

Characteristic Boundary Conditions for Numerical Magnetohydrodynamic (MHD) Simulation of Solar and Laboratory Plasma Dynamics

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I. Abstract

We discuss the self-consistent time-dependent numerical boundary conditions for the magnetohydrodynamic (MHD) simulation to study the solar and laboratory plasma dynamics. It is well known that these plasma flows cover from the sub-sonic and sub-Alfvénic to super-Alfvénic region. Hence the characteristic boundary conditions must be used because the information propagating according to these characteristics will affect the solutions. To illustrate the importance of this set of boundary conditions, two examples are presented; one for solar plasma which is to study the energy and magnetic flux transport from the sub-photosphere to the corona, and the other one concerns laboratory plasma flow for the investigation of the sub-Alfvénic inlet boundary conditions for an MHD nozzle.

II. Mathematical Model

The governing equations are a set of conservation laws of mass, momentum, and energy. In addition, the magnetic induction equation, resulting from Maxwell's equation, is included to account for the nonlinear interactions between the plasma flows and magnetic field. We have cast these governing equations in a spherical coordinate for solar atmospheric dynamics and a cylindrical coordinate for MHD nozzle problem.

For convenience of development of a numerical model and characteristic boundary conditions, these equations are casted in a conservation form in spherical coordinates and cgs units.

$$\frac{\partial U}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta G) + \frac{1}{r \sin \theta} \frac{\partial H}{\partial \varphi} = S$$

where

$$U = \begin{bmatrix} \rho \\ \rho v_r \\ \rho v_\theta \\ \rho v_\varphi \\ B_r \\ B_\theta \\ B_\varphi \\ \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{8\pi} \end{bmatrix}$$

$$F = \begin{bmatrix} \rho v_r \\ p + \rho v_r^2 \frac{1}{8\pi} (B_\theta^2 + B_\phi^2 - B_r^2) \\ \rho v_r v_\theta - \frac{B_r B_\theta}{4\pi} \\ \rho v_r v_\phi - \frac{B_r B_\phi}{4\pi} \\ 0 \\ v_r B_\theta - v_\theta B_r \\ v_r B_\phi - v_\phi B_r \\ v_r \left(\frac{\mathcal{P}}{\gamma - 1} + \frac{\rho v^2}{2} \frac{B_\theta^2 + B_\phi^2}{8\pi} \right) - \frac{B_r}{4\pi} (v_\theta B_\theta + v_\phi B_\phi) \end{bmatrix}$$

$$G = \begin{bmatrix} \rho v_\theta \\ \rho v_r v_\theta - \frac{B_r B_\theta}{4\pi} \\ p + \rho v_\theta^2 \frac{1}{8\pi} (B_r^2 + B_\phi^2 - B_\theta^2) \\ \rho v_\theta v_\phi - \frac{B_\theta B_\phi}{4\pi} \\ v_\theta B_r - v_r B_\theta \\ 0 \\ v_\theta B_\phi - v_\phi B_\theta \\ v_\theta \left(\frac{\mathcal{P}}{\gamma - 1} + \frac{\rho v^2}{2} \frac{B_r^2 + B_\phi^2}{8\pi} \right) - \frac{B_\theta}{4\pi} (v_r B_r + v_\phi B_\phi) \end{bmatrix}$$

$$H = \begin{bmatrix} \rho v_\varphi \\ \rho v_r v_\varphi - \frac{B_r B_\varphi}{4\pi} \\ \rho v_\theta v_\varphi - \frac{B_\theta B_\varphi}{4\pi} \\ p + \rho v_\varphi^2 + \frac{1}{8\pi} (B_r^2 + B_\theta^2 - B_\varphi^2) \\ v_\varphi B_r - v_r B_\varphi \\ v_\varphi B_\theta - v_\theta B_\varphi \\ 0 \\ v_\varphi \left(\frac{\mathcal{P}}{\gamma - 1} + \frac{\rho v^2}{2} \frac{B_r^2 + B_\theta^2}{8\pi} \right) - \frac{B_\varphi}{4\pi} (v_r B_r + v_\theta B_\theta) \end{bmatrix}$$

$$S = \left[\begin{array}{c} 0 \\ \frac{1}{r} \left[2p + \rho(v_\theta^2 + v_\varphi^2) + \frac{B_r^2}{4\pi} \right] - \rho g \\ \left(p + \frac{B_r^2 + B_\theta^2 + B_\varphi^2}{8\pi} + \rho v_\varphi^2 \right) \frac{\text{ctg} \theta}{r} - \frac{1}{r} (\rho v_r v_\theta - \frac{B_r B_\theta}{4\pi}) \\ \left(\frac{B_\theta B_\varphi}{4\pi} - \rho v_\varphi v_\theta \right) \frac{\text{ctg} \theta}{r} + \frac{1}{r} (\frac{B_r B_\varphi}{4\pi} - \rho v_r v_\varphi) \\ - \frac{\eta}{r \sin \theta} \frac{\partial}{\partial \theta} \left[\frac{\sin \theta}{r} \left(\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta} \right) \right] + \frac{\eta}{r^2 \sin \theta} \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin \theta} \frac{\partial B_r}{\partial \varphi} - \frac{\partial(rB_\varphi)}{\partial r} \right) \\ \frac{B_\theta v_r - B_r v_\theta}{r} - \frac{\eta}{r^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \left(\frac{\partial(\sin \theta B_\varphi)}{\partial \theta} - \frac{\partial B_\theta}{\partial \varphi} \right) + \frac{\eta}{r} \frac{\partial}{\partial r} \left(\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta} \right) \\ \frac{v_r B_\varphi - v_\varphi B_r + \text{ctg} \theta (B_\varphi v_\theta - v_\varphi B_\theta)}{r} - \frac{\eta}{r} \frac{\partial}{\partial r} \left(\frac{1}{\sin \theta} \frac{\partial B_r}{\partial \varphi} - \frac{\partial(rB_\varphi)}{\partial r} \right) + \frac{\eta}{r^2} \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \left(\frac{\partial(\sin \theta B_\varphi)}{\partial \theta} - \frac{\partial B_\theta}{\partial \varphi} \right) \right] \\ - v_r \rho g - \eta j^2 - \nabla \cdot \vec{q} - Q_R + H \end{array} \right]$$

In these equations, ρ is the plasma mass density, v_r , v_θ , and v_φ are three components of the plasma flow velocity in spherical coordinates, B_r , B_θ , and B_φ are the three components of magnetic field, $p = \rho RT$ is the plasma thermal pressure, with T and R being the temperature and universal gas constant, respectively, j is the electric current, η is the coefficients of the magnetic diffusion, \vec{q} is the thermal conduction and Q_R is the radiation flux.

