

# *High-order ADER schemes for hyperbolic equations: a review*

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I am indebted to my collaborators:

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**We are interested in developing numerical methods to solve**

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = S(Q) + D(Q)$$

$$\partial_t Q + A(Q)\partial_x Q + B(Q)\partial_y Q + C(Q)\partial_z Q = S(Q) + D(Q)$$

Source terms  $S(Q)$  may be stiff

Advection terms may not admit a conservative form  
(nonconservative products)

Meshes are assumed unstructured

Very high order of accuracy in both space and time

May use upwind or centred approaches for numerical fluxes

*Design constraints:*

*We want the schemes to be*

*Conservative*

*(in view of the Lax-Wendroff theorem)*

*Non-linear*

*(in view of the Godunov's theorem)*

**In view of Godunov's theorem our numerical approach relies on:**

- 1.A monotone numerical flux (at most first order accurate)**
- 2.A framework to construct non-linear numerical methods of high-order of accuracy in both space and time in which  
The monotone flux is the building block**

## **Part A:**

**FORCE-type centred, monotone schemes for  
hyperbolic equations in multiple space  
dimensions solved on general meshes**

## **Conservative schemes in 1D**

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}]$$

**Task: define numerical flux**

$$F_{i+1/2}$$

Basic property required: MONOTONICITY

**There are two approaches:**

**I: Upwind approach (Godunov). Solve the Riemann problem**

$$\left. \begin{array}{l} \partial_t Q + \partial_x F(Q) = 0 \\ Q(x,0) = \begin{cases} Q_i^n & \text{if } x < 0 \\ Q_{i+1}^n & \text{if } x > 0 \end{cases} \end{array} \right\} \Rightarrow F_{i+1/2}$$

(Toro E F. *Riemann solvers and numerical methods for fluid dynamics*. Springer, 3º edition 2009)

**II: Centred approach. The numerical flux is**

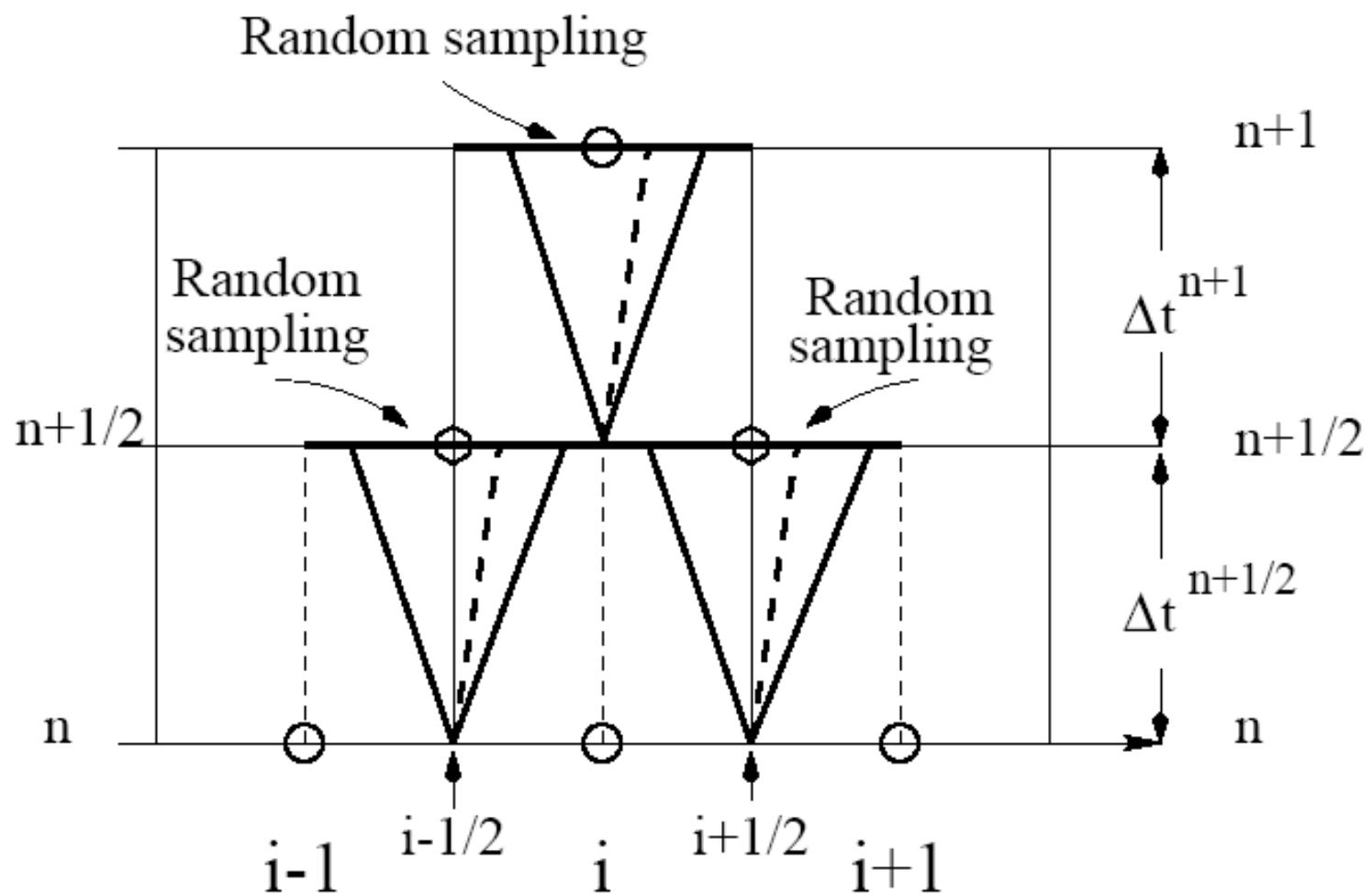
$$F_{i+1/2} = H(Q_i^n, Q_{i+1}^n)$$

# **The FORCE flux (First ORder CEntred)**

## **Brief review of 1D case**

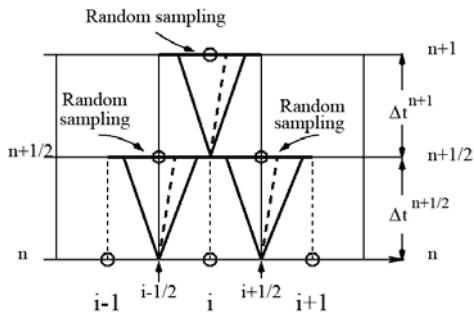
TORO E F, BILLETT S J. (2000). Centred TVD schemes for hyperbolic conservation laws. IMA JOURNAL OF NUMERICAL ANALYSIS. vol. 20, pp. 47-79 ISSN: 0272-4979.

# Glimm's method on a staggered mesh



# FORCE: replace random state by averaged state

Recall that the integral form of the conservation laws



$$\partial_t Q + \partial_x F(Q) = 0$$

in a control volume  $[x_L, x_R] \times [t_1, t_2]$

is

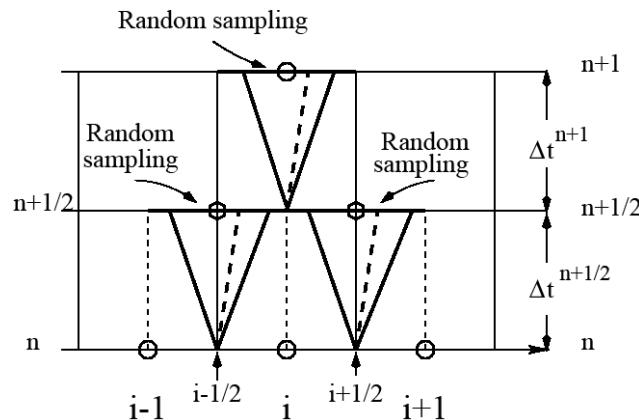
$$\frac{1}{\Delta x} \int_{x_L}^{x_R} Q(x, t_2) dx = \frac{1}{\Delta x} \int_{x_L}^{x_R} Q(x, t_1) dx - \frac{1}{\Delta x} \left[ \int_{t_1}^{t_2} F(Q(x_R, t)) dt - \int_{t_1}^{t_2} F(Q(x_L, t)) dt \right]$$

## Averaging operator

## Step I

$$Q_{i-1/2}^{n+1/2} = \frac{1}{2} (Q_i^n + Q_{i-1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_i^n) - F(Q_{i-1}^n))$$

$$Q_{i+1/2}^{n+1/2} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$



## Step II

$$Q_i^{n+1} = \frac{1}{2} (Q_{i-1/2}^{n+1/2} + Q_{i+1/2}^{n+1/2}) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1/2}^{n+1/2}) - F(Q_{i-1/2}^{n+1/2}))$$

### Step III: One-step conservative form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (F_{i+1/2}^{\text{force}} - F_{i-1/2}^{\text{force}})$$

with numerical flux

$$F_{i+1/2}^{\text{force}} = \frac{1}{2} (F_{i+1/2}^{\text{LW}} + F_{i+1/2}^{\text{LF}})$$

$$F_{i+1/2}^{\text{LW}} = F(Q_{i+1/2}^{\text{lw}}), \quad Q_{i+1/2}^{\text{lw}} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

$$F_{i+1/2}^{\text{LF}} = \frac{1}{2} (F(Q_i^n) + F(Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\Delta t} (Q_{i+1}^n - Q_i^n)$$

The numerical flux is in fact

$$F_{i+1/2}^{\text{force}} = \frac{1}{2}(F_{i+1/2}^{\text{LW}} + F_{i+1/2}^{\text{LF}})$$

with

$$F_{i+1/2}^{\text{LW}} = F(Q_{i+1/2}^{\text{lw}}), \quad Q_{i+1/2}^{\text{lw}} = \frac{1}{2}(Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

$$F_{i+1/2}^{\text{LF}} = \frac{1}{2}(F(Q_i^n) + F(Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\Delta t} (Q_{i+1}^n - Q_i^n)$$

## Properties of the FORCE scheme

*Stable*     $0 \leq |c| \leq 1$

*Monotone*

*Modified equation*  $\partial_t q + \lambda \partial_x q = \alpha_{force} \partial_x^{(2)} q$

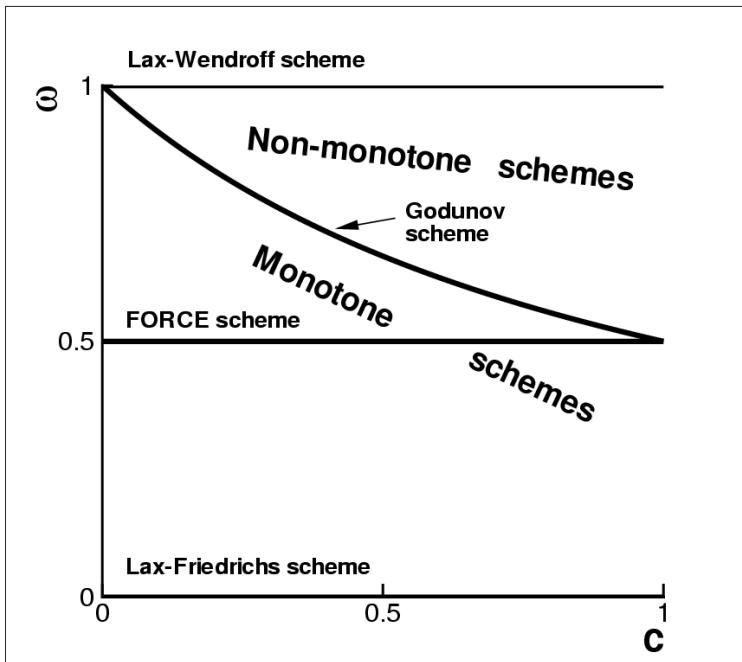
$$\alpha_{force} = \frac{1}{4} \lambda \Delta x \left( \frac{1 - c^2}{c} \right) = \frac{1}{2} \alpha_{lf}$$

Proof of convergence of FORCE scheme in:

*Chen C Q and Toro E F.  
Centred schemes for non-linear hyperbolic equations.  
Journal of Hyperbolic Differential Equations 1 (1), pp 531-566,  
2004.*

# The FORCE flux for the scalar case: more general averaging.

$$F_{i+1/2}^\omega = \omega F_{i+1/2}^{LW} + (1 - \omega) F_{i+1/2}^{LF}, \quad 0 < \omega < 1$$



Special cases:

$\omega = 0$  (Lax - Friedrichs)

$\omega = 1$  (Lax - Wendroff)

$\omega = 1/2$  (FORCE)

$$\omega = \frac{1}{1+c} \text{ (GFORCE)}$$

## ***Related works on centred schemes include***

**H Nessyahu and E Tadmor.** *Non-oscillatory central differencing for hyperbolic conservation Laws.* *J. Computational Physics*, Vol 87, pp 408-463, 1990.

