Spectral methods for homogeneous sheared turbulence

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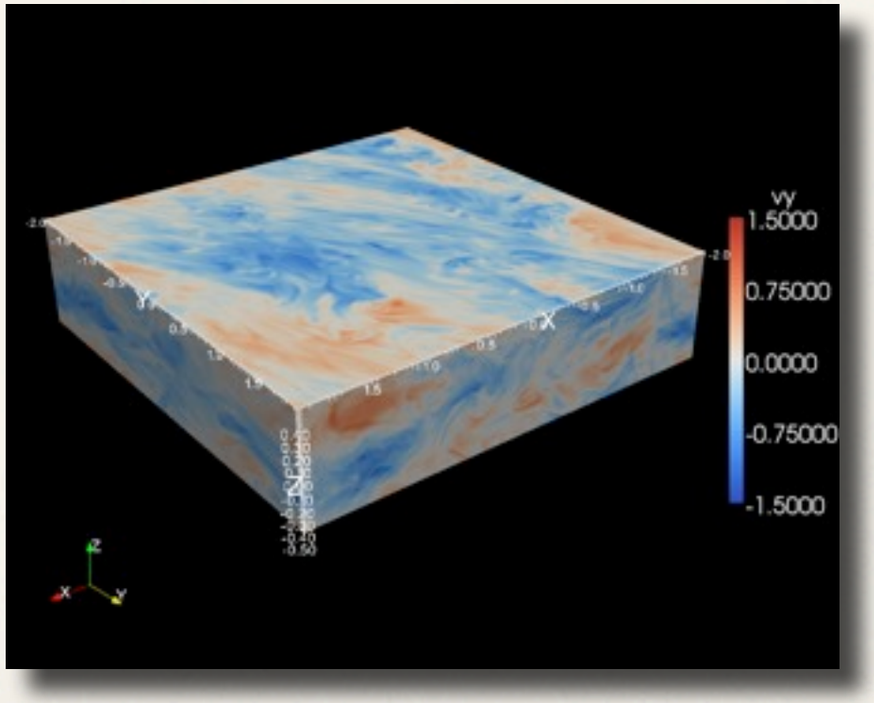
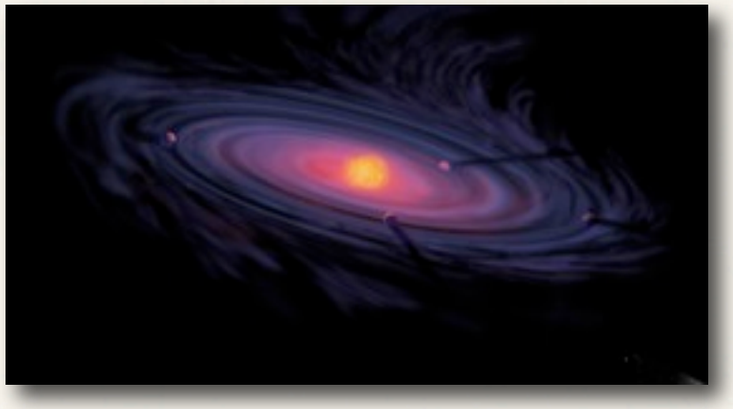


Outline

- Modeling homogeneous sheared flow
  - Sheared flows in astrophysics
  - Shearing box and sheared frames
  - Spectral representation
  - Parallel simulations
- Example: the magnetorotational instability in accretion discs
  - The accretion problem
  - The magnetorotational instability
  - Impact of non ideal MHD

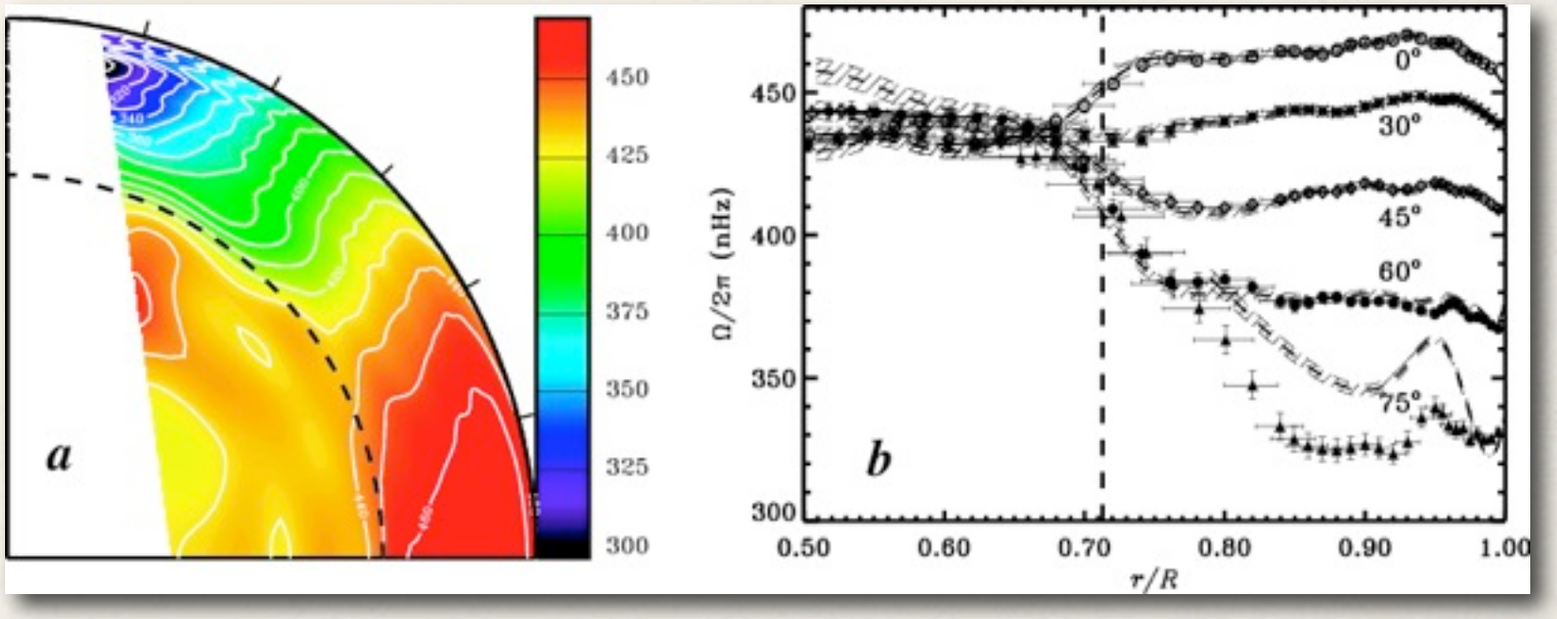
Sheared flow in astrophysics

- Accretion discs



- Protoplanetary discs

- Stellar interiors



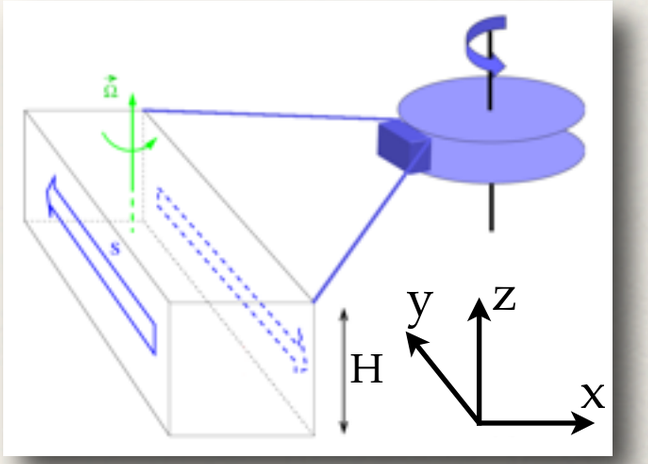
The solar tachocline
(Thompson *et al.* 2003)

