APPROXIMATE SOLUTION FOR TRANSIENT HEAT TRANSFER IN STATIC TURBULENT HE II

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ABSTRACT

Analytical solution in one dimension of the heat diffusion equation in static turbulent superfluid helium (He II) is proposed by mean of integral method. Although this is an approximate method, it has proven that it gives solutions with fairly good accuracy in non-linear fluid dynamics and heat transfer, especially in boundary layer theory. This analytical method is adequate for this class of equations because of its capability of solving non-linear problems and it proposes also a simpler alternative method to numerical calculation. To present the method and compare its accuracy, a simple case solution is compared with the exact solution and experimental data. A more general solution, taking account of the temperature dependence of the thermodynamic properties is also proposed.

INTRODUCTION

Analytical treatment of transient heat transfer in He II has received not enough attention, considering the substantial interest as it relates to the cooling and stability of magnet systems. Dresner using similarity solutions method has developed three analytical solutions[1,2]. These cases deal with linear boundary conditions and temperature independent properties for semi-infinite media. Several solutions are still of interest of designer to investigate the cooling performance and stability of magnet systems, such as solutions in a finite media and temperature dependent properties. An adequate method in the solution of heat diffusion problems is the integral method because of its capability of solving non-linear problems where the non-linearity can be found either in the differential equation itself or in the boundary conditions. This method is analogous to the method employed to solve thermal and momentum boundary layer in fluid mechanics[3]. With exact method, the resulting solution satisfies locally the system over the entire range of space and time. Such solutions are rather difficult to obtain when the differential equation is non-linear or if the boundary conditions involved are non-linear. Integral method, in the solution of time-dependent boundary-value problems, gives solutions, which satisfy the differential system only on the average over the region considered rather than considering a local solution. It is often sufficient for engineering calculations in which many more approximations are used to model complex cryogenics systems.

SOLUTION IN A SEMI-INFINITE MEDIA WITH CONSTANT PROPERTIES

Case for an Clamped Heat Flux

In this paper we examine the solution of a system in which the differential equation is non-linear but not the boundary conditions. The simplest case that can be studied is where the thermodynamic properties are temperature independent and the media is considered semi-infinite. We detail, to present the method, the case of a heat flux step where at t=0 a constant heat flux q_0 is applied at the boundary x=0. It has been already proven that the diffusion equation is able to model He II transient heat transfer[1,2]. The main reason is because for sufficient heat flux and length (~1 m) the heat transfer is dominated by the enthalpy variation of the He II. For a fully developed turbulent state, the heat flux is given by the Gorter-Mellink law[4], neglecting the dissipation effects in He II, the partial differential equation modeling our system for one space dimension is,

$$\frac{\partial T}{\partial t} = \frac{f^{1/3}}{\rho C_p}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)^{1/3} \text{ in } 0\leqslant x\leqslant\infty \text{ and for } t>0, \tag{1}$$

where ρ is the density, C_p the specific heat at constant pressure and f the He II turbulent thermal conductivity function. The first boundary condition is

$$-\left(f\frac{\partial T}{\partial x}\right)^{1/3} = q_0 \text{ at } x=0 \text{ and for } t>0, \tag{2}$$

where q_0 is the heat flux at x=0. At the initial time, the entire media is at constant temperature, so the initial condition is

$$T=T_b \text{ in } 0\leqslant x\leqslant\infty \text{ and at } t=0. \tag{3}$$

As it is a semi-infinite media, the necessary second boundary conditions is a constant temperature when $x\to\infty$ or practically for large x, i.e. the temperature field is not disturbed for large x. This conditions is expressed by

$$T=T_b \text{ for } x\to\infty \text{ and for } t>0. \tag{4}$$

We are only interested by the solution of the disturbed temperature field which is limited by a distance $\delta(t)$, called the thermal layer, after which the temperature field is not disturbed. For a semi-infinite media, the thermal layer is defined as being always inferior to the length of the system. From this definition we can modify the boundary condition Eq. (4),

$$T=T_b \text{ at } x=\delta(t) \text{ and for } t>0. \tag{5}$$

Introducing a set of non-dimensional variables as

$$\theta = \frac{T-T_b}{T_\lambda - T_b},\ \chi = \frac{x}{L} \text{ and } \tau = \frac{f^{1/3}}{\rho C_p L^{4/3}\left(T_\lambda - T_b\right)^{2/3}}t, \tag{6}$$

where T_λ is the temperature corresponding to the lambda transition, L the length of the domain which is supposed to be thermally semi-infinite in this case, θ, χ and τ respectively the non-dimensional temperature, space dimension and time, the system is transformed into