\vec{q}

III. Characteristic Boundary Conditions

Method of Characteristics

A path in a two-dimensional plane used to transform partial differential equations into systems of ordinary differential equations is called a characteristic. The method of characteristics can be used to solve first order PDE's

Fundamentals:

Partial differential equation of two variables x and y .

$$\frac{\partial f(x,t)}{\partial t} + u(x,t) \frac{\partial f(x,t)}{\partial x} = g(x,t)$$

$$\frac{df}{ds} = g \quad \text{Total differential equation, since}$$

$$\frac{df}{ds} = \frac{\partial f}{\partial t} \frac{\partial t}{\partial s} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial s}$$

$$\frac{dx}{dt} = \frac{\partial s / \partial t}{\partial s / \partial x} = u$$

$$f(s) = f(s=0) + \int_0^s g \, ds$$

$$x \frac{\partial f(x,t)}{\partial x} + y \frac{\partial f(x,t)}{\partial y} = (f^2 - R^2) / f$$

$$\frac{df}{ds} = (f^2 - R^2) / f \quad f(s) = f(x(s), y(s))$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} = (f^2 - R^2) / f$$

$$\frac{dx}{ds} = x \quad \frac{dy}{ds} = y$$

$$x(s) = a e^s \quad y(s) = b e^s$$

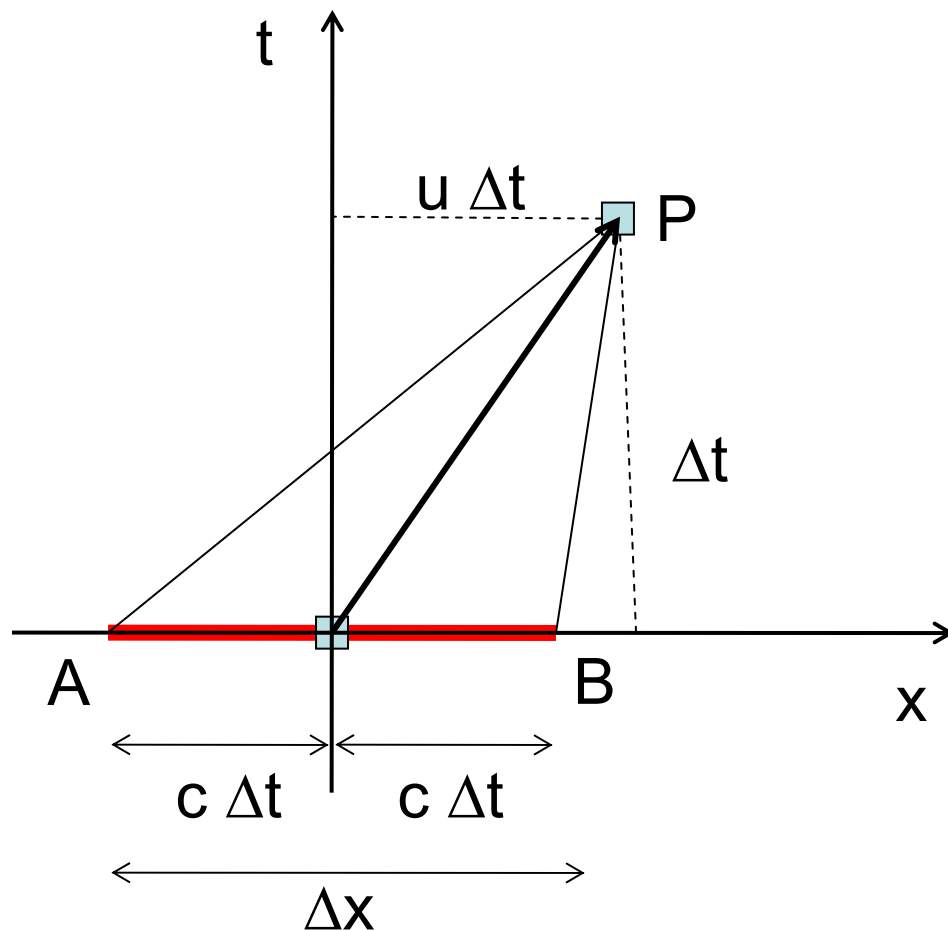
$$s = \ln(\sin \varphi) \quad a = \cos(\theta) \quad b = \sin(\theta)$$

$$\frac{dy}{dx} = \frac{\partial s / \partial x}{\partial s / \partial y} = y / x = \tan(\theta)$$

$$f(s) = f(s=0) + \int_0^s (f^2 - R^2) / f \, ds$$

$$f(s) = R \cos(\varphi) = R(1 - e^{2s})^{1/2}$$

Region of Influence for Plasma Element



- = plasma element
- u = flow velocity
- c = wave speed
- = pts affecting P

Time Step

$$\Delta t =$$

$$\Delta x / 2c_{\max}$$

(Incompressible plasma moving at velocity u)

Surface Boundary Conditions

In order to accommodate the observations at the bottom boundary, the evolutionary boundary conditions must be used, thus, the method of “*projected*” characteristics are used for the derivation of such boundary conditions. For a three-dimensional magnetohydrodynamic problem, the governing equations have been cast in a vectoral form,

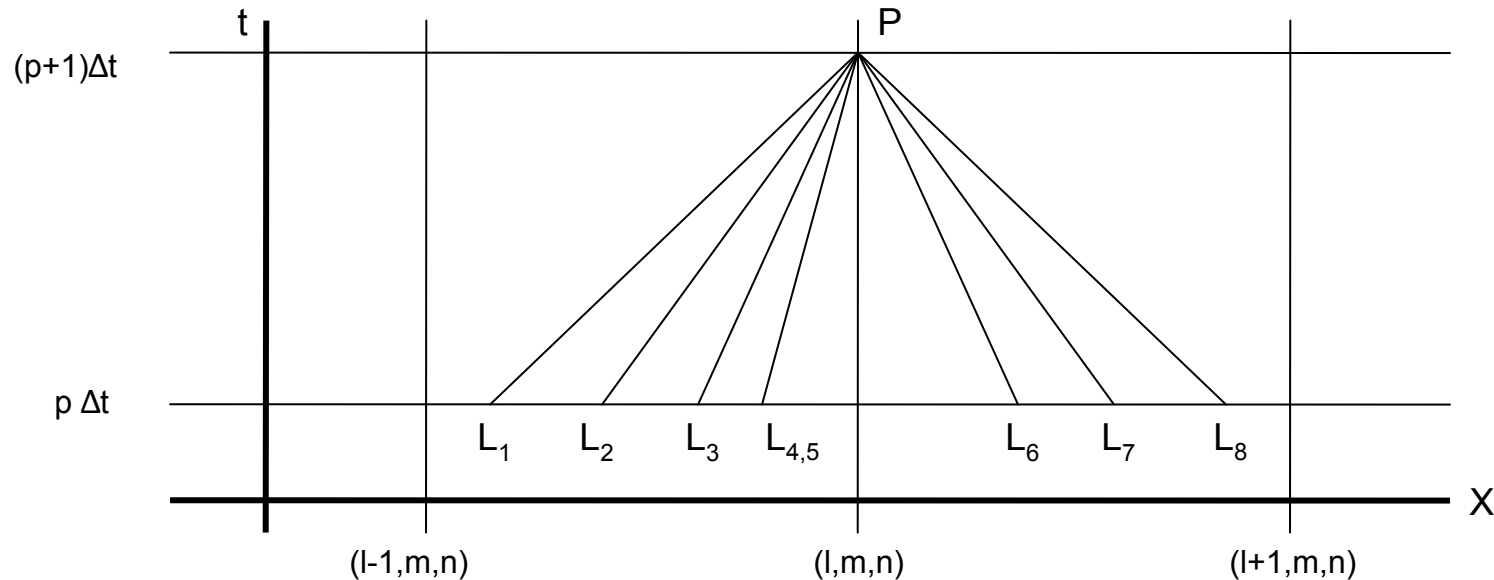
$$\frac{\partial \vec{U}}{\partial t} = -\vec{A} \frac{\partial \vec{U}}{\partial x_1} - \vec{B} \frac{\partial \vec{U}}{\partial x_2} - \vec{C} \frac{\partial \vec{U}}{\partial x_3} + \vec{S}$$

where \vec{U} and $\vec{A}, \vec{B}, \vec{C}, \vec{S}$ are column vectors consisting of primary physical quantities such as, density, temperature and three components of velocity and magnetic field. To obtain these characteristics on the boundary, we have chosen the unit normal on the boundary is along certain coordinates, e.g. along the z-direction (radial). Thus, the characteristics along the projected normal will be found in the z-t plane (r-t plane). For example, at z-t plane for a small time difference Δt the projected characteristics given by:

$$\frac{dz}{dt} = \lambda_i \quad i = 1, 2, \dots, 8$$

can be approximated by straight lines. (reference: S. T. Wu, USTC, 2007)