P. Arminjon and A. St-Cyr. Nessyahu-Tadmor-type central finite volume methods without predictor for 3D Cartesian and unstructured tetrahedral grids. *Applied Numerical Mathematics*, 46:135–155, 2003.

B. Haasdonk, D. Kröner, and C. Rohde. Convergence of a staggered-lax-friedrichs-scheme on unstructured grids in 2d. *Numerische Mathematik*, 88:459–484, 2001.

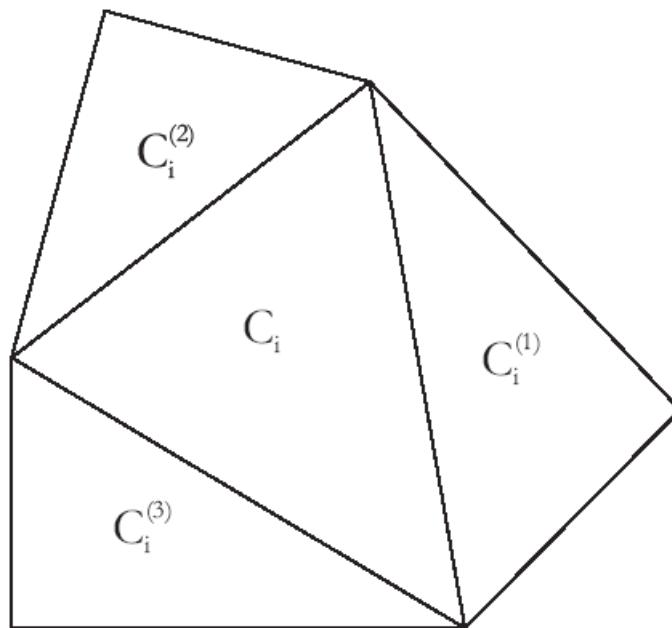
# *Recent extensions of* **FORCE**

- **Multiple space dimensions**
- **Unstructured meshes**
- **Path-conservative version**
- **High-order non-oscillatory (FV, DG, PC)**

**Toro E F, Hidalgo A and Dumbser M.**  
*FORCE schemes on unstructured meshes I:  
Conservative hyperbolic systems.*  
**(Journal of Computational Physics, Vol. 228, pp 3368-3389, 2009)**

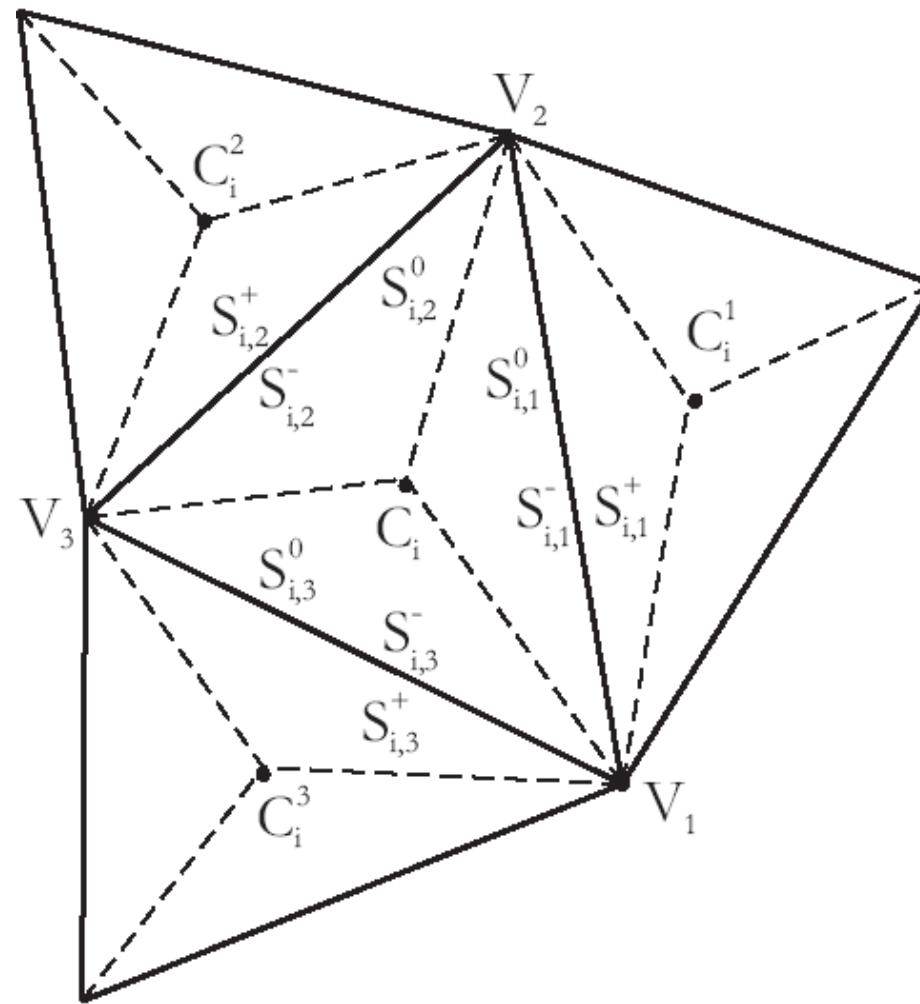
# *Illustration in 2D*

Consider, for example, triangular meshes in 2 space dimensions



Compact stencil

## Definition of secondary mesh for intercell fluxes



# Stage I

Initial condition: integral averages at time n       $\bar{Q}_i^n$

Averaging operator applied on edge-base control volume gives

$$Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\bar{Q}_i^n V_i^- + \bar{Q}_j^n V_j^+}{V_j^- + V_j^+} - \frac{1}{2} \frac{\Delta t S_j}{V_j^- + V_j^+} \left( \underline{\underline{F}}(\bar{Q}_j^n) - \underline{\underline{F}}(\bar{Q}_i^n) \right) \cdot \vec{n}_j$$

$V_j^-$       Portion of j edge-base volume inside cell i

$V_j^+$       Portion of j edge-base volume outside cell i

$S_j$       Area of face j (between cells i and j)

$n_j$       Unit outward normal vector to of face j

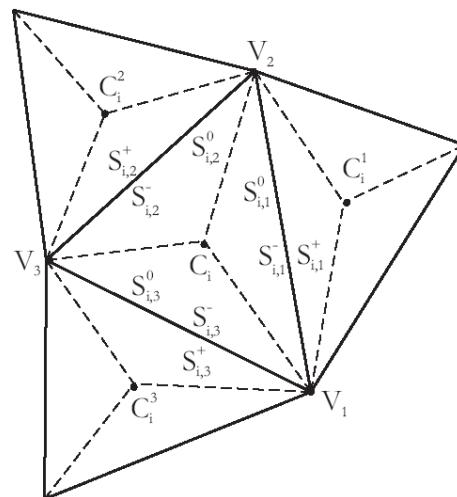
$$\underline{\underline{F}} = (F, G, H)$$

# Stage II

Initial condition: integral averages at time  $n+1/2$      $\mathbf{Q}_{j+1/2}^{n+1/2}$

Averaging operator applied on primary mesh gives

$$\mathbf{Q}_i^{n+1} = \frac{1}{|T_i|} \sum_{j=1}^{n_f} \left( \mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} V_j^- - \frac{1}{2} \Delta t S_j \underline{\mathbf{F}} \left( \mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right) \cdot \vec{n}_j \right)$$



## Stage III: one-step conservative scheme

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{|T_i|} \sum_{j=1}^{n_f} S_j \underline{\underline{\mathbf{F}}}{}^{\text{FORCE}\alpha}_{j+\frac{1}{2}} \cdot \vec{n}_j$$

$$\underline{\underline{\mathbf{F}}}{}^{\text{FORCE}\alpha}_{j+\frac{1}{2}} = \frac{1}{2} \left( \underline{\underline{\mathbf{F}}}{}^{LW\alpha}_{j+\frac{1}{2}} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) + \underline{\underline{\mathbf{F}}}{}^{LF\alpha}_{j+\frac{1}{2}} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) \right)$$

# Numerical flux in conservative scheme

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{\text{FORCE}\alpha} = \frac{1}{2} \left( \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) + \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) \right)$$

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) = \underline{\underline{\mathbf{F}}} \left( \mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right) ,$$

$$\mathbf{Q}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{\mathbf{Q}_i^n V_i^- + \mathbf{Q}_j^n V_j^+}{V_j^- + V_j^+} - \frac{1}{2} \frac{\Delta t S_j}{V_j^- + V_j^+} \left( \underline{\underline{\mathbf{F}}} (\mathbf{Q}_j^n) - \underline{\underline{\mathbf{F}}} (\mathbf{Q}_i^n) \right) \cdot \vec{n}_j$$

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) = \frac{V_j^- \underline{\underline{\mathbf{F}}} (\mathbf{Q}_j^n) + V_j^+ \underline{\underline{\mathbf{F}}} (\mathbf{Q}_i^n)}{V_j^- + V_j^+} - \frac{V_j^- V_j^+}{V_j^- + V_j^+} \frac{2}{\Delta t S_j} (\mathbf{Q}_j^n - \mathbf{Q}_i^n) \vec{n}_j^T .$$

# **The Cartesian case**

# Three space dimensions

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = 0$$

**Conservative one-step scheme:**

$$Q_{i,j,k}^{n+1} = Q_{i,j,k}^n - \frac{\Delta t}{\Delta x} [F_{i+1/2,j,k} - F_{i-1/2,j,k}] - \frac{\Delta t}{\Delta y} [G_{i,j+1/2,k} - G_{i,j-1/2,k}] - \frac{\Delta t}{\Delta z} [H_{i,j,k+1/2} - H_{i,j,k-1/2}]$$

**Task: to define numerical fluxes**

$$F_{i+1/2,j,k} \quad ; \quad G_{i,j+1/2,k} \quad ; \quad H_{i,j,k+1/2}$$

The FORCE flux in 3D is

$$F_{i+1/2,j,k}^{\text{force}\alpha} = \frac{1}{2}(F_{i+1/2,j,k}^{\text{lwa}} + F_{i+1/2,j,k}^{\text{lfa}})$$

Lax-Wendroff type flux

$$F_{i+1/2,j}^{\text{lwa}} = F(Q_{i+1/2,j}^{\text{lwa}}),$$

$$Q_{i+1/2,j,k}^{\text{lwa}} = \frac{1}{2}(Q_{i,j,k}^n + Q_{i+1,j,k}^n) - \frac{1}{2} \frac{\alpha \Delta t}{\Delta x} (F(Q_{i+1,j,k}^n) - F(Q_{i,j,k}^n))$$

Lax-Friedrichs type flux

$$F_{i+1/2,j,k}^{\text{lfa}} = \frac{1}{2}(F(Q_{i,j,k}^n) + F(Q_{i+1,j,k}^n)) - \frac{1}{2} \frac{\Delta x}{\alpha \Delta t} (Q_{i+1,j,k}^n - Q_{i,j,k}^n)$$

$$\alpha = 3$$

## *One-dimensional interpretation*

$$\partial_t Q + \partial_x F(Q) = 0$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}]$$

$$F_{i+1/2}^{\text{force}\alpha} = \frac{1}{2} (F_{i+1/2}^{\text{lwa}} + F_{i+1/2}^{\text{lfa}})$$

$$F_{i+1/2}^{\text{lwa}} = F(Q_{i+1/2}^{\text{lwa}})$$

$$Q_{i+1/2}^{\text{lwa}} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{1}{2} \frac{\alpha \Delta t}{\Delta x} (F(Q_{i+1}^n) - F(Q_i^n))$$

$$F_{i+1/2}^{\text{lfa}} = \frac{1}{2} (F(Q_i^n) + F(Q_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\alpha \Delta t} (Q_{i+1}^n - Q_i^n)$$