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial\chi}\left(\frac{\partial\theta}{\partial\chi}\right)^{1/3} \text{ in } 0\leqslant\chi\leqslant\infty \text{ and for } \tau>0, \tag{7-a}$$

$$-\left(\frac{\partial\theta}{\partial\chi}\right)^{1/3}=\phi \quad \text{at } \chi=0 \text{ and for } \tau>0, \tag{7-b}$$

$$\theta=0 \text{ at } \chi=\Delta \text{ and for } \tau>0, \tag{7-c}$$

$$\theta=0 \text{ in } 0\leq\chi\leq\infty \text{ and at } \tau=0, \tag{7-d}$$

where we define a non-dimensional heat flux ϕ and thermal layer Δ as

$$\phi^3=\frac{q_0^3 L}{f(T_\lambda-T_b)} \text{ and } \Delta=\frac{\delta}{L}. \tag{8}$$

If Eq. (7-a) is integrated with respect to space over the thermal layer the resulting equation is called the Heat-Integral Equation. With this integration, terms in space gradient can be removed from the energy equation. Following these directions, the energy equation is then transformed into

$$\int_0^\Delta \frac{\partial\theta}{\partial\tau}d\chi=\int_0^\Delta \frac{\partial}{\partial\chi}\left(\frac{\partial\theta}{\partial\chi}\right)^{1/3}d\chi=\left(\left.\frac{\partial\theta}{\partial\chi}\right|_\Delta\right)^{1/3}-\left(\left.\frac{\partial\theta}{\partial\chi}\right|_0\right)^{1/3}. \tag{9}$$

With the use of the boundary conditions (7-b) and noticing that in our system $\left.\frac{\partial\theta}{\partial\chi}\right|_\Delta$ is null because of the definition of the thermal boundary Δ, Eq. (9) is reduced to

$$\int_0^\Delta \frac{\partial\theta}{\partial\tau}d\chi=-\left(\left.\frac{\partial\theta}{\partial\chi}\right|_0\right)^{1/3}. \tag{10}$$

When the rule of differentiation is used on Eq. (10), the integral on the left hand-side is transformed into

$$\frac{d}{d\tau}\int_0^\Delta \theta d\chi-\left[\theta\frac{d\chi}{d\tau}\right]_0^\Delta=-\left(\left.\frac{\partial\theta}{\partial\chi}\right|_0\right)^{1/3}. \tag{11}$$

One can notice that due to the boundary conditions the second term of the right hand-side of Eq. (11) is null which reduces it to a simpler formulation,

$$\frac{d}{d\tau}\int_0^\Delta \theta d\chi=-\left(\left.\frac{\partial\theta}{\partial\chi}\right|_0\right)^{1/3}=\phi. \tag{12}$$

Eq. (12) is the Heat-Integral Equation for the clamp heat flux problem; it could be used to treat non-linear boundary condition too. Let assume that the temperature has a polynomial form as $\theta=a+b\chi+c\chi^2+d\chi^3$ where the coefficients a, b, c and d are function of the thermal layer Δ. Obviously, θ is an approximate solution of the system and to find the different coefficients, we need to use different boundary conditions: the natural conditions, which ensues from the problem, and derived conditions, which are constructed from either the differential equation or the natural boundary conditions. For this expression of the solution we need two extra boundary conditions. The first one we choose is straightforward and comes from the definition of the thermal layer,

$$\frac{\partial\theta}{\partial\chi} = 0 \text{ at } \chi=\Delta \text{ for } \tau>0. \quad (13)$$