The Projected Characteristics Straight Lines



Here $\lambda_i = (u_z, u_z, u_z + V_A, u_z - V_A, u_z + V_f, u_z - V_f, u_z + V_s, u_z - V_s)$ are eigenvalues. The projected characteristics passing the point P at the time $(p+1)\Delta t$ and the spatial location (l, m, n) then intersect the $p\Delta t$ axis at $L_1, L_2, L_3, L_4, L_5, L_6, L_7$ and L_8 . For $V_f > V_A > V_s > u_z$, the projected characteristic PL_1 can be identified with $\lambda_5 = u_z + V_f$, PL_2 with $\lambda_3 = u_z + V_A$, PL_3 with $\lambda_7 = u_z + V_s$, $PL_{4,5}$ with $\lambda_{1,2} = u_z$, PL_6 with $\lambda_8 = u_z - V_s$, PL_7 with $\lambda_4 = u_z - V_A$, PL_8 with $\lambda_6 = u_z - V_f$.

Projected Characteristic Equations

With their left-eigenvectors, the projected normal characteristic equations (i.e. compatibility equations) are :

$$\bar{\eta}_j \frac{\partial U}{\partial t} + \lambda_j \bar{\eta}_j \frac{\partial U}{\partial z} = -\bar{\eta}_j \bar{A} \frac{\partial U}{\partial x} - \bar{\eta}_j \bar{B} \frac{\partial U}{\partial y} + \bar{\eta}_j S \quad j=1,2,\dots,8.$$

They are as follows:

$$\begin{aligned} \frac{dz}{dt} = u_z & \quad a^2 \frac{\partial \rho}{\partial t} - \frac{\partial p}{\partial t} = -a^2 \vec{u} \cdot \nabla \rho + \vec{u} \cdot \nabla p \\ & \quad \frac{\partial B_z}{\partial t} = -\vec{u} \cdot \nabla B_z - B_z \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + B_x \frac{\partial u_z}{\partial x} + B_y \frac{\partial u_z}{\partial y} \\ & \quad - B_y B_z \frac{\partial u_x}{\partial t} + B_x B_z \frac{\partial u_y}{\partial t} + B_y V_A \frac{\partial B_x}{\partial t} - B_x V_A \frac{\partial B_y}{\partial t} = A_+ \\ & \quad + B_y B_z \frac{\partial u_x}{\partial t} - B_x B_z \frac{\partial u_y}{\partial t} + B_y V_A \frac{\partial B_x}{\partial t} - B_x V_A \frac{\partial B_y}{\partial t} = A_- \\ \frac{dz}{dt} = u_z + V_A & \quad - B_x B_z V_f \frac{\partial u_x}{\partial t} - B_y B_z V_f \frac{\partial u_y}{\partial t} + \rho V_f (V_f^2 - V_A^2) \frac{\partial u_z}{\partial t} + (V_f^2 - V_A^2) \frac{\partial p}{\partial t} + B_x V_f^2 \frac{\partial B_x}{\partial t} + B_y V_f^2 \frac{\partial B_y}{\partial t} = B_+ \\ & \quad + B_x B_z V_f \frac{\partial u_x}{\partial t} + B_y B_z V_f \frac{\partial u_y}{\partial t} - \rho V_f (V_f^2 - V_A^2) \frac{\partial u_z}{\partial t} + (V_f^2 - V_A^2) \frac{\partial p}{\partial t} + B_x V_f^2 \frac{\partial B_x}{\partial t} + B_y V_f^2 \frac{\partial B_y}{\partial t} = B_- \\ \frac{dz}{dt} = u_z + V_f & \quad + B_x B_z V_s \frac{\partial u_x}{\partial t} + B_y B_z V_s \frac{\partial u_y}{\partial t} - \rho V_s (V_s^2 - V_A^2) \frac{\partial u_z}{\partial t} - (V_s^2 - V_A^2) \frac{\partial p}{\partial t} - B_x V_s^2 \frac{\partial B_x}{\partial t} - B_y V_s^2 \frac{\partial B_y}{\partial t} = C_+ \\ & \quad - B_x B_z V_s \frac{\partial u_x}{\partial t} - B_y B_z V_s \frac{\partial u_y}{\partial t} + \rho V_s (V_s^2 - V_A^2) \frac{\partial u_z}{\partial t} - (V_s^2 - V_A^2) \frac{\partial p}{\partial t} - B_x V_s^2 \frac{\partial B_x}{\partial t} - B_y V_s^2 \frac{\partial B_y}{\partial t} = C_- \\ \frac{dz}{dt} = u_z + V_s & \end{aligned}$$

Projected Characteristic Equations

For the bottom boundary, if the eigenvalue (i.e. wave speed) is negative, that means the boundary condition will be affected by computational domain, and we have to use the compatibility equation to determine the physical parameter. The number of compatibility equations that have to use is determined by the number of negative eigenvalues.

As an example, if $u_z > 0$ and $|u_z| \leq V_A, V_s, V_f$, then only $u_z - V_A, u_z - V_s, u_z - V_f$ will be negative. In this case, three out of eight physical parameters have to be determined by three compatibility equations, and five could be given or using compatibility equations to solve. If relaxing the condition above to only $|u_z| \leq V_A, V_s, V_f$, then (note that this is two degenerated eigenvalues),

$u_z - V_A, u_z - V_s, u_z - V_f$ could be negative. In this case, five out of eight physical parameters have to be determined by five compatibility equations, and three could be given or using compatibility equations to solve.

The set of evolutionary boundary conditions in spherical coordinates is:

$$a^2 \frac{\partial \rho}{\partial t} - \frac{\partial p}{\partial t} = 0 \Rightarrow \frac{p}{\rho^\gamma} = \text{const.}$$

$$\begin{aligned} \frac{\partial B_r}{\partial t} = & \frac{B_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_r}{r} \frac{\partial B_\theta}{\partial \theta} - \frac{v_\theta}{r} \frac{\partial B_r}{\partial \theta} - \frac{B_r}{r} \frac{\partial v_\theta}{\partial \theta} \\ & + \frac{\cot \theta}{r} (v_r B_\theta - v_\theta B_r) - \frac{v_\varphi}{r \sin \theta} \frac{\partial B_r}{\partial \varphi} \\ & - \frac{B_r}{r \sin \theta} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r \sin \theta} \frac{\partial B_\varphi}{\partial \varphi} + \frac{B_\theta}{r \sin \theta} \frac{\partial v_r}{\partial \varphi} \end{aligned}$$

$$\frac{\partial p}{\partial t} = \frac{v_s^2 B + v_f^2 C}{2v_a^2 (v_f^2 - v_s^2)}$$

$$\frac{\partial v_r}{\partial t} = \frac{v_s^2 B + v_f C}{2\rho v_s v_f (v_f^2 - v_s^2)}$$

$$\frac{\partial v_\theta}{\partial t} = \frac{B_\theta [v_s (v_a^2 - v_s^2) B - v_f (v_f^2 - v_a^2) C]}{2B_r (B_\theta^2 + B_\varphi^2) v_s v_f (v_f^2 - v_s^2)} + \frac{B_\varphi A}{2B_r (B_\theta^2 + B_\varphi^2)}$$