- …

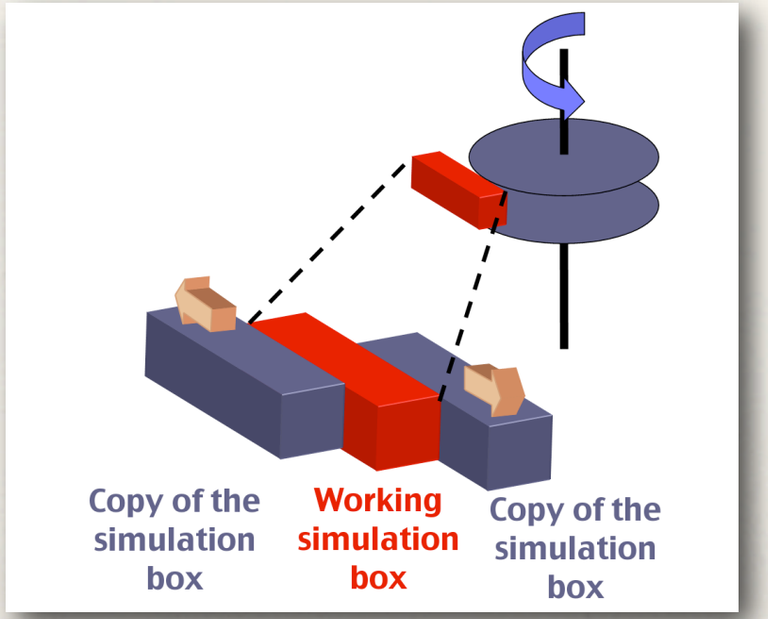
Modeling homogeneous sheared flows

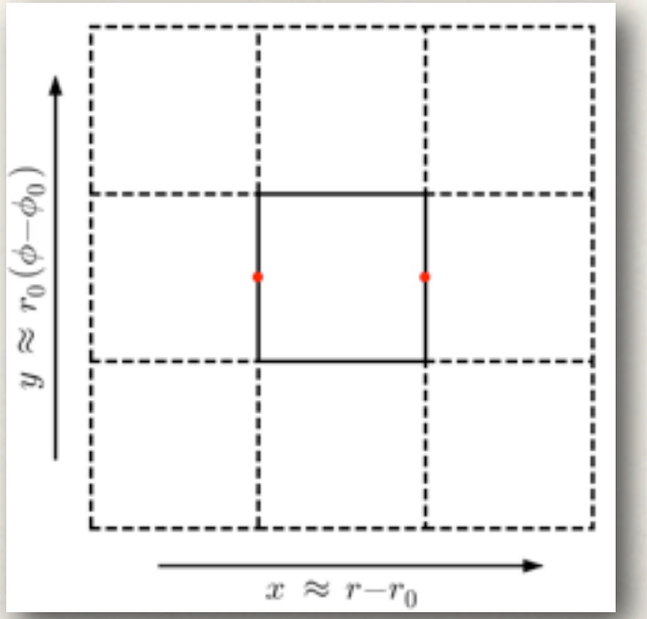
- Assume incompressible flow (accretion disc turbulence is subsonic)
- Neglect vertical stratification («small» shearing box approximation)
- Equations with a background sheared flow $\mathbf{U} = -Sx\mathbf{e_y}$

$$
\begin{aligned}
\partial_t \mathbf{v} - Sx\partial_y \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} &= -\nabla\psi + (S-2\Omega)v_x\mathbf{e_y} + 2\Omega v_y \mathbf{e_x} \\
&\quad + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\Delta\mathbf{v} \\
\partial_t \mathbf{B} - Sx\partial_y \mathbf{B} &= -SB_x\mathbf{e_y} + \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\Delta\mathbf{B} \\
\nabla\cdot\mathbf{v} &= 0 \\
\nabla\cdot\mathbf{B} &= 0
\end{aligned}
$$



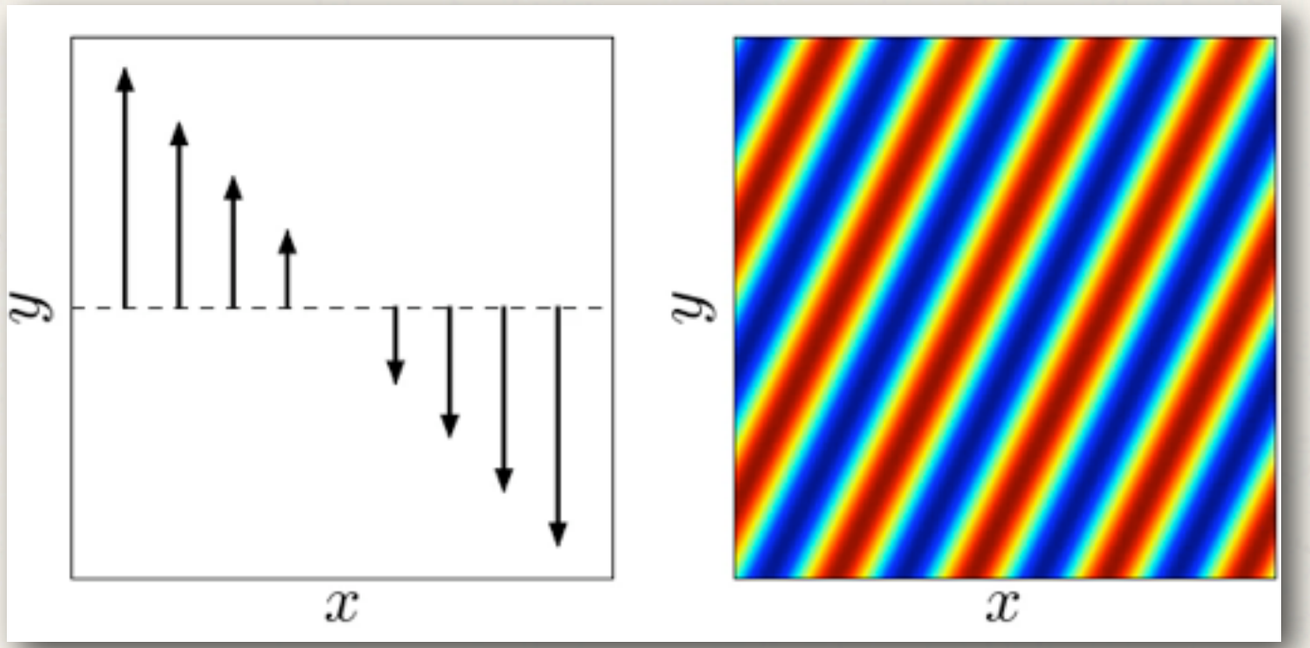
- Shearing-sheet boundary conditions (Goldreich & Lynden-Bell 1965 - Hawley *et al.* 1995)





Shearing waves and sheared coordinates

- Let's consider the evolution of a non axisymmetric «wave» ($k_y \neq 0$)



- The natural basis of the flow is made of «shearing waves»

$$\mathbf{Q}(x,y,z,t) = \sum_{\mathbf{k}} \widehat{\mathbf{Q}}_{\mathbf{k}}(t) \exp(i\mathbf{k}(t)\cdot\mathbf{x})$$

with

$$\mathbf{k}(t) = (k_x^0 + Stk_y)\mathbf{e_x} + k_y\mathbf{e_y} + k_z\mathbf{e_z}$$