$\alpha$  : parameter

# One-dimensional interpretation

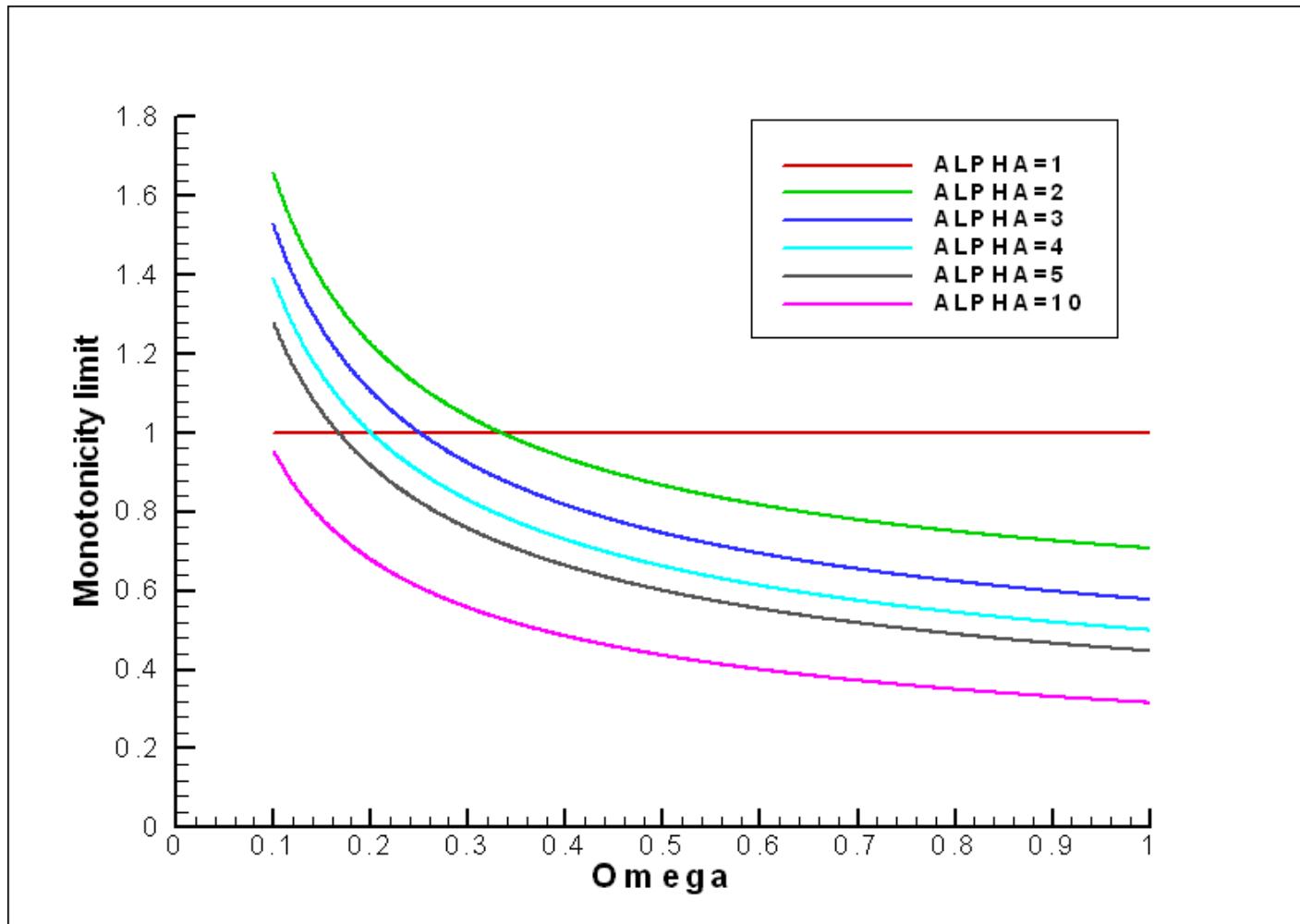
$$\partial_t q(x, t) + \lambda \partial_x q(x, t) = 0$$

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} (f_{i+1/2} - f_{i-1/2})$$

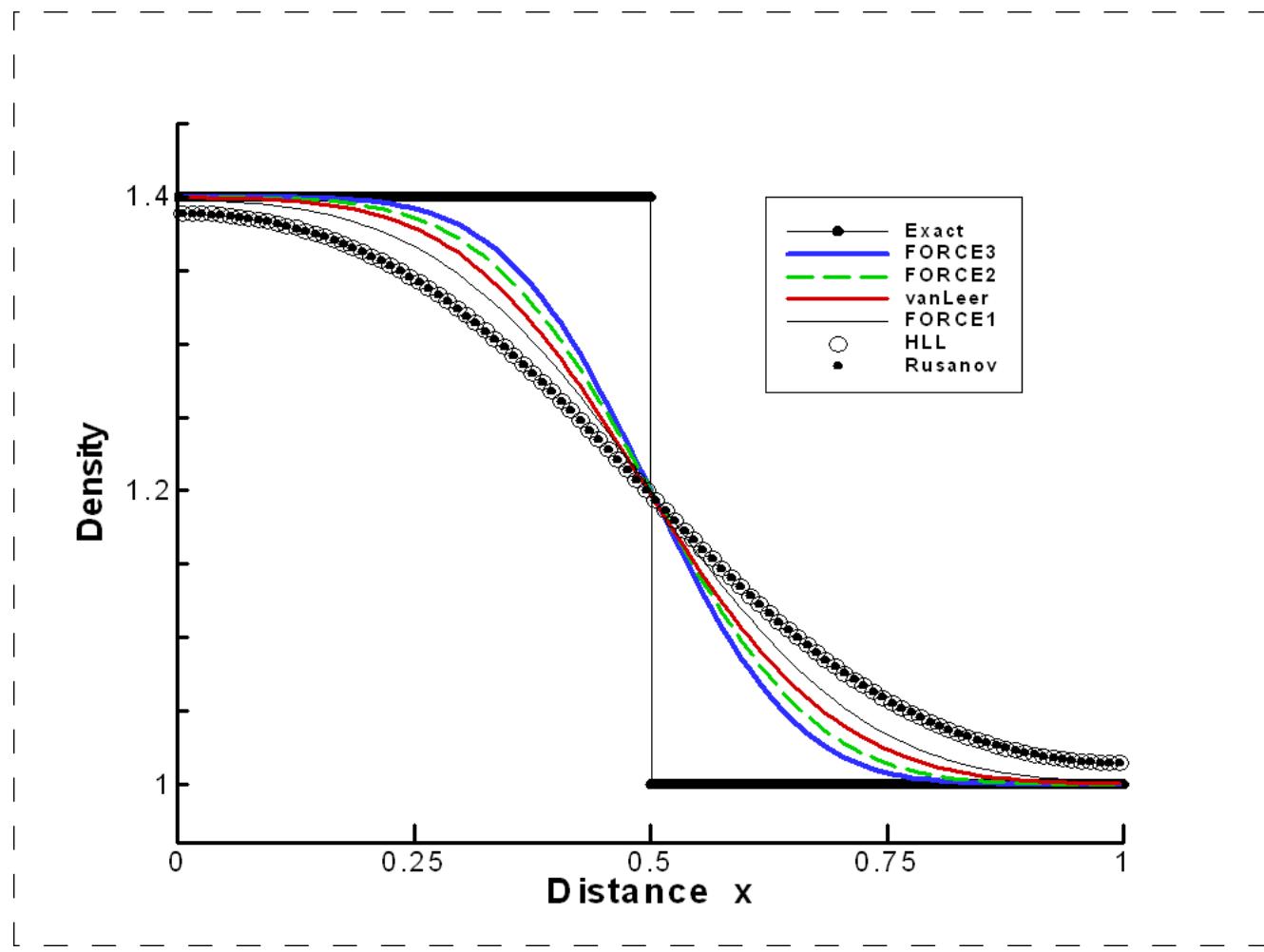
The  $\alpha$  scheme is monotone and linearly stable under the condition

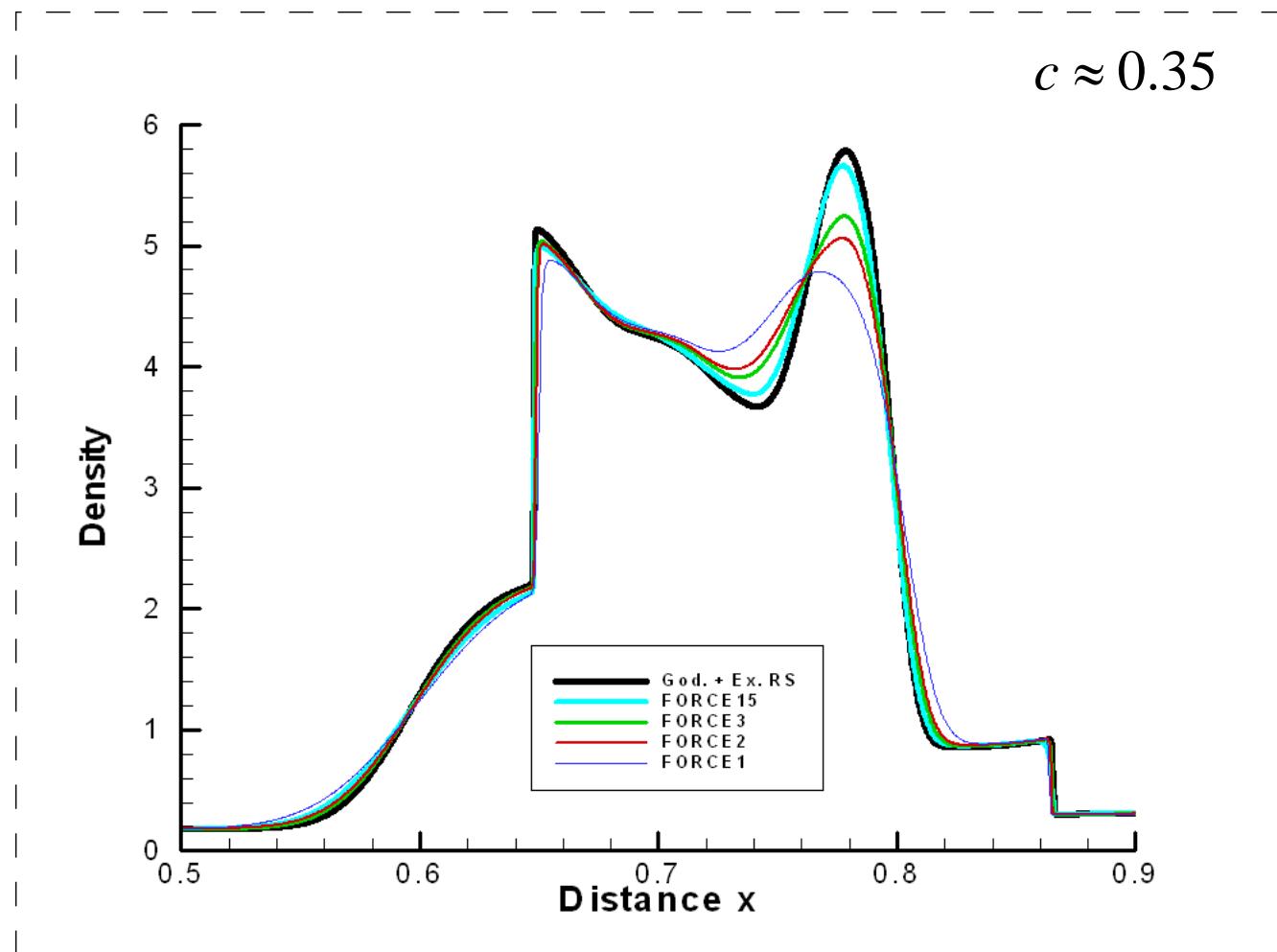
$$c \leq \frac{\sqrt{2\alpha - 1}}{\alpha}$$

$$\underline{\underline{F}}_{j+\frac{1}{2}}^{\text{GFORCE}\alpha} = \omega \underline{\underline{F}}_{j+\frac{1}{2}}^{LW\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) + (1 - \omega) \underline{\underline{F}}_{j+\frac{1}{2}}^{LF\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n)$$



*Numerical results  
for the 1D Euler equations*





*Analysis of the FORCE  
schemes in 1,2 and 3 space  
dimensions*

# Consider model hyperbolic equation

$$\partial_t q + \partial_x f(q) + \partial_y g(q) + \partial_z h(q) = 0$$

$$f(q) = \lambda_1 q ; \quad g(q) = \lambda_2 q ; \quad h(q) = \lambda_3 q ;$$

$$q_{i,j,k}^{n+1} = q_{i,j,k}^n - \frac{\Delta t}{\Delta x} (f_{i+1/2,j,k} - f_{i-1/2,j,k}) - \frac{\Delta t}{\Delta y} (g_{i+1/2,j,k} - g_{i-1/2,j,k}) - \frac{\Delta t}{\Delta z} (h_{i+1/2,j,k} - h_{i-1/2,j,k})$$

Monotonicity, linear stability and numerical viscosity

# FORCE-type fluxes

$$\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{\text{GFORCE}\alpha} = \omega \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LW\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n) + (1 - \omega) \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}^{LF\alpha} (\mathbf{Q}_i^n, \mathbf{Q}_j^n)$$

Stability and monotonicity results

$\omega$	1D	2D	3D
$0 \leq \omega < \frac{1}{2}$	$ c  \leq \frac{1}{\alpha}$	$ c_x ,  c_y  \leq \frac{1}{2}$	$ c_x ,  c_y ,  c_z  \leq \frac{1}{3}$
$\omega = \frac{1}{2}$	$ c  \leq \frac{\sqrt{2\alpha-1}}{\alpha}$	$c_x^2 + c_y^2 \leq \frac{1}{2}$	$c_x^2 + c_y^2 + c_z^2 \leq \frac{1}{3}$
$\frac{1}{2} < \omega < 1$	$ c  \leq \left  \frac{-1+\omega}{\omega\alpha} \right $	$ c_x ,  c_y  \leq \left  \frac{-1+\omega}{2\omega} \right $	$ c_x ,  c_y ,  c_z  \leq \left  \frac{-1+\omega}{3\omega} \right $

## **Part A. Summary:**

**I have presented a new conservative, centred monotone scheme for hyperbolic systems in three space dimensions, solved on general unstructured meshes**

**The flux can be used in the following schemes:**

- **Finite volume**
- **Discontinuous Galerkin FE methods (ADER, RK)**
- **Path-conservative methods (Pares et al. 2006)**

**The flux is the building block of high-order non-linear methods**

# **Part B:**

# **High-Order Methods**

# *The ADER approach*

***Toro E F, Millington R and Nejad L (1999).***

Towards Very High-Order Godunov Schemes. In Godunov Methods: Theory and Applications, Edited Review, E F Toro (Editor), Kluwer/Plenum Academic Publishers, 2001, pp 907-940.