$$\frac{\partial v_\varphi}{\partial t} = \frac{B_\varphi [v_s (v_a^2 - v_s^2) B - v_f (v_f^2 - v_a^2) C]}{2B_r (B_\theta^2 + B_\varphi^2) v_s v_f (v_f^2 - v_s^2)} - \frac{B_\theta A}{2B_r (B_\theta^2 + B_\varphi^2)}$$

$$\frac{\partial B_\theta}{\partial t} = \frac{B_\theta [B(v_a^2 - v_s^2) - C(v_f^2 - v_a^2)]}{2v_a^2 (v_f^2 + v_s^2) (B_\theta^2 - B_\varphi^2)} + \frac{B_\varphi A}{2v_a (B_\theta^2 + B_\varphi^2)}$$

$$\frac{\partial B_\varphi}{\partial t} = \frac{B_\varphi [B(v_a^2 - v_s^2) - C(v_f^2 - v_a^2)]}{2v_a^2 (v_f^2 + v_s^2) (B_\theta^2 - B_\varphi^2)} - \frac{B_\theta A}{2v_a (B_\theta^2 + B_\varphi^2)}$$

where

$$\begin{aligned}
A = & -B_r B_\varphi (v_r - v_a) \frac{\partial v_\theta}{\partial r} + B_r B_\theta (v_r - v_a) \frac{\partial v_\varphi}{\partial r} + B_\theta v_a (v_r - v_f) \frac{\partial B_\varphi}{\partial r} \\
& - \frac{B_\theta v_\theta v_f^2}{r} \frac{\partial B_\theta}{\partial \theta} + (B_\theta v_a - B_r v_\theta) \frac{B_\varphi}{r} \frac{\partial v_\theta}{\partial \theta} - (B_\theta v_a - B_r v_\theta) \frac{B_\theta}{r} \frac{\partial v_\varphi}{\partial \theta} \\
& - \frac{B_r^2 B_\varphi}{\rho r} \frac{\partial B_r}{\partial \theta} - \frac{B_r B_\varphi}{\rho r} \frac{\partial p}{\partial \theta} - \frac{B_\varphi v_\theta v_a}{r} \frac{\partial B_\theta}{\partial \theta} + \left[\frac{B_\varphi v_\theta v_a}{r} - \frac{B_r (B_\theta^2 + B_\varphi^2)}{\rho r} \right] \frac{\partial B_\varphi}{\partial \theta} \\
& - \frac{B_r B_\varphi}{\rho r} (B_\theta^2 + B_\varphi^2) \operatorname{ctg} \theta + \frac{B_r v_\varphi}{r} (B_\theta v_\theta + B_\varphi v_\varphi) \operatorname{ctg} \theta - \frac{B_r v_r}{r} (B_\varphi v_\theta - B_\theta v_\varphi) \\
& - \frac{v_a}{r} (B_r + B_\theta \operatorname{ctg} \theta) (B_\varphi v_\theta - B_\theta v_\varphi) + \frac{B_r}{\rho r \sin \theta} (B_\theta^2 + B_\varphi^2) \frac{\partial B_\theta}{\partial \varphi} + \frac{B_\theta B_r^2}{\rho r \sin \theta} \frac{\partial B_r}{\partial \varphi} \\
& + \frac{B_\varphi v_a - B_r v_\varphi}{r \sin \theta} (B_\varphi \frac{\partial v_\theta}{\partial \varphi} - B_\theta \frac{\partial v_\varphi}{\partial \varphi}) - \frac{B_\theta B_r}{\rho r \sin \theta} \frac{\partial p}{\partial \varphi} + \frac{v_a v_\varphi}{r \sin \theta} (B_\theta \frac{\partial B_\varphi}{\partial \varphi} - B_\varphi \frac{\partial B_\theta}{\partial \varphi}) \\
& - B_\varphi v_a (v_r - v_a) \frac{\partial B_\theta}{\partial r}
\end{aligned}$$

$$\begin{aligned}
B = & \rho(v_f^2 - v_a^2)v_f(v_r - v_f)\frac{\partial v_r}{\partial r} - B_r B_\theta v_f(v_r - v_f)\frac{\partial v_\theta}{\partial r} \\
& - B_r B_\phi v_f(v_r - v_f)\frac{\partial v_\phi}{\partial r} - B_\phi v_f^2(v_r - v_f)\frac{\partial B_\phi}{\partial r} - (v_f^2 - v_a^2)(v_r - v_f)\frac{\partial p}{\partial r} \\
& - B_\theta v_f^2(v_r - v_f)\frac{\partial B_\theta}{\partial r} - \frac{B_\theta v_\theta v_f^2}{r}\frac{\partial B_\theta}{\partial \theta} + \rho(v_f^2 - v_a^2)v_f v_\theta \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\
& - v_f[v_f(\rho(v_f^2 - v_a^2) - B_\theta^2) + B_r B_\theta v_\theta]\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{B_\theta v_f^3}{r}\frac{\partial B_r}{\partial \theta} \\
& + (B_\theta v_f - B_r v_\theta)\frac{B_\phi v_f}{r}\frac{\partial v_\phi}{\partial \theta} - [v_\theta(v_f^2 - v_a^2) + \frac{B_r B_\theta v_f}{\rho}]\frac{1}{r}\frac{\partial p}{\partial \theta} \\
& - \frac{B_\phi v_\theta v_f^2}{r}\frac{\partial B_\phi}{\partial \theta} - \frac{v_r v_f^2}{r}[2\rho(v_f^2 - v_a^2) - B_\theta^2 - B_\phi^2] \\
& - \frac{v_f^2}{r}(B_\theta v_\theta + B_\phi v_\phi)(B_r + B_\theta \operatorname{ctg} \theta) + \frac{v_f^3}{r}(B_\theta^2 + B_\phi^2) \\
& - \frac{v_\theta v_f^2}{r}[\rho(v_f^2 - v_a^2) - B_\theta^2 - B_\phi^2]\operatorname{ctg} \theta - \frac{B_r v_r v_f}{r}(B_\theta v_\theta + B_\phi v_\phi) \\
& - \rho(v_f^2 - v_a^2)v_f g - \frac{\rho v_f}{r}(v_f^2 - v_a^2)(v_\theta^2 + v_\phi^2) \\
& - \frac{v_\phi}{r}(B_\phi v_\theta - B_\theta v_\phi)B_r v_f \operatorname{ctg} \theta \\
& - \frac{B_\phi v_\phi v_f^2}{r \sin \theta}\frac{\partial B_\phi}{\partial \phi} \pm B_\phi v_f^3 \frac{\partial B_r}{\partial \phi} + \rho(v_f^2 - v_a^2)\frac{v_\phi v_f}{r \sin \theta}\frac{\partial v_r}{\partial \phi} \\
& + \frac{B_\theta v_f}{r \sin \theta}(B_\phi v_f - B_r v_\phi)\frac{\partial v_\theta}{\partial \phi} \\
& - \frac{B_\theta v_\phi v_f^2}{r \sin \theta}\frac{\partial B_\theta}{\partial \phi} - \frac{1}{r \sin \theta}[\rho a^2(v_f^2 - v_a^2) + B_\theta^2 v_f^2 + B_r B_\phi v_\phi v_f]\frac{\partial v_\phi}{\partial \phi} \\
& - \frac{1}{r \sin \theta}[v_\phi(v_f^2 - v_a^2) + \frac{B_r B_\phi v_f}{\rho}]\frac{\partial p}{\partial \phi}
\end{aligned}$$