One can notice that this condition has been already used to construct the Heat-Integral Equation. The second one comes from the differential equation at $\chi=\Delta$ where derivative of the temperature with respect to space is null because of condition Eq. (7-c). We have, what it is called a derived condition,

$$\frac{\partial^2\theta}{\partial\chi^2} = 0 \text{ at } \chi=\Delta \text{ for } \tau>0. \quad (14)$$

By the use of the natural boundary conditions Eq. (7-b), (7-c) and (13) and the derived one, Eq. (14), we can formulate a solution of θ as a function of Δ,

$$\theta = \frac{\phi^3\Delta}{3}\left(1-\frac{\chi}{\Delta}\right)^3. \quad (15)$$

By substituting Eq. (15) into the Heat-Integral Equation, Eq. (12), we obtain a first order ordinary differential equation for the thermal layer thickness Δ,

$$\frac{d}{d\tau}\Delta^2 = \frac{12}{\phi^2}. \quad (16)$$

The solution of Eq. (16) subjected to the initial condition Eq. (7-d) gives

$$\Delta = \frac{2\sqrt{3}}{\phi}\sqrt{\tau}. \quad (17)$$

The negative solution of Eq. (17), which has no physical meaning in our problem, has been eliminated. The general solution is composed of Eq. (15) and Eq. (17). The time constant of the system is $\phi^2/12\times t$, when transformed into dimensional variables it is consistent with a dimensional analysis of Eq. (1), giving a time constant of $\rho C_p L^2 q_0^2/12f$.

Comparison with Existing Solution and Experimental Data

If we compare the solution of Eq. (15) at $\chi=0$ which is given by

$$\theta_0 = \frac{2\sqrt{3}}{3}\phi^2\sqrt{\tau}, \quad (18)$$

and can be expressed as a function of the dimensional variables as

$$\frac{T_0 - T_b}{T_\lambda - T_b} = \frac{2\sqrt{3}}{3}\frac{q_0^{\,2}}{\sqrt{\rho C_p f}(T_\lambda - T_b)}\sqrt{t}, \quad (19)$$

one can notice that this formulation is similar to Dresner's model with the exception of the coefficient $2\sqrt{3}/3$ (1.15), which is 0.83 in his solution[1]. We can formulate also the time where the temperature of the helium reaches the lambda temperature at the boundary. According to this, $\theta_0 = 1$ and the time t_λ is defined by

$$t_\lambda = \frac{3}{4}\frac{f\rho C_p (T_\lambda - T_b)^2}{q_0^{\,4}}. \quad (20)$$

This formulation is also similar to Dresner's formulation with the exception of the coefficient 3/4 which is 1.43 in his model but it agrees on the quartic dependence on the heat flux with experimental results reported by Van Sciver[5]. Dresner's coefficients are found by identification with experimental results reported by Van Sciver which means that these coefficients are only valid for the thermodynamic conditions of Van Sciver's experiment. A comparison with experimental data is encouraging, when we look at the proportional function between the time t_λ and q_0^4. The experimental results of Van Sciver give a value of 110 $W^4 s\ cm^{-8}$ for T_b=1.802 K whereas Eq. (20) gives a value comprised between 52 and 141 $W^4 s\ cm^{-8}$ for a bath temperature comprises between 1.8 K and 2.0 K.

The approximate solution Eq. (15) is plotted on Figure 1 with the data obtained by Van Sciver. As we are assuming that the thermodynamic properties are constant, we need to define an average temperature to evaluate ρ, C_p and f. For this plot, the best match have been found to be for an average temperature of 1.99 K. There is a good agreement for small temperature variation and small x whatever the time but for large x the solution reaches a null temperature variation too soon. It is the limitation of the model and in fact the null temperature variation space location is a function of time and corresponds to the thermal layer. The model underestimates the length of the thermal layer which comes from the profile of the approximate solution and also the associated boundary conditions taken to calculate the solution. Other profile and boundary conditions have been investigated in the following paragraph.

Other Solutions

For heat conduction problems, it has been shown that taking a polynomial form with a degree higher than three does not improve necessarily the accuracy of the solution. The reason is that for each of the polynomial coefficient, which are time dependent i.e. function of the thermal layer, a boundary condition has to be provided. For four coefficients, we have to provide two supplementary conditions, one natural and another derived, whereas for a fourth degree polynomial, another derived boundary condition has to be used for the fifth coefficients and the choice of the extra condition can reduce the accuracy.