- Equivalent to a Fourier basis in sheared coordinates (y'=y+Sxt)

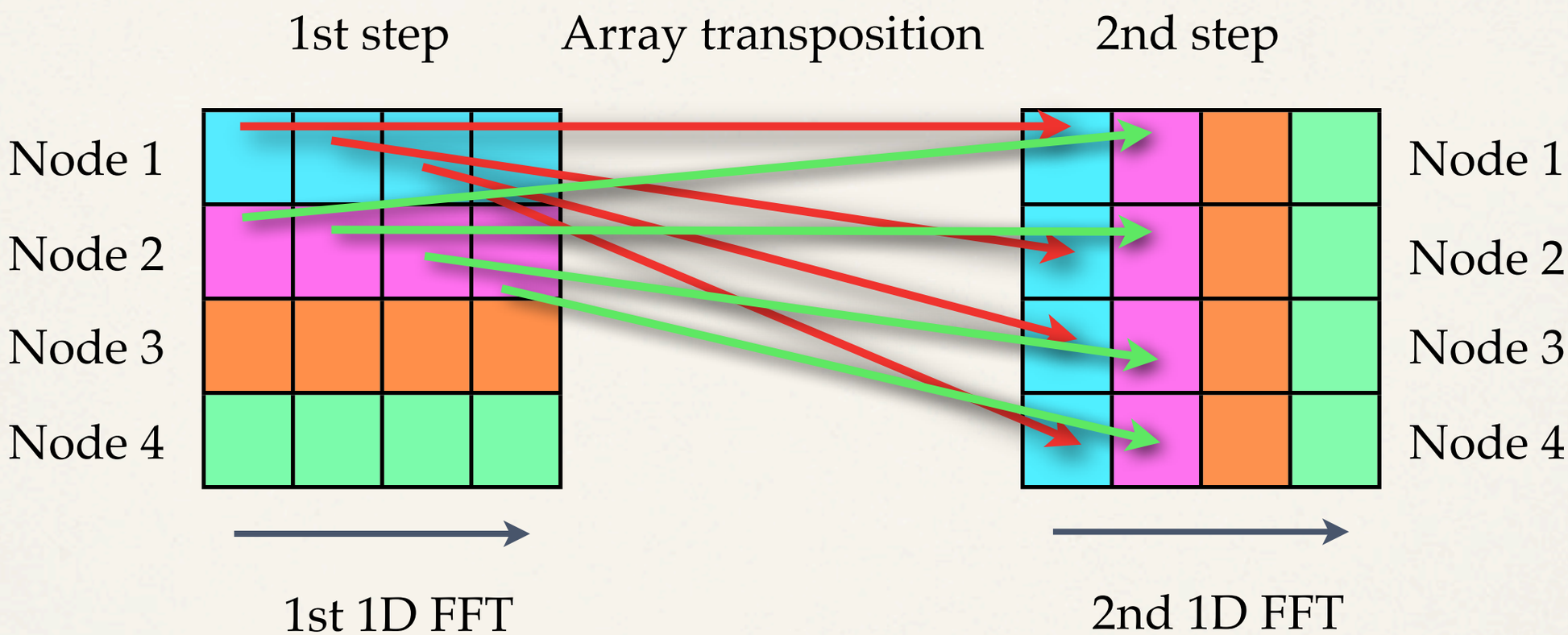
The Snoopy code

- MHD equations solved in the sheared frame
- Use the Fourier basis to compute derivatives: $\partial_{x'} \leftrightarrow ik_x(t)$ ...
- Compute non linear terms using a pseudo spectral representation
- 3rd order low storage Runge-Kutta integrator
- OpenMP and/or MPI parallelization
- Written in C

- Advantages:
  - Shearing waves are computed exactly (natural basis)
  - Exponential convergence when resolution is increased
  - Magnetic flux conserved to machine precision
  - Sheared frame & incompressible approximation: no CFL constrain due to the background sheared flow/sound speed.
  - Very weak numerical dissipation: tight control on physical dissipation processes
- Disadvantages:
  - Slower than finite differences for the same resolution (number of real grid points)
  - Shocks/discontinuities can't be treated spectrally (Gibbs oscillations)
  - Strongly parallel codes are not very efficient

Parallel Fourier Transforms

- Example: 2D FFT on a distributed memory cluster (e.g. using MPI)



- To compute 1 FFT, the complete array has to transit through the network.
- Typically, the transposition step represent 40% of the total computation time (Vargas, IDRIS with QDR Infiniband network)
- Parallelism allowed in only one direction (OpenMP can help to overcome this limitation)

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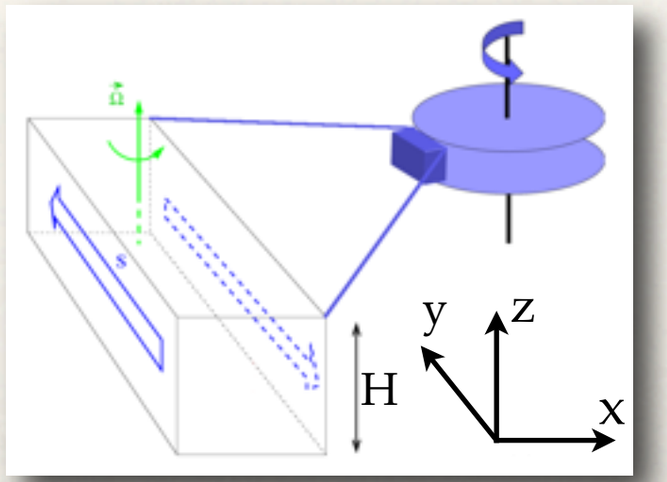
A problem of accretion...

- Observations, dynamics and lifetime of discs suggest they are turbulent.
- Turbulence generates an anomalous viscosity ν_t :

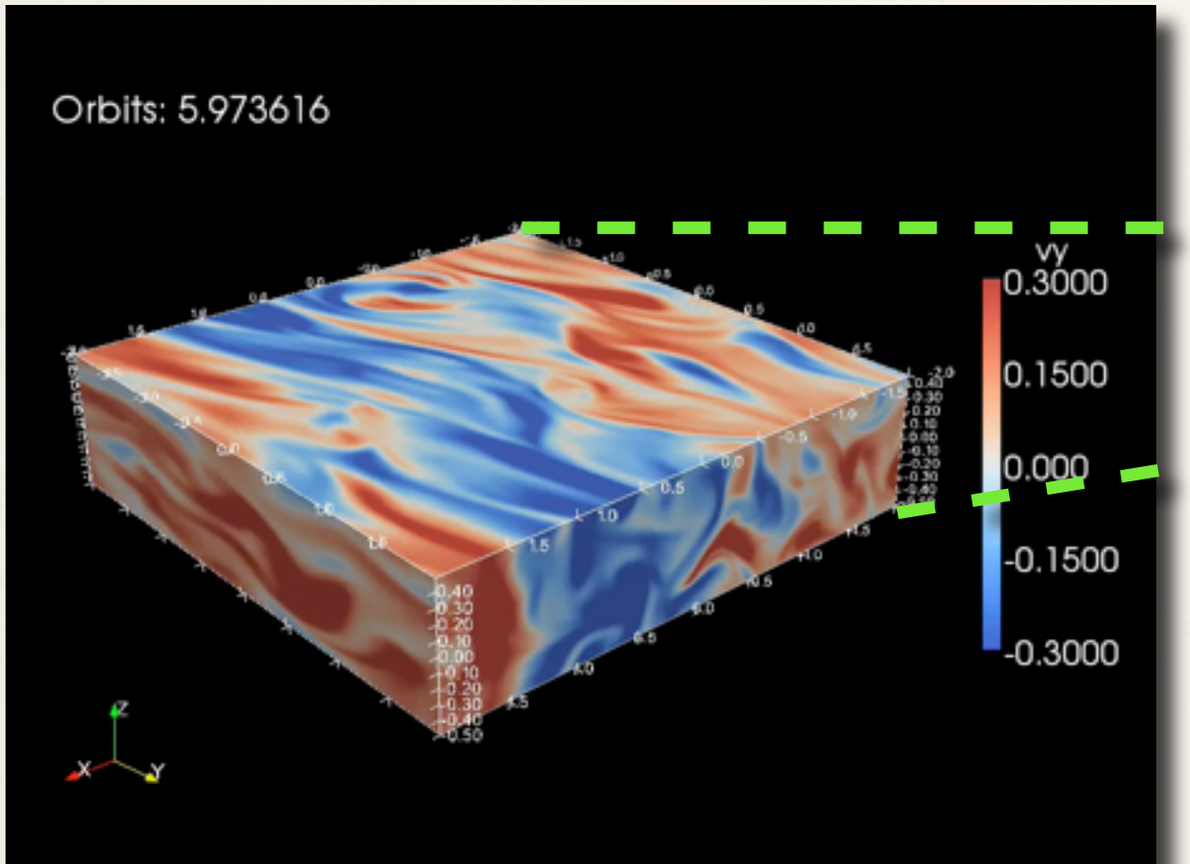
$$\nu_t = \alpha c_s H \qquad 10^{-3} < \alpha < 1 \quad \text{(observations)}$$