$$\begin{aligned}
C = & \rho(v_a^2 - v_s^2)v_s(v_r - v_s)\frac{\partial v_r}{\partial r} + B_r B_\theta v_s(v_r - v_s)\frac{\partial v_\theta}{\partial r} \\
& + B_r B_\phi v_s(v_r - v_s)\frac{\partial v_\phi}{\partial r} + B_\phi v_s^2(v_r - v_s)\frac{\partial B_\phi}{\partial r} - (v_a^2 - v_s^2)(v_r - v_f)\frac{\partial p}{\partial r} \\
& + B_\theta v_s^2(v_r - v_s)\frac{\partial B_\theta}{\partial r} + \frac{B_\theta v_\theta v_s^2}{r}\frac{\partial B_\theta}{\partial \theta} + \rho(v_a^2 - v_s^2)v_s v_\theta \frac{1}{r}\frac{\partial v_r}{\partial \theta} \\
& + v_s[v_s(\rho(v_s^2 - v_a^2) - B_\theta^2) + B_r B_\theta v_\theta]\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{B_\theta v_s^3}{r}\frac{\partial B_r}{\partial \theta} \\
& - (B_\theta v_s - B_r v_\theta)\frac{B_\phi v_s}{r}\frac{\partial v_\phi}{\partial \theta} - [v_\theta(v_a^2 - v_s^2) - \frac{B_r B_\theta v_s}{\rho}]\frac{1}{r}\frac{\partial p}{\partial \theta} \\
& + \frac{B_\phi v_\theta v_s^2}{r}\frac{\partial B_\phi}{\partial \theta} + \frac{v_r v_s^2}{r}[2\rho(v_s^2 - v_a^2) - B_\theta^2 - B_\phi^2] \\
& + \frac{v_s^2}{r}(B_\theta v_\theta + B_\phi v_\phi)(B_r + B_\theta \text{ctg} \theta) - \frac{v_s^3}{r}(B_\theta^2 + B_\phi^2) \\
& + \frac{v_\theta v_s^2}{r}[\rho(v_s^2 - v_a^2) - B_\theta^2 - B_\phi^2]\text{ctg} \theta + \frac{B_r v_r v_s}{r}(B_\theta v_\theta + B_\phi v_\phi) \\
& - \rho(v_s^2 - v_a^2)v_s g - \frac{\rho v_s}{r}(v_a^2 - v_s^2)(v_\theta^2 + v_\phi^2) + \frac{v_\phi}{r}(B_\phi v_\theta - B_\theta v_\phi)B_r v_s \text{ctg} \theta \\
& + \frac{B_\phi v_\phi v_s^2}{r \sin \theta}\frac{\partial B_\phi}{\partial \phi} + B_\phi v_s^3\frac{\partial B_r}{\partial \phi} + \rho(v_a^2 - v_s^2)\frac{v_\phi v_s}{r \sin \theta}\frac{\partial v_r}{\partial \phi} \\
& - \frac{B_\theta v_s}{r \sin \theta}(B_\phi v_s - B_r v_\phi)\frac{\partial v_\theta}{\partial \phi} + \frac{B_\theta v_\phi v_s^2}{r \sin \theta}\frac{\partial B_\theta}{\partial \phi} \\
& - \frac{1}{r \sin \theta}[\rho a^2(v_a^2 - v_s^2) - B_\theta^2 v_s^2 - B_r B_\phi v_\phi v_s]\frac{\partial v_\phi}{\partial \phi} \\
& - \frac{1}{r \sin \theta}[v_\phi(v_a^2 - v_s^2) - \frac{B_r B_\phi v_s}{\rho}]\frac{\partial p}{\partial \phi}
\end{aligned}$$

Numerical Methods

The numerical method we used is simple TVD Lax-Friedrichs formulation. This scheme achieves the second order accuracy both temporally and spatially. To achieve second order temporal accuracy, the Hancock predictor step and corrector step are used.

Predictor Step:

$$U_{i,j,k}^{n+\frac{1}{2}} = U_{i,j,k}^n - \Delta t \left\{ \frac{F(U_{i,j,k}^n + \frac{1}{2} \delta U_i^n) - F(U_{i,j,k}^n - \frac{1}{2} \delta U_i^n)}{\Delta x_1} \right\} - \Delta t \left\{ \frac{G(U_{i,j,k}^n + \frac{1}{2} \delta U_j^n) - G(U_{i,j,k}^n - \frac{1}{2} \delta U_j^n)}{\Delta x_2} \right\} \\ - \Delta t \left\{ \frac{H(U_{i,j,k}^n + \frac{1}{2} \delta U_k^n) - H(U_{i,j,k}^n - \frac{1}{2} \delta U_k^n)}{\Delta x_3} \right\} + \frac{\Delta t}{2} S(U_{i,j,k}^n)$$

Corrector Step:

$$U_{i,j,k}^{n+1} = U_{i,j,k}^T + \frac{1}{2} (\Phi_{i+\frac{1}{2}}^F - \Phi_{i-\frac{1}{2}}^F) + \frac{1}{2} (\Phi_{j+\frac{1}{2}}^G - \Phi_{j-\frac{1}{2}}^G) + \frac{1}{2} (\Phi_{k+\frac{1}{2}}^H - \Phi_{k-\frac{1}{2}}^H) + \frac{\Delta t}{2} S(U_{i,j,k}^{n+\frac{1}{2}})$$

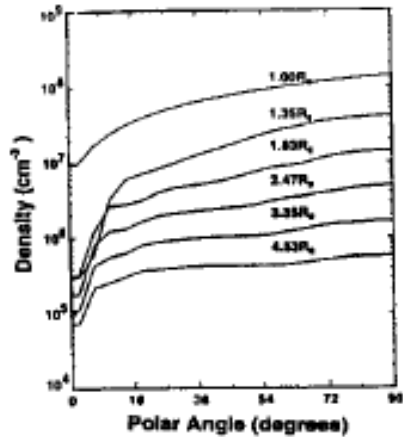
$$U_{i,j,k}^T = U_{i,j,k}^n + \Delta t \left[\frac{1}{2} (F_{i+\frac{1}{2}}^{LR} - F_{i-\frac{1}{2}}^{LR}) + \frac{1}{2} (G_{j+\frac{1}{2}}^{LR} - G_{j-\frac{1}{2}}^{LR}) + \frac{1}{2} (H_{k+\frac{1}{2}}^{LR} - H_{k-\frac{1}{2}}^{LR}) \right]$$

IV. Examples

IV.1. Solar Atmospheric Dynamics

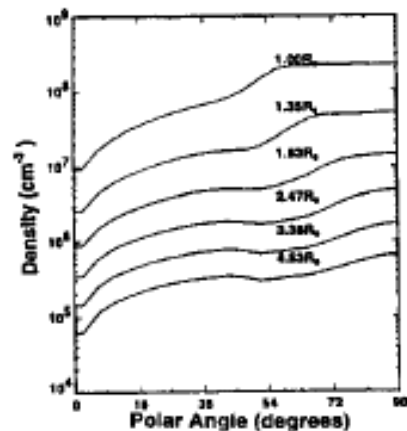
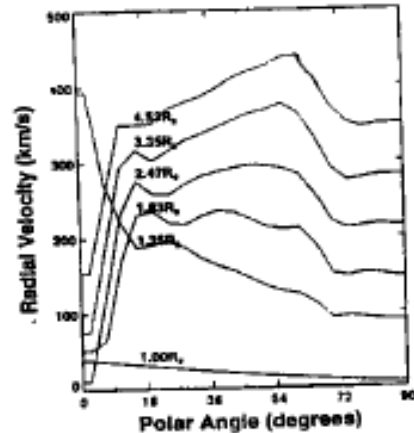
- Energy and Magnetic Flux Transport from the Sub-photosphere to the Corona
- Solar Wind

