Relaxation solvers for hydro- and magnetohydrodynamics and their application

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Entropy condition and conservation laws

Many phenomena in continuum mechanics may be modeled as systems of hyperbolic conservation laws:

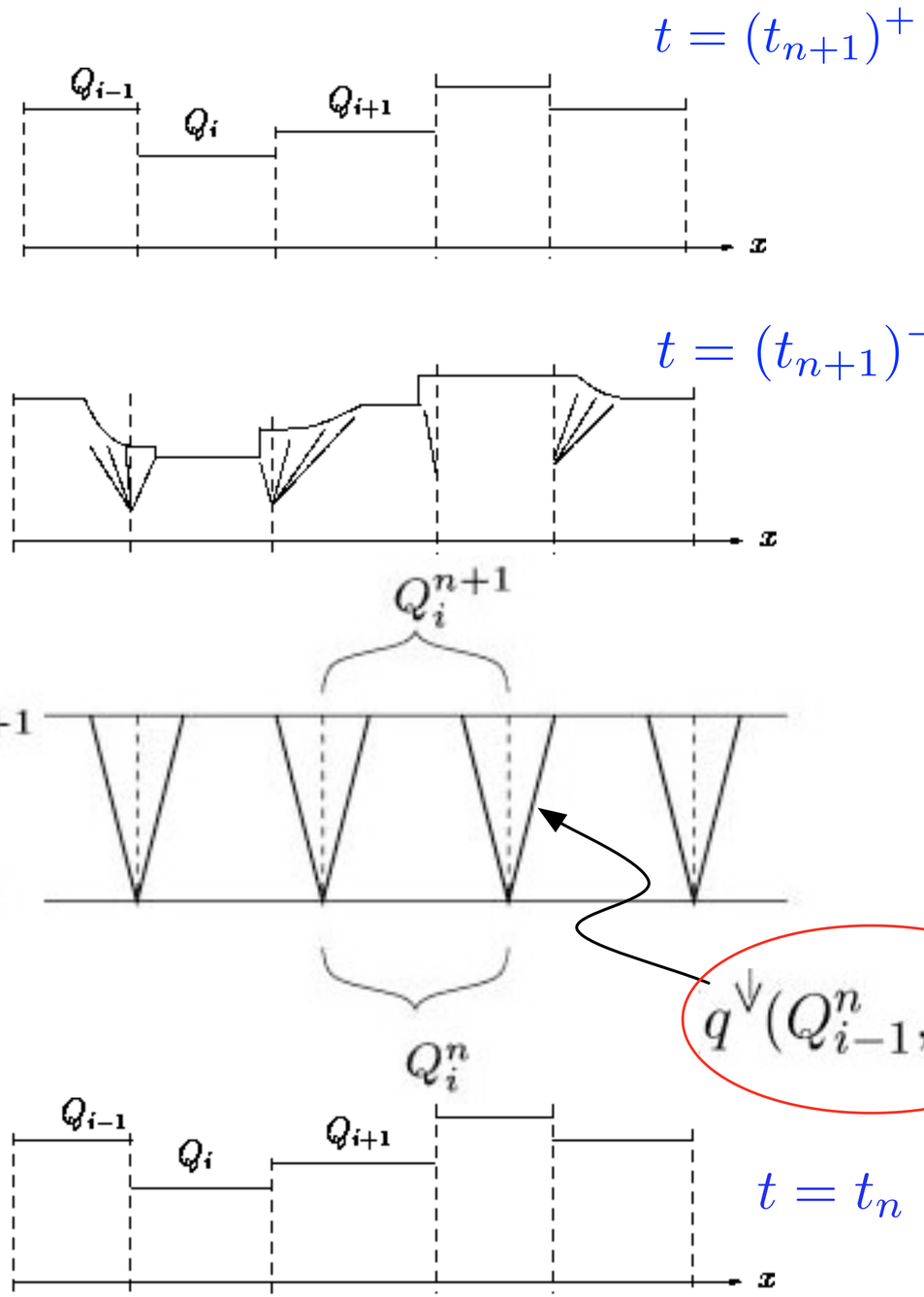
$$\frac{\partial U(x,t)}{\partial t} + \nabla F\big(U(x,t)\big) = 0$$

Their solutions need to be considered together with some *admissibility* condition,
also called *entropy* condition.

This will be done through appropriate approximate Riemann solvers.

Godunov's method (for nonlinear equation)

$$q_t + f(q)_x = 0$$



Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.

$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i) \quad \text{for } t > t_n$$

$$F_{i-1/2}^n = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\, dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

Riemann solution

Linear hyperbolic systems

$$q_t + Aq_x = 0$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p, for $p = 1,\ 2,\ ,\ldots,\ m$.

Let R be matrix of right eigenvectors and $v = R^{-1}q$.

$$R^{-1}q_t + R^{-1}AR\,R^{-1}q_x = 0$$

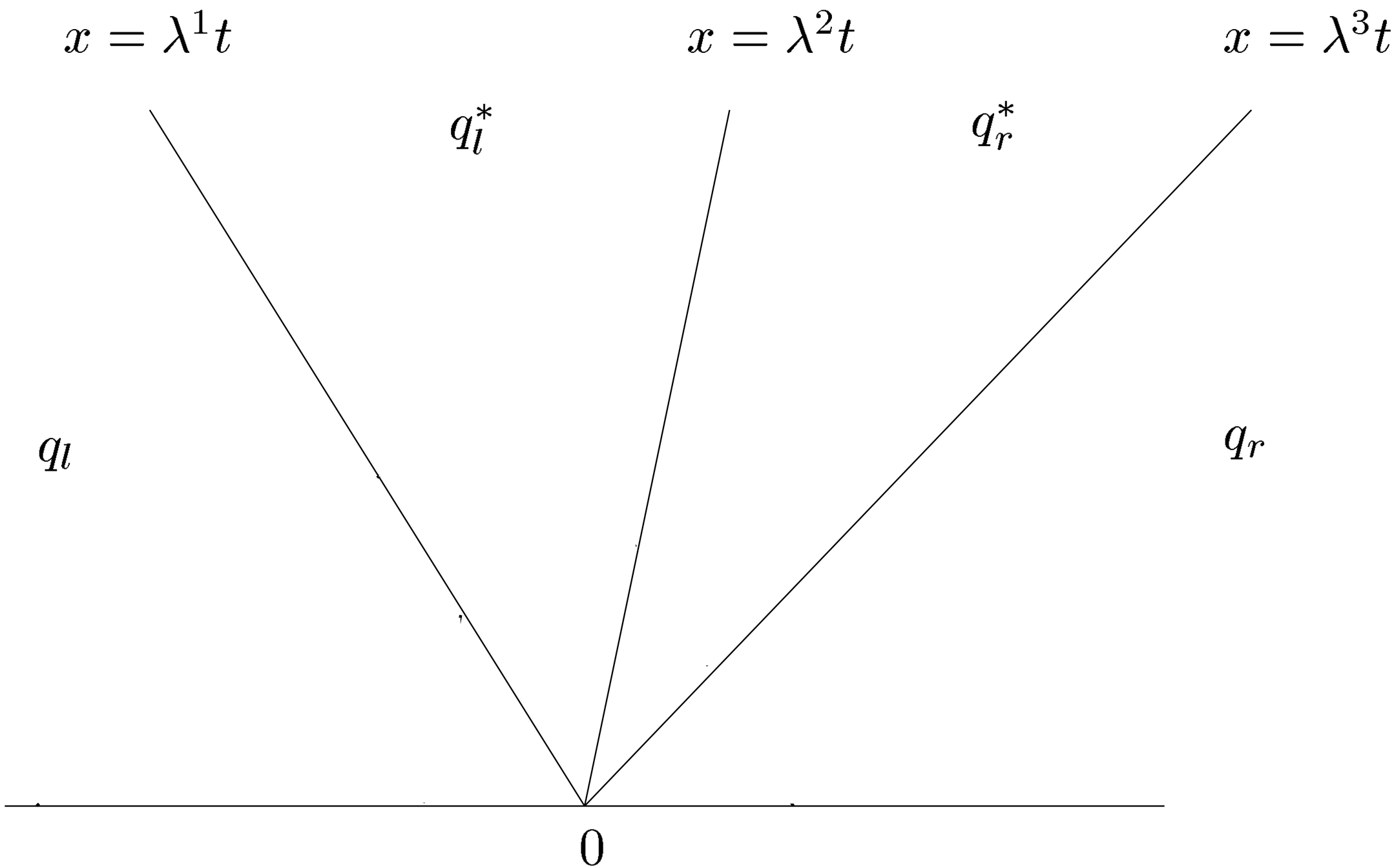
Since $R^{-1}AR = \Lambda$, this diagonalizes the system:

$$v_t + \Lambda v_x = 0.$$

This is a system of m decoupled advection equations

$$v_t^p + \lambda^p v_x^p = 0.$$

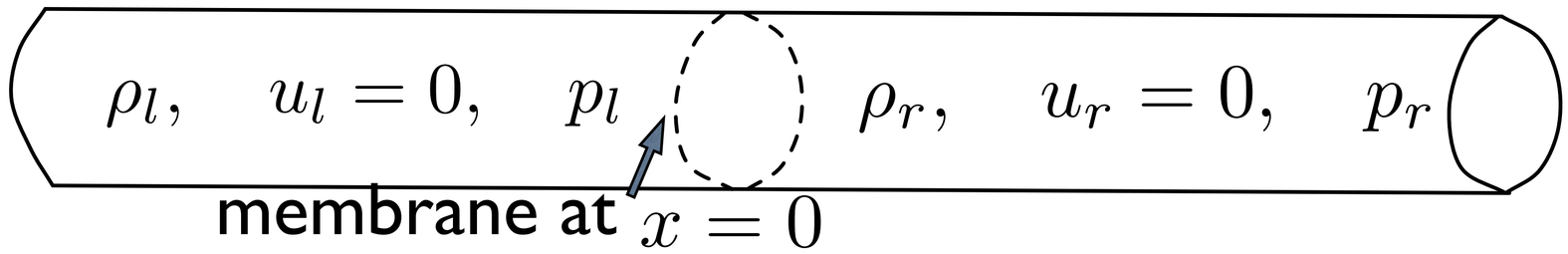
Riemann solution for linear systems



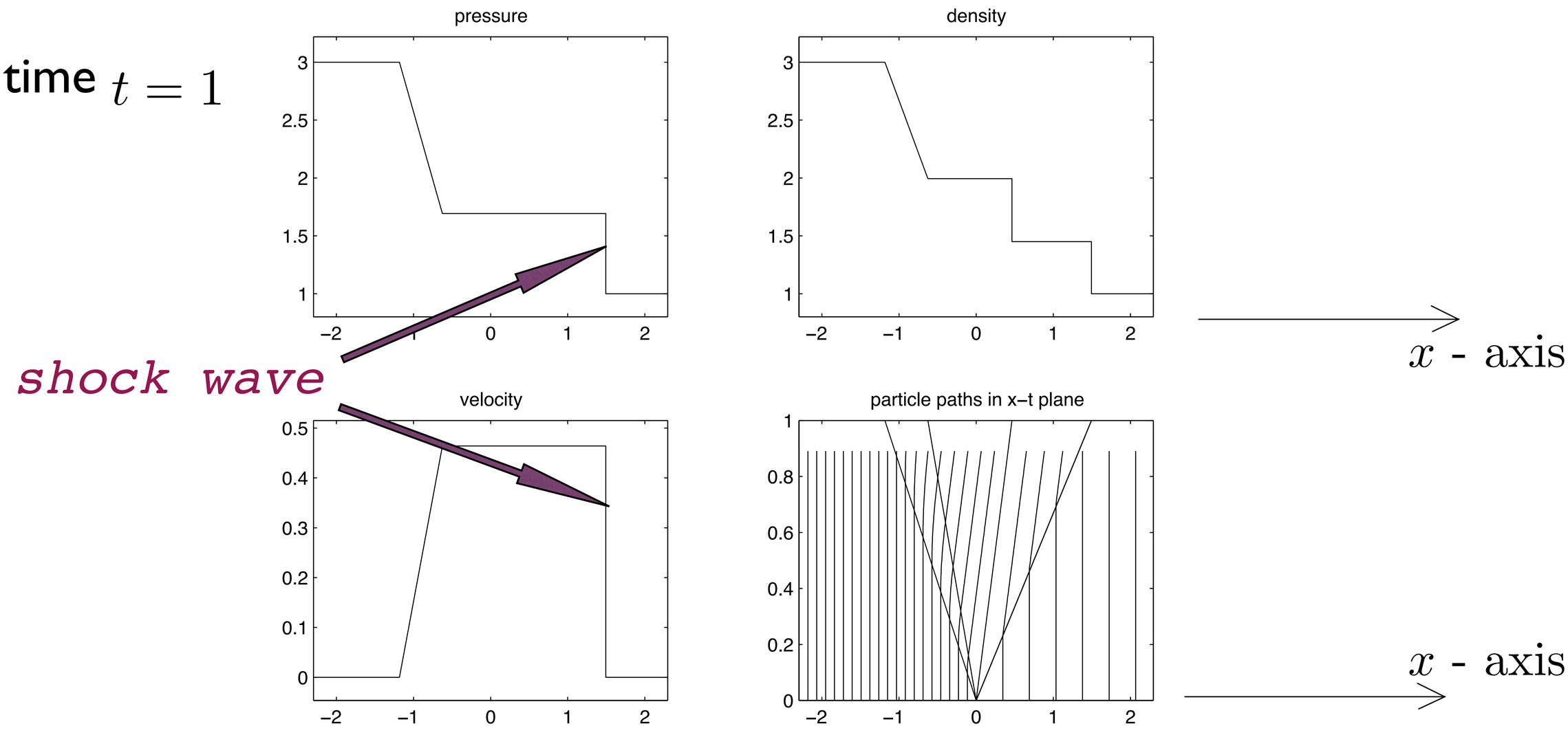
Shock tube problem for the Euler equations of compressible gas dynamics:

at time

$t = 0$



at time $t = 1$



This is the exact solution of the Riemann problem.

