

Turbulence: simulation, statistical theory and physical intuition

Wolf-Christian Müller

Max-Planck-Institut für Plasmaphysik, 85748 Garching, Germany

Mathematical Framework

- ▶ Reducing physical complexity close to bare minimum:
Incompressible MHD description in v and b
(nonlinear system, deterministic-chaotic, many degrees of freedom)
- ▶ **Assuming statistical homogeneity:** Still too difficult for analytical treatment
- ▶ Direct numerical simulation in Fourier space
(pseudo-spectral, periodic boundaries, „clean” and boring numerics)

Incompressible MHD

Vorticity $\omega = \nabla \times \mathbf{v}$

$$\partial_t \omega = \nabla \times [\mathbf{v} \times \omega - \mathbf{b} \times (\nabla \times \mathbf{b})] + Re^{-1} \Delta \omega ,$$

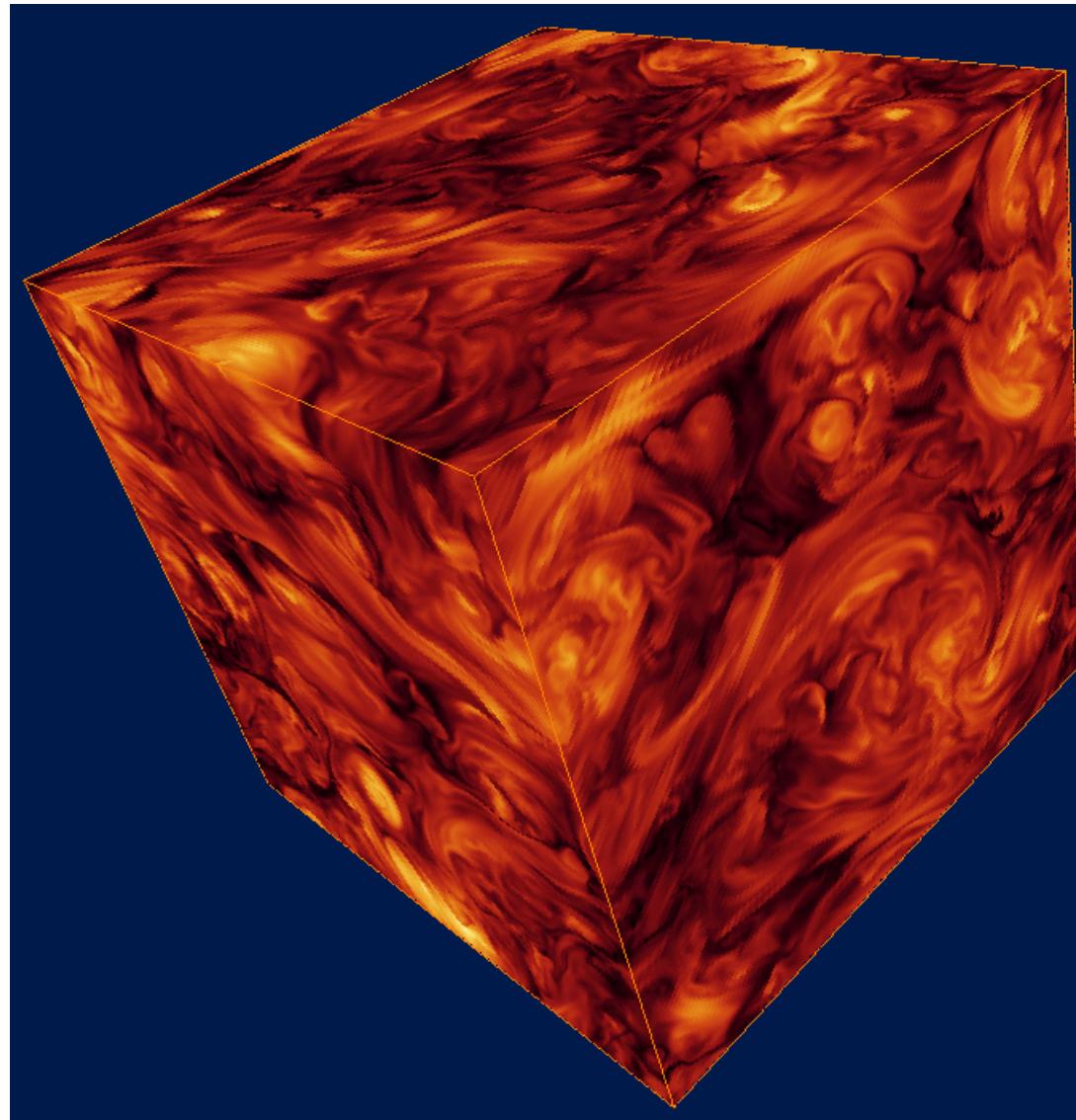
$$\partial_t \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}) + Rm^{-1} \Delta \mathbf{b} ,$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = 0 .$$