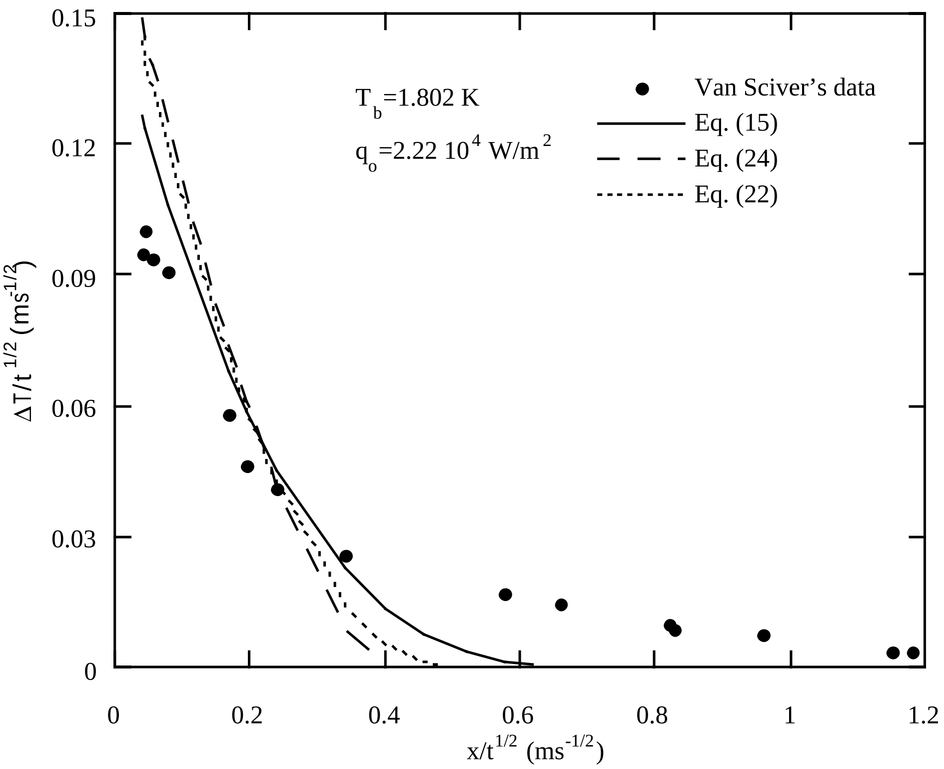

Figure 1. Comparison between different solutions and Van Sciver's experiment[5].

We solved the same problem with a quadric polynomial form. In order to do so, we have to use an extra boundary condition which is,

$$\frac{\partial^2\theta}{\partial\chi^2} = 0 \text{ at } \chi=0 \text{ for } \tau>0, \tag{21}$$

which comes from the derivation of the boundary condition at $\chi=0$ Eq. (7-b). The solution for that case is

$$\theta = \frac{\phi^3\Delta}{2}\left(1 - 2\frac{\chi}{\Delta} + 2\left(\frac{\chi}{\Delta}\right)^3 - \left(\frac{\chi}{\Delta}\right)^4\right) \text{ with } \Delta = 2\sqrt{\frac{5}{3}}\frac{\sqrt{\tau}}{\phi}. \tag{22}$$

Another solution can be calculated with a cubic polynomial form to illustrate the effect of the boundary condition on the solution. We use a different boundary condition then Eq. (14) derived from the boundary condition Eq. (7-b), that is to say,

$$\frac{\partial^2\theta}{\partial\chi^2} = 0 \text{ at } \chi=0 \text{ for } \tau>0. \tag{23}$$

The solution found is

$$\theta = \frac{2}{3}\phi^3\Delta\left(1 - \frac{3}{2}\frac{\chi}{\Delta} + \frac{1}{2}\left(\frac{\chi}{\Delta}\right)^3\right) \text{ with } \Delta = 2\frac{\sqrt{\tau}}{\phi}. \tag{24}$$

These two solutions are also plotted on Figure 1 and it is interesting to note that the solutions do not differ by a lot even if we can note that the quartic polynomial form is less accurate that the others. Solutions given by Eq. (22) has a lower accuracy than solution given by Eq. (15). Accuracy of these solutions depends on the boundary conditions and the profile of the approximate solution and is hard to predict unless by comparison with the exact solutions.