Consider equations of ideal MHD:

$$
\begin{aligned}
&\partial_t \rho + \partial_x(\rho v_n) = 0,\\
&\partial_t \rho v_n + \partial_x\left(\rho v_n^2 + p + \tfrac{1}{2}\mathbf{B}_t^2\right) = 0,\\
&\partial_t \rho \mathbf{v}_t + \partial_x(\rho v_n \mathbf{v}_t - B_n \mathbf{B}_t) = 0,\\
&\partial_t \mathbf{B}_t + \partial_x(v_n \mathbf{B}_t - B_n \mathbf{v}_t) = 0,\\
&\partial_t E + \partial_x\left(\left(E + p + \tfrac{1}{2}\mathbf{B}_t^2\right) v_n - B_n \mathbf{B}_t \cdot \mathbf{v}_t\right) = 0.
\end{aligned}
$$

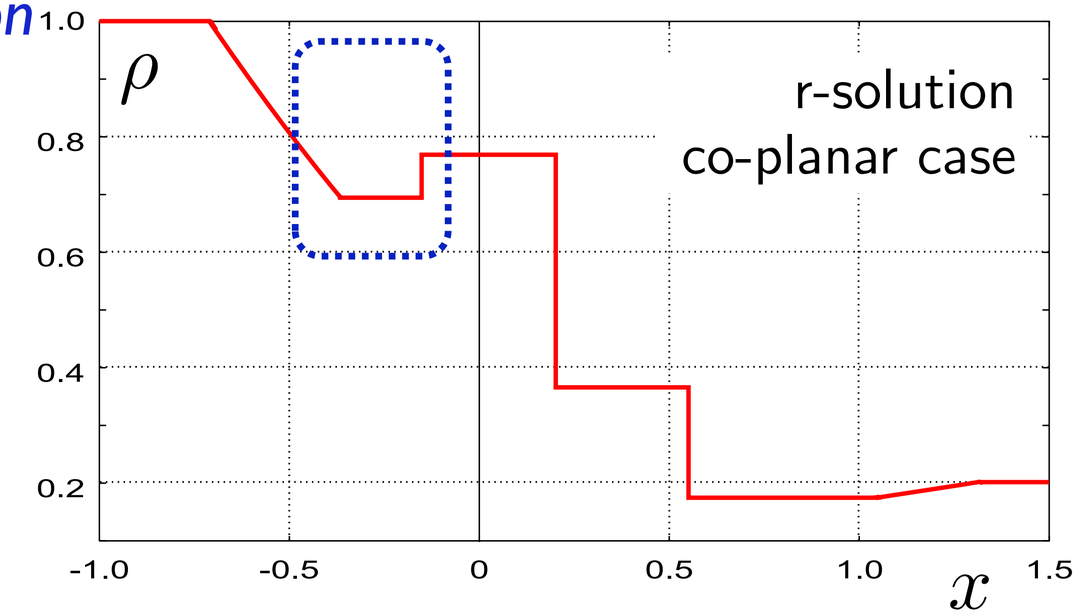
with Riemann initial data:

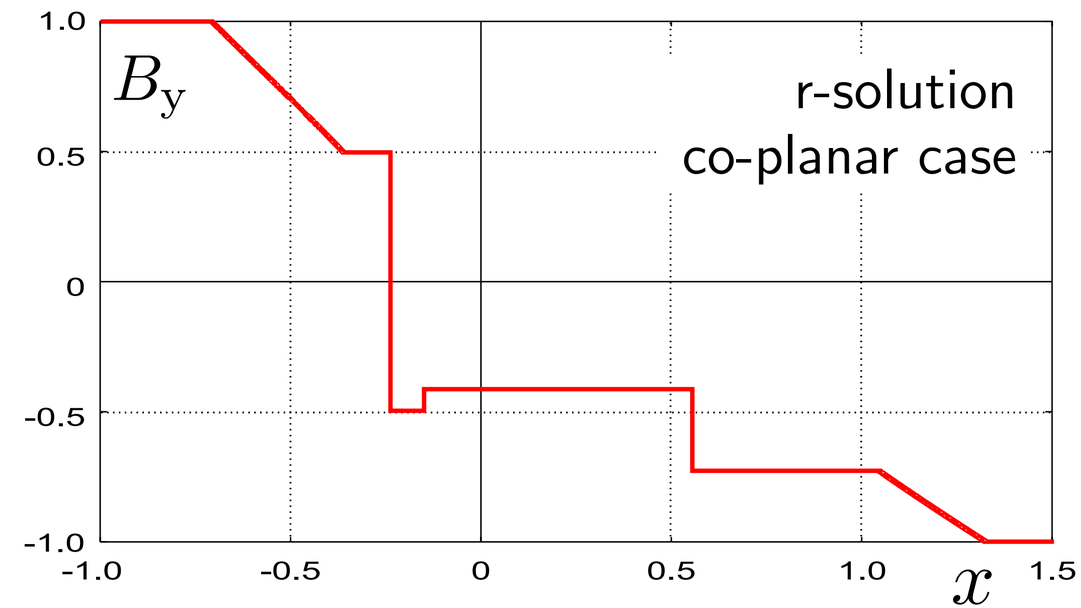
$$
\left(\rho_1, v_n^{(1)}, \mathbf{v}_t^{(1)}, B_n, \mathbf{B}_t^{(1)}, p_1\right) = \left(1, 0, \mathbf{0}, 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1\right),
$$

$$
\left(\rho_0, v_n^{(0)}, \mathbf{v}_t^{(0)}, B_n, \mathbf{B}_t^{(0)}, p_0\right) = \left(0.2, 0, \mathbf{0}, 1, \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, 0.2\right)
$$

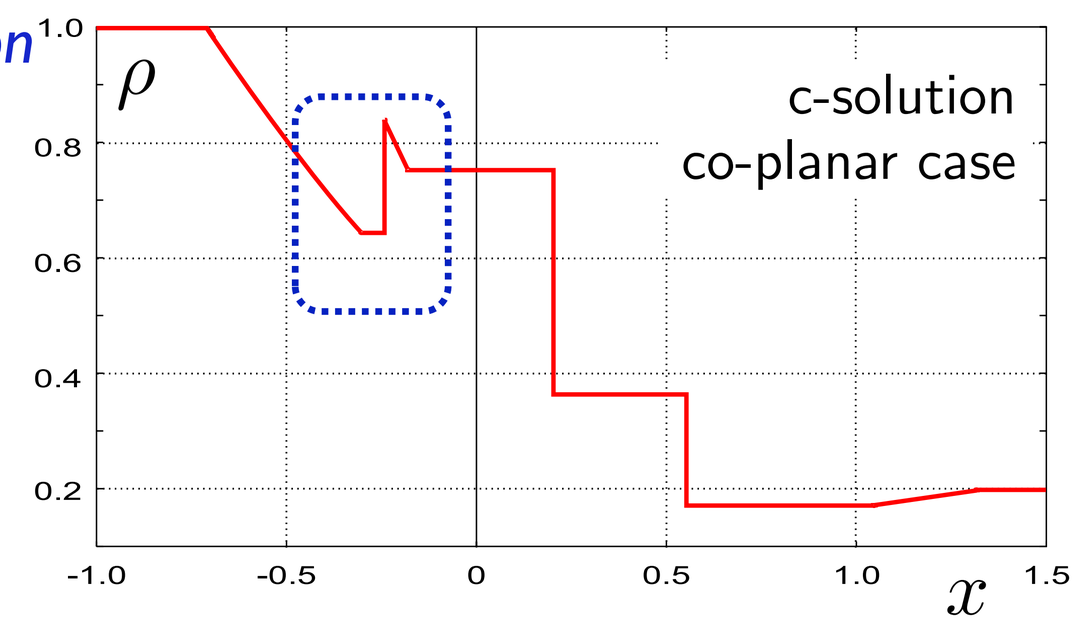
The same initial data allows for two solutions:

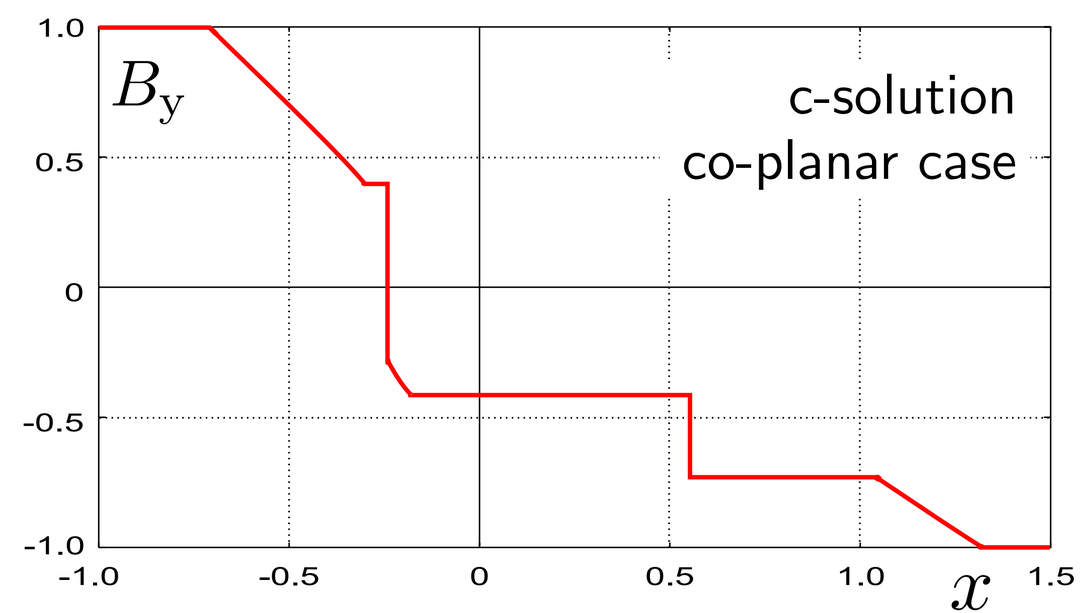
solution 1:





solution 2:





We shall use the following admissibility (or entropy) condition:

$$(\rho\phi(s))_t + \mathrm{div}(\rho\mathbf{u}\phi(s)) \leq 0$$

where ϕ is an appropriately chosen convex functional.

An entropy condition implies for gas dynamics it preserves nonnegative density and internal energy.