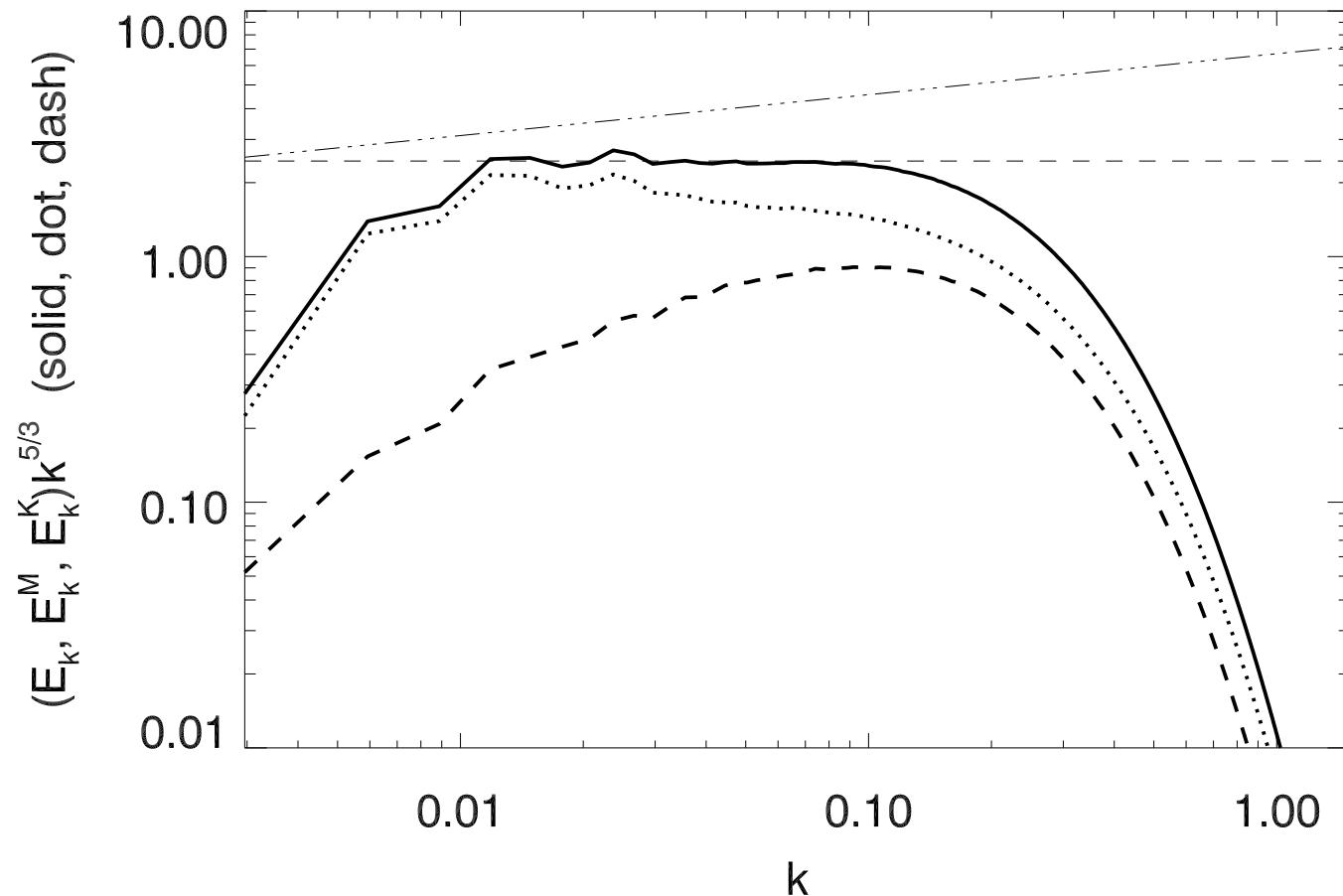
Reynolds numbers:

- Atmosphere ($Re \sim 10^8$)
- Solar convection zone ($Re \sim 10^{13}$, $Rm \sim 10^8$)
- Earth's liquid core ($Re \sim 10^9$, $Rm \sim 10^2$)
- Direct numerical simulation ($Re \sim Rm \sim 10^3$)

Turbulent Magnetic Field



Energy Spectra



1024^3 collocation points

Initially: stochastic fluctuations $\sim \exp[-k^2/(2k_0^2)]$, $k_0 = 4$

Statistical Closure Theory

Equation hierarchy for statistical moments of v (schematical):

$$\partial_t \langle v \rangle = \langle vv \rangle \quad (\text{MHD})$$

$$\partial_t \langle vv \rangle = \langle vvv \rangle \quad (\text{EDQNM})$$

$$\partial_t \langle vvv \rangle = \langle vvvv \rangle$$

:

- ▶ Quasinormal approximation
- ▶ Phenomenological damping of third order moments (e.g. energy flux)
- ▶ Neglecting long time correlations (Markovianization)
- ▶ Eddy-damped-quasi-normal-Markovian-approximation (EDQNM)