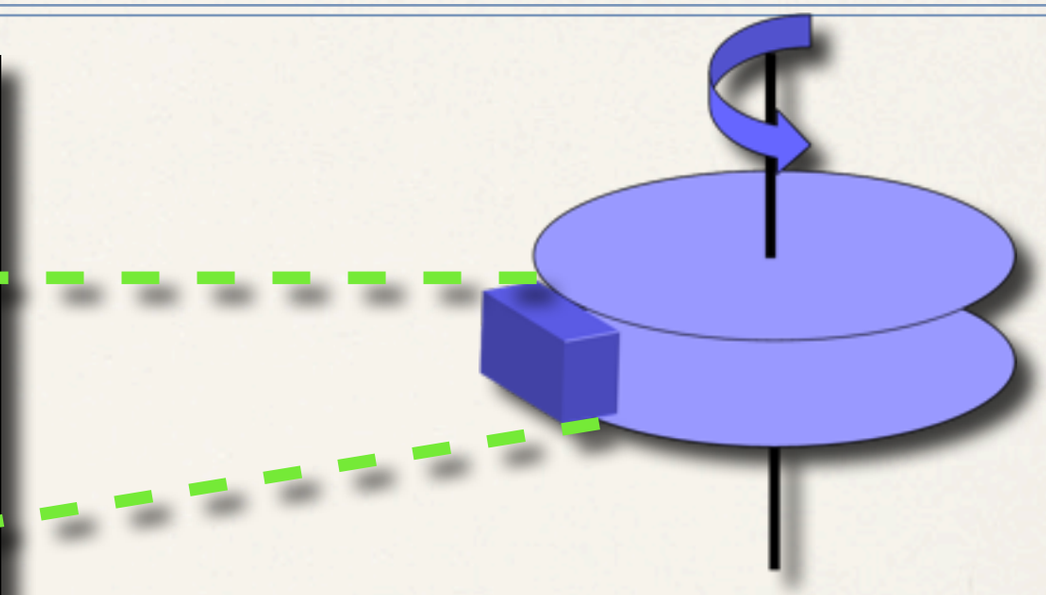
- The magnetorotational instability (Velikhov 1958, Balbus & Hawley 1991) is an obvious candidate to produce turbulence in discs
- Can we explain the observed anomalous viscosity with turbulence generated by the MRI?
- Measure the anomalous viscosity in local models:

$$\alpha = \frac{\langle \delta V_x \delta V_y \rangle - \langle \delta B_x \delta B_y \rangle / 4\pi\rho}{S^2 H^2}$$



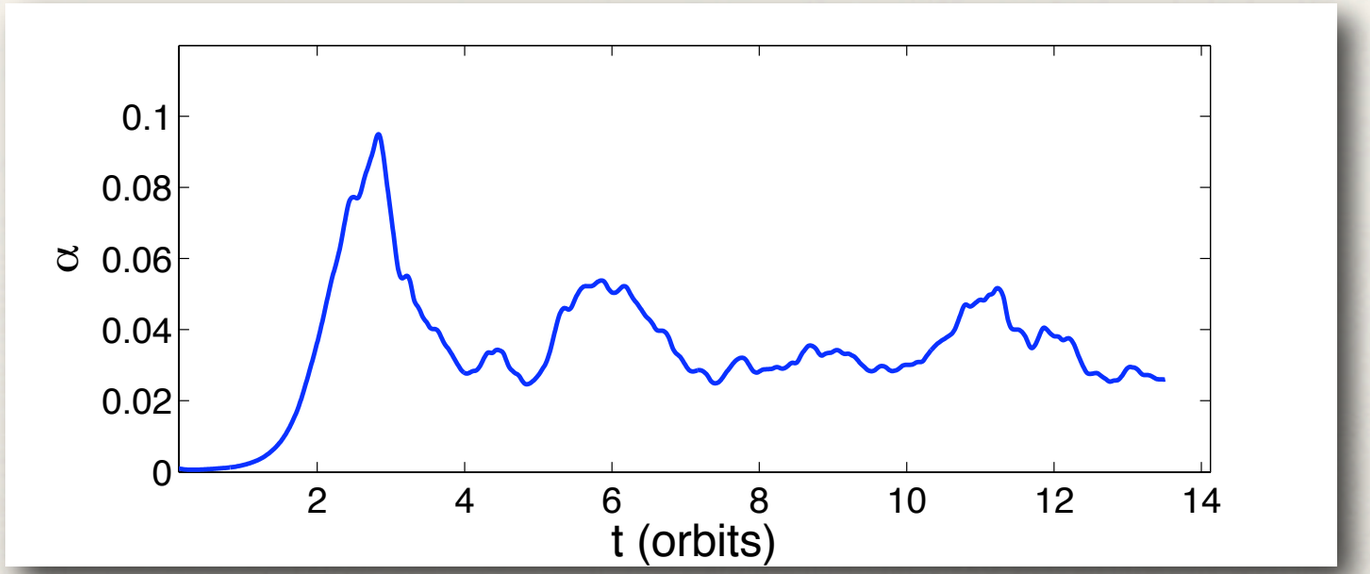
Simulation example





Simulation parameters: Re=1000, Pm=1, β=1000

3D map of v_y (azimuthal velocity)