Jin - Xin relaxation:

replace: $$u_t + f(u)_x = 0$$

by:

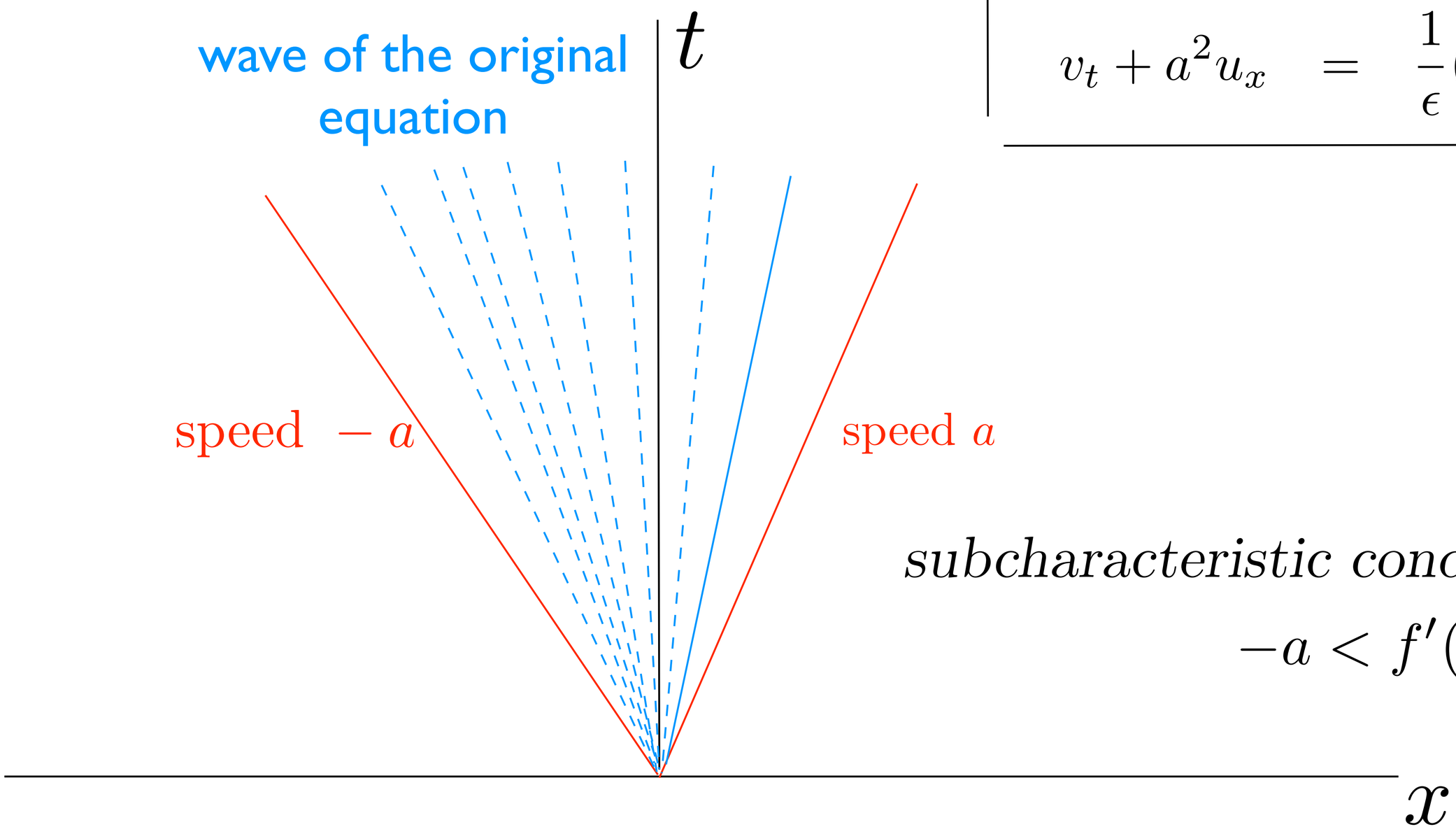
$$\begin{aligned} u_t + \quad v_x &= 0 \\ v_t + a^2 u_x &= \frac{1}{\epsilon}(f(u) - v)) \end{aligned}$$

the role of a^2

consider Riemann problem for the two sets of equations:

$$u_t + f(u)_x = 0$$

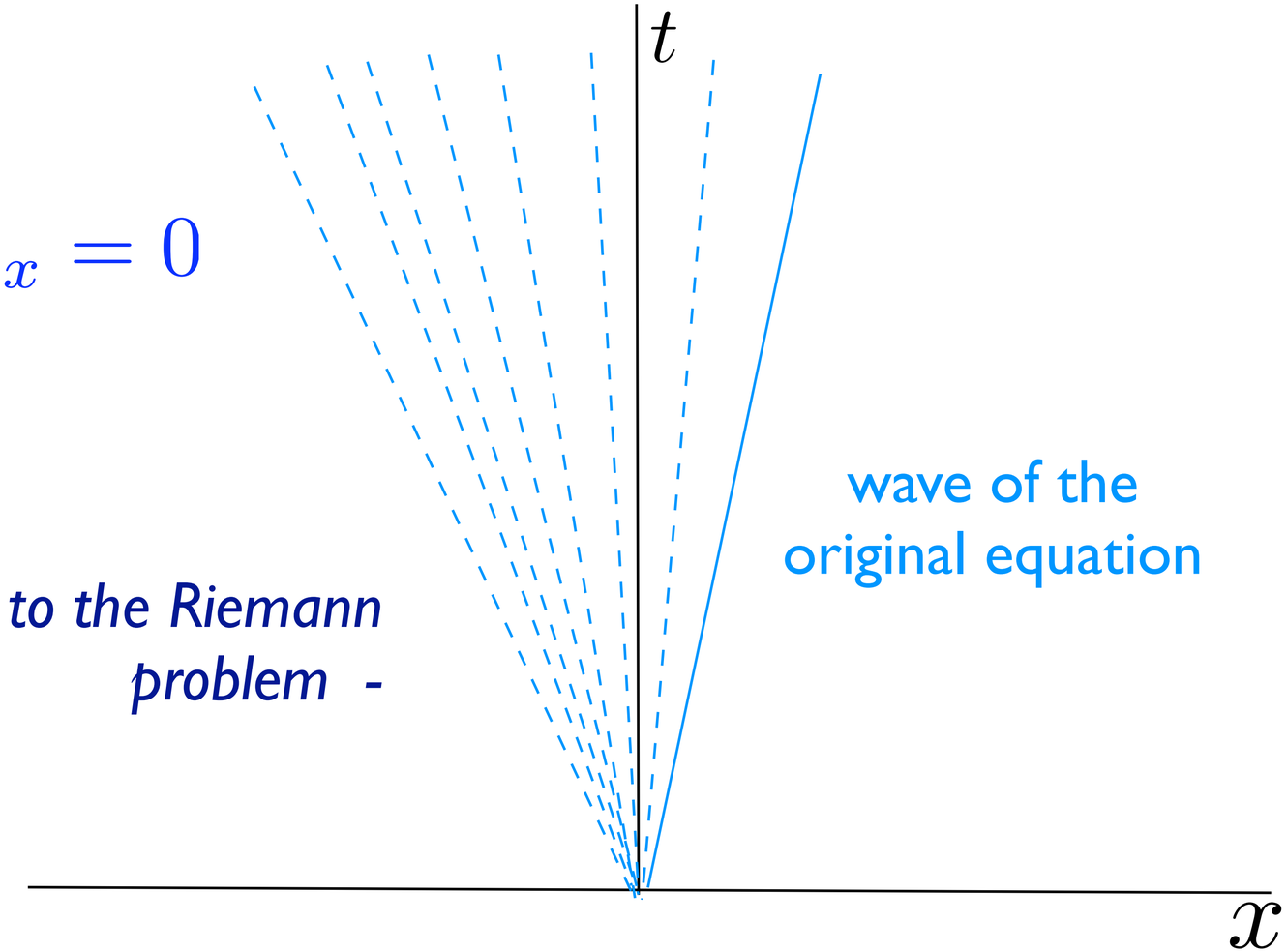
$$\begin{aligned} u_t + \quad v_x &= 0 \\ v_t + a^2 u_x &= \frac{1}{\epsilon}(f(u) - v)) \end{aligned}$$



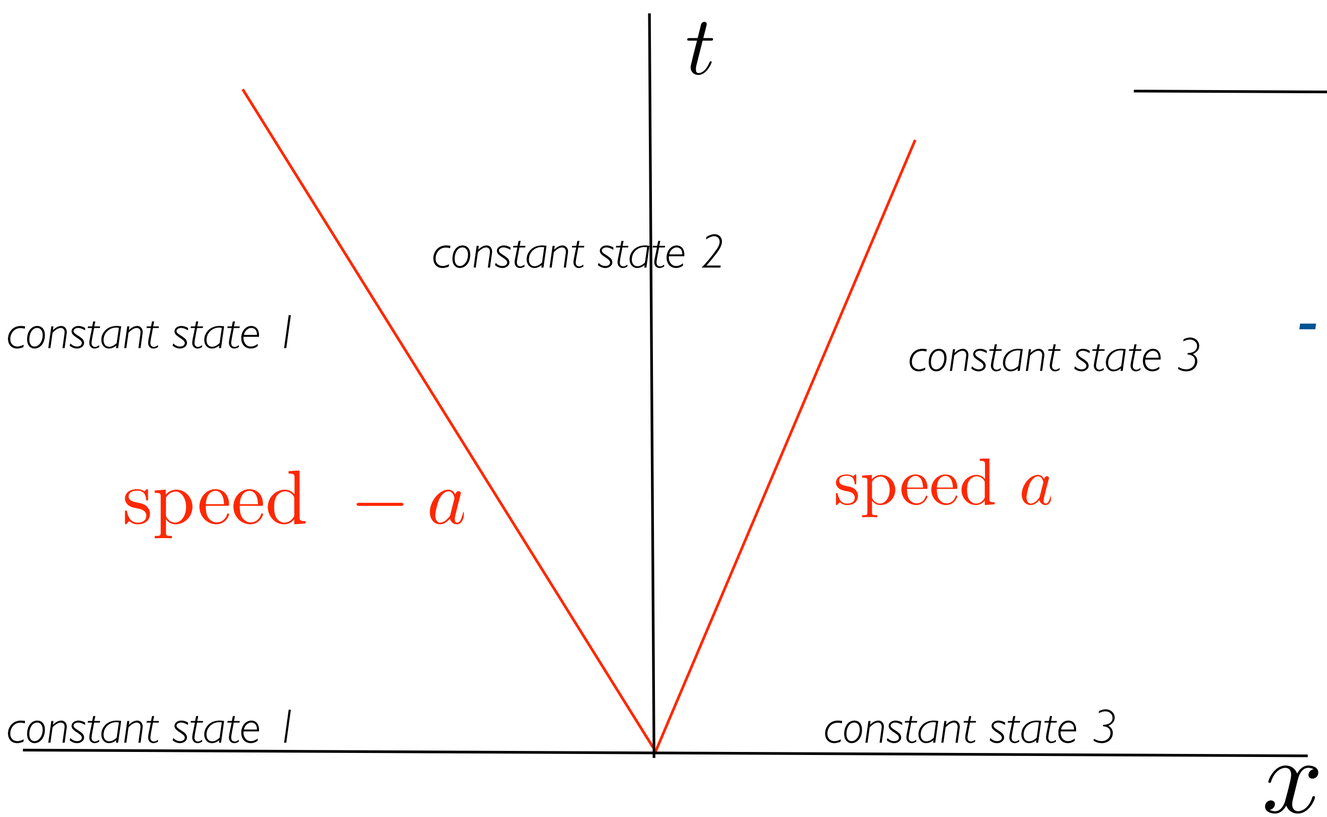
subcharacteristic condition:

$$-a < f'(u) < a$$

Goal:
find approximation to
Riemann problem of $u_t + f(u)_x = 0$



approximate the exact solution to the Riemann problem -



- by the Riemann solution of the relaxation system

The Riemann solution to the relaxation system is easy to find because it is linearly degenerate.

In addition it is possible to determine the speeds of the approximate Riemann solver quite accurately while still maintaining the subcharacteristic condition.

Brief history of approximate Riemann solvers

In the

1. **Reconstruct**
2. **Evolve**
3. **Average**

Algorithm

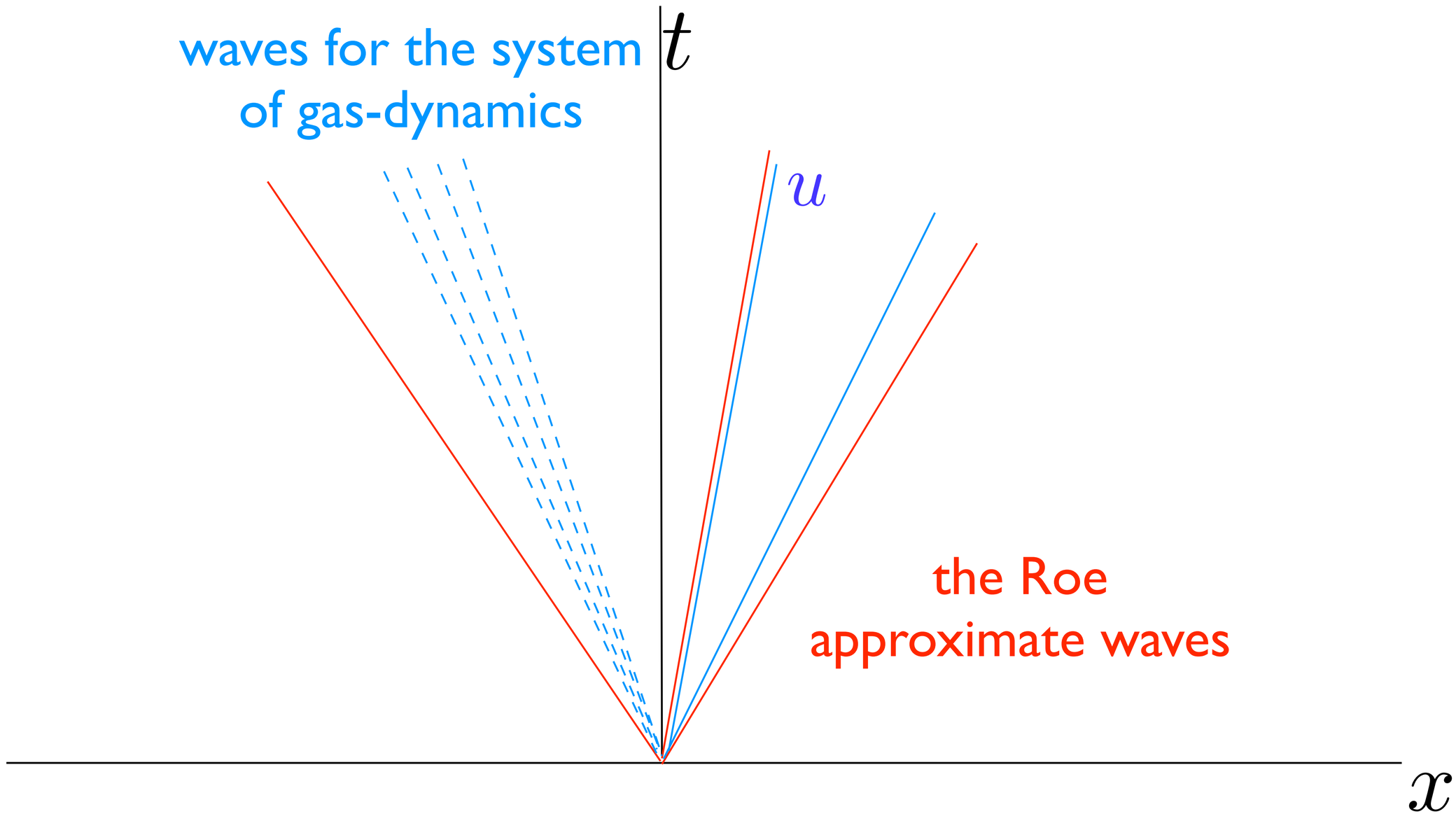
Phil Roe in 1981 noticed that it is not necessary to do the evolution step (2.)(the Riemann solution) exact, because we loose quite a bit of information in the averaging step (3.).