Spectral MHD-EDQNM Equations

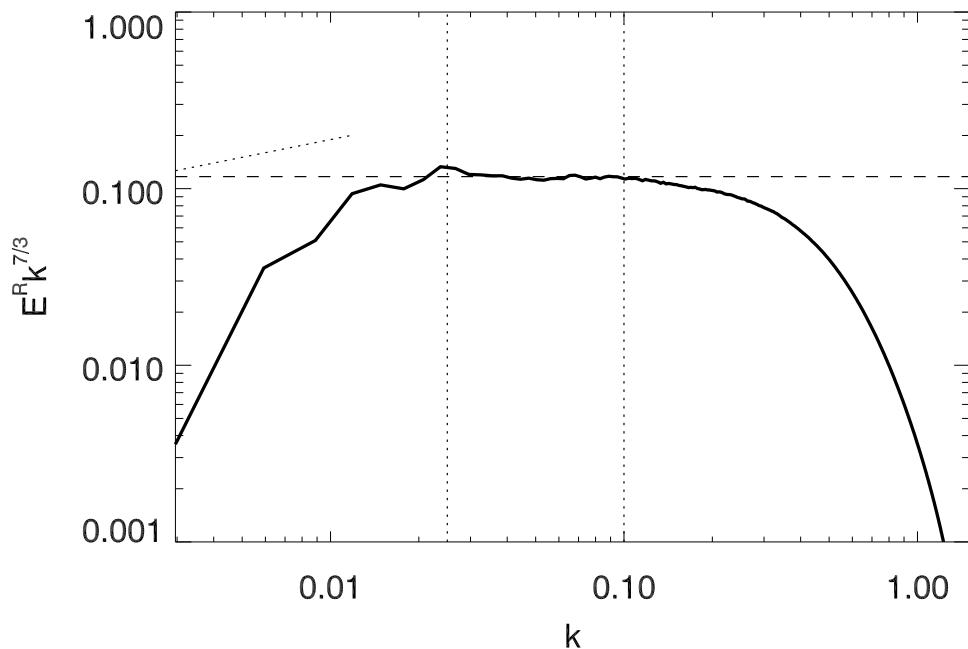
$$\left(\frac{\partial}{\partial t} + 2Re^{-1}k^2 \right) E_k = \iint_{\Delta} dp dq \Theta_{kpq} T_{kpq}(E_p E_q)$$

- ▶ Nonlinear relaxation timescale Θ_{kpq} , **phenomenological**
- ▶ Integrand complex spectral flux density (two counter-acting ingredients)
- ▶ $T_k^{\uparrow} \sim k^3 E_k^2$ — Small-scale dynamo by field-line deformation
- ▶ $T_k^{\downarrow} \sim k^3 E_k^M |E_k^K - E_k^M|$ — Energetic equipartition by Alfvén-effect
- ▶ Eliminating Θ_{kpq} by dynamical equilibrium assumption, $T_k^{\uparrow} \sim T_k^{\downarrow}$

Residual Energy, $E_k^R = |E_k^K - E_k^M|$

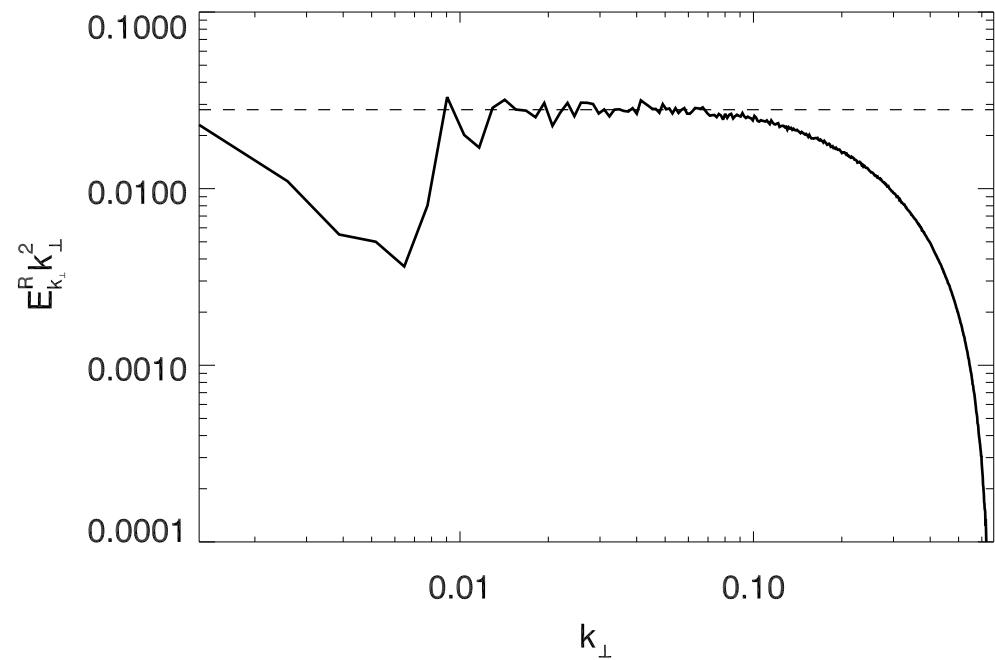
$$T_k^\uparrow \sim T_k^\downarrow \Rightarrow E_k^R \sim \left(\frac{\tau_A}{\tau_{NL}} \right)^2 E_k$$

K41: $E_k \sim k^{-5/3} \Rightarrow E_k^R \sim k^{-7/3}$



1024^3 Simulation, $B_0 = 0$

IK : $E_k \sim k^{-3/2} \Rightarrow E_k^R \sim k^{-2}$



$1024^2 \times 256$ Simulation, $B_0 = 5$

Magnetic Helicity

$$H^M = \langle \mathbf{a} \cdot \mathbf{b} \rangle \quad \mathbf{b} = \nabla \times \mathbf{a}$$