DENSITY

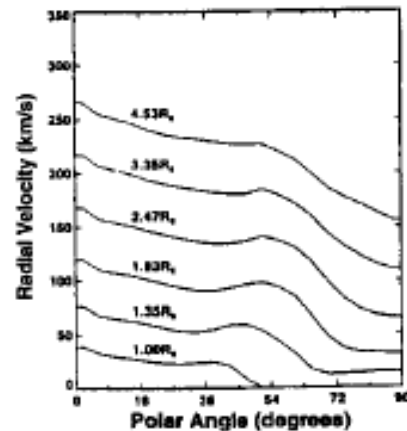


CASE 1

VELOCITY



CASE 2



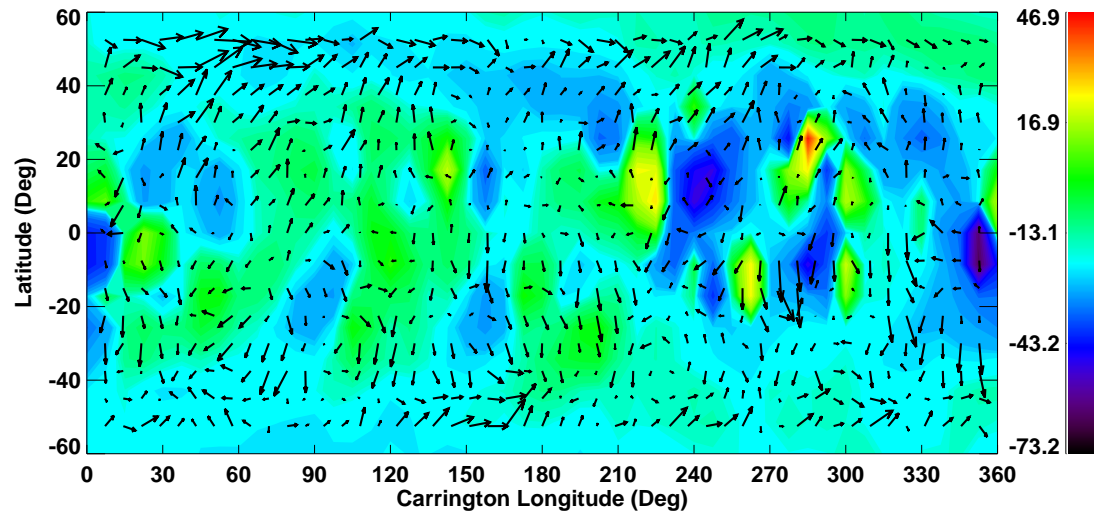
To illustrate the usage of characteristic boundary, we use two techniques:

(i) Test of Boundary Conditions

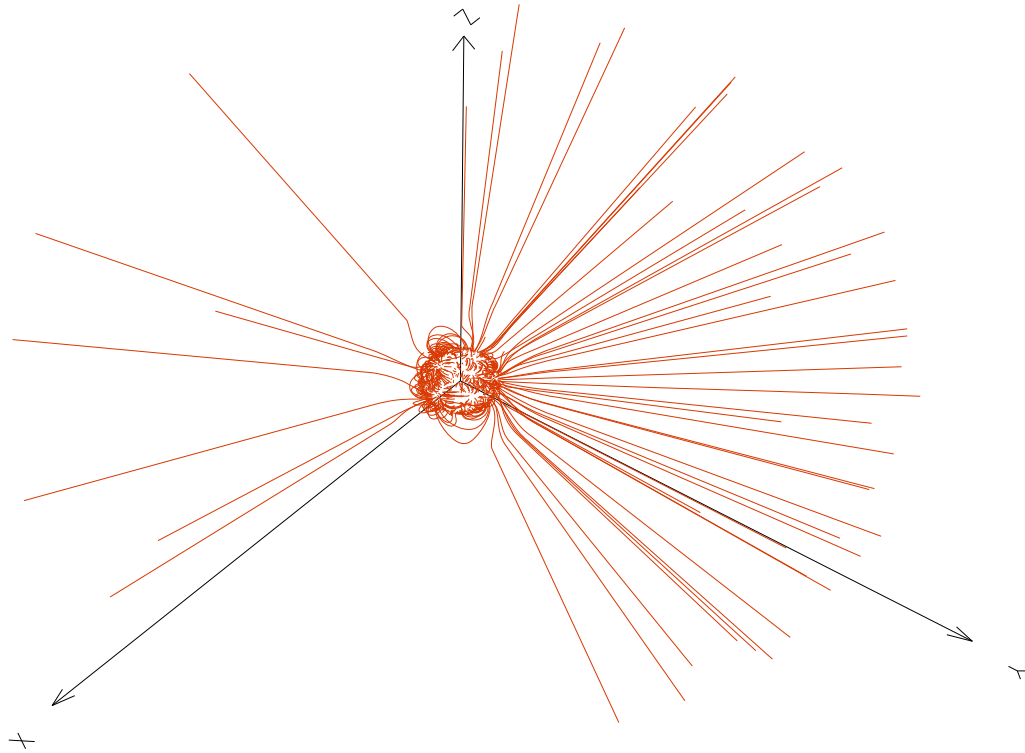
Three types of boundary conditions are tested:

- All fixed values (i.e. density, temperature, velocity, and magnetic field strength) at the lower boundary
- Time –dependent boundary conditions using characteristics
- Time-dependent boundary conditions using numerical procedures (i.e. linear extrapolation)

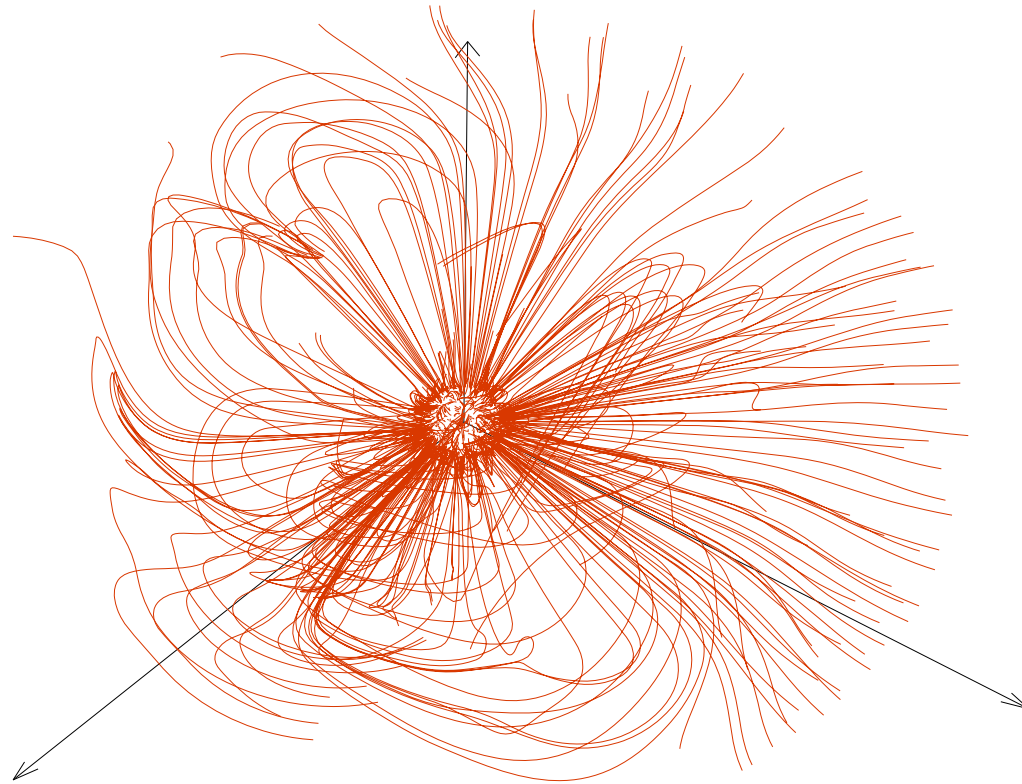
(ii) Using observations together with characteristic boundary conditions at the lower boundary