He thus suggested to introduced an

approximate Riemann solver.

He introduced a local linearization of the flux which is consistent and conservative.

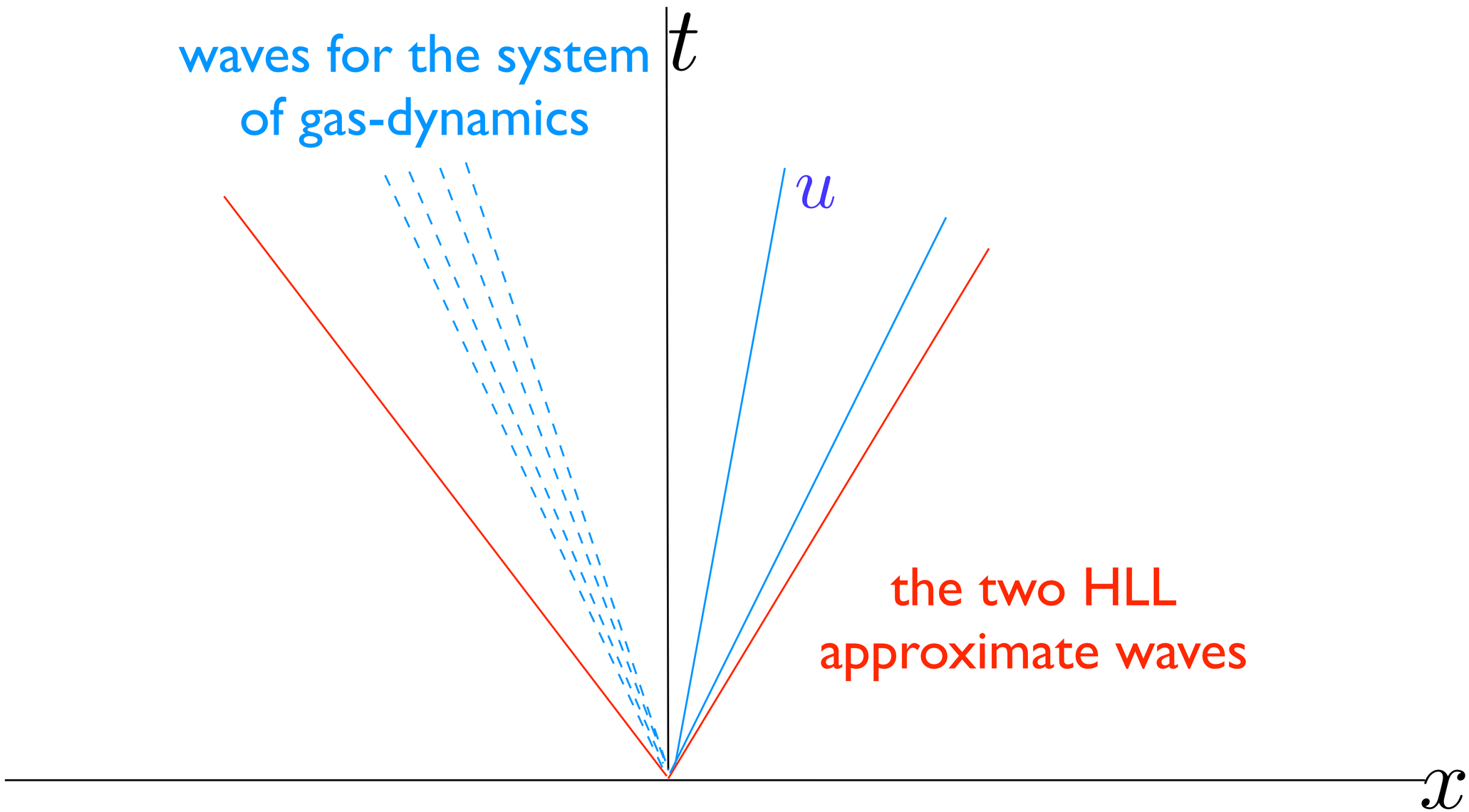
For the Euler equations Roe's approximate Riemann solver consists of three constant states separated by jumps.



advantage: can be made quite accurate

disadvantage: poor stability

Harten, Lax, van Leer 1983: even simpler approximate Riemann solver with only two waves, called the "HLL" solver.

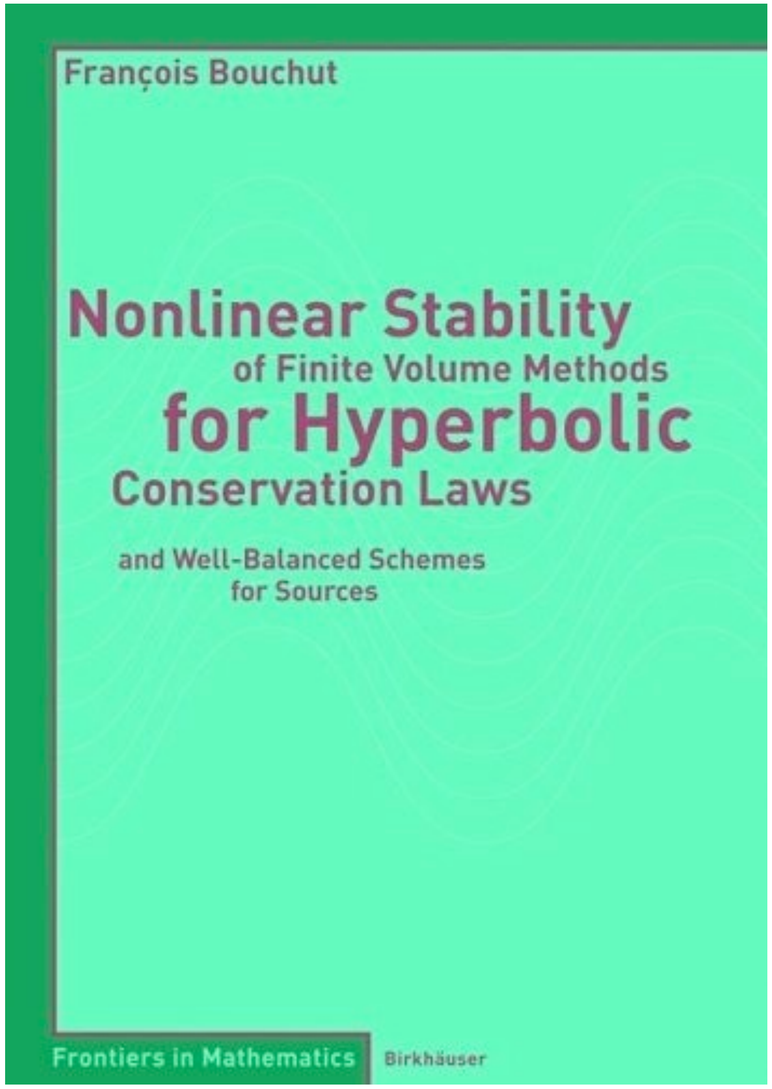


advantage: good stability, entropy consistent

disadvantage: poor accuracy

Toro et. al. (1994) for gas dynamics improved the HLL solver by introducing a middle wave, the "HLLC" solver.

Siliciu (1990), Tzavaras (1999) Coquel (1999), Coquel & Kl. (1999) and others noticed that the HLLC solver could be improved by a relaxation approach.



This opened the way for precise tools to analyze these schemes, see book by Bouchut (2004):

- relaxation solvers -

which are entropy consistent (stable), accurate and allow for rigorous analysis

In this spirit we embed system of compressible gas dynamics into a more "complete model".

For smooth solutions of the Euler equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x &= 0 \\ E_t + \big(u(E+p)\big)_x &= 0 \end{aligned}$$

we can write an evolution equation for the pressure:

$$(\rho p)_t + (\rho u p)_x + \rho^2 p'(\rho) u_x = 0$$

Replace p by a new dependant variable π and let c replace the soundspeed $\rho\sqrt{p'(\rho)}$

$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p-\pi}{\epsilon}$$

Siliciu (1990), Coquel, et.al. (1999)

the enlarged system has a small parameter $\epsilon > 0$ s.th.

$\epsilon > 0$ enlarged system

$\epsilon = 0$ original system

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + \pi)_x = 0$$

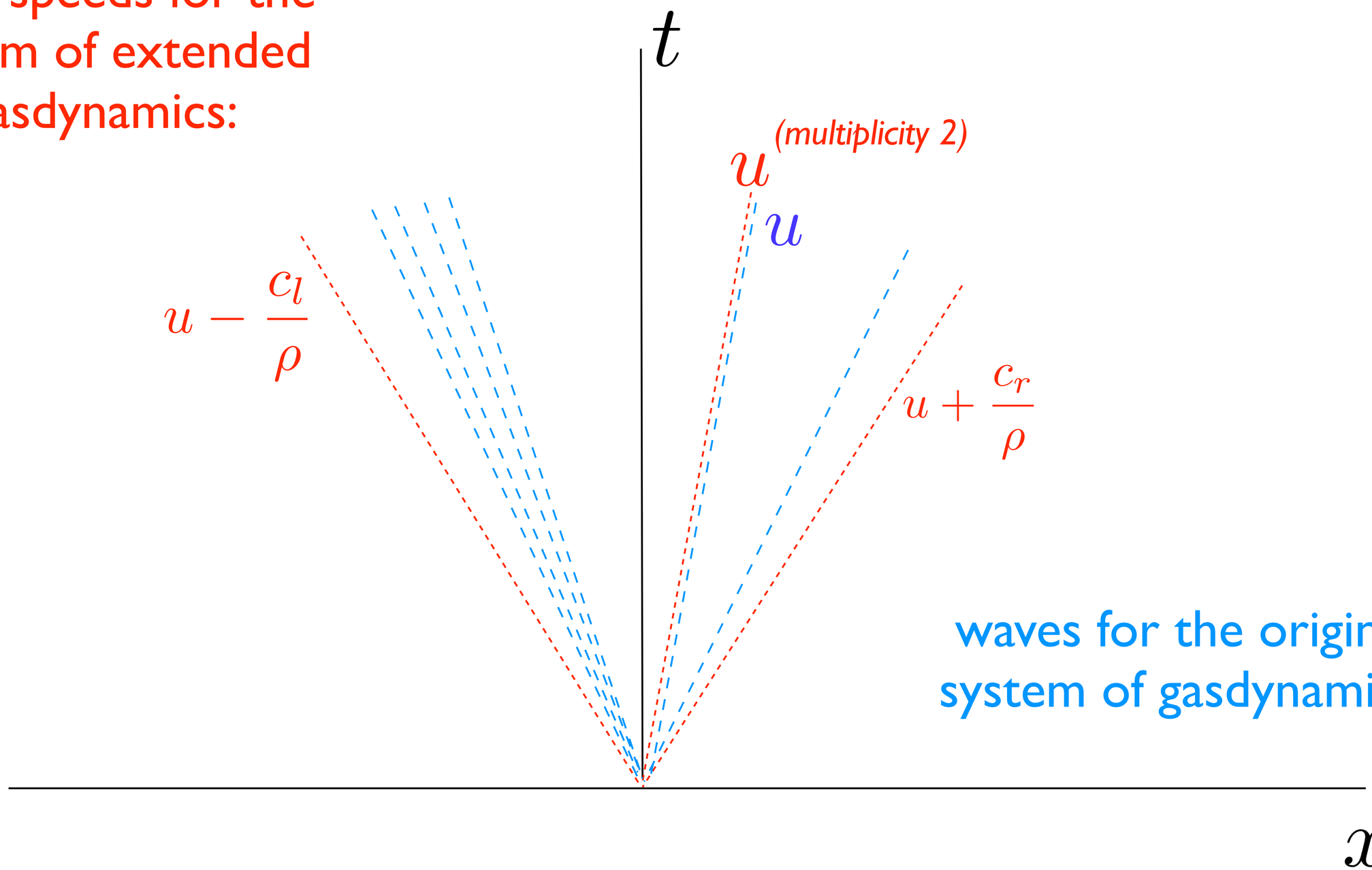
$$E_t + [(E + \pi)u]_x = 0$$

$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p - \pi}{\epsilon}$$

The constant c replaces the sound speed, which is a nonlinear function.