***First written account of schemes for linear problems in 1D,  
2D and 3D on structured meshes.***

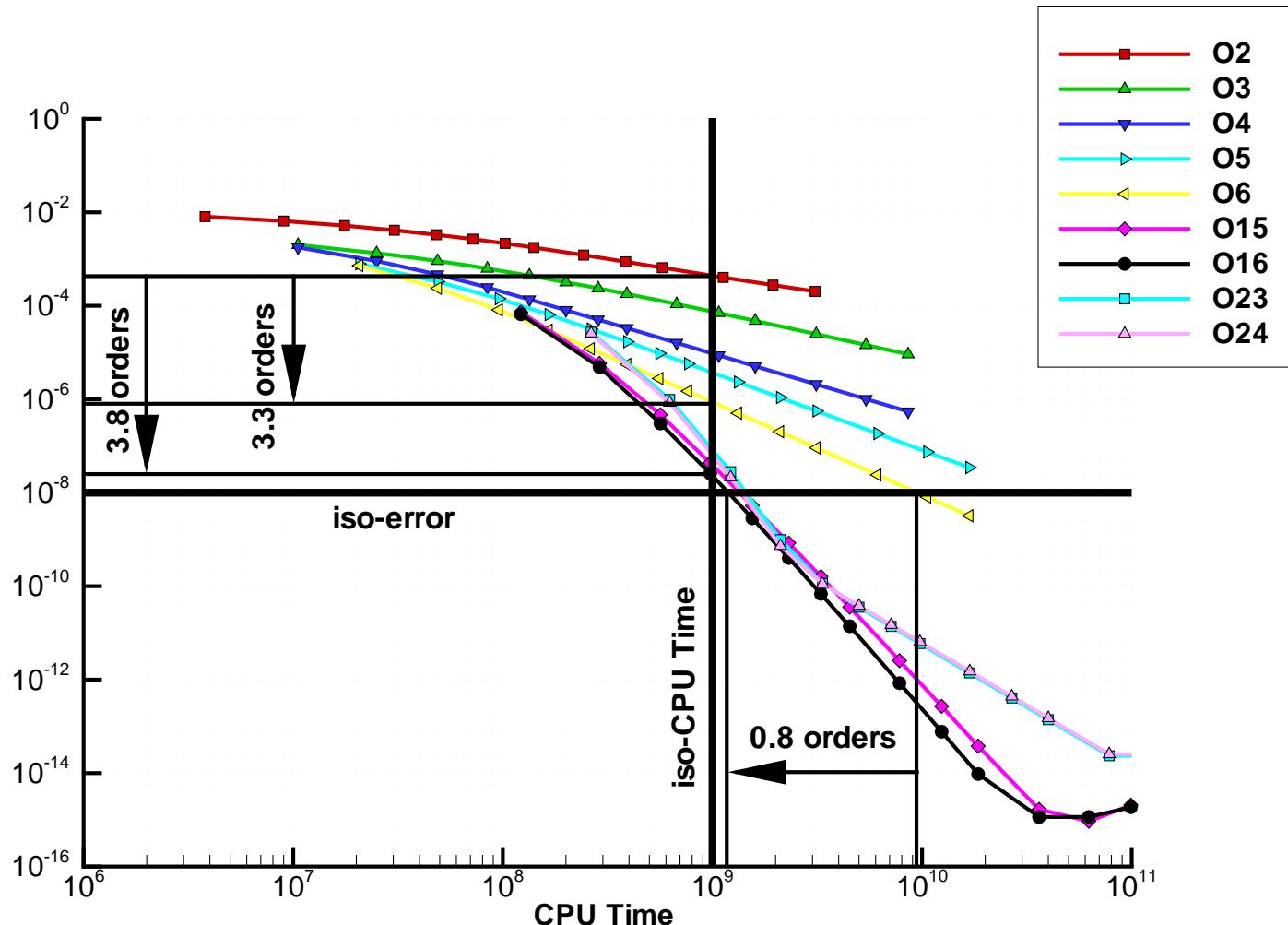
***Further developments:***

***Toro, Titarev, Schwartzkopff, Munz, Dumbser, Kaeser,  
Takakura, Castro, Vignoli, Hidalgo, Enaux, Grosso, Russo,  
Felcman, ...***

*High accuracy.*  
*But why ?*

# Test for acoustics

*ADER*



Collaborators: Munz, Schwatzkoppf (Germany), Dumbser (Trento)

# Exact relation between integral averages

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Integration in space and time  
on control volume

$$[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}] + \Delta t S_i \quad \text{Exact relation}$$

$$\left. \begin{aligned} Q_i^n &= \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} Q(x, 0) dx \\ F_{i+1/2} &= \frac{1}{\Delta t} \int_0^{\Delta t} F(Q_{i+1/2}(\tau)) d\tau \\ S_i &= \frac{1}{\Delta t} \frac{1}{\Delta x} \int_0^{\Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x, t)) dx dt \end{aligned} \right\} \text{Integral averages}$$

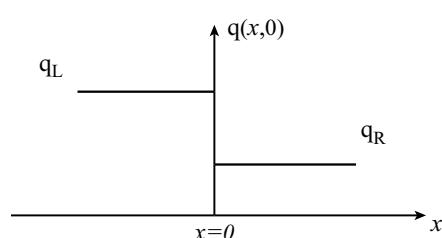
# Godunov's finite volume scheme in 1D

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}] \quad \text{Conservative formula}$$

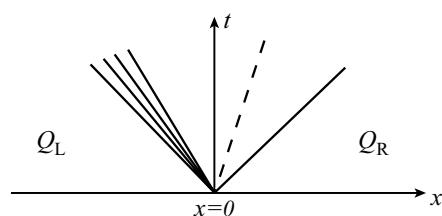
$$F_{i+1/2} = \frac{1}{\Delta t} \int_0^{\Delta t} F(Q_{LR}(\tau)) d\tau \quad \text{Godunov's numerical flux}$$

$Q_{LR}(\tau)$  : Solution of classical Riemann problem

Classical Riemann Problem



$$\left. \begin{aligned} \partial_t Q + \partial_x F(Q) &= 0 \\ Q(x,0) &= \begin{cases} Q_i^n & \text{if } x < 0 \\ Q_{i+1}^n & \text{if } x > 0 \end{cases} \end{aligned} \right\} \Rightarrow Q^{(0)}(x/t)$$



$$F_{i+1/2} = \frac{1}{\Delta t} \int_0^{\Delta t} F(Q^{(0)}(0)) d\tau = F(Q^{(0)}(0))$$

## ***Illustration of ADER finite volume method***

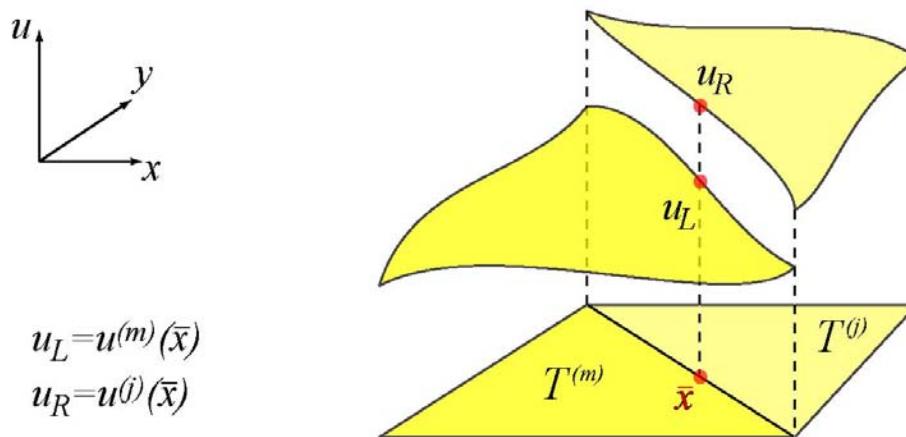
$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Control volume in  
computational domain       $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}] + \Delta t S_i \quad \text{Update formula}$$

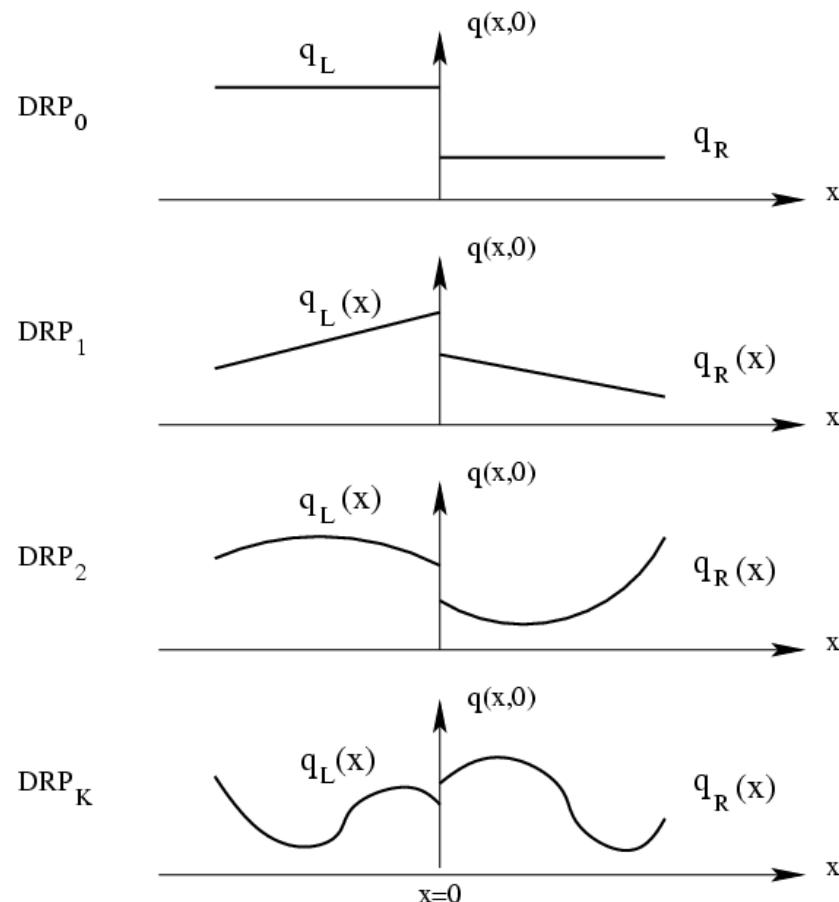
Integral average at time n	$Q_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} Q(x, 0) dx$	}
Numerical flux	$F_{i+1/2} = \frac{1}{\Delta t} \int_0^{\Delta t} F(Q_{i+1/2}(\tau)) d\tau$	
Numerical source	$S_i = \frac{1}{\Delta t} \frac{1}{\Delta x} \int_0^{\Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x, t)) dx dt$	

# *ADER on 2D unstructured meshes*



The numerical flux requires the calculation of an integral in space along  
The volume/element interface and in time.