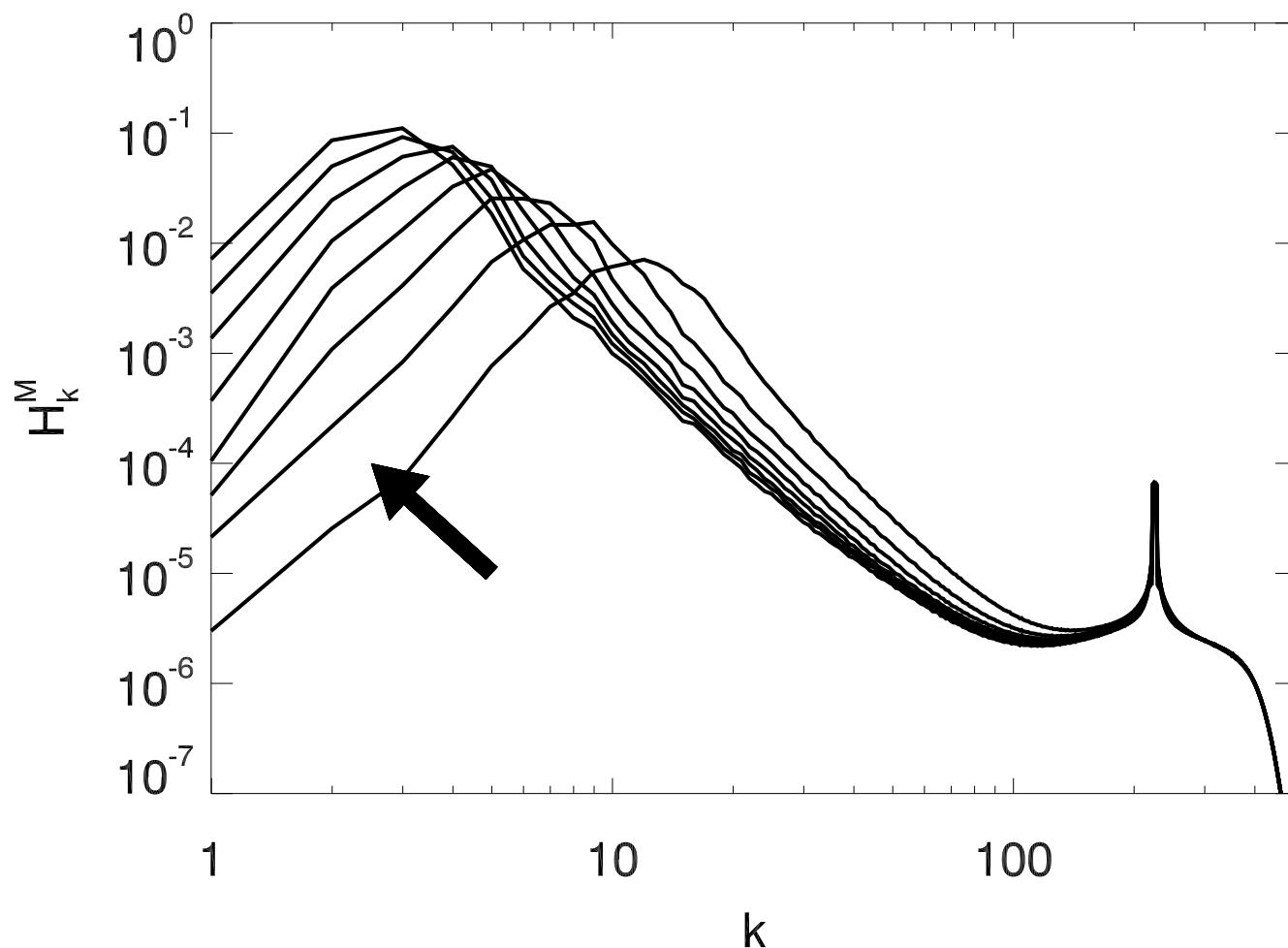
- „Knottedness“ of magnetic field-lines
- In 3D: Inverse cascade \rightarrow Large-scale structure formation
- No Dynamo: $E_k^M \sim k H_k^M, \quad k \sim \ell^{-1}$

Dimensional analysis à la K41 (constant flux of H^M):