The advantage of the extended system is that by making the pressure a new dependent variable it easy to solve the Riemann problem for the homogeneous part of the extended system (all eigenvalues are degenerate).

wave speeds for the system of extended gasdynamics:



waves for the original system of gasdynamics:

Absolutely essential is the choice of the constant c (replacing the sound speed).

$$c > \rho\sqrt{p'(\rho)}$$

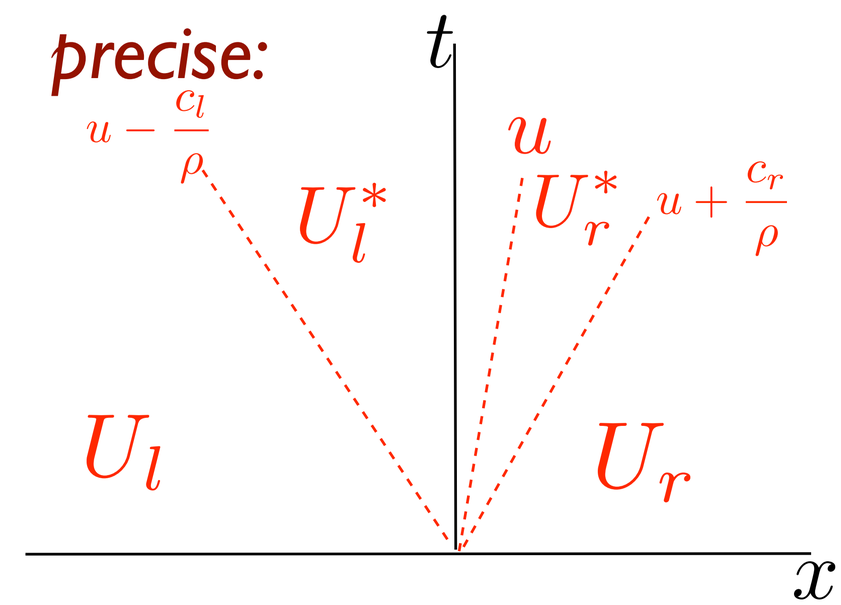
"subcharacteristic condition"

The choice of c determines the "stability' of this relaxation.

It ensures an entropy inequality.

This is analyzed à la Chen, Levermore, Liu (1994) allowing for rigorous justification.

more precise:



$$\forall \rho \in [\rho_l, \rho_l^*], \quad \rho^2 \left(\frac{\partial p}{\partial \rho}\right)_s \leq c_l^2$$

$$\forall \rho \in [\rho_r, \rho_r^*], \quad \rho^2 \left(\frac{\partial p}{\partial \rho}\right)_s \leq c_r^2$$

For practical purposes, in order to devise a formula for a numerical scheme, one has to choose a particular value for c out of the possible values the inequality allows for.

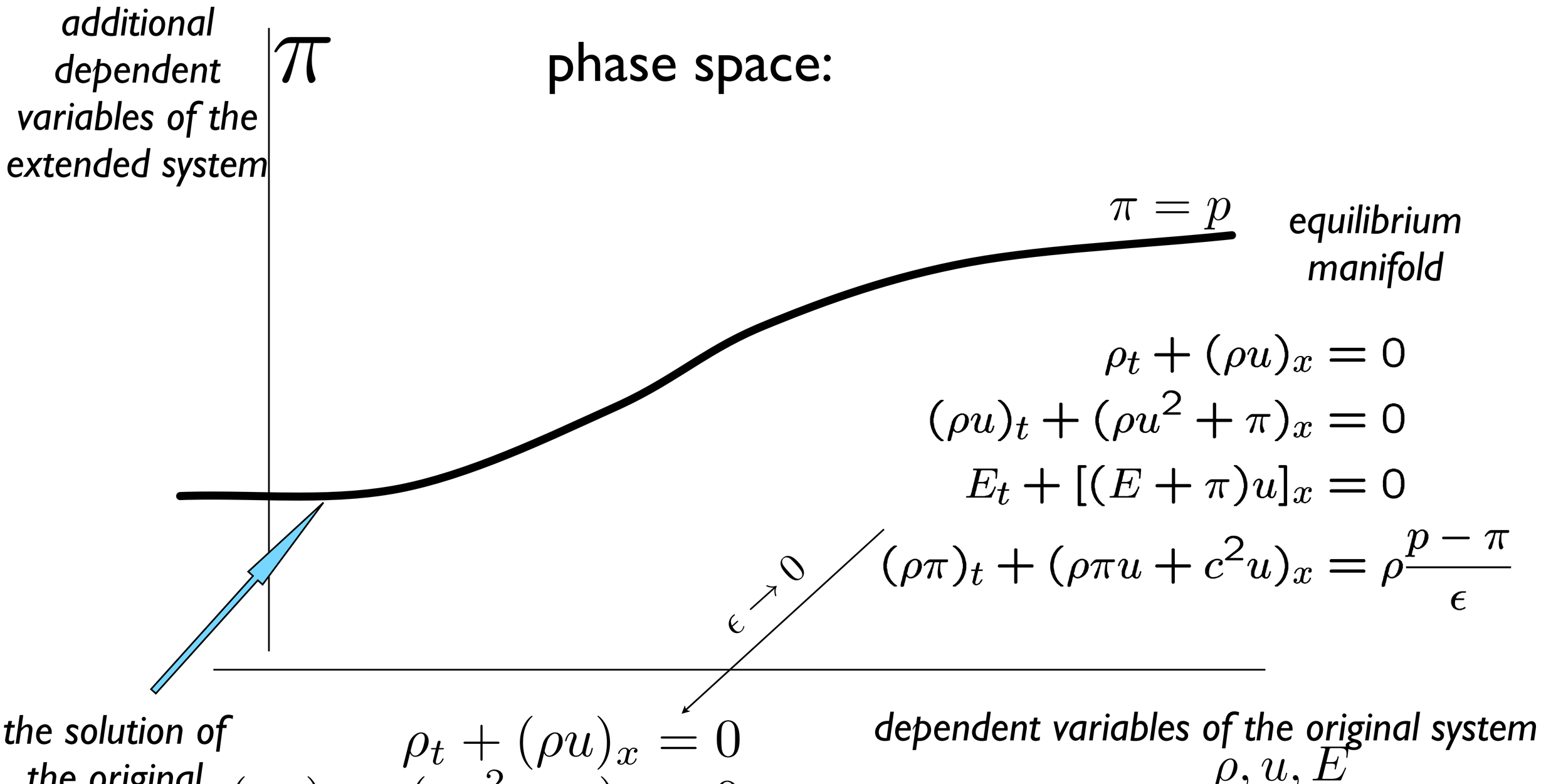
$$\text{if } p_r - p_l \geq 0, \quad \begin{cases} \dfrac{c_l}{\rho_l} = \sqrt{p'(\rho_l)} + \alpha \left(\dfrac{p_r - p_l}{\rho_r \sqrt{p'(\rho_r)}} + u_l - u_r \right)_+ , \\ \dfrac{c_r}{\rho_r} = \sqrt{p'(\rho_r)} + \alpha \left(\dfrac{p_l - p_r}{c_l} + u_l - u_r \right)_+ , \end{cases}$$

$$\text{if } p_r - p_l \leq 0, \quad \begin{cases} \dfrac{c_r}{\rho_r} = \sqrt{p'(\rho_r)} + \alpha \left(\dfrac{p_l - p_r}{\rho_l \sqrt{p'(\rho_l)}} + u_l - u_r \right)_+ , \\ \dfrac{c_l}{\rho_l} = \sqrt{p'(\rho_l)} + \alpha \left(\dfrac{p_r - p_l}{c_r} + u_l - u_r \right)_+ . \end{cases}$$

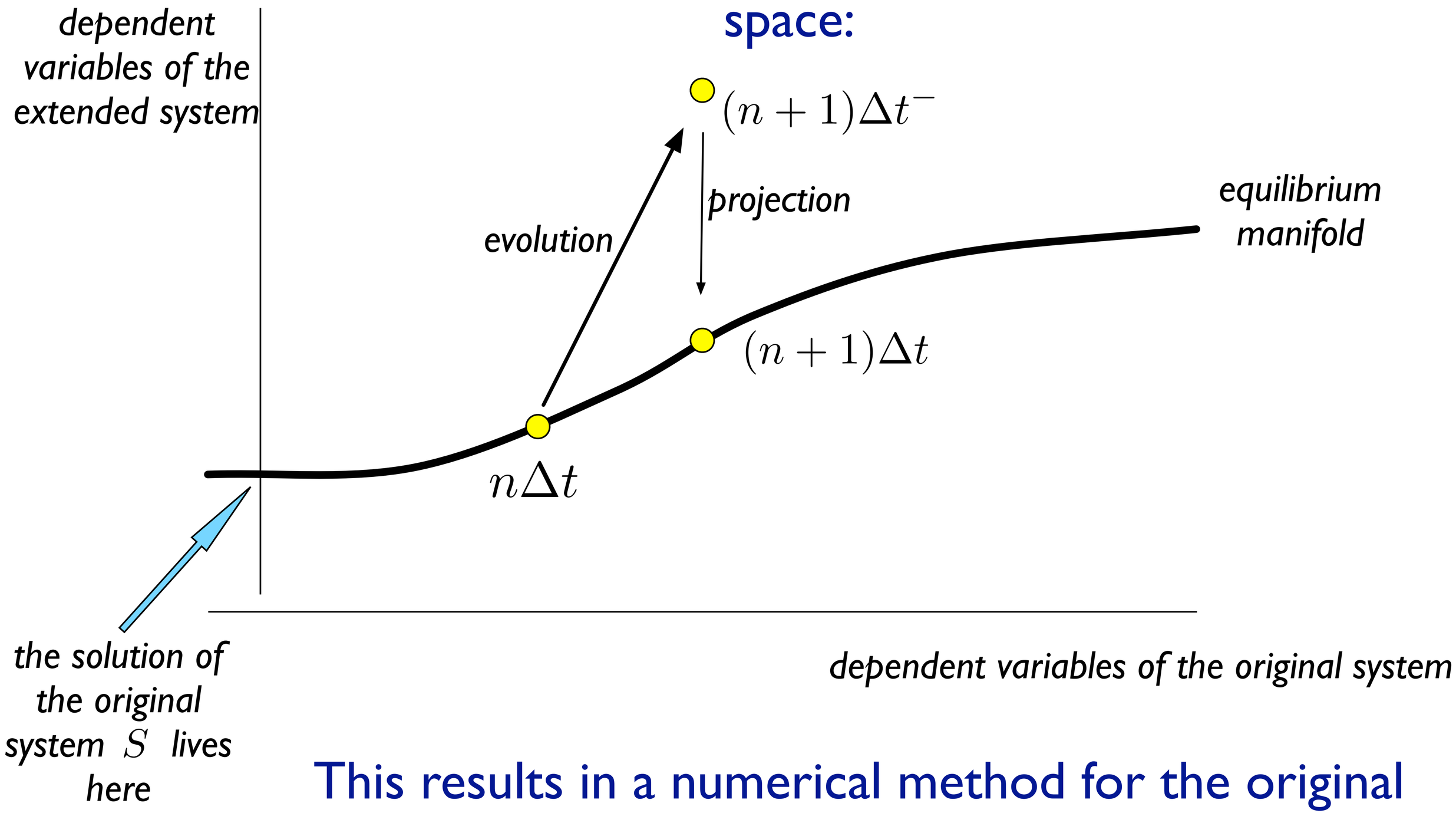
Bouchut (2004)

This ensures the optimal properties of this approximate Riemann solver.