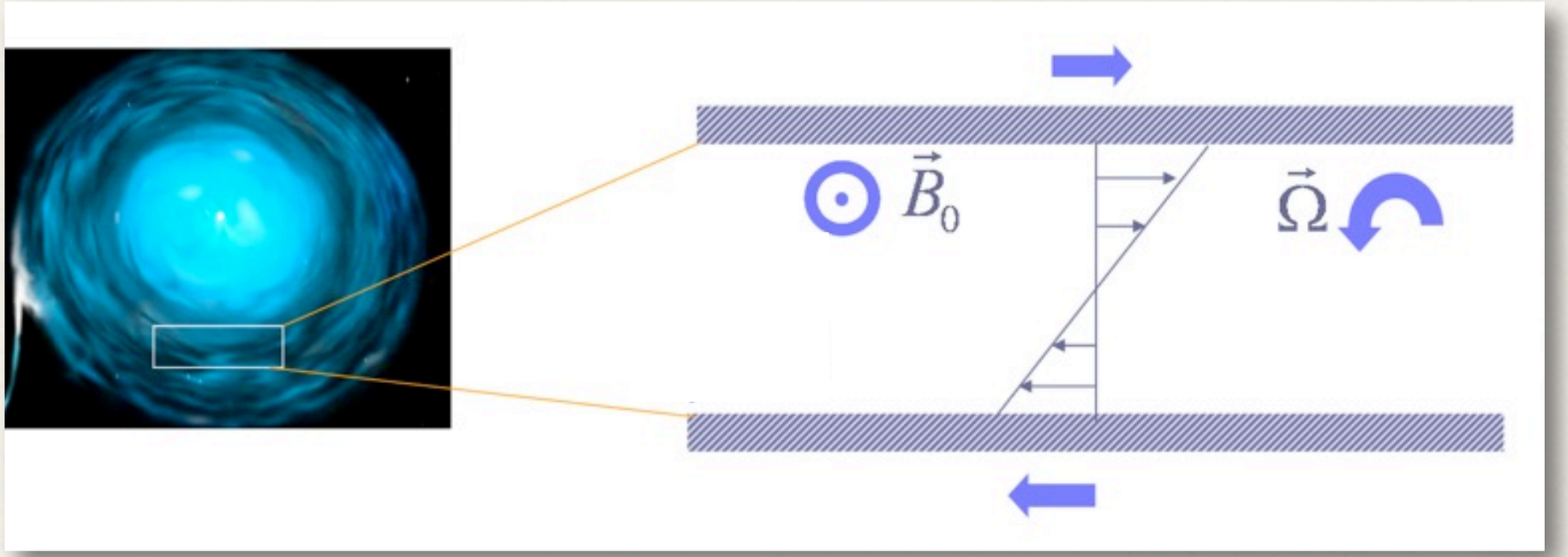


The Magnetorotational instability: Open questions

- «Efficient» instability (for astrophysicists) found for weak fields.
- First 3D non linear numerical simulations by Hawley *et al.* (1995) and Stone *et al.* (1996)
  - Second order finite differences
  - Numerical dissipation (convergence? see Fromang & Papaloizou 2007)
  - No properly defined Re/Rm/Pm
- In several known examples (e.g. small scale dynamos, see Schekochihin *et al.* 2004), properties of MHD turbulence depends on the dissipation coefficients (for instance *Pm*)...

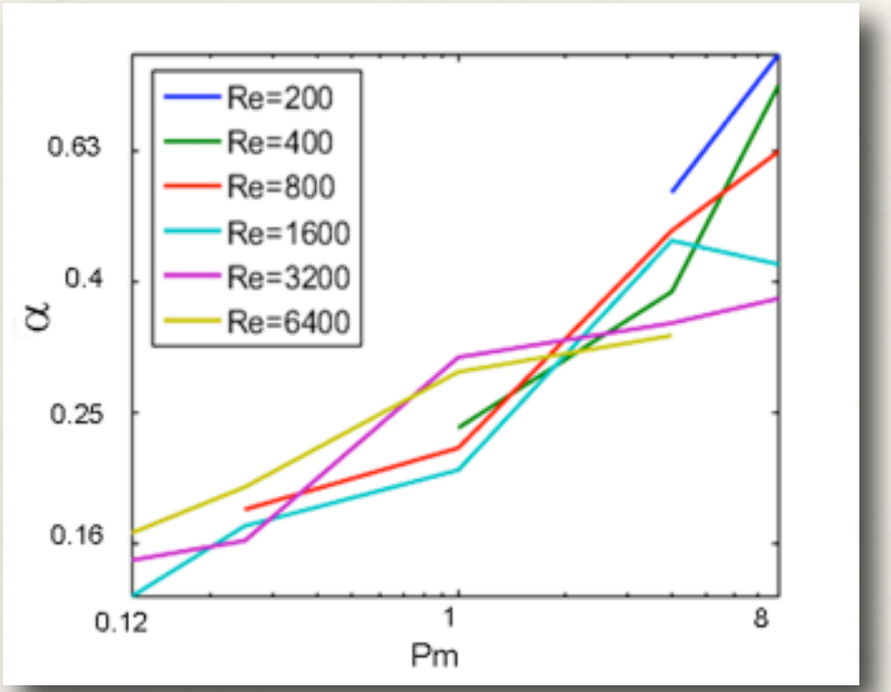
Does MRI turbulence depend on dissipation coefficients?
If yes, can we explain it?
What is the impact for astrophysical objects?

Parameter space

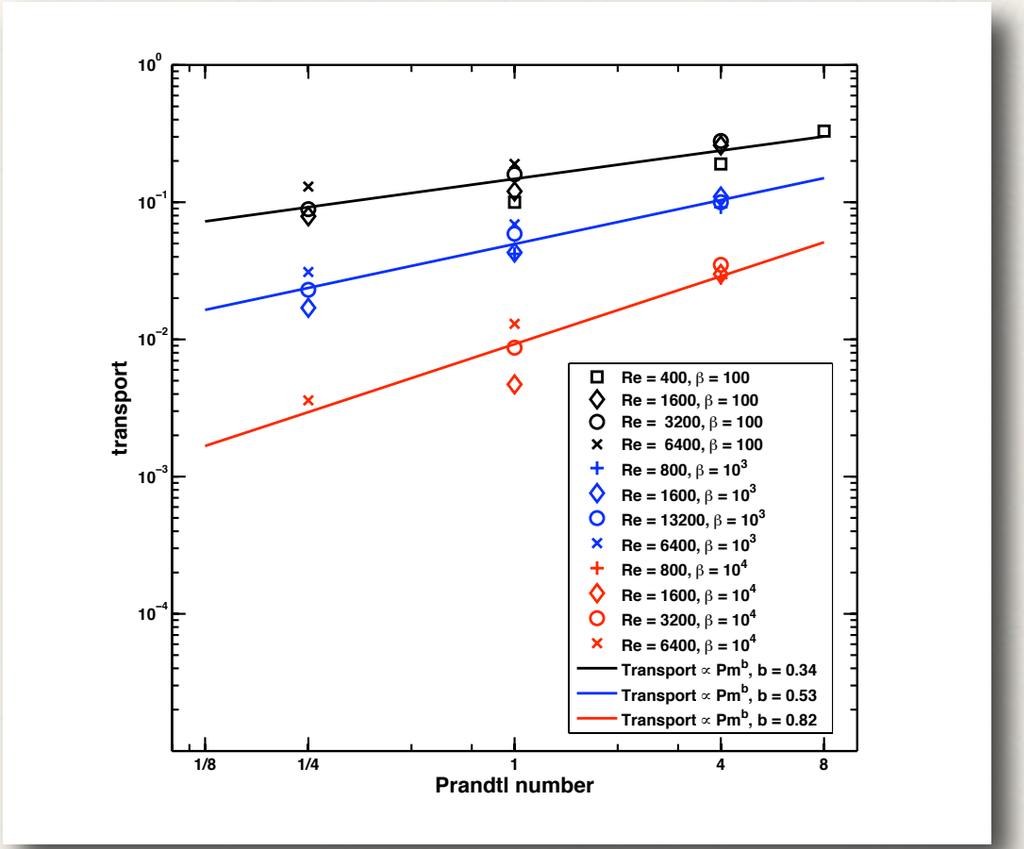


- Includes a mean vertical magnetic field B_0.
  - β-like parameter: $\beta = \left(\frac{SH}{V_A}\right)^2$
  - Reynolds number $Re = \frac{SH^2}{\nu}$ (Re~10^{10}-10^{15})
  - Magnetic Reynolds number: $Rm = \frac{SH^2}{\eta}$ (Rm~1-10^{15})
  - Magnetic Prandtl number: $Pm = \frac{\nu}{\eta} = \frac{Rm}{Re}$ (Pm~10^{-4}-10^{5})