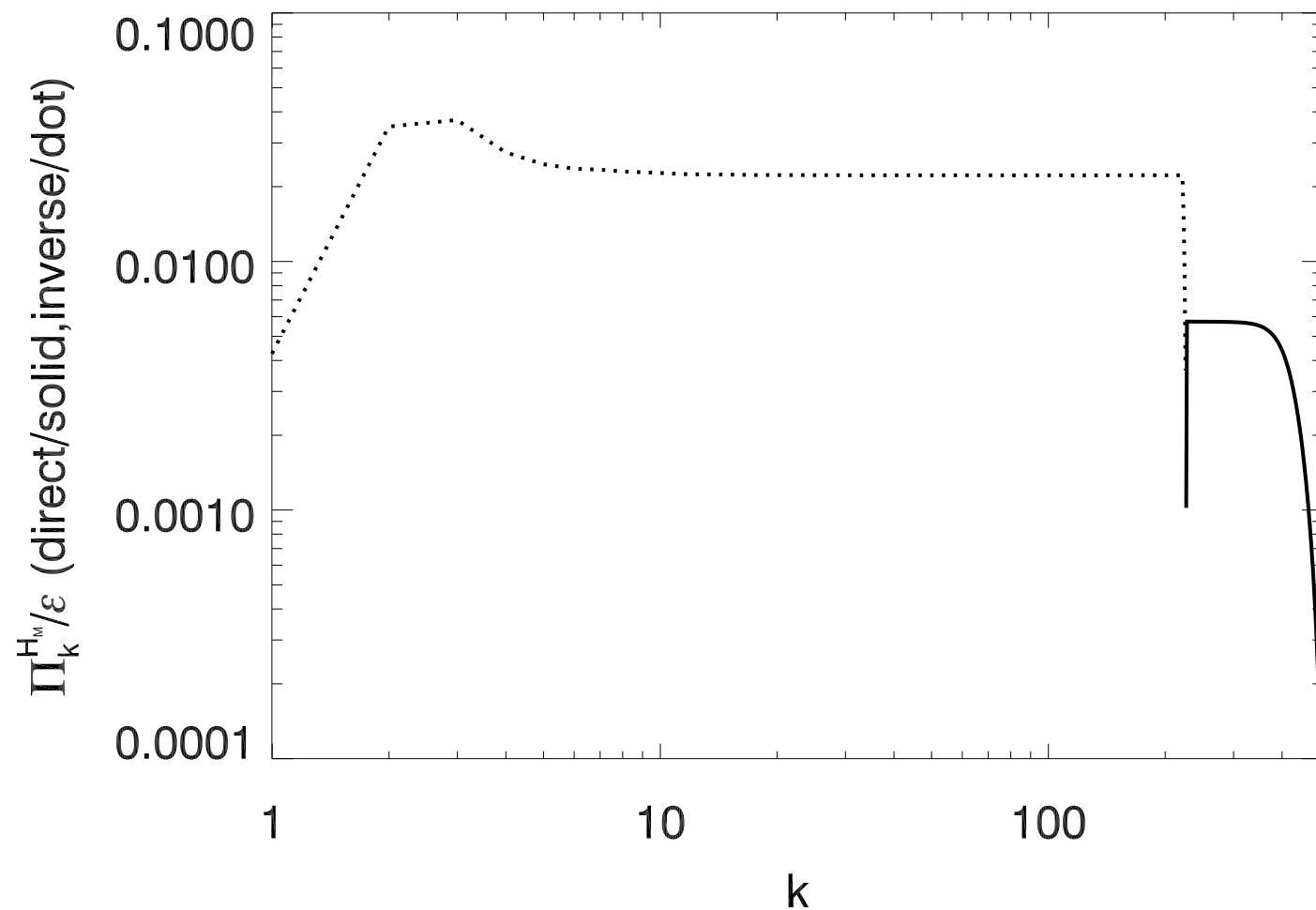
$$H_k^M \sim \varepsilon_H^{2/3} k^{-2}$$

Found in model calculations [Pouquet et al. 1976]