Illustrate relaxation solver in phase space



Numerical procedure in phase space:



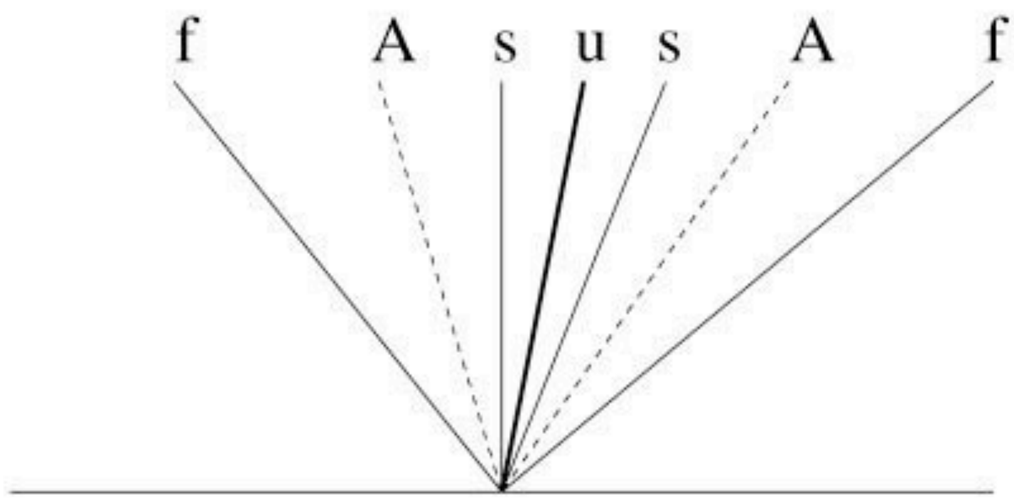
This results in a numerical method for the original system.

A relaxation solver for magnetohydrodynamics

Bouchut, Klingenberg, Waagan: *A multi-wave approximate Riemann solver for ideal MHD based on relaxation I - theoretical framework*, Numerische Mathematik (2008)

Waves in one-dimensional MHD

u	entropy waves — contact discontinuities
$u \pm c_s$	slow magnetosonic waves
$u \pm c_A$	Alfvén waves
$u \pm c_f$	fast magnetosonic waves



Designing the relaxation solver for MHD

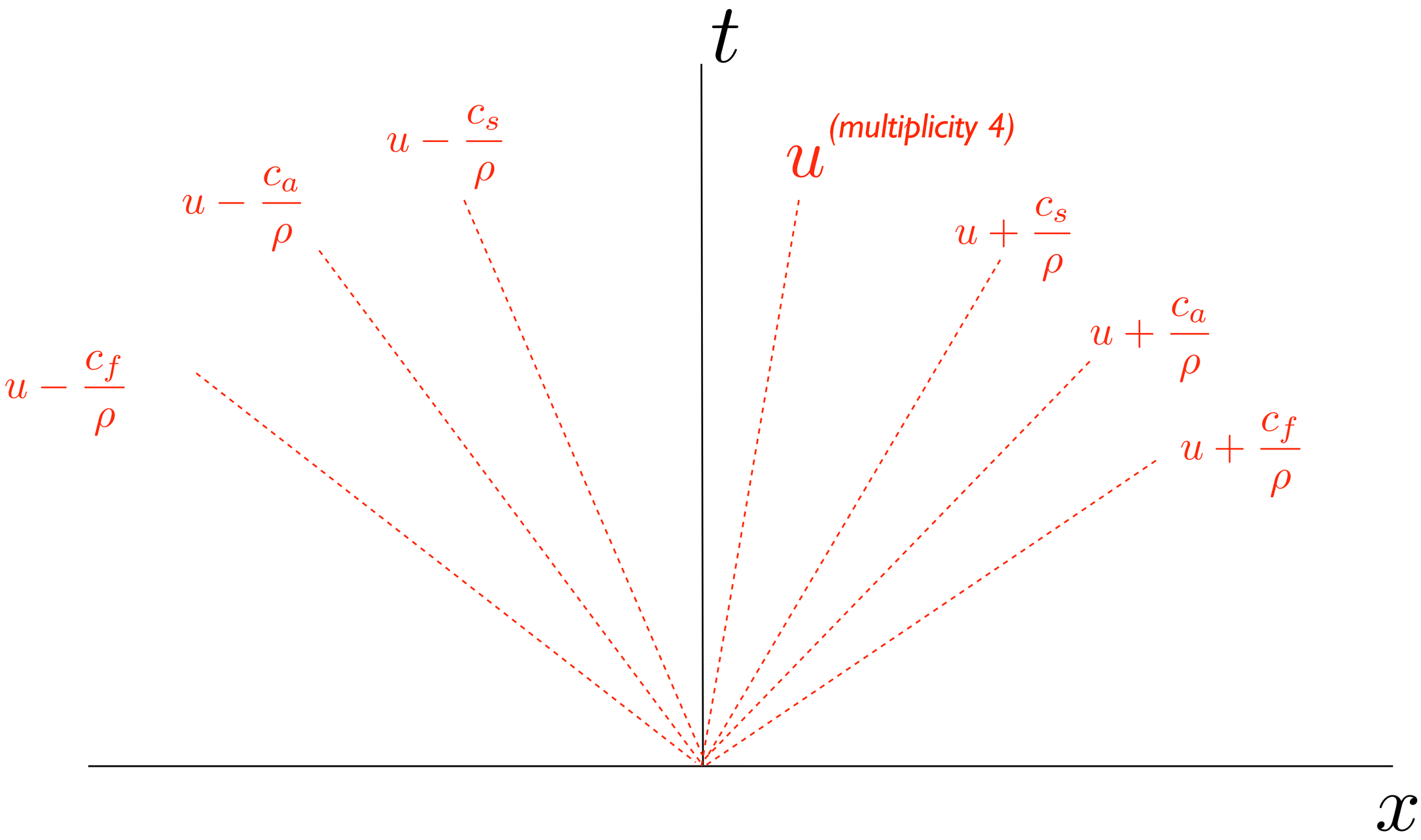
The extended system for MHD:

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + \pi)_x &= 0 \\
(\rho u_\perp)_t + (\rho u v + \pi_\perp)_x &= 0 \\
E_t + [(E + \pi)u + \pi_\perp \cdot u_\perp]_x &= 0 \\
(B_\perp)_t + (B_\perp u - B_x u_\perp)_x &= 0 \\
(\rho\pi)_t + [\rho\pi u + (c_s^2 + c_f^2 - c_a^2)u - c_a b \cdot u_\perp]_x &= \rho \frac{p + \frac{1}{2}B_\perp^2 - \frac{1}{2}B_x^2 - \pi}{\epsilon} \\
(\rho\pi_\perp)_t + (\rho\pi_\perp u + c_a^2 u - c_a b u)_x &= \rho \frac{-B_x B_\perp - \pi_\perp}{\epsilon}
\end{aligned}
$$

$$\pi = p + |B|^2/2 - B_x^2$$

with: $\pi_\perp = -B_x B_\perp$

wave speeds for the system of extended magnetohydrodynamics:



A three wave approximate Riemann solver is obtained by:

$$\text{Set } c_s = c_a = c_f$$

Theorem

The approximate Riemann solver defined by this 3-wave relaxation is positive and defines a discrete entropy inequality if for all intermediate states we have:

$$\frac{1}{\rho_2} - \frac{B_x^2}{c_a^2} \geq 0$$

$$\left| \frac{B_\perp^1 + B_\perp^2}{2} - \frac{B_x b}{c_a} \right|^2 \leq \left(\frac{c_s^2 c_f^2}{c_a^2} - (\rho^2 p')_{1,2} \right) \left(\frac{1}{\rho_2} - \frac{B_x^2}{c_a^2} \right)$$

The proof of the discrete entropy inequality

$$\rho_i^{n+1}\phi(s(\rho_i^{n+1}, e_i^{n+1})) - \rho_i^n\phi(s(\rho_i^n, e_i^n)) + \frac{\Delta t}{h}\left(G^s_{i+\frac{1}{2}} - G^s_{i-\frac{1}{2}}\right) \leq 0$$

is given in Bouchut, Kl., Waagan (2008).

A formal derivation of this for smooth solutions is available by a Chapman-Enscog expansion.

Write $\pi = p + \frac{1}{2}B_\perp^2 - \frac{1}{2}B_x^2 + g(\epsilon) + O(\epsilon^2)$ $\qquad \pi_\perp = -B_x B_x + + g_\perp \epsilon + O(\epsilon^2)$

Insert this into the extended system

$$\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + \pi)_x &= 0 \\
(\rho u_\perp)_t + (\rho u v + \pi_\perp)_x &= 0 \\
E_t + [(E+\pi)u + \pi_\perp \cdot u_\perp]_x &= 0 \\
(B_\perp)_t + (B_\perp u - B_x u_\perp)_x &= 0 \\
(\rho\pi)_t + [\rho\pi u + (c_s^2 + c_f^2 - c_a^2)u - c_a b \cdot u_\perp]_x &= \rho\frac{p + \frac{1}{2}B_\perp^2 - \frac{1}{2}B_x^2 - \pi}{\epsilon} \\
(\rho\pi_\perp)_t + (\rho\pi_\perp u + c_a^2 u - c_a b u)_x &= \rho\frac{-B_x B_\perp - \pi_\perp}{\epsilon}
\end{aligned}$$

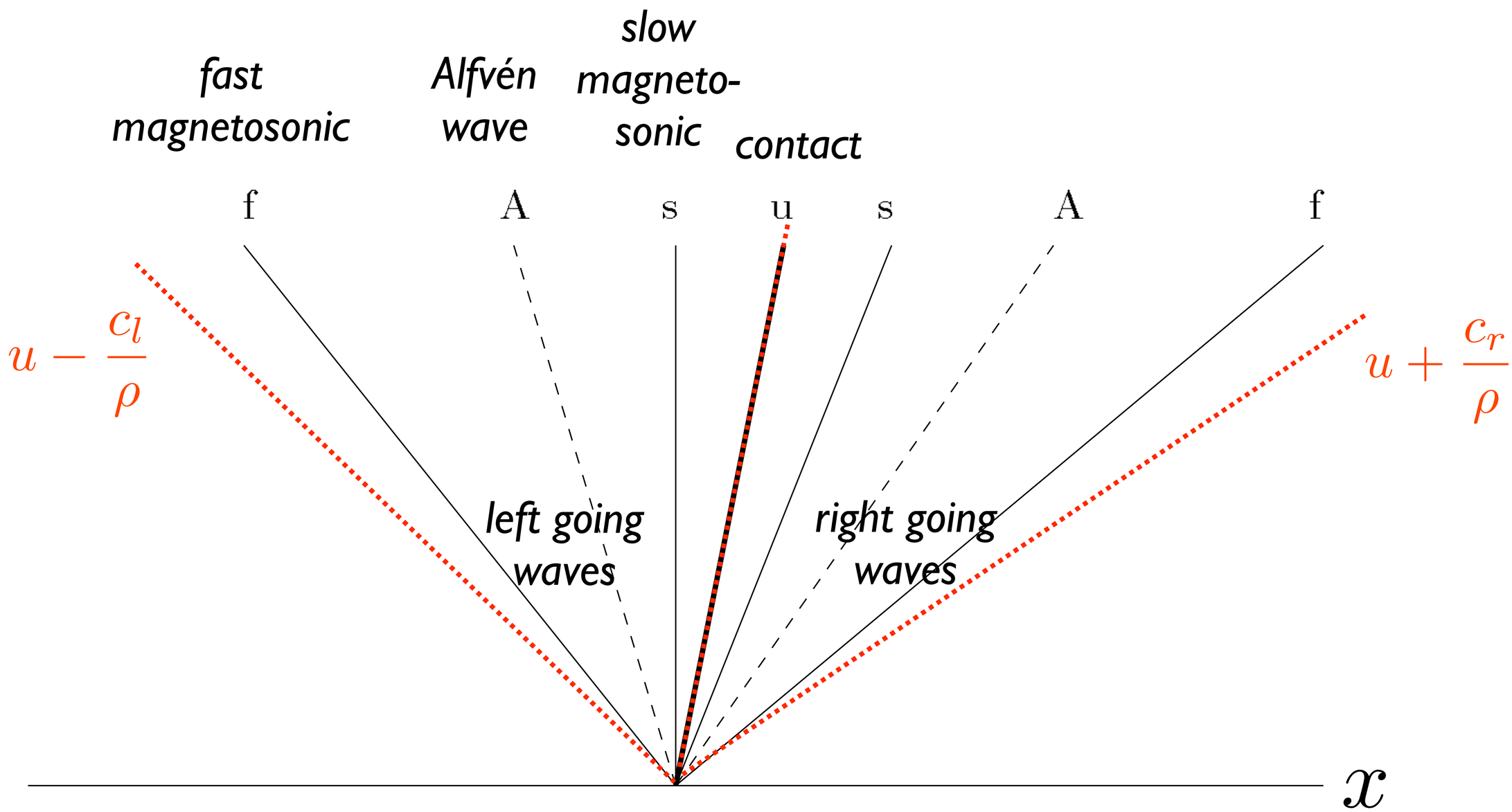
This gives

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + \pi)_x &= \epsilon \left[\left(\frac{c_s^2 + c_f^2 - c_a^2}{\rho} - (\rho p' + B_\perp^2) \right) u_x + (B_x B_\perp - \frac{B_x b}{c_a})(u_\perp)_x \right]_x + O(\epsilon^2) \\
(\rho u_\perp)_t + (\rho u v + \pi_\perp)_x &= \epsilon \left[(B_x B_\perp - \frac{B_x b}{c_a}) u_x + (\frac{c_a^2}{\rho} - B_x^2)(u_\perp)_x \right]_x + O(\epsilon^2) \\
E_t + [(E + \pi)u + \pi_\perp \cdot u_\perp]_x &= \epsilon \Bigg[u \left(\frac{c_s^2 + c_f^2 - c_a^2}{\rho} - (\rho p' + B_\perp^2) \right) u_x + u(B_x B_\perp - \frac{B_x b}{c_a}) \cdot (u_n)_x \\
&\qquad + u_\perp \cdot (B_x B_\perp - \frac{B_x b}{c_a}) u_x + u_\perp \cdot (\frac{c_a^2}{\rho} - B_x^2)(u_\perp)_x \Bigg]_x + O(\epsilon^2) \\
(B_\perp)_t + (B_\perp u - B_x u_\perp)_x &= 0
\end{aligned}
$$

The entropy is evolved by an equation of the type

$$
\eta(U)_t + G(U)_x - \epsilon[\eta'(U) D(U) U_x]_x = -\epsilon D(U)^t \eta''(U) U_x \cdot U_x
$$

The conditions of the theorem then ensure entropy dissipation.



the three wave solver superimposed onto the exact 8-wave solution

When devising a numerical scheme we need to get concrete speeds of the waves out of the inequality in the theorem.