SOLUTION IN A SEMI-INFINITE MEDIA WITH TEMPERATURE DEPENDENT PROPERTIES

The system to solve is similar to the one defined by the system of Eq. (1), Eq. (2), Eq. (3) and Eq. (5), such as the differential equation,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(f\frac{\partial T}{\partial x}\right)^{1/3} \text{ in } 0\leqslant x\leqslant\infty \text{ and for } t>0, \tag{25}$$

where in this case we consider the thermodynamic properties ρ, C_p and f temperature dependent. The boundary conditions are identical to the previous case. By applying the Kirchhoff transformation

$$\Theta = \int_{T_b}^{T} f(T)dT, \tag{26}$$

the system Eq. (25), Eq. (2), Eq. (3) and Eq. (5) is transformed into,

$$\frac{1}{\alpha}\frac{\partial\Theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial\Theta}{\partial x}\right)^{1/3} \text{ in } 0\leqslant x\leqslant\infty \text{ and for } t>0, \tag{27-a}$$

$$-\left(\frac{\partial\Theta}{\partial x}\right)^{1/3} = q_0 \text{ at x=0 and for t>0,} \tag{27-b}$$

$$\Theta=0 \text{ in } 0\leqslant x\leqslant\infty \text{ and for t=0.} \tag{27-c}$$

$$\Theta=0 \text{ at } x=\delta(t) \text{ and for t>0,} \tag{27-d}$$

where $\alpha=f/\rho C_p$. One can remark that $\alpha=\alpha(\Theta)$. The Heat-Integral Equation is now written

$$\frac{d}{dt}\int_0^\delta \Theta dx = \alpha q_0, \tag{28}$$

where in a first approximation we consider $\alpha(\Theta)$ constant. If we use the same polynomial form $\Theta=a+bx+cx^2+dx^3$ for the expression of the temperature, the solution is expressed as

$$\Theta = q_0{}^3\frac{\delta}{3}\left(1-\frac{x}{\delta}\right)^3, \tag{29}$$

when the following natural and derived boundary conditions used are,

$$\left.\frac{\partial\Theta}{\partial x}\right|_\delta = 0 \text{ and } \left.\frac{\partial^2\Theta}{\partial x^2}\right|_\delta = 0. \tag{30}$$

One can observe that the solution is again a function of the boundary condition at x=0, which is defined as $\Theta_0=q_0{}^3\delta/3$. The expression of δ is obtained by inserting Eq. (29) in Eq. (28) and using Eq. (27-d) is

$$\delta = \frac{2\sqrt{3}}{q_0}\sqrt{\alpha t}. \tag{31}$$

Since the boundary surface temperature Θ_0 is not yet know, Eq. (31) cannot be directly used to evaluate δ but we can eliminate the thermal layer in the expression of Θ_0 and have a transcendental equation for Θ_0 when α_0 is given as a function of Θ_0,

$$\Theta_0\sqrt{\alpha(\Theta_0)} = \frac{2\sqrt{3}}{3}q_0{}^2\sqrt{t}. \tag{32}$$

It is possible to have the expression of $\alpha_0=f_0/\rho_0C_{p0}$ as a function of Θ_0 by using analytical expressions of the He II turbulent thermal conductivity function and the specific heat as a function of the temperature[6],

$$f = \frac{\rho^2 s_\lambda^4 T_\lambda^4}{A_\lambda}\left[\left(\frac{T}{T_\lambda}\right)^{5.7}\left(1-\left(\frac{T}{T_\lambda}\right)^{5.7}\right)\right]^3 \text{ and } C_p=KT^{5.6}. \tag{33),(34}$$