Magnetic Prandtl number and MRI turbulence



Lesur & Longaretti (2007) 1x4x1 box, β=100



Lesur & Longaretti (2009) *in prep* 4x4x1 box

- The Magnetic Prandtl number modifies the transport efficiency (independently of the aspect ratio)

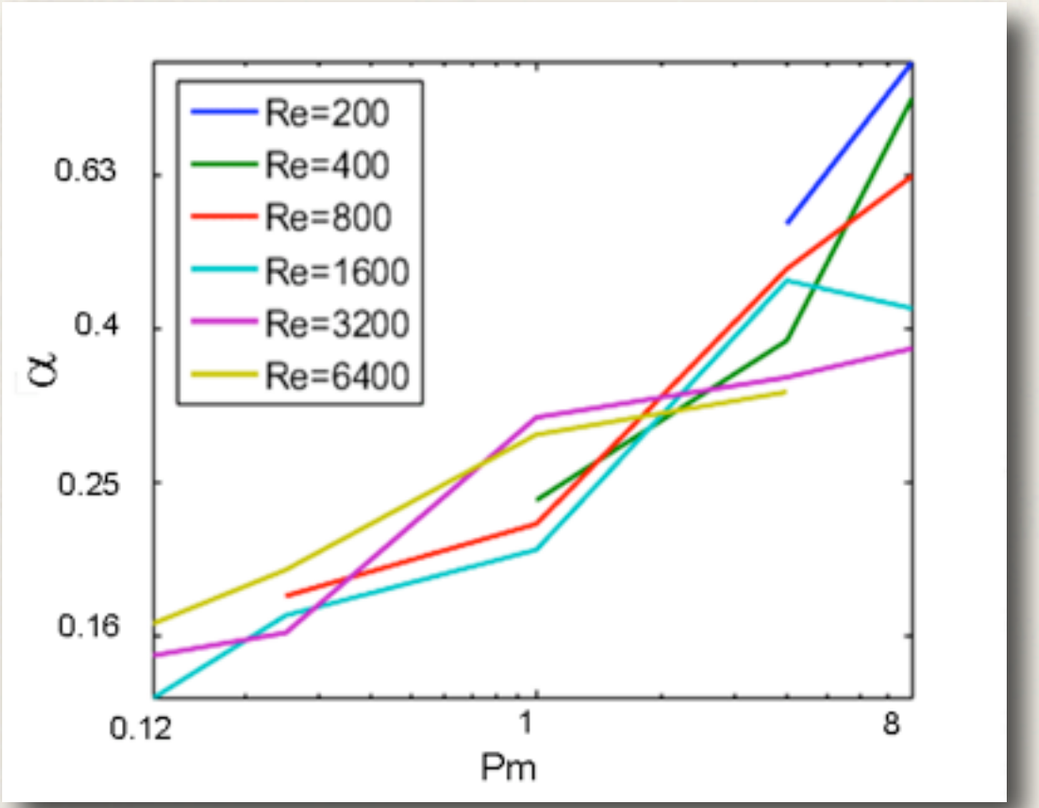
$$\alpha \propto Pm^{\delta} \quad 0.25 < \delta < 0.8$$

- Points out a reaction of the small scales (dissipative scales) on the large scales (responsible for the turbulent transport).
- In real astrophysical objects, *Pm* varies by many orders of magnitude.

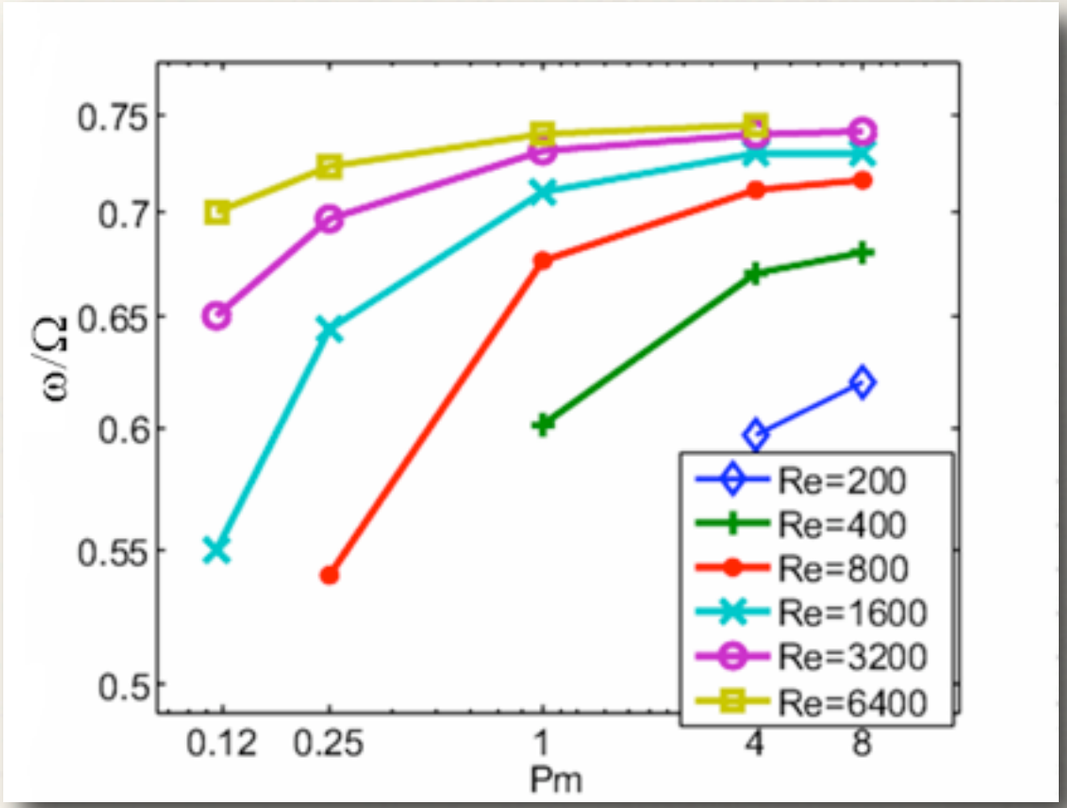
➡ The actual efficiency of MRI turbulence in astrophysical discs is not known...

Is it a linear effect?