Theorem:

For the three wave solver the following relaxation speeds are sufficient to guarantee positivity and entropy stability:

$$c_l = \rho_l a_l^0 + \alpha \rho_l \left((u_l - u_r)_+ + \frac{(\pi_r - \pi_l)_+}{\rho_l \sqrt{p_l'} + \rho_r a_{qr}} \right)$$

$$c_r = \rho_r a_r^0 + \alpha \rho_r \left((u_l - u_r)_+ + \frac{(\pi_l - \pi_r)_+}{\rho_r \sqrt{p_r'} + \rho_l a_{ql}} \right)$$

where $\alpha = \frac{\gamma + 1}{2}$ and α_l^0 α_r^0 are given by a complicated formula.

Bouchut, Klingenberg, Waagan: *A multiwave approximate Riemann solver for ideal MHD based on relaxation II - numerical aspects*, submitted (2009)

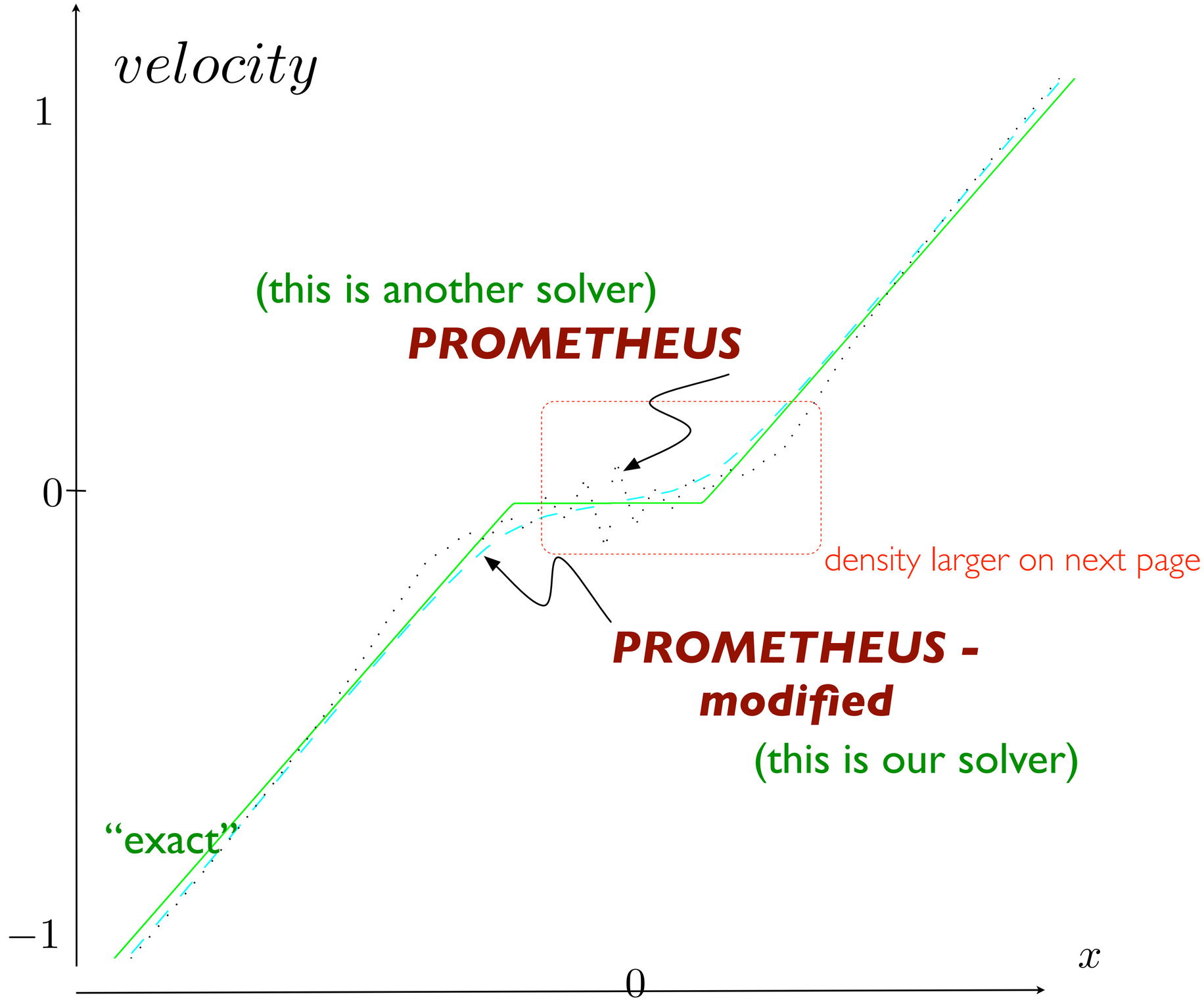
We have also found a five and seven wave approximate solver.

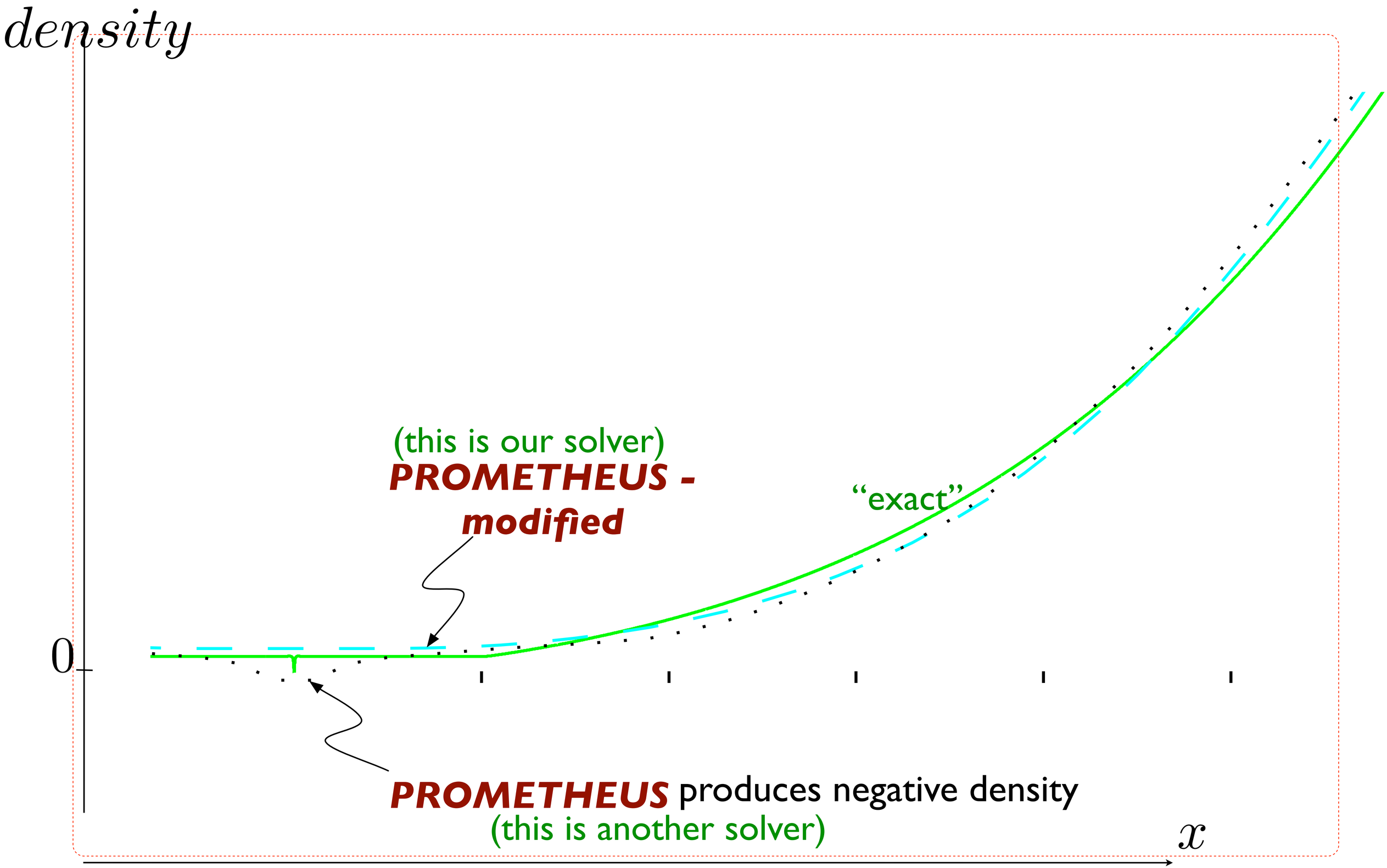
Again we can prove entropy consistency under some complicated "subcharacteristic" condition

We have explicit formulas for the speeds.

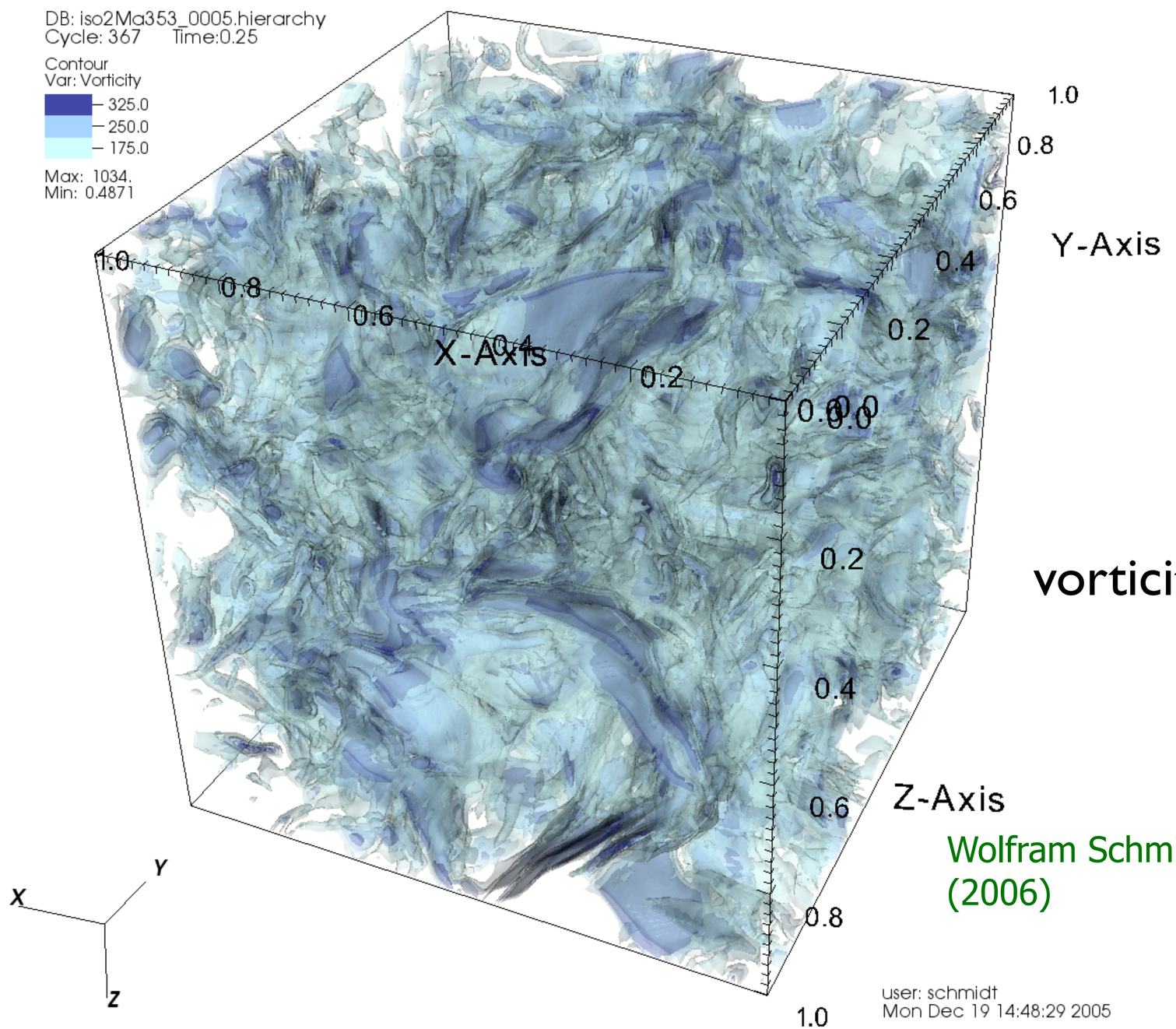
Showing how our Riemann solver compares to other solvers

This Riemann problem has two strong rarefaction waves going apart creating a low density region.



