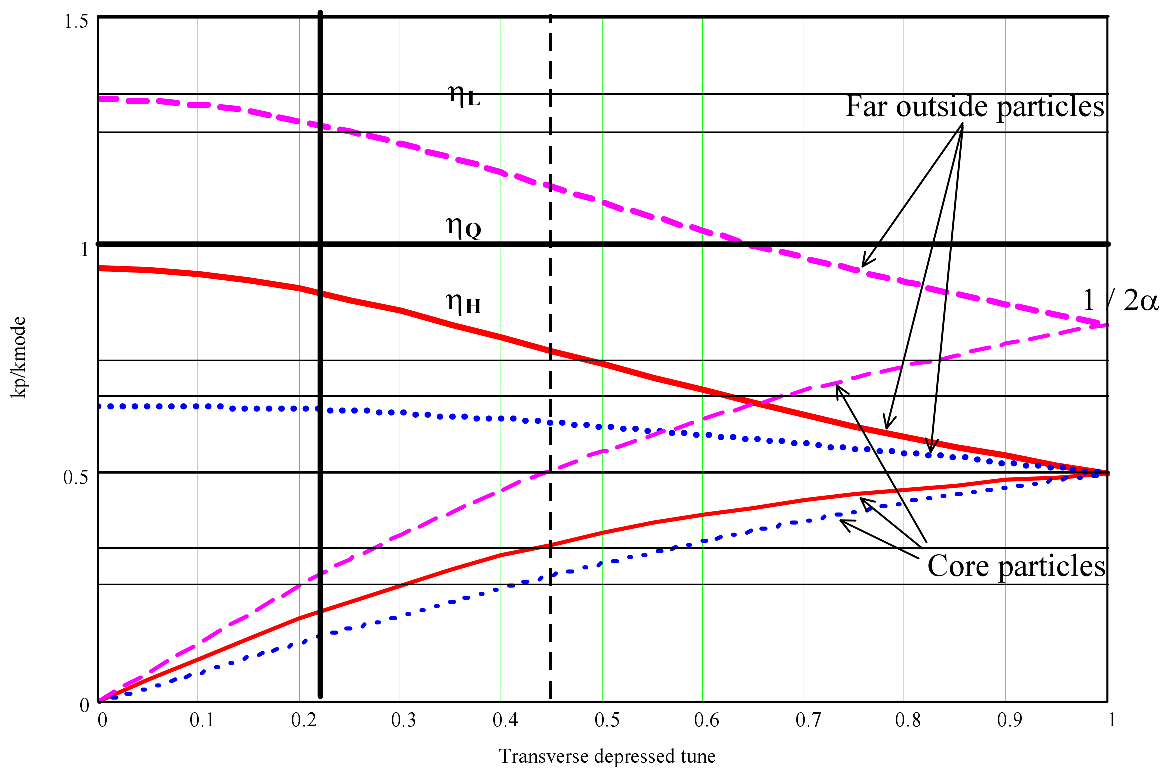

Figure 4 : Evolution of the ratio between particles betatron wave number and mode wave numbers according to the transverse depressed tune.

Beam particles have a phase advance per meter between this of the core and this of the far outside. 1, ½, 1/3 and ¼-order resonance positions have been represented.