Simulations



Linear growth rate

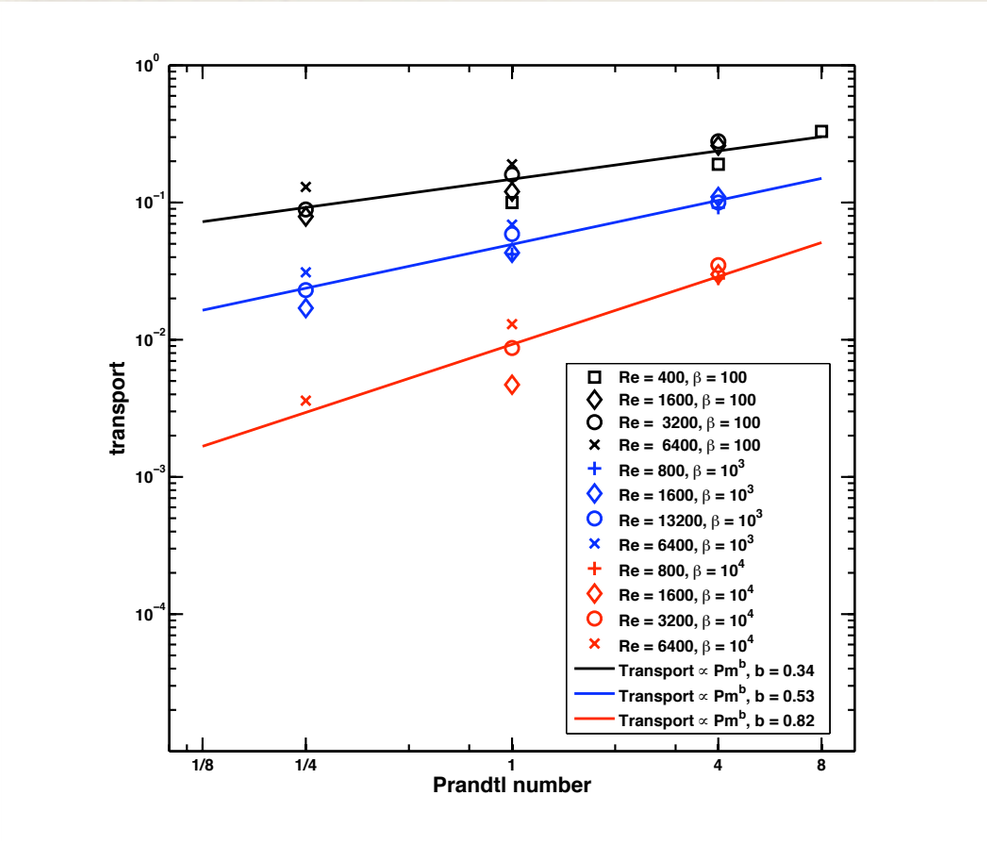


- No! The linear analysis is unable to explain the Pm-α correlation.
- Turbulent transport ≠ Linear theory
- One should look for a non linear theory...

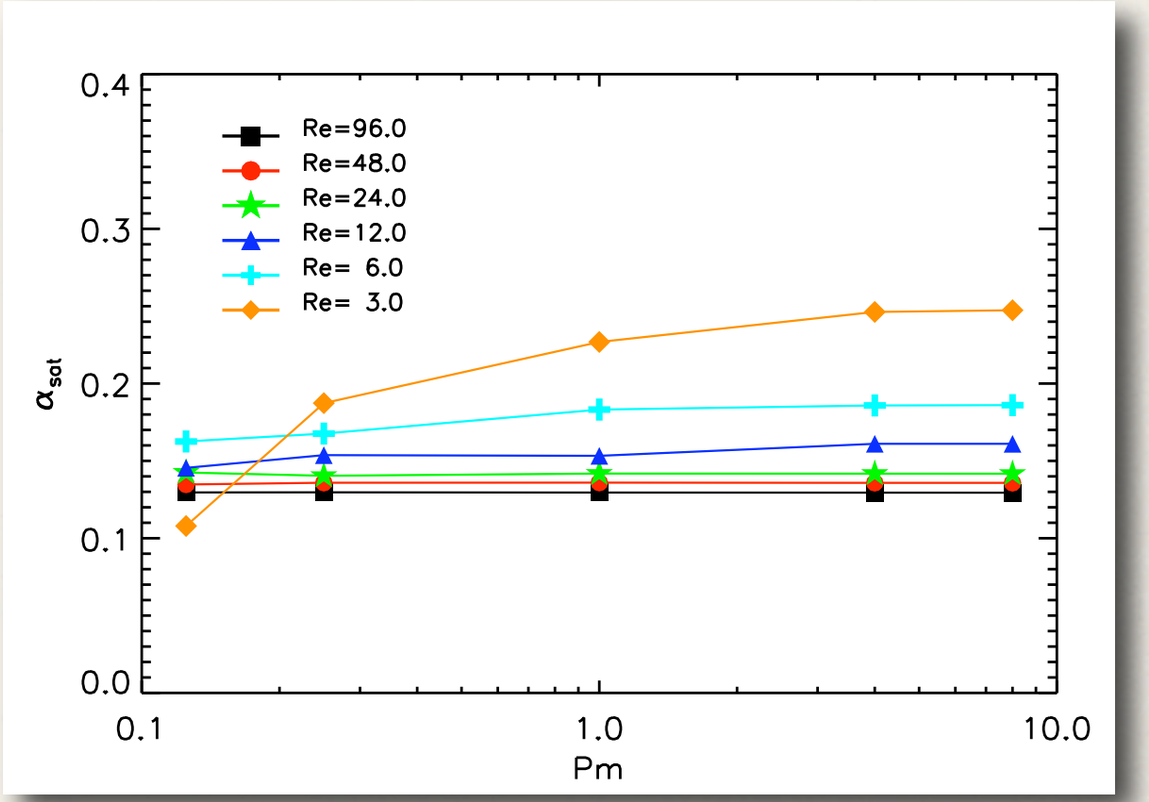
Parasitic instabilities?

- Parasitic instabilities are secondary instabilities of MRI modes
- Possible to predict a saturation level of the MRI thanks to these instabilities...