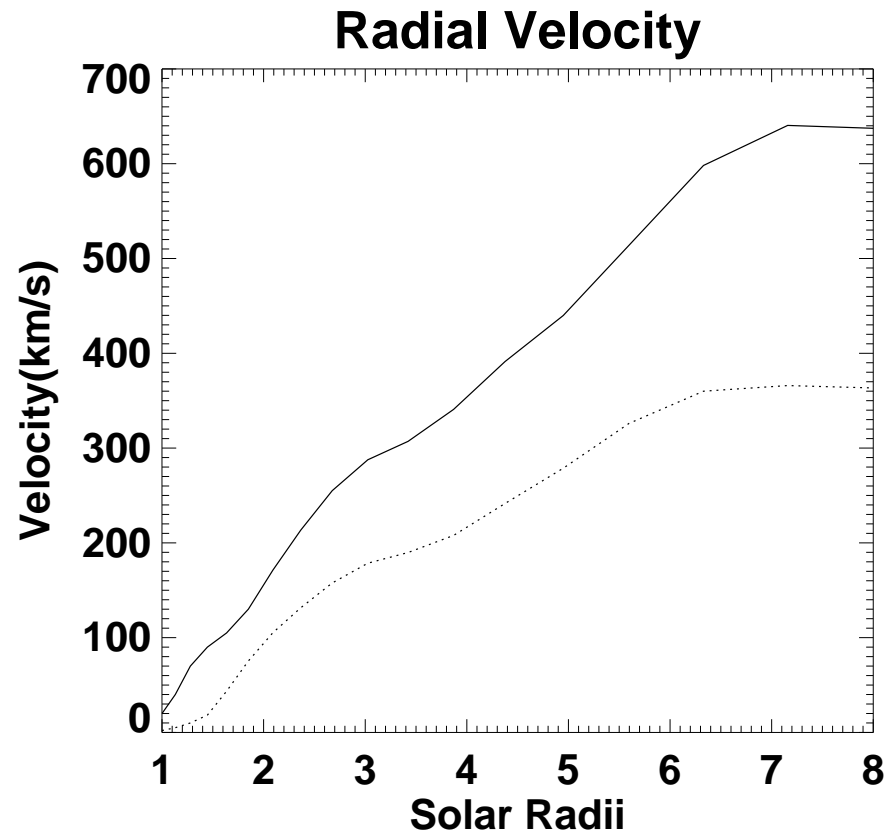
GONG's observed data which include line-of-sight magnetic field of active region CR2009, and transverse velocity field. The color contours show the line-of-sight magnetic field, and the black arrows the velocity field.



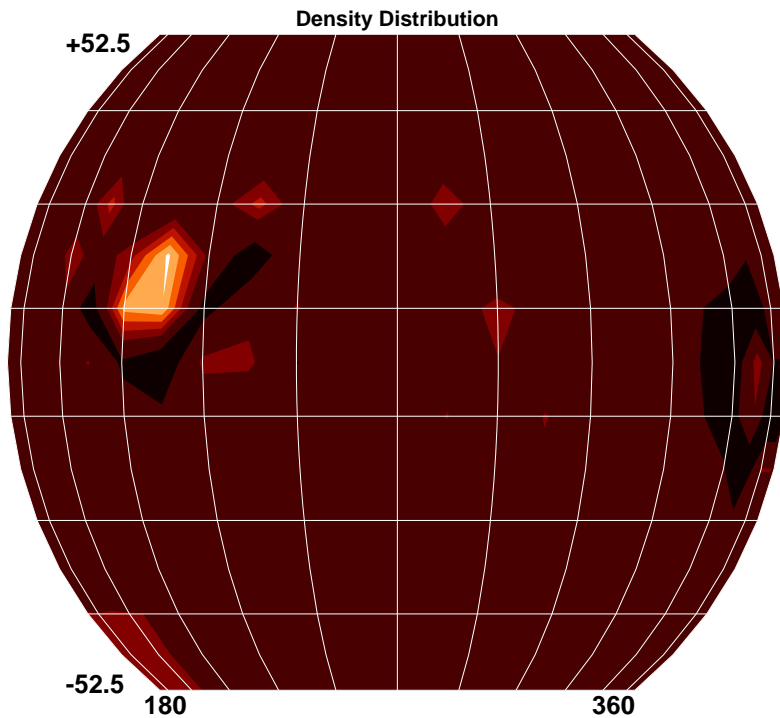
Pre-initial 3D magnetic field configuration. Inside the computational domain, it is a potential field derived from the line-of-sight GONG field.



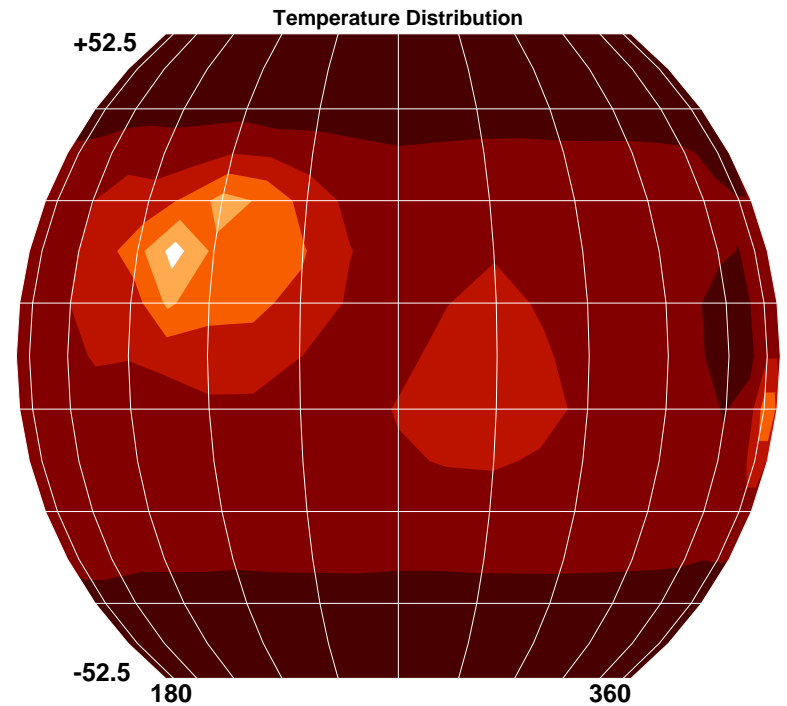
An equilibrium magnetic field configuration after relaxation from the pre-initial state.



GONG's transverse velocity in red arrows and the MHD simulation transverse velocity without using GONG's velocity in blue arrows.