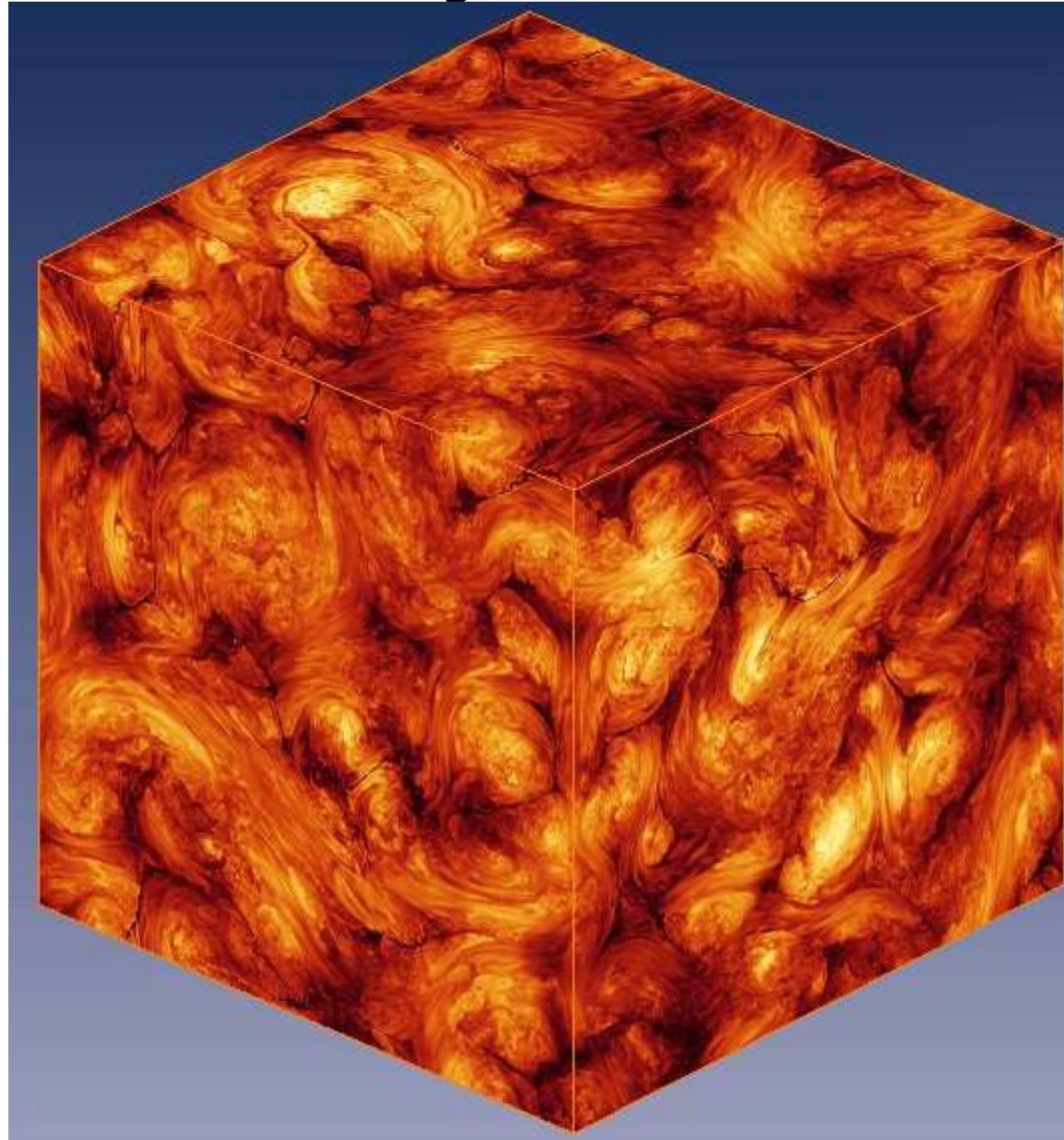
Inverse Cascade of H^M



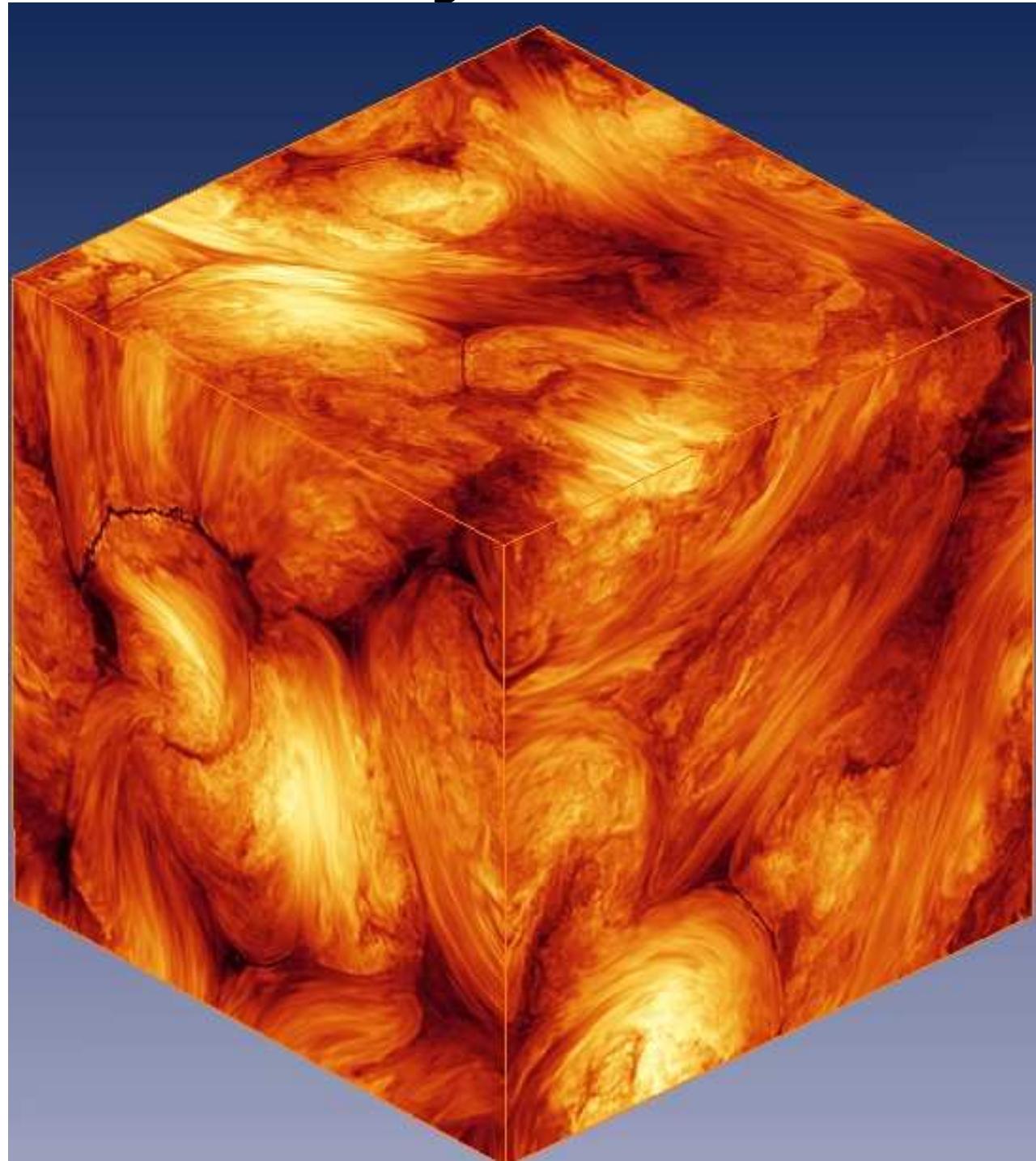
\mathcal{H}^M Flux

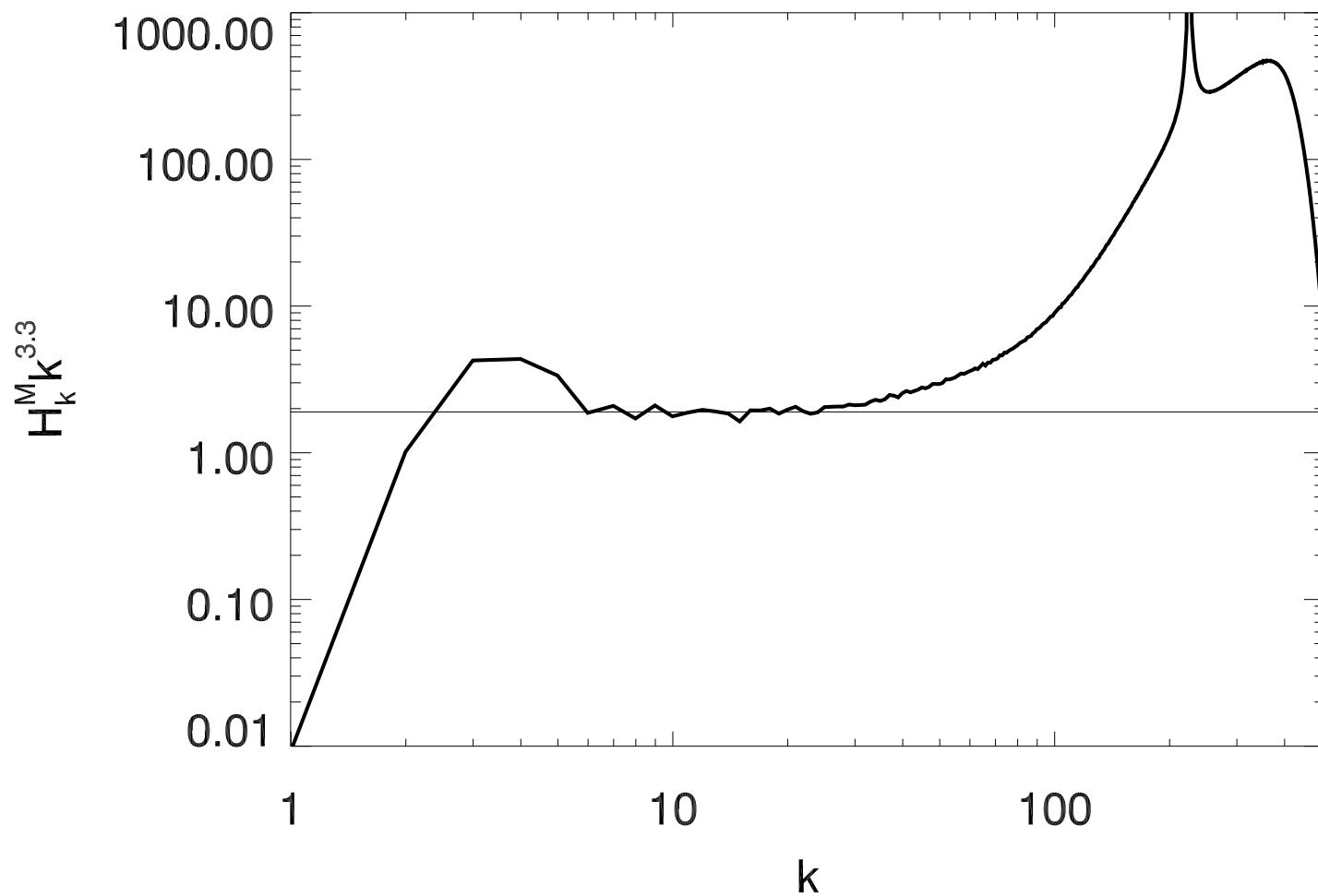


Magnetic field



Magnetic field



H_k^M 

$$H_k^M \sim k^{-2} ?$$

EDQNM Equation

Magnetic helicity H_k^M spectrally:

$$(\partial_t + 2Re^{-1}k^2)H_k^M = \iint_{\Delta} dp dq \Theta_{kpq} T_{kpq}$$

Nonlinear flux density contributions in T_{kpq} :

- ▶ $T_1 \sim k H_k^M E_k^K$ (Turbulent advection)
- ▶ $T_2 \sim H_k^K E_k^M / k$ (Flow-Twisting)