Simulations



Predicted saturation

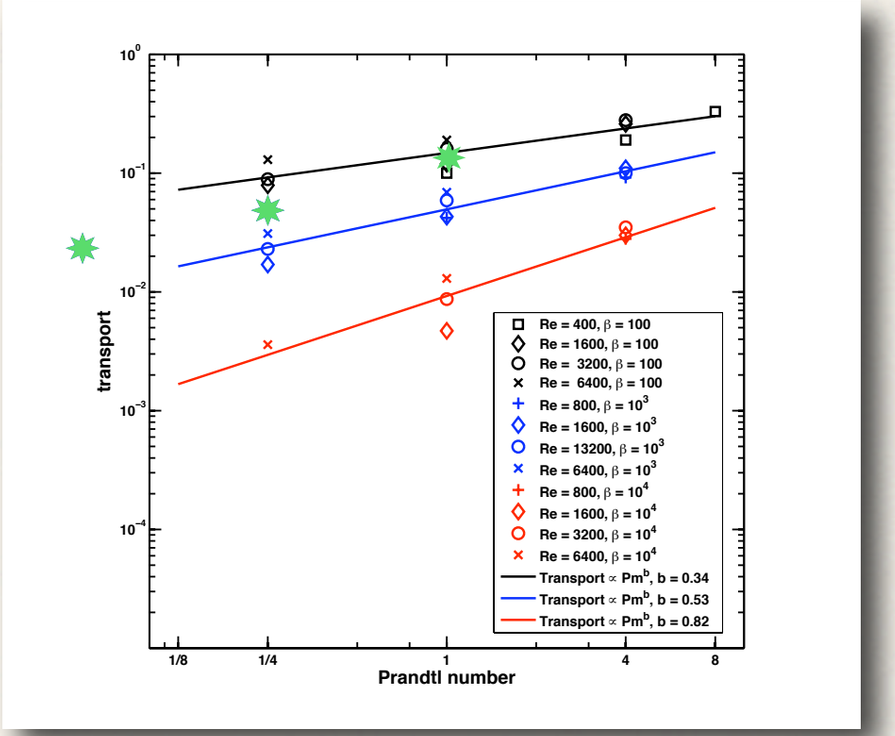


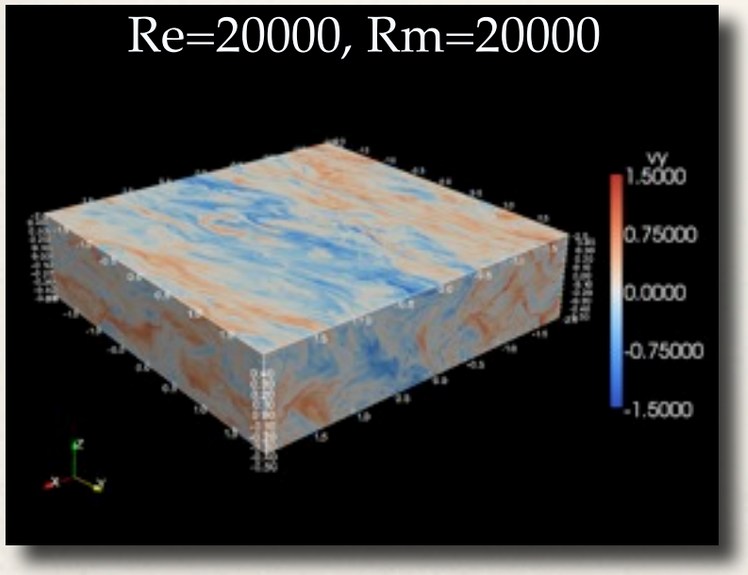
Pessah & Goodman (2009)

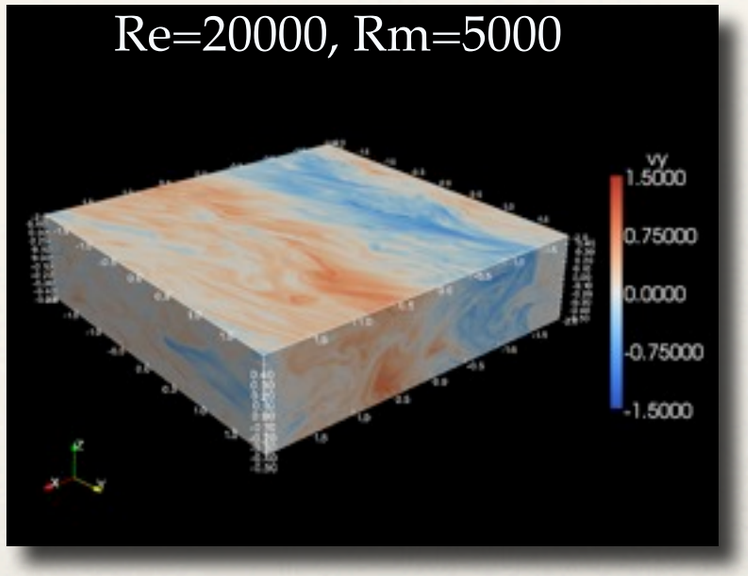
➡ «Quasi-linear» theory is still not enough...

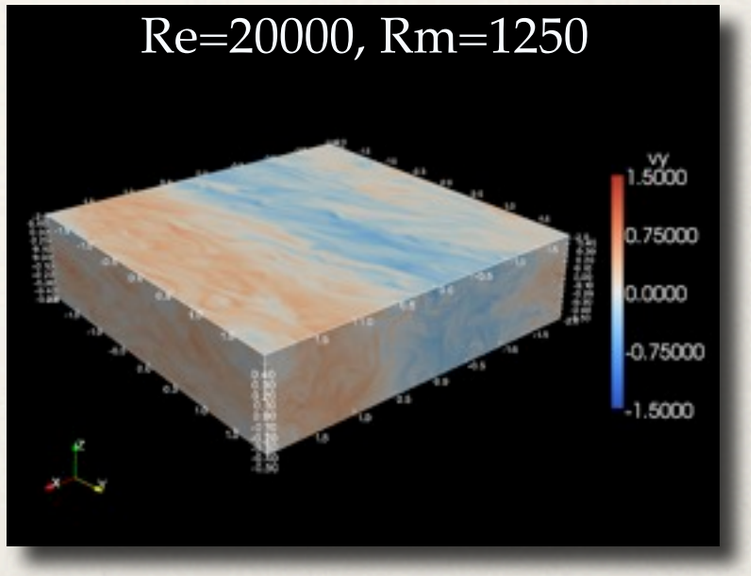
High Reynolds number: saturation?

- High resolution runs (768x384x192 points) with Re=20,000 and β=1000
  - Same slope as a function of Pm
  - Transport increases at large Re for a given Pm
  - No saturation observed (yet?)





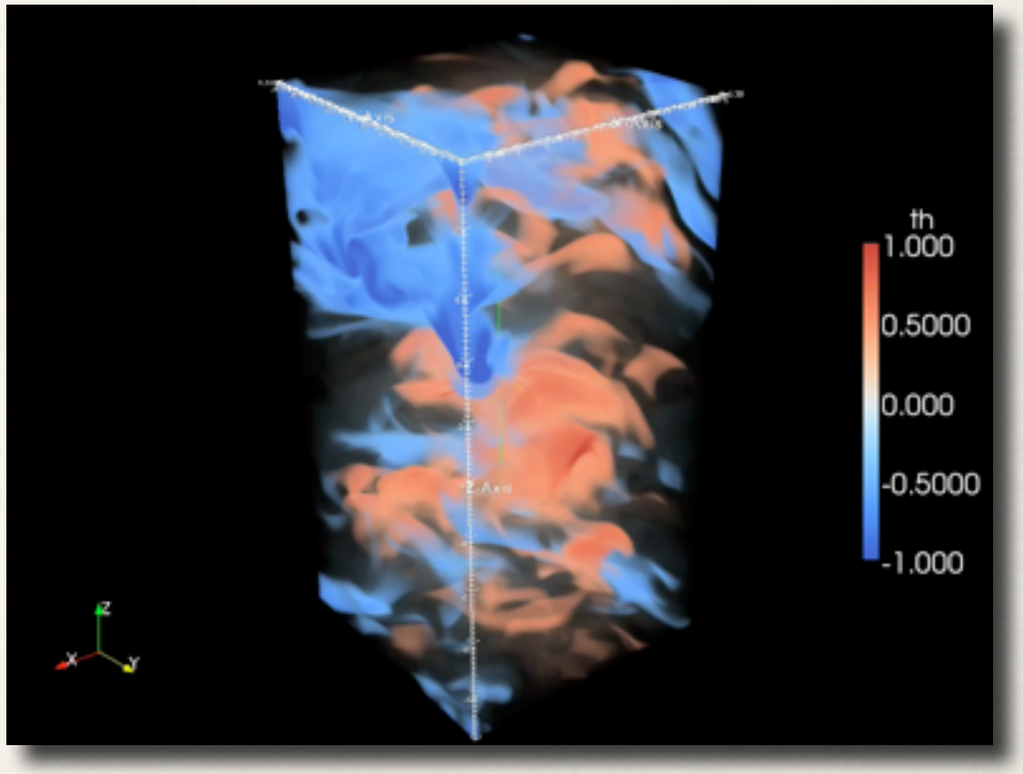




Conclusions

- The magnetorotational instability in discs:
  - The impact of dissipative process on MRI turbulence is strong
  - No saturation has been observed at large *Re*.
  - Linear/parasitic modes theories do *not* agree with simulations

- Spectral methods for sheared flow:
  - Simulations done with the Snoopy code (*http://www.damtp.cam.ac.uk/user/glesur/*)
  - Also includes modules for:
    - Boussinesq convection
    - Time-dependant shear
    - MHD



Turbulent convection in a sheared flow