## Local Riemann problems from high-order representation of data



*Key ingredient:*  
*the high-order*  
*(or generalized)*  
*Riemann problem*

**The high-order (or derivative, or generalized)  
Riemann problem:**

$$\left. \begin{aligned} \partial_t Q + \partial_x F(Q) &= S(Q) \\ Q(x,0) &= \begin{cases} Q_L(x) & \text{if } x < 0 \\ Q_R(x) & \text{if } x > 0 \end{cases} \end{aligned} \right\} \text{GRP}_K$$

**Initial conditions: two smooth functions**

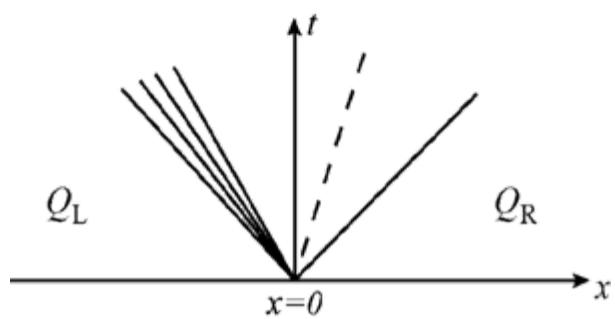
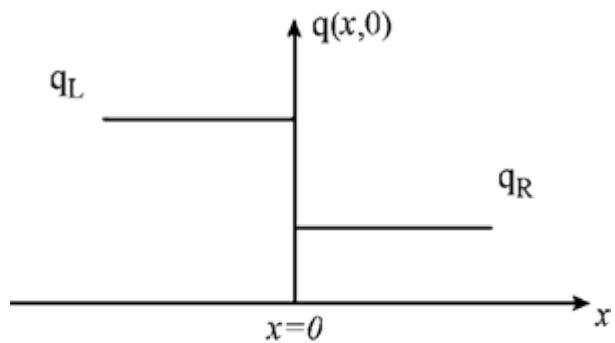
$$Q_L(x), Q_R(x)$$

**For example, two polynomials of degree K**

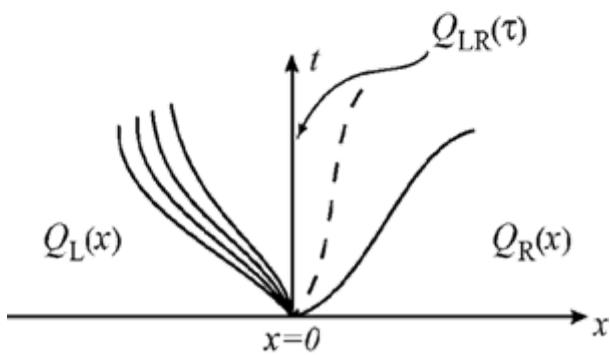
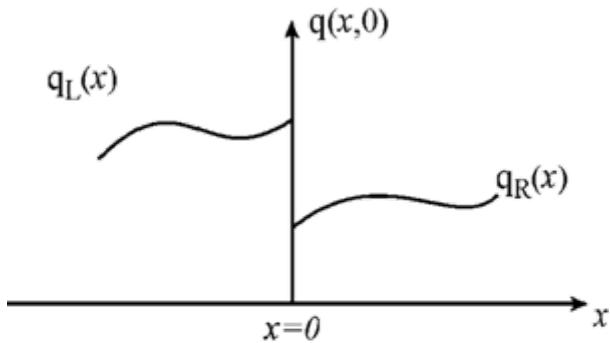
The generalization is twofold:

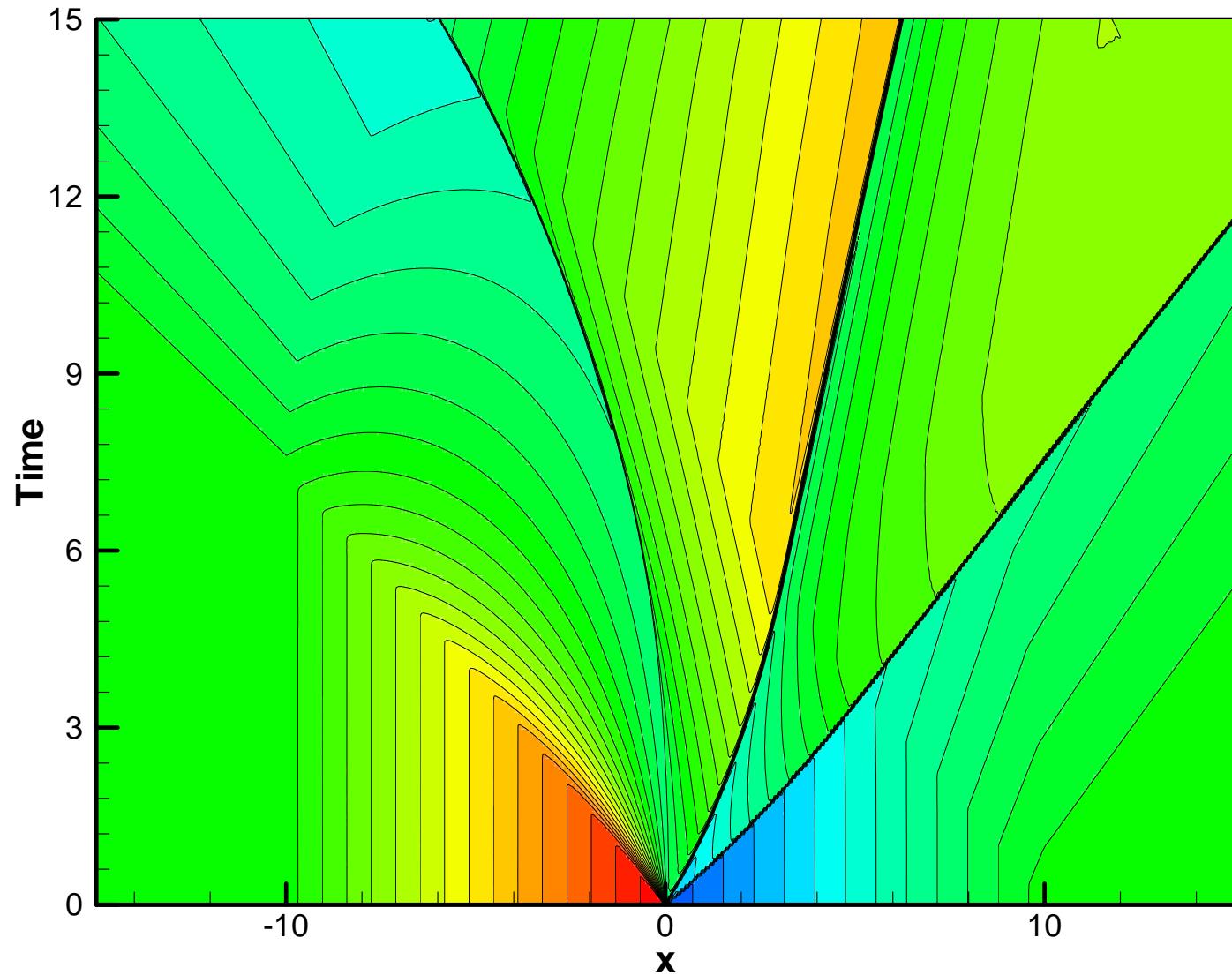
- (1) the intial conditions are two polynomials of arbitrary degree
- (2) The equations include source terms

Classical Riemann Problem



Derivative Riemann Problem





# *Four solvers:*

E F Toro and V A Titarev. Solution of the generalized Riemann problem for advection-reaction equations. Proc. Royal Society of London, A, Vol. 458, pp 271-281, 2002.

E F Toro and V A Titarev. Derivative Riemann solvers for systems of conservation laws and ADER methods. Journal Computational Physics Vol. 212, pp 150-165, 2006

C E Castro and E F Toro. Solvers for the high-order Riemann problem for hyperbolic balance laws. Journal Computational Physics Vol. 227, pp 2482-2513,, 2008

M Dumbser, C Enaux and E F Toro. Finite volume schemes of very high order of accuracy for stiff hyperbolic balance laws . Journal of Computational Physics, Vol 227, pp 3971-4001, 2008.

# Solver 1

*Toro E. F. and Titarev V. A. Proc. Roy. Soc. London. Vol. 458, pp 271-281, 2002*

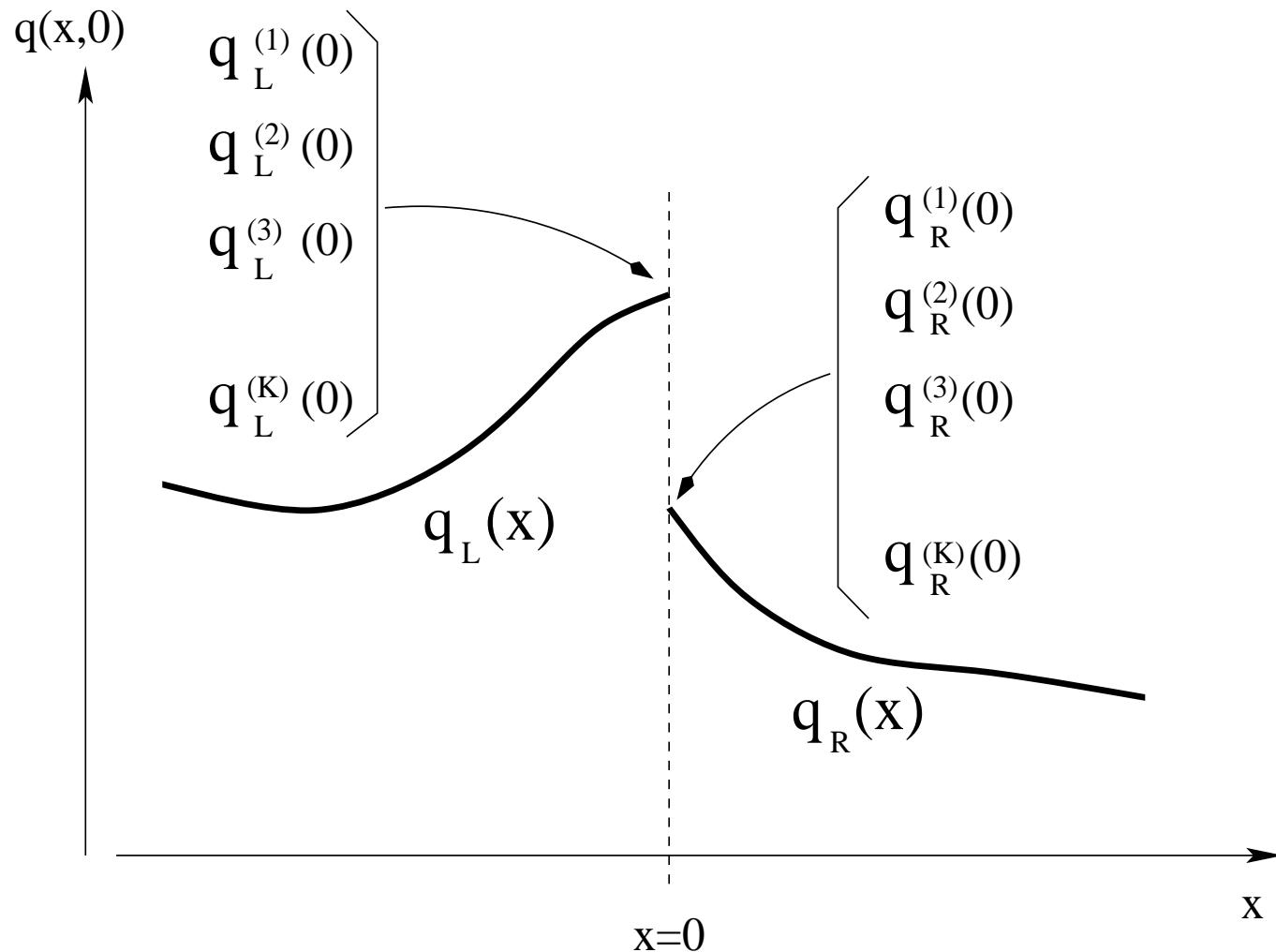
*Toro E. F. and Titarev V. A. J. Comp. Phys. Vol. 212, No. 1, pp. 150-165, 2006.*

$$Q_{LR}(\tau) = Q(0,0_+) + \sum_{k=1}^K \partial_t^{(k)} Q(0,0_+) \frac{\tau^k}{k!}$$

(Based on work of Ben-Artzi and Falcovitz, 1984, see also Raviart and LeFloch 1989)

**The leading term  
and  
higher-order terms**

# Initial conditions



# Computing the leading term:

Solve the *classical* RP

$$\left. \begin{array}{l} \partial_t Q + \partial_x F(Q) = 0 \\ Q(x,0) = \begin{cases} Q_L(0) & \text{if } x < 0 \\ Q_R(0) & \text{if } x > 0 \end{cases} \end{array} \right\}$$

**Solution:**  $D^{(0)}(x/t)$

**Take Godunov state at  $x/t=0$**

**Leading term:**  $Q(0,0_+) = D^{(0)}(0)$

# Computing the higher-order terms:

First use the Cauchy-Kowalewski (\*) procedure yields

$$\partial_t^{(k)} Q(x, t) = G^{(k)}(\partial_x^{(0)} Q, \dots, \partial_x^{(k)} Q)$$

**Example:**

$$\partial_t q + \lambda \partial_x q = 0 \Rightarrow \begin{cases} \partial_t q &= -\lambda \partial_x q \\ \partial_t^{(2)} q &= (-\lambda)^2 \partial_x^{(2)} q \\ \partial_t^{(m)} q &= (-\lambda)^m \partial_x^{(m)} q \end{cases}$$

Must define spatial derivatives at  $x=0$  for  $t>0$

(\*) Cauchy-Kowalewski theorem. One of the most fundamental results in the theory of PDEs. Applies to problems in which all functions involved are analytic.