where $A_\lambda\approx145$ ms/kg and $K\approx117$ J/kgK$^{6.6}$. The solution is easy to find when the boundary temperature difference (at x=0) is set, we can evaluate α_0 and Θ_0 and calculate the time needed to reach this temperature difference. As the solution is defined, by evaluating the thermal layer defined from the initial temperature, it is also easy to find the value of the temperature for location different than the boundary, i.e. $x\neq0$ with

$$\Theta = \Theta_0\left(1-q_0{}^3\frac{x}{3\Theta_0}\right)^3 \tag{35}$$

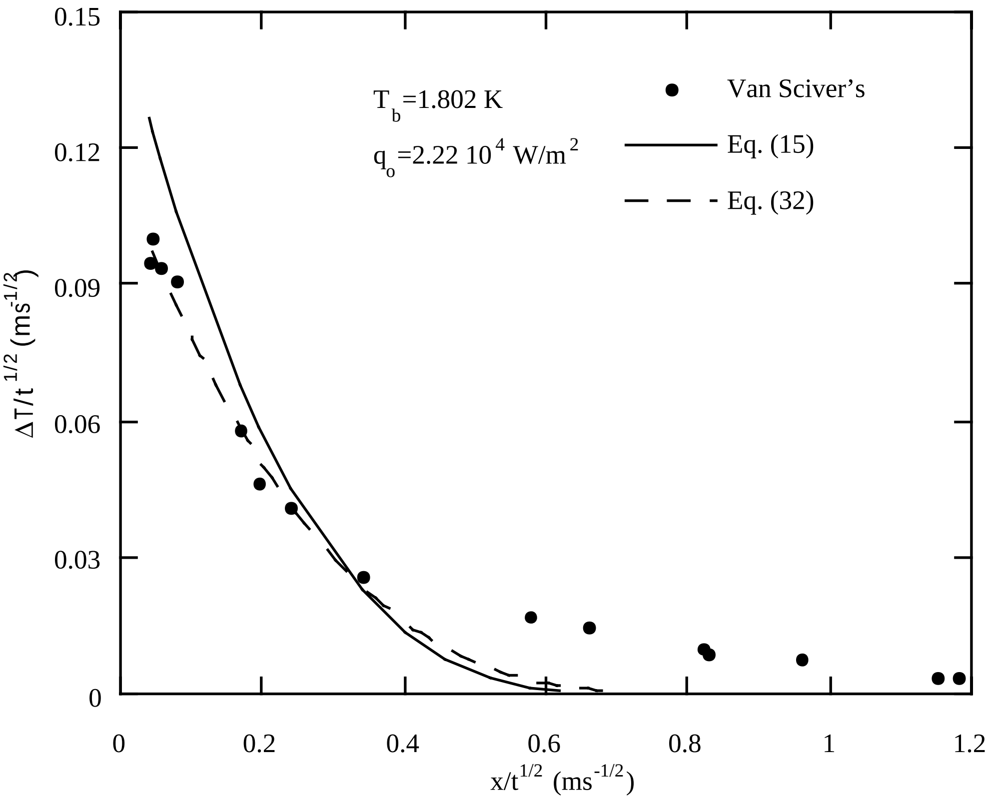

Figure 2. Comparison between Eq. (15), Eq. (32) and Van Sciver's experiment[5].

The solution is plotted on Figure 2 and compare with Eq. (15) and data obtained by Van Sciver. Not only the solution is more accurate than the solution given by Eq. (15), this solution gives a direct result, even with the constant α approximation, without the need to evaluate an average temperature to find the best fit of the experimental data. For the same reason than the others solutions, this one predicts with less accuracy the evolution of the temperature for large x. It comes from the profile of the approximate solution which gives a thermal layer shorter, as a function of time, than the experimental data's one.

CONCLUSION

Integral method are suited to solved the non-linear heat diffusion equation for superfluid helium with acceptable accuracy, but further work is needed to improve it, especially in the choice of the approximate solution profile and the associated boundary conditions. Taking account of the temperature dependence of the thermodynamics properties gives better accuracy and a direct result without the need of the evaluation of an average temperature to define the thermodynamic properties. Further work should involve other boundary conditions such as clamped temperature or pulsed-source problem.

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