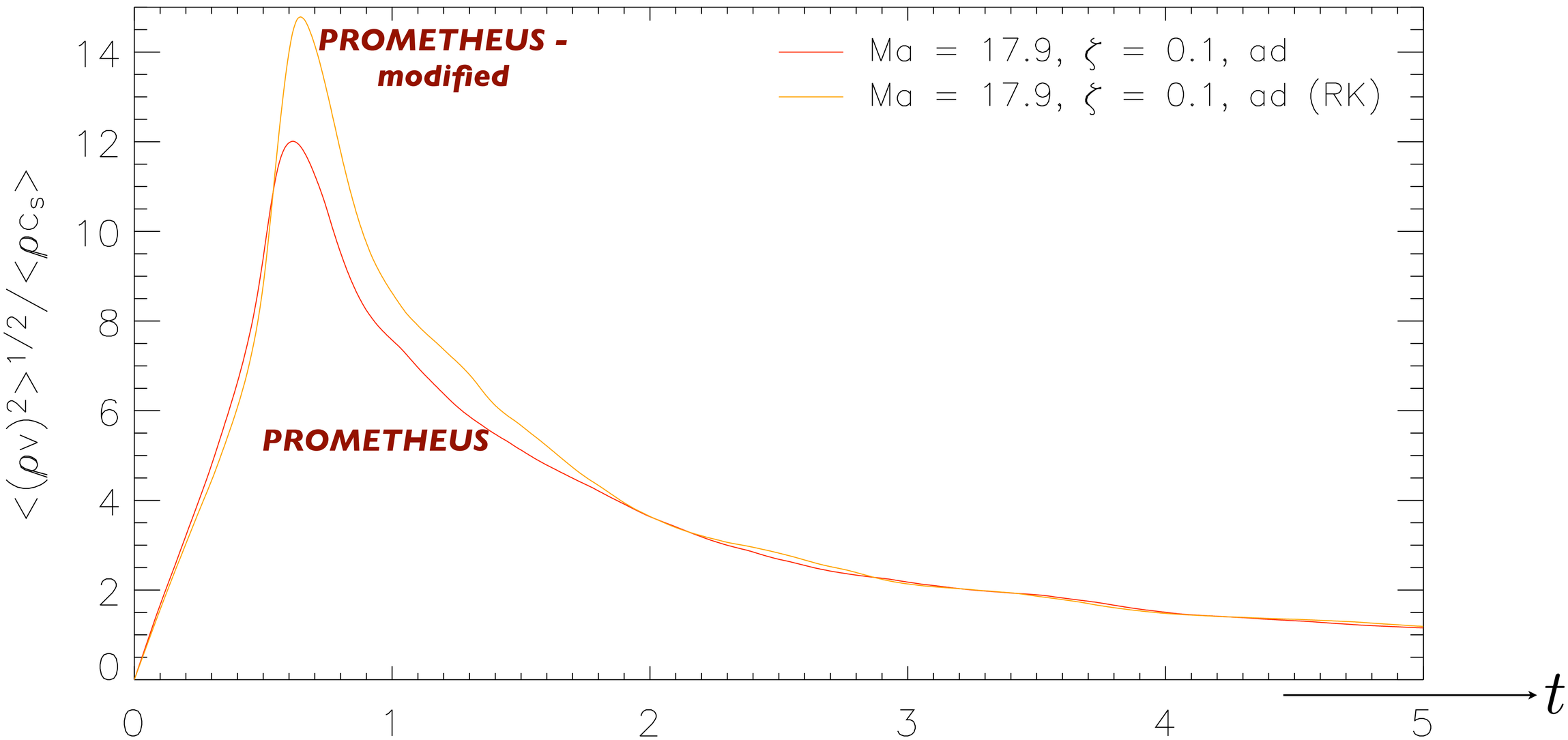

density
(this is our solver)
PROMETHEUS - modified
"exact"
0
PROMETHEUS produces negative density
(this is another solver)
x

three space dimensions, turbulence simulations:

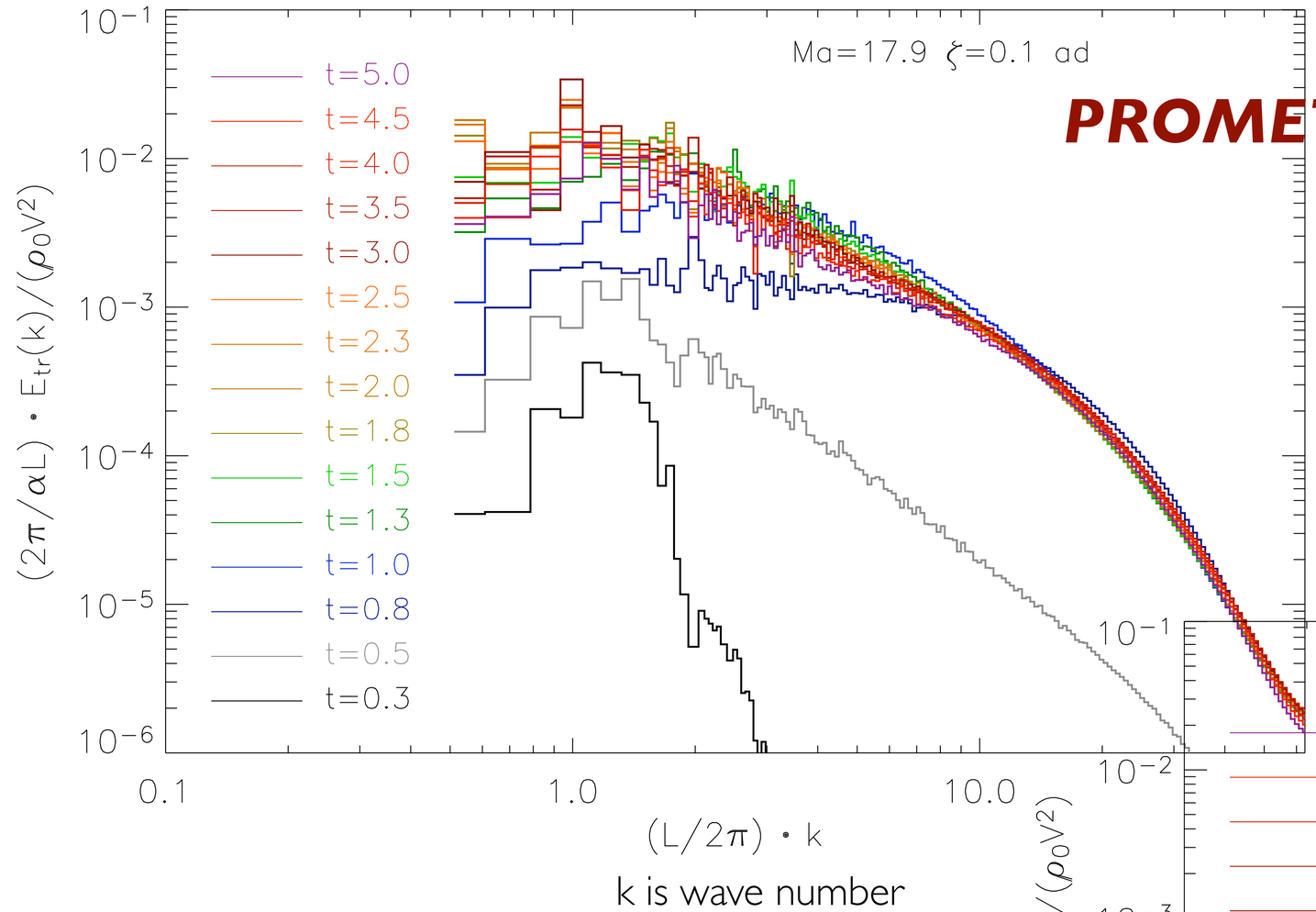


vorticity

Wolfram Schmidt, J. Niemeyer (2006)

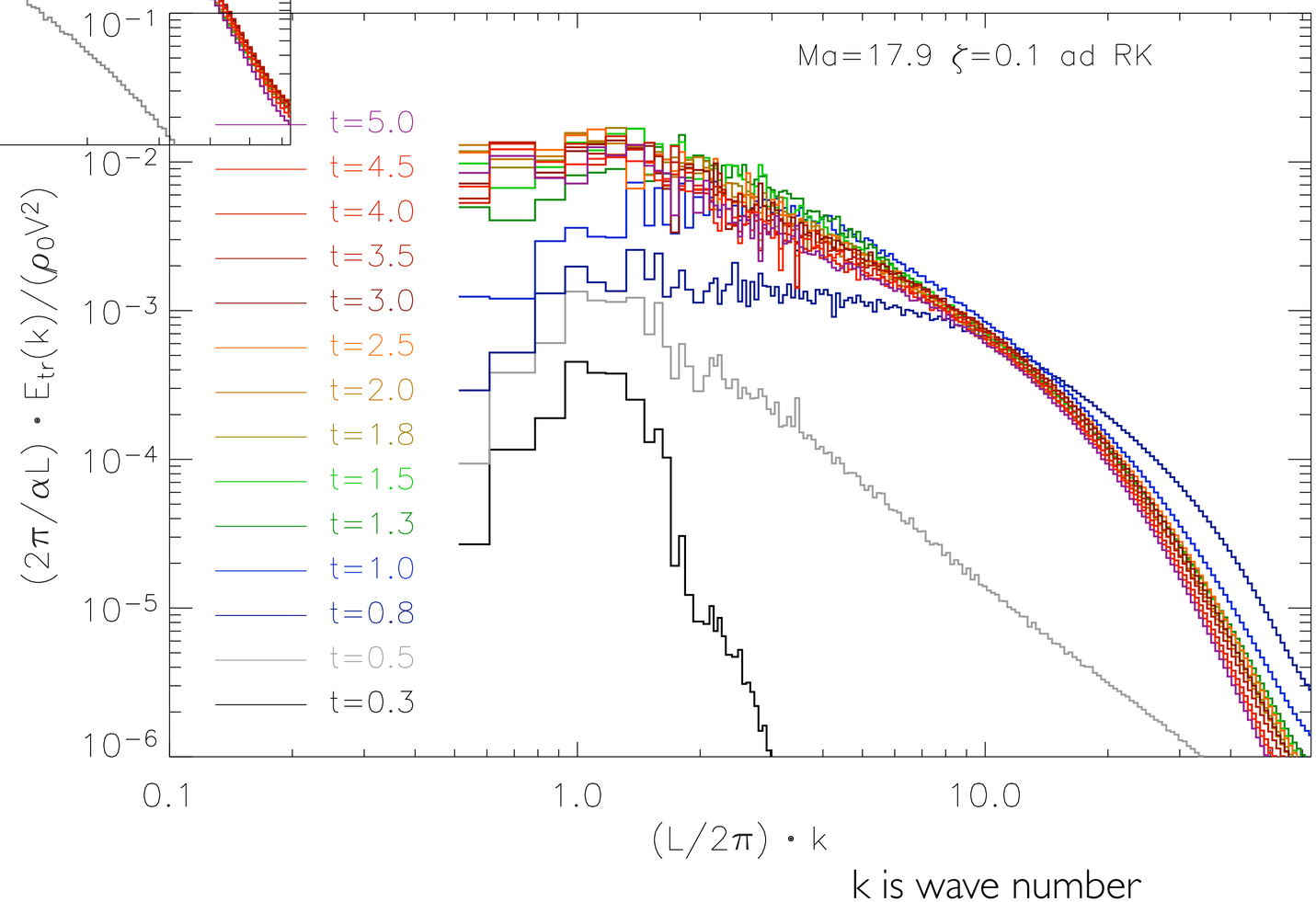


time evolution of root mean squared Mach number



PROMETHEUS

PROMETHEUS - modified



time histories of transversal energy spectra

conclusion:

dissipativity of ***PROMETHEUS*** is independent of Mach number

dissipativity of ***PROMETHEUS-modified*** is less for higher than for lower Mach numbers

The PPM method is widely used in the astrophysics community. Thus there was a question on how much their results depend on this algorithm

We conclude that PPM is accurate with respect to the Riemann solver.

PROMETHEUS-modified is at least 20% faster than ***PROMETHEUS***.