# Computing the higher-order terms

Then construct evolution equations for the variables:

$$\partial_x^{(k)} Q(x, t)$$

Note:

$$\partial_t q + \lambda \partial_x q = 0 \Rightarrow \partial_t (\partial_x q) + \lambda \partial_x (\partial_x q) = 0$$

For the general case it can be shown that:

$$\partial_t (\partial_x^{(k)} Q) + A(Q) \partial_x (\partial_x^{(k)} Q) = H^{(k)}(\partial_x^{(0)} Q, \partial_x^{(1)} Q, \dots, \partial_x^{(k)} Q)$$

Neglecting source terms and linearizing we have

$$\partial_t (\partial_x^{(k)} Q) + A(Q(0, 0_+)) \partial_x (\partial_x^{(k)} Q) = 0$$

# Computation of higher-order terms

For each  $k$  solve *classical* Riemann problem:

$$\left. \begin{aligned} \partial_t(\partial_x^{(k)} Q) + A(Q(0,0_+))\partial_x(\partial_x^{(k)} Q) &= 0 \\ \partial_x^{(k)} Q(x,0) &= \begin{cases} \partial_x^{(k)} Q_L(0) & \text{if } x < 0 \\ \partial_x^{(k)} Q_R(0) & \text{if } x > 0 \end{cases} \end{aligned} \right\}$$

**Similarity solution**  $D^{(k)}(x/t)$

**Evaluate solution at  $x/t=0$**

**All spatial derivatives at  $x=0$  are now defined**

$$\partial_x^{(k)} Q(0,0_+) = D^{(k)}(0)$$

# Computing the higher-order terms

All time derivatives at  $x=0$  are then defined

$$\partial_t^{(k)} Q(0,0_+) = G^{(k)}(\partial_x^{(0)} Q(0,0_+), \dots, \partial_x^{(k)} Q(0,0_+))$$

Solution of DRP is

$$Q_{LR}(\tau) = Q(0,0_+) + \sum_{k=1}^K \partial_t^{(k)} Q(0,0_+) \frac{\tau^k}{k!}$$

**GRP-K = 1 (non-linear RP) + K (linear RPs)**

Options: state expansion and flux expansion

## ***Illustration of ADER finite volume method***

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Control volume in  
computational domain       $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} [F_{i+1/2} - F_{i-1/2}] + \Delta t S_i \quad \text{Update formula}$$

Integral average at time n	$Q_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} Q(x, 0) dx$	}
Numerical flux	$F_{i+1/2} = \frac{1}{\Delta t} \int_0^{\Delta t} F(Q_{i+1/2}(\tau)) d\tau$	
Numerical source	$S_i = \frac{1}{\Delta t} \frac{1}{\Delta x} \int_0^{\Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x, t)) dx dt$	

# The latest solver

M Dumbser, C Enaux and E F Toro. Finite volume schemes of very high order of accuracy for stiff hyperbolic balance laws. *Journal of Computational Physics*, Vol 227, pp 3971-4001, 2008.

## **Extends Harten's method (1987)\*\***

A. Harten, B. Engquist, S. Osher, and S.R. Chakravarthy. Uniformly high order accurate essentially non-oscillatory schemes III. *Journal of Computational Physics*, 71:231–303, 1987.

- **Evolves data left and right prior to “time-interaction”**
- **Evolution of data is done numerically by an implicit space-time DG method**
- **The solution of the LOCAL generalized Riemann problem is therefore implicit**
- **The scheme remains globally explicit**
- **Stiff source terms can be treated adequately**
- **Reconciles stiffness with high accuracy in both space and time**

\*\*C E Castro and E F Toro. Solvers for the high-order Riemann problem for hyperbolic balance laws. *Journal of Computational Physics* Vol. 227, pp 2482-2513, 2008

# *Main features of ADER schemes*

*One-step fully discrete schemes*

*Reconstruction done once per time step*

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = S(Q)$$

$$\partial_t Q + A\partial_x Q + B\partial_y Q + C\partial_z Q = S(Q)$$

*Accuracy in space and time is arbitrary*

*Unified framework*

*General meshes*

*Finite volume, DG finite element and Path-conservative  
formulations*

Part C:

# APPLICATIONS

# *Main applications so far*

*1, 2, 3 D Euler equations on unstructured meshes*

*3D Navier-Stokes equations*

*Reaction-diffusion (parabolic equations)*

*Sediment transport in water flows (single phase)*

*Two-phase sediment transport (Pitman and Le model)*

*Two-layer shallow water equations*

*Aeroacoustics in 2 and 3D*

*Seismic wave propagation in 3D*

*Tsunami wave propagation*

*Magnetohydrodynamics*

*3D Maxwell equations*

*3D compressible two-phase flow, etc.*

## Two and three-dimensional Euler equations

# Two space dimensions

2D Euler equations on unstructured triangular meshes

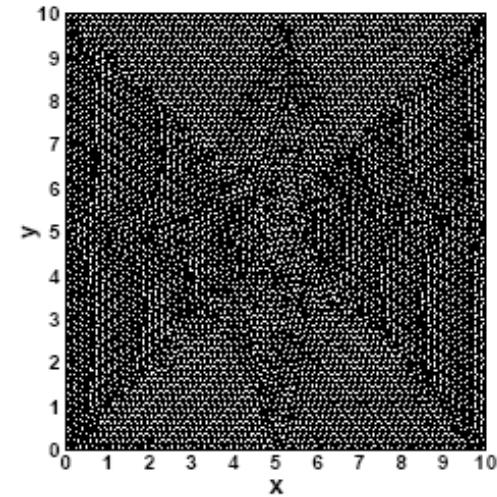
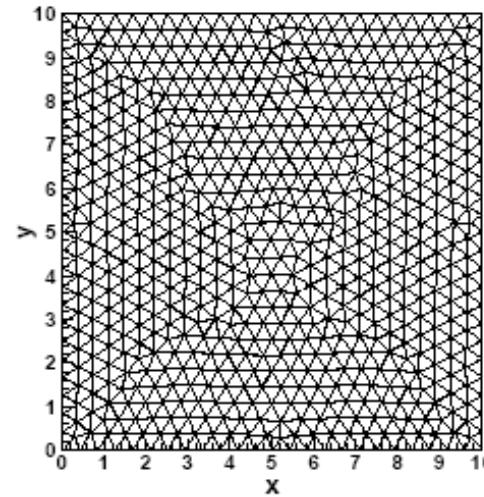
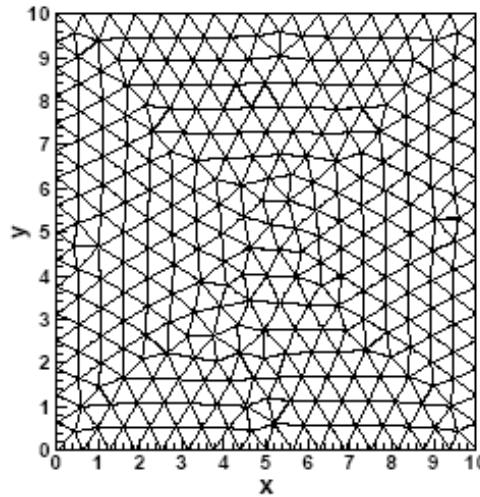


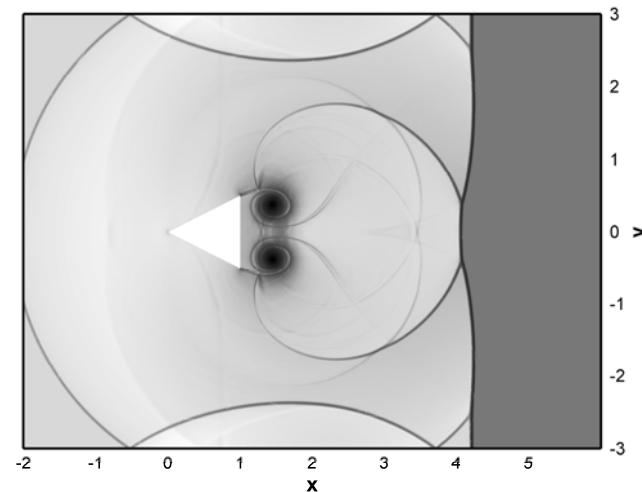
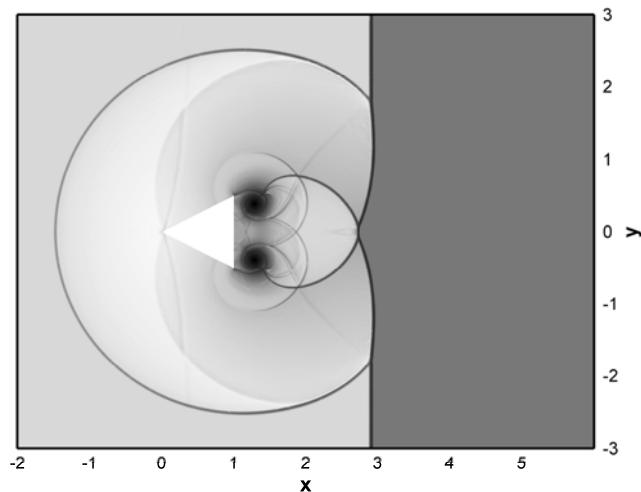
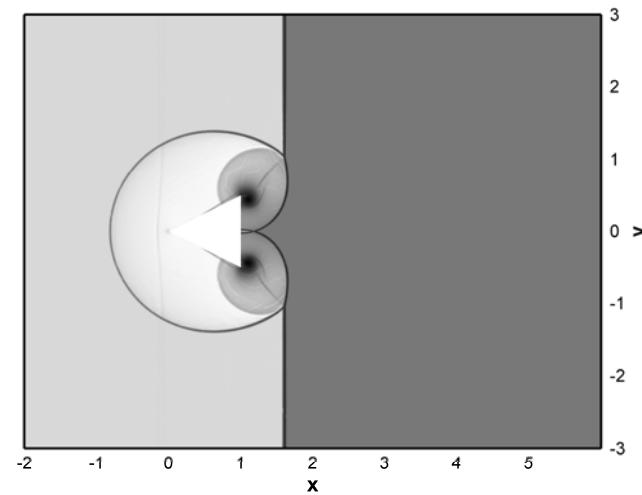
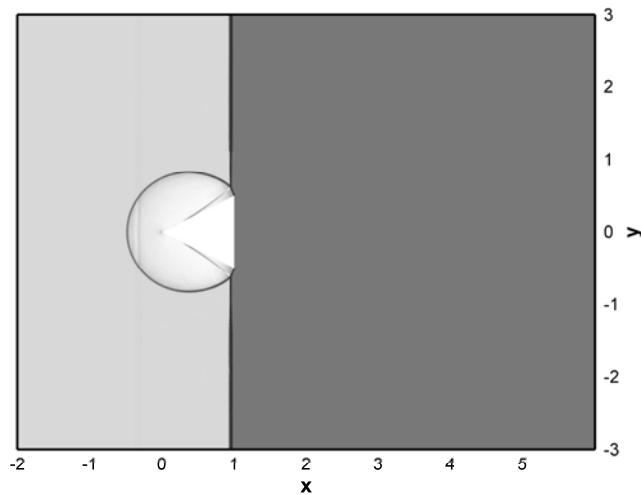
Table 3

Numerical convergence results obtained with  $P_0P_M$  finite volume schemes with the new unstructured FORCE flux (left), the Rusanov scheme (middle) and the HLLE flux (right). Second to sixth order in space and time.