Kinetic helicity $H^K = \langle \boldsymbol{v} \cdot \boldsymbol{\omega} \rangle$

Dynamical Equilibrium

Turbulent scrambling vs. Average flow-twist, $T_1 \sim T_2$:

$$H_k^K \sim \underbrace{\left(\frac{E_k^K}{E_k^M} \right)}_{=\alpha} k^2 H_k^M$$

oder equivalently

$$E_k^M \sim \underbrace{\left(\frac{k^2 H_k^M}{H_k^K} \right)}_{=\beta} E_k^K$$

- $\alpha = 1 \Rightarrow H_k^R = H_k^K - k^2 H_k^M = 0$ (residual helicity)
- $\beta = 1 \Rightarrow E_k^R = |E_k^M - E_k^K| = 0$ (residual energy)

Additionally:

$$|E_k^M - E_k^K| \sim k^2 E_k$$

Summary



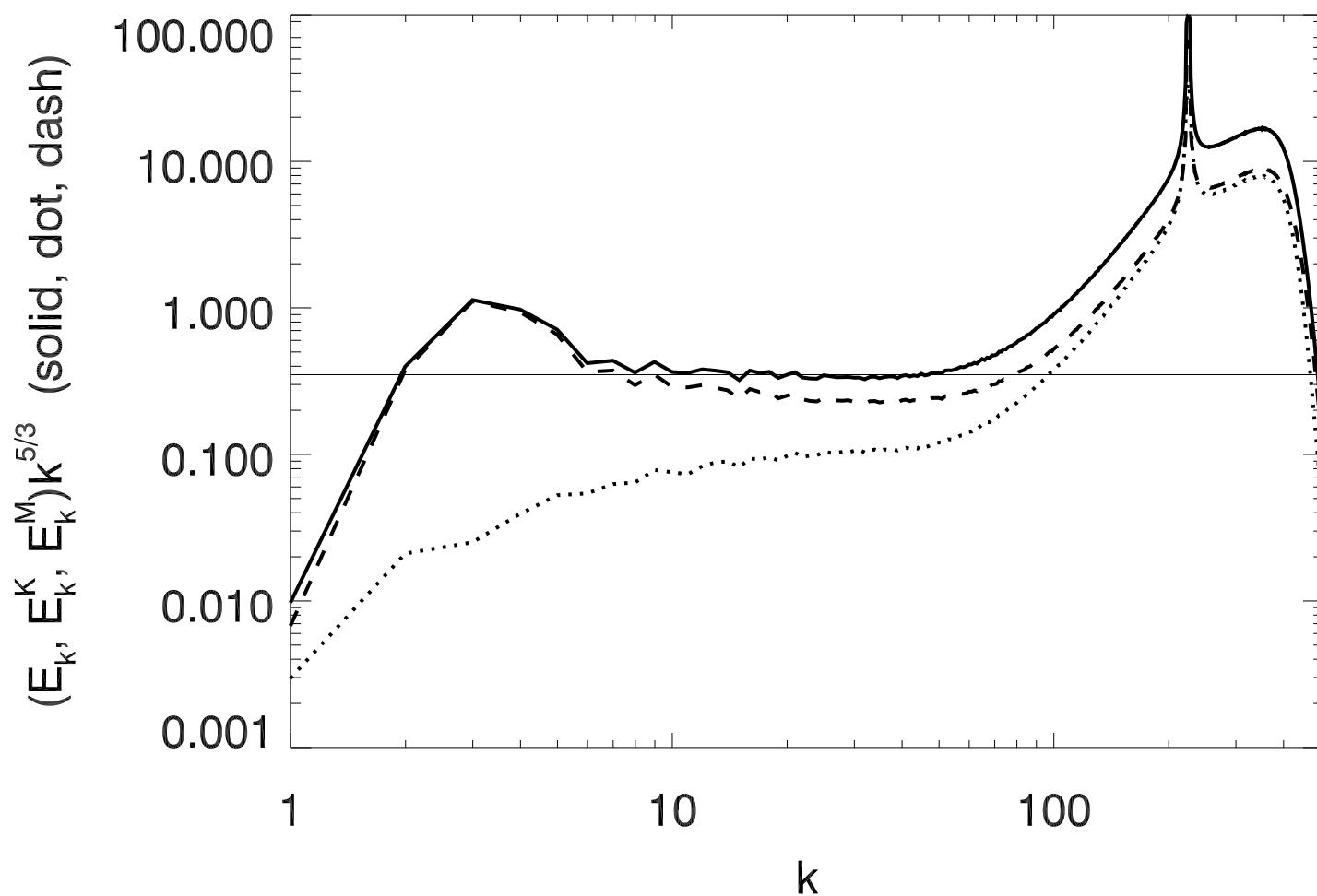
Dimensional analysis of EDQNM allows discovery of complex nonlinear relations between second order moments of turbulent MHD fields.

Examples:

- ▶ Excess of magnetic energy in MHD-turbulence
Dynamical equilibrium of small-scale dynamo and Alfvén-effect
- ▶ Scaling in inverse cascade of magnetic helicity
Dynamical equilibrium between turbulent field-line deformation and average twisting

Energy Spectra

IPP



Energy Flux