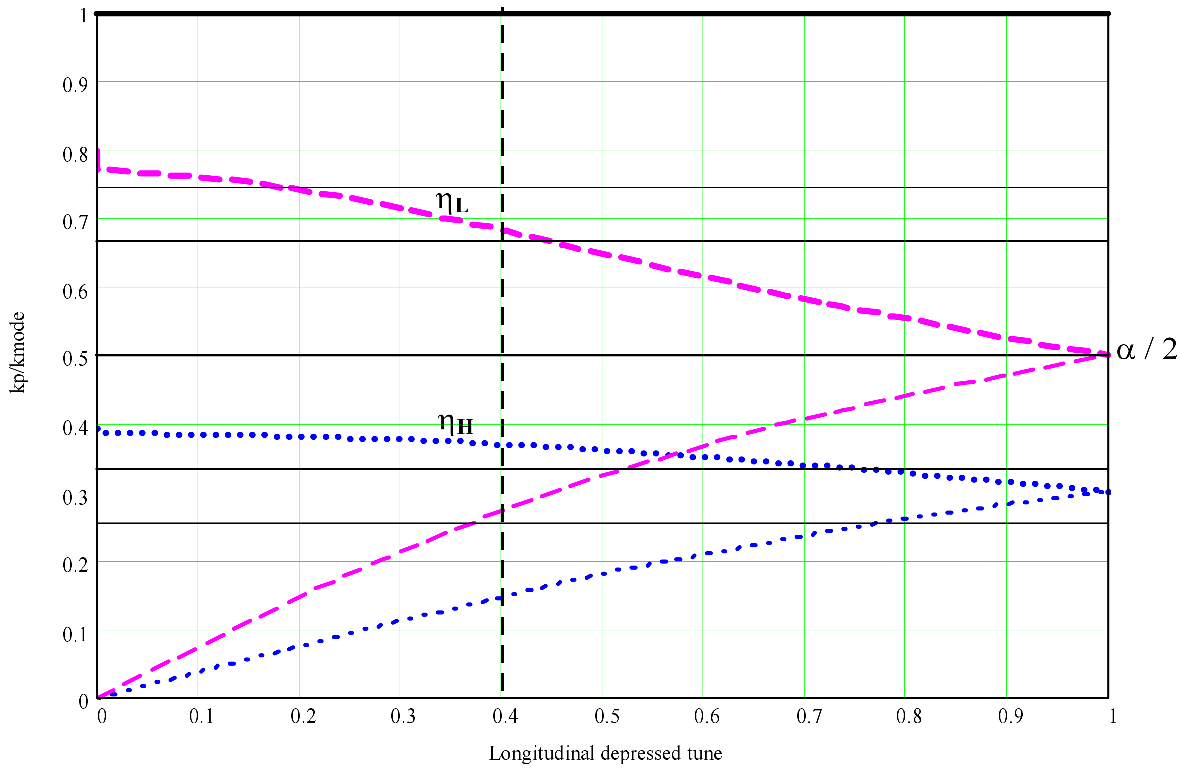

Figure 5 : Evolution of the ratio between particles synchrotron wave number and mode wave numbers according to the longitudinal depressed tune.
Beam particles have a phase advance per meter between this of the core and this of the far outside.

3.4 Case of strong mismatch

When the mismatch is strong, envelope perturbation equation is no longer linear **(Eq. 14)**. In that case, as with unbunched beam [5, p.158], other frequencies k_{qlh}, linear combinations of the three main frequencies, appear (**Figure 6**).

$$k_{q\,h} = q\,k_Q + l\,k_L + h\,k_H\,.$$

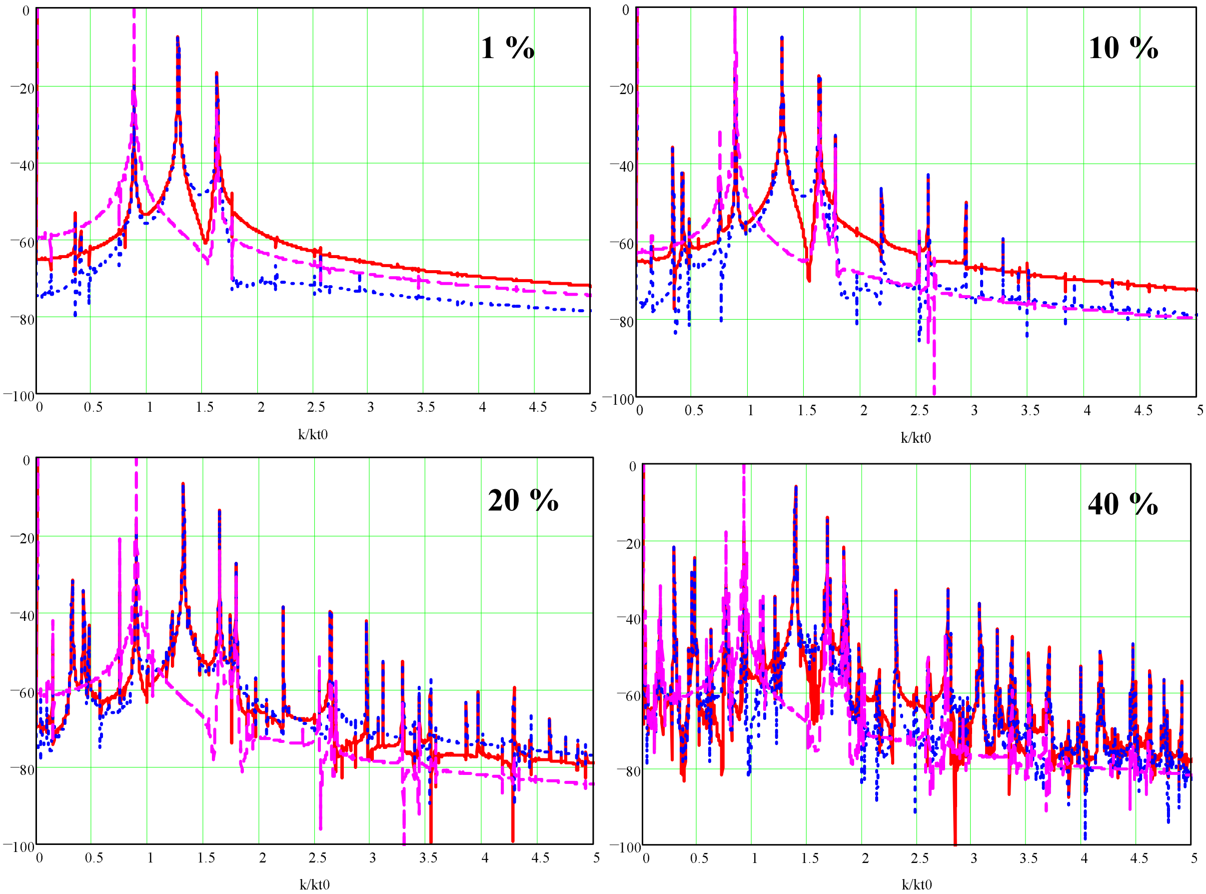

Figure 6 : Envelope spectrum for different mismatches
Mx = Mz = M, My = -M; M = 1%, 10%, 20%, 40%.

These non-linearity induced modes can have very low frequencies and then excite very dangerous frequencies. Fortunately, for small mismatches, they carry a small power, and their effect should not be very sensitive.

4. Conclusion

The mismatched modes of envelope oscillations have been calculated in the case of a bunched beam. The calculus shows that, as with unbunched beam, some particles will always feel the ½ order resonance with the mismatch-modes in the transverse as in the longitudinal directions. Moreover, as the transverse focusing is different of the longitudinal one, the 1st order resonance with one mode can be excited in the transverse (α<1) or in the longitudinal (α>1) direction for high depressed tune. The effect of this resonance has to be investigated.

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