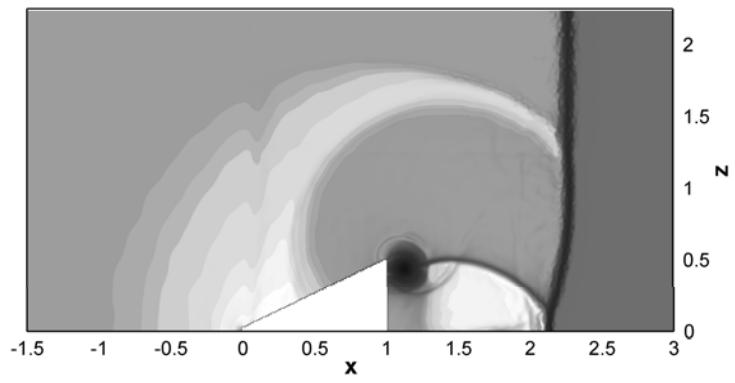
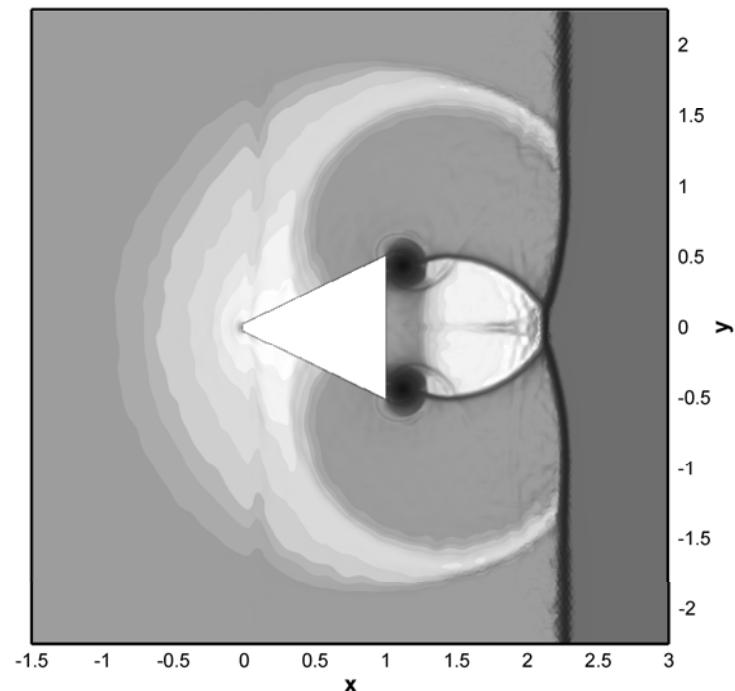
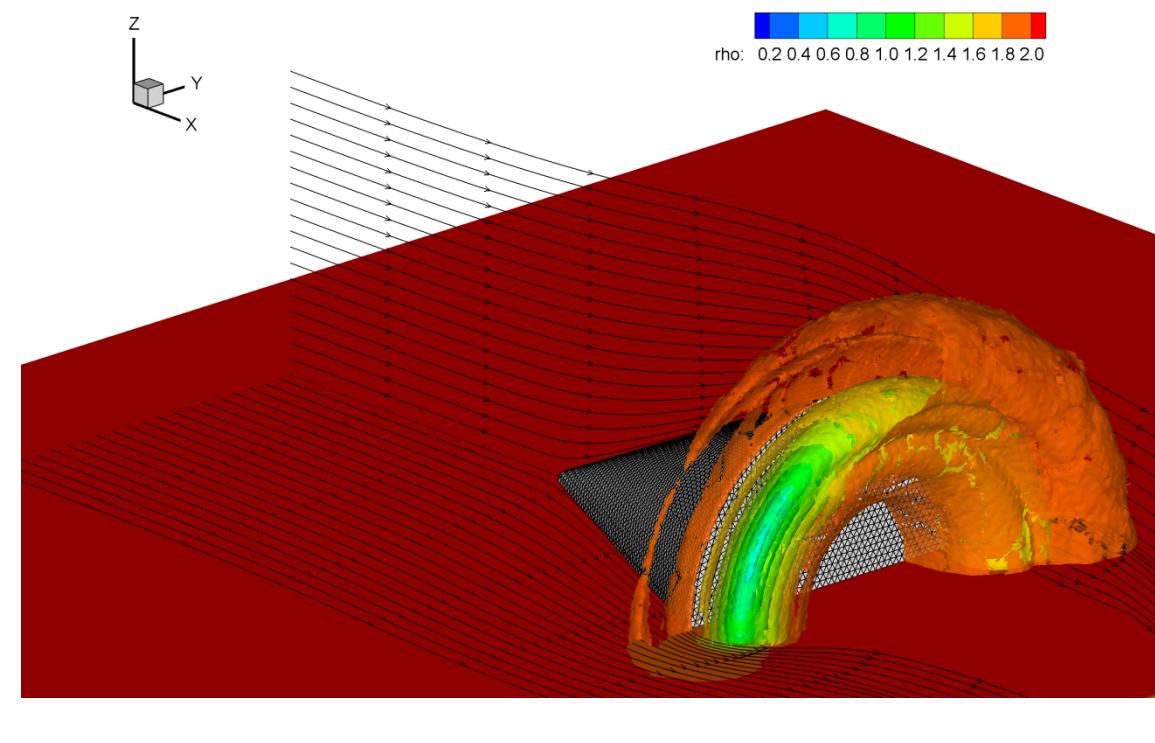
$N_G$	$L^2$	$\mathcal{O}_{L^2}$	$t_{\text{CPU}}[s]$	$L^2$	$\mathcal{O}_{L^2}$	$t_{\text{CPU}}[s]$	$L^2$	$\mathcal{O}_{L^2}$	$t_{\text{CPU}}[s]$
FORCE				Rusanov				HLLE	
$P_0P_1 (\mathcal{O}2)$									
16	3.40E-01		1.0	3.87E-01		1.1	2.68E-01		1.1
32	1.06E-01	1.7	9.2	1.27E-01	1.6	9.0	7.84E-02	1.8	9.1
64	3.13E-02	1.8	67.6	3.78E-02	1.7	71.7	2.17E-02	1.9	72.3
128	9.08E-03	1.8	626.1	9.97E-03	1.9	659.7	5.31E-03	2.0	671.1
$P_0P_2 (\mathcal{O}3)$									
16	3.04E-01		1.8	3.46E-01		1.9	2.47E-01		2.0
32	7.17E-02	2.1	14.5	8.67E-02	2.0	17.1	5.14E-02	2.3	15.3
64	1.35E-02	2.4	112.0	1.66E-02	2.4	120.0	9.91E-03	2.4	120.4
128	2.13E-03	2.7	1053	2.40E-03	2.8	1139	1.37E-03	2.9	1075

$P_0 P_3$ ( $\mathcal{O}4$ )								
16	7.01E-02		3.5	8.44E-02		3.5	5.89E-02	3.7
32	1.57E-02	2.2	27.8	1.91E-02	2.1	27.0	1.15E-02	2.4
64	1.10E-03	3.8	199.0	1.41E-03	3.8	207.5	8.27E-04	3.8
128	7.62E-05	3.8	1994	8.78E-05	4.0	1989	5.13E-05	4.0
$P_0 P_4$ ( $\mathcal{O}5$ )								
16	6.41E-02		7.2	7.54E-02		6.8	5.32E-02	8.0
32	1.30E-02	2.3	50.0	1.58E-02	2.3	51.7	1.01E-02	2.4
64	6.62E-04	4.3	390.0	8.31E-04	4.2	397.7	5.18E-04	4.3
128	2.75E-05	4.6	3501	3.14E-05	4.7	3604	1.88E-05	4.8
$P_0 P_5$ ( $\mathcal{O}6$ )								
16	4.79E-02		13.7	5.18E-02		13.5	5.27E-02	18.1
32	3.07E-03	4.0	108.3	3.56E-03	3.9	105.8	2.39E-03	4.5
64	8.73E-05	5.1	767.1	1.08E-04	5.0	783.4	6.58E-05	5.2
128	1.37E-06	6.0	6329	1.49E-06	6.2	6450	9.06E-07	6.2

## *2D Euler equations: reflection from triangle*



# *3D Euler equations: reflection from cone*



# Baer-Nunziato equations

# The Baer-Nunziato equations in 1D

$$\partial_t Q + \partial_x F(Q) + T(Q) \partial_x \bar{\alpha} = S(Q)$$

$$Q = \begin{bmatrix} \bar{\alpha}\rho \\ \bar{\alpha}\rho\bar{u} \\ \bar{\alpha}\rho\bar{E} \\ \bar{\alpha} \\ \hat{\alpha}\hat{\rho} \\ \hat{\alpha}\hat{\rho}\hat{u} \\ \hat{\alpha}\hat{\rho}\hat{E} \end{bmatrix} \quad F(Q) = \begin{bmatrix} \bar{\alpha}\rho\bar{u} \\ \bar{\alpha}\rho\bar{u}^2 + \bar{\alpha}p \\ \bar{u}(\bar{\alpha}\rho\bar{E} + \bar{\alpha}p) \\ 0 \\ \hat{\alpha}\hat{\rho}\hat{u} \\ \hat{\alpha}\hat{\rho}\hat{u}^2 + \hat{\alpha}\hat{p} \\ \hat{u}(\hat{\alpha}\hat{\rho}\hat{E} + \hat{\alpha}\hat{p}) \end{bmatrix} \quad T(Q) = \begin{bmatrix} 0 \\ -P_i \\ -PV_i \\ V_i \\ 0 \\ P_i \\ PV_i \end{bmatrix} \quad S(Q) = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{bmatrix}$$

$\bar{Q}$  : Solid phase       $\hat{Q}$  : Gas phase

$$V_i = \bar{u} \quad P_i = \hat{p}$$

# *Application of ADER to the 3D Baer-Nunziato equations*

$$\left. \begin{aligned} \frac{\partial}{\partial t} (\phi_1 \rho_1) + \nabla \cdot (\phi_1 \rho_1 \mathbf{u}_1) &= 0, \\ \frac{\partial}{\partial t} (\phi_1 \rho_1 \mathbf{u}_1) + \nabla \cdot (\phi_1 \rho_1 \mathbf{u}_1 \otimes \mathbf{u}_1) + \nabla \phi_1 p_1 &= p_I \nabla \phi_1 + \lambda (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} (\phi_1 \rho_1 E_1) + \nabla \cdot ((\phi_1 \rho_1 E_1 + \phi_1 p_1) \mathbf{u}_1) &= -p_I \partial_t \phi_1 + \lambda \mathbf{u}_I \cdot (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} (\phi_2 \rho_2) + \nabla \cdot (\phi_2 \rho_2 \mathbf{u}_2) &= 0, \\ \frac{\partial}{\partial t} (\phi_2 \rho_2 \mathbf{u}_2) + \nabla \cdot (\phi_2 \rho_2 \mathbf{u}_2 \otimes \mathbf{u}_2) + \nabla \phi_2 p_2 &= p_I \nabla \phi_2 - \lambda (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} (\phi_2 \rho_2 E_2) + \nabla \cdot ((\phi_2 \rho_2 E_2 + \phi_2 p_2) \mathbf{u}_2) &= p_I \partial_t \phi_1 - \lambda \mathbf{u}_I \cdot (\mathbf{u}_2 - \mathbf{u}_1), \\ \frac{\partial}{\partial t} \phi_1 + \mathbf{u}_I \nabla \phi_1 &= 0. \end{aligned} \right\} \quad (54)$$

12 PDES  
 Stiff source terms: relaxation terms

## **EXTENSION TO NONCONSERVATIVE SYSTEMS: Path-conservative schemes**

DUMBESER M, HIDALGO A, CASTRO M, PARES C, TORO E F. (2009).  
FORCE Schemes on Unstructured Meshes II: Nonconservative Hyperbolic Systems.  
(Under review)

**Also published (NI09005-NPA) in pre-print series of the  
Newton Institute for Mathematical Sciences  
University of Cambridge, UK.**

**It can be downloaded from**  
<http://www.newton.ac.uk/preprints2009.html>

CASTRO M, PARDO A, PARES C, TORO E F (2009).  
ON SOME FAST WELL-BALANCED FIRST ORDER SOLVERS FOR  
NONCONSERVATIVE SYSTEMS.  
MATHEMATICS OF COMPUTATION. ISSN: 0025-5718. Accepted.

*Three space dimensions*

*Unstructured meshes*

*Path-conservative method*

*Centred non-conservative FORCE is building block*

*ADER: high-order of accuracy in space and time*

*(implemented upto 6-th order in space and time)*

# **Reference solutions to the BN equations**

**Exact Riemann solver of Schwendemann et al.  
(2006) (1D)**

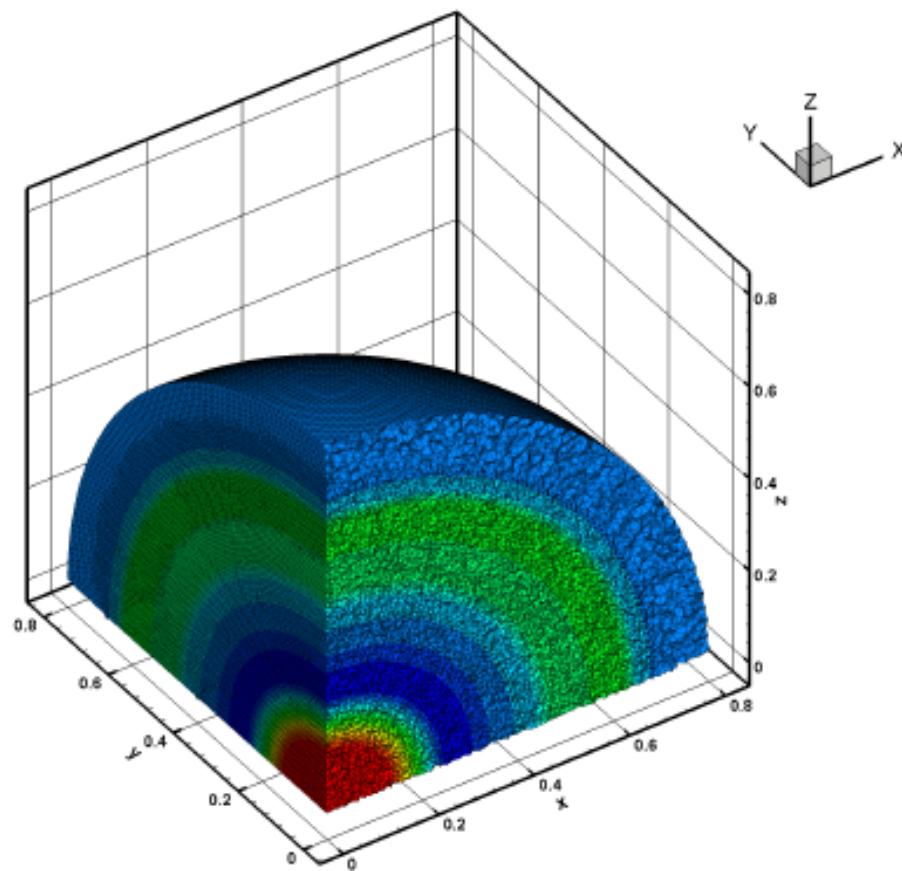
**Exact smooth solution the 2D BN equations to be  
used in convergence rate studies (Dumbser et al.  
2009)**

**Spherically symmetric 3D BN equations reduced to  
1D system with geometric source terms. This is used  
to test 2 and 3 dimensional solutions with shocks  
(Dumbser et al. 2009)**

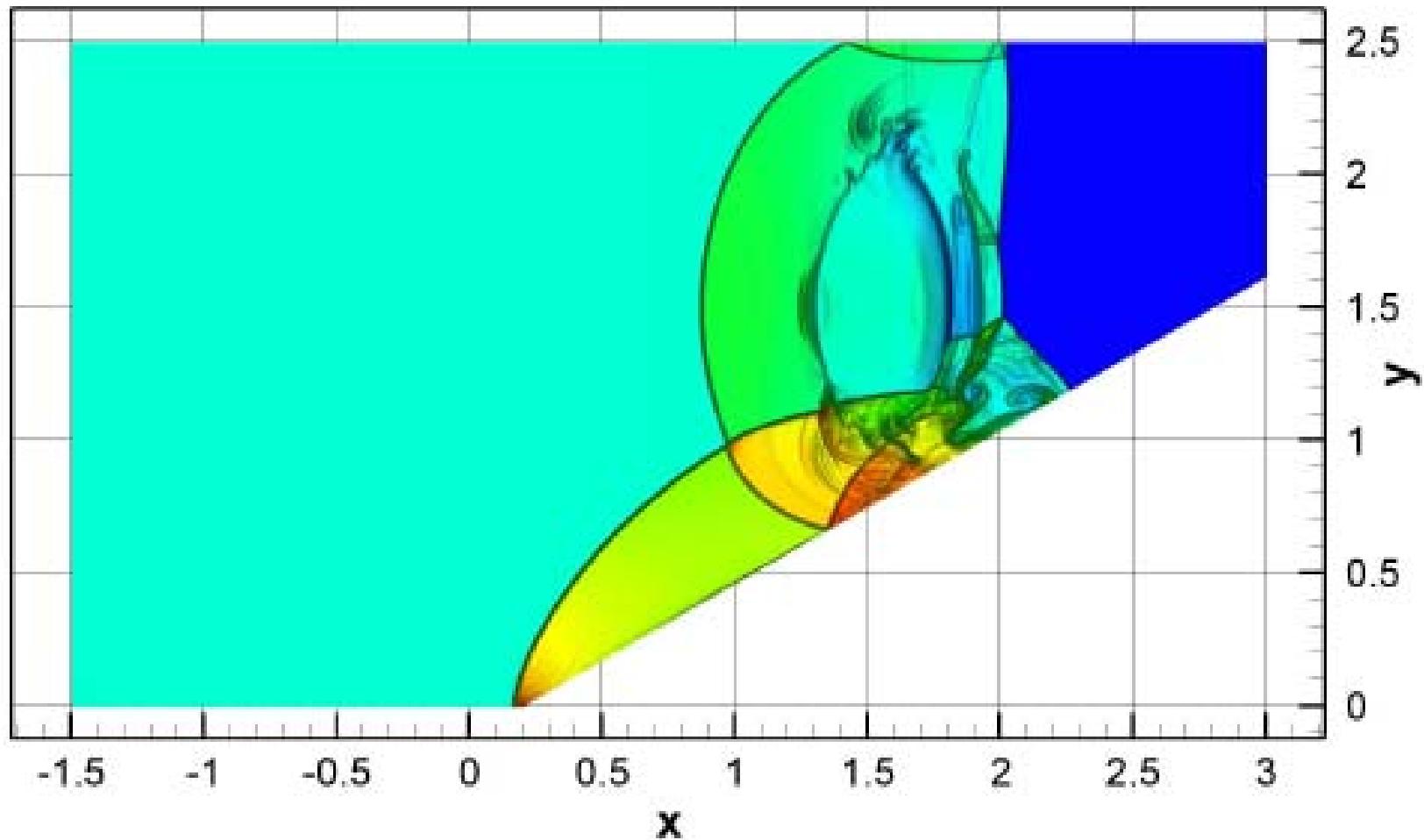
## Convergence rates study in 2D unstructured meshes

$N_G$	$L^2$	$\mathcal{O}_{L^2}$										
$\mathcal{O}2$		$P_0P_1$		$P_1P_1$								
64/24	1.86E-01		2.04E-01									
128/48	5.94E-02	1.7	3.04E-02	2.7								
192/64	2.80E-02	1.9	1.45E-02	2.6								
256/128	1.75E-02	1.6	1.92E-03	2.9								
$\mathcal{O}3$		$P_0P_2$		$P_1P_2$		$P_2P_2$						
32 /16	5.09E-01		2.77E-01		5.59E-02							
64 /24	1.63E-01	1.6	8.97E-02	2.8	1.67E-02	3.0						
128/32	3.50E-02	2.2	2.91E-02	3.9	6.56E-03	3.2						
192/64	1.16E-02	2.7	2.07E-03	3.8	7.84E-04	3.1						
$\mathcal{O}4$		$P_0P_3$		$P_1P_3$		$P_2P_3$		$P_3P_3$				
32 /16	1.71E-01		1.95E-01		2.14E-02		1.77E-02					
64 /24	1.71E-02	3.3	4.95E-02	3.4	3.79E-03	4.3	2.46E-03	4.9				
128/32	1.28E-03	3.7	1.45E-02	4.3	8.95E-04	5.0	5.61E-04	5.1				
192/64	2.80E-04	3.7	5.16E-04	4.8	3.94E-05	4.5	2.07E-05	4.8				
$\mathcal{O}5$		$P_0P_4$		$P_1P_4$		$P_2P_4$		$P_3P_4$		$P_4P_4$		
32 /16	2.09E-01		9.85E-02		9.70E-03		5.22E-03		1.79E-03			
64 /24	2.30E-02	3.2	1.75E-02	4.3	1.18E-03	5.2	5.56E-04	5.5	2.24E-04	5.1		
128/32	1.16E-03	4.3	3.27E-03	5.8	2.09E-04	6.0	8.36E-05	6.6	4.36E-05	5.7		
192/64	1.63E-04	4.8	4.53E-05	6.2	7.23E-06	4.9	2.28E-06	5.2	1.75E-06	4.6		
$\mathcal{O}6$		$P_0P_5$		$P_1P_5$		$P_2P_5$		$P_3P_5$		$P_4P_5$		$P_5P_5$
32 / 8	8.45E-02		5.50E-01		1.49E-01		6.22E-02		5.90E-02		2.76E-02	
64 /16	3.09E-03	4.8	8.72E-02	2.7	5.90E-03	4.7	1.73E-03	5.2	6.12E-04	6.6	4.69E-04	5.9
128/24	5.95E-05	5.7	1.46E-02	4.4	6.18E-04	5.6	1.39E-04	6.2	4.18E-05	6.6	3.72E-05	6.2
192/32	5.39E-06	5.9	2.39E-03	6.3	8.31E-05	7.0	2.17E-05	6.5	5.12E-06	7.3	4.99E-06	7.0

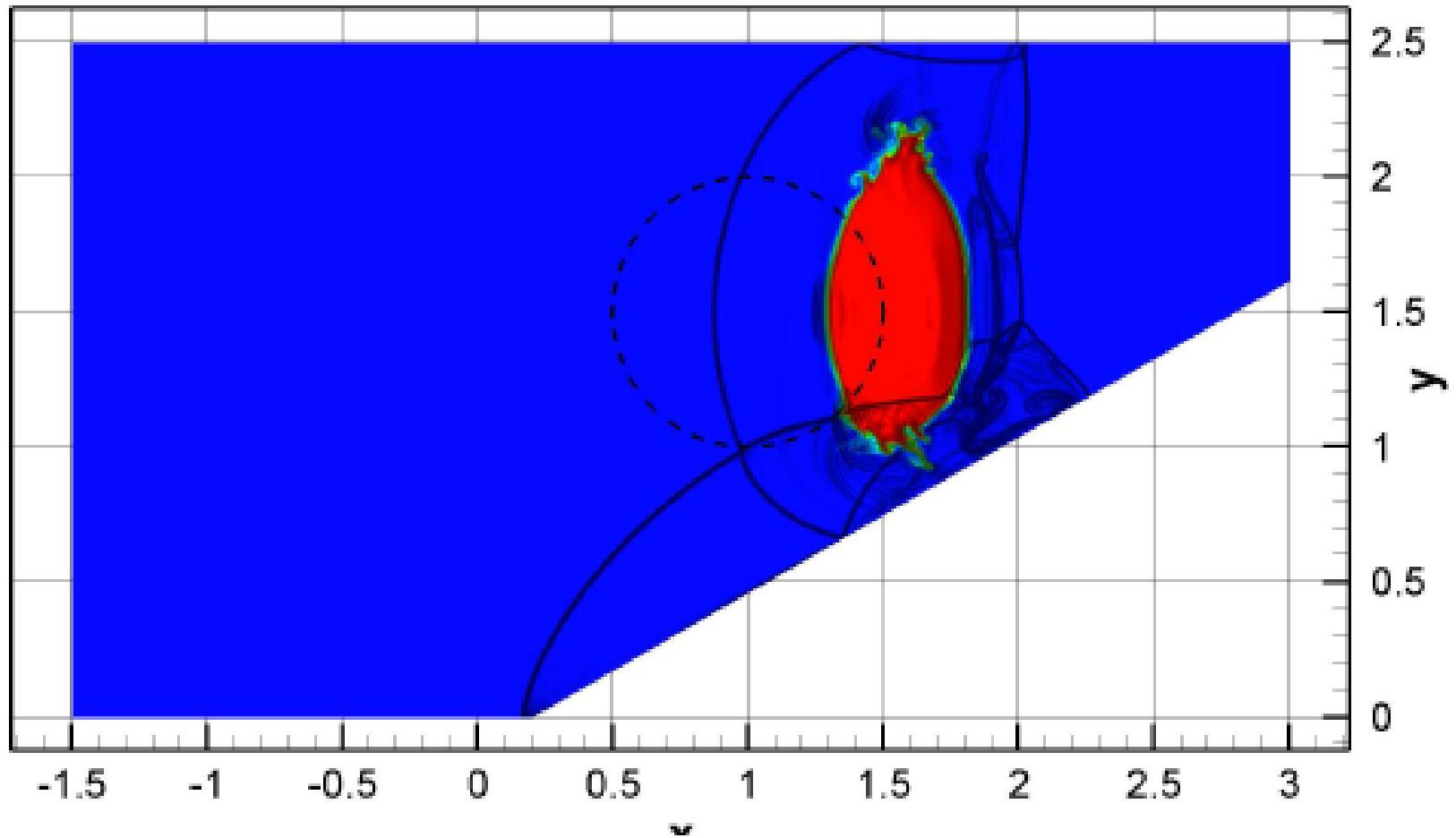
# BN equations: spherical explosion test



# Double Mach reflection for the 2D Baer-Nunziato equations



# Double Mach reflection for the 2D Baer-Nunziato equations



## **Summary:**

**Part A: FORCE scheme in 2 and 3D on general meshes.**

**Part B: ADER high-order schemes**

**Part C: Application to the 3D Euler and Baer-Nunziato two-phase flow equations**

*Thank you*