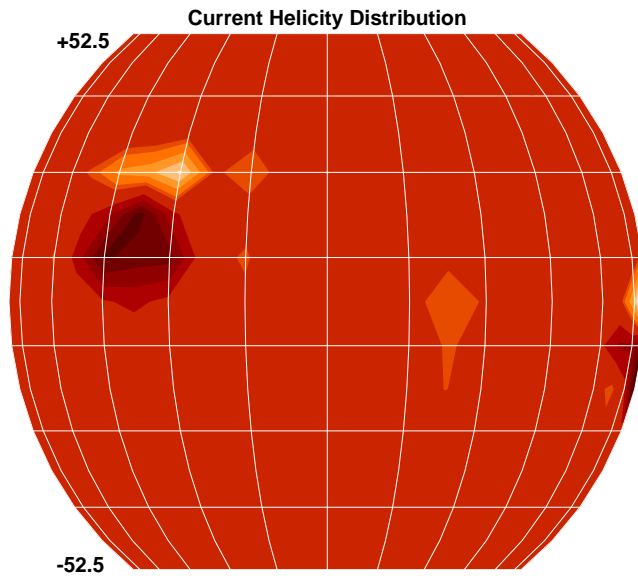


(a)

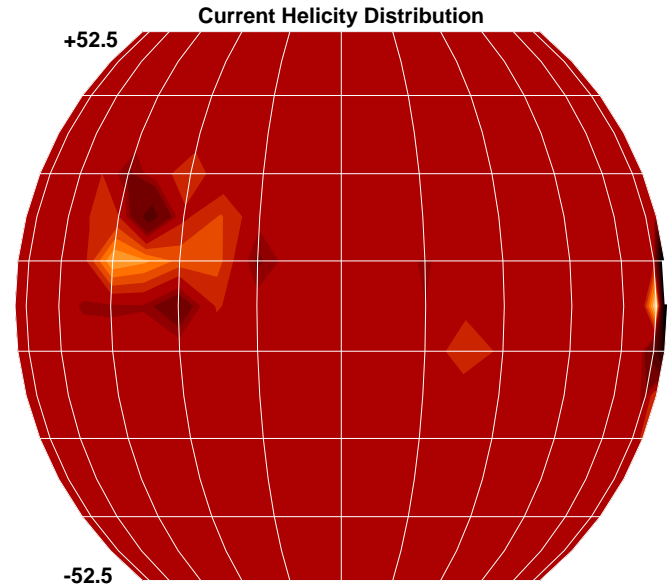


(b)

The density (a) and temperature (b) are from the MHD equilibrium state with the GONG's observed velocity data.

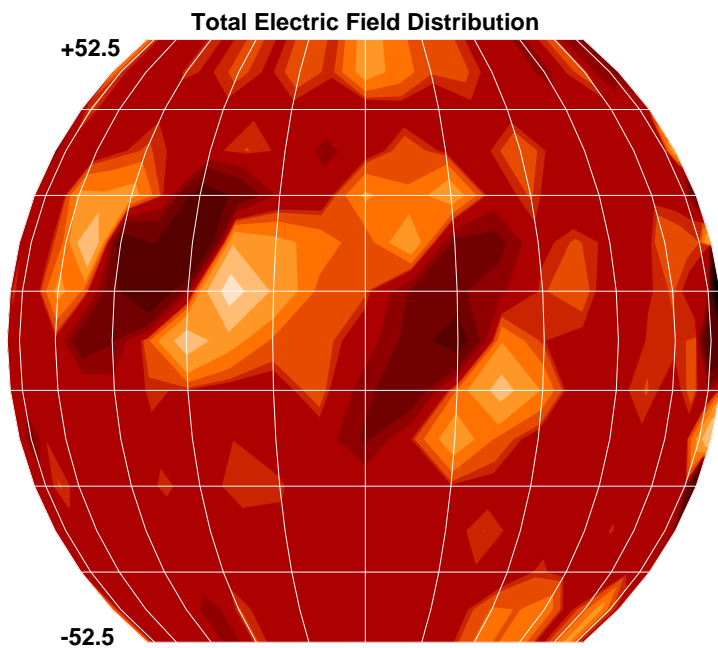


(a)

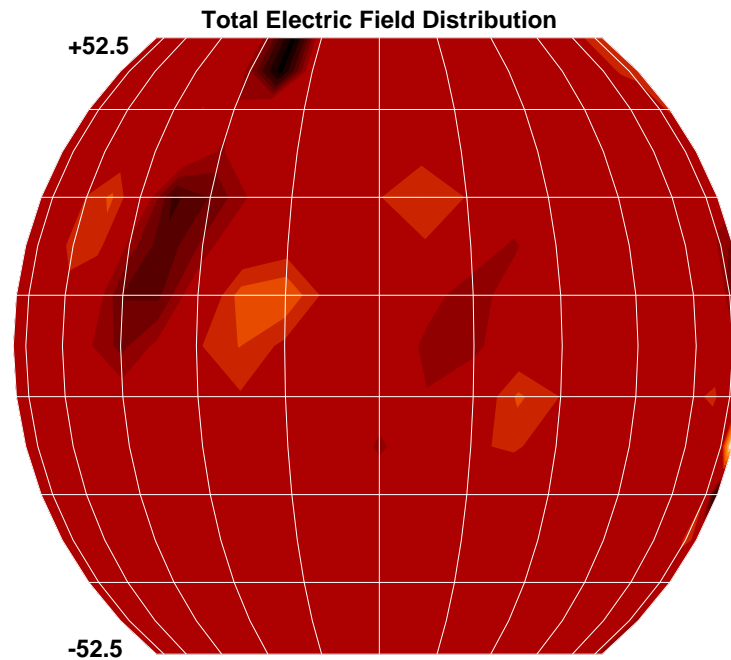


(b)

The total current helicities from the MHD equilibrium state (a) with the GONG's observed velocity data;
(b) without the GONG's observed velocity data



(a)



(b)

The total current from the MHD equilibrium state (a) with the GONG's observed velocity data; (b) without the GONG's observed velocity data.

IV.1. Sub-Alfvenic inlet Boundary for an MHD Nozzle Flow

This simulation is performed using a 2 ½ D multi-block, Arbitrary Lagrangian Eularin (ALE), resistive single fluid MHD code (MACH 2: Peterllin et al. J., Comp. Phys. 1998, 104, 1, 48-71). MACH 2 solves the mass, momentum and electron energy, ion energy, radiation energy density and magnetic induction equations in a fractional time-split manner for the flow variables. The equations are closed with an ideal gas equation of state, Braginski thermal conductivity and Spitzer resistivity are used. The simulation was performed in two steps. Firstly, a steady vacuum field was established using MACH 2's magnetic field diffusion solver. The calculation was allowed to proceed until changes in values of test cells in the domain were less than $\sim 10^{-4}\%$. Secondly, the flux was frozen into the domain by turning off the diffusion solver and allowing plasma to flow through the inlet. To avoid sharp discontinuities at the inlet, the density and velocity were ramped up to the intended value over 1 μ s.

Description of the boundary condition to be tested

Three sets of boundary were tested in order to evaluate the necessity of using the characteristic boundary conditions. These three cases are: In Case 1, characteristic boundary conditions. By counting the characteristics, we have computed B_z , B_r , and v_r using compatibility relations. In Case 2, B_z , B_r , and v_r are computed using first order extrapolation from the computational domain. Finally, in Case 3, B_z and B_r are “frozen in” from the calculation of the vacuum field and v_r is set to 0.0

Magnetic Nozzle Description

The inlet flow parameters and geometry are designed such that the plasma will transition from sub-Alfvénic to super-Alfvénic inside the computational domain (cylindrical symmetry: r & z). See Figure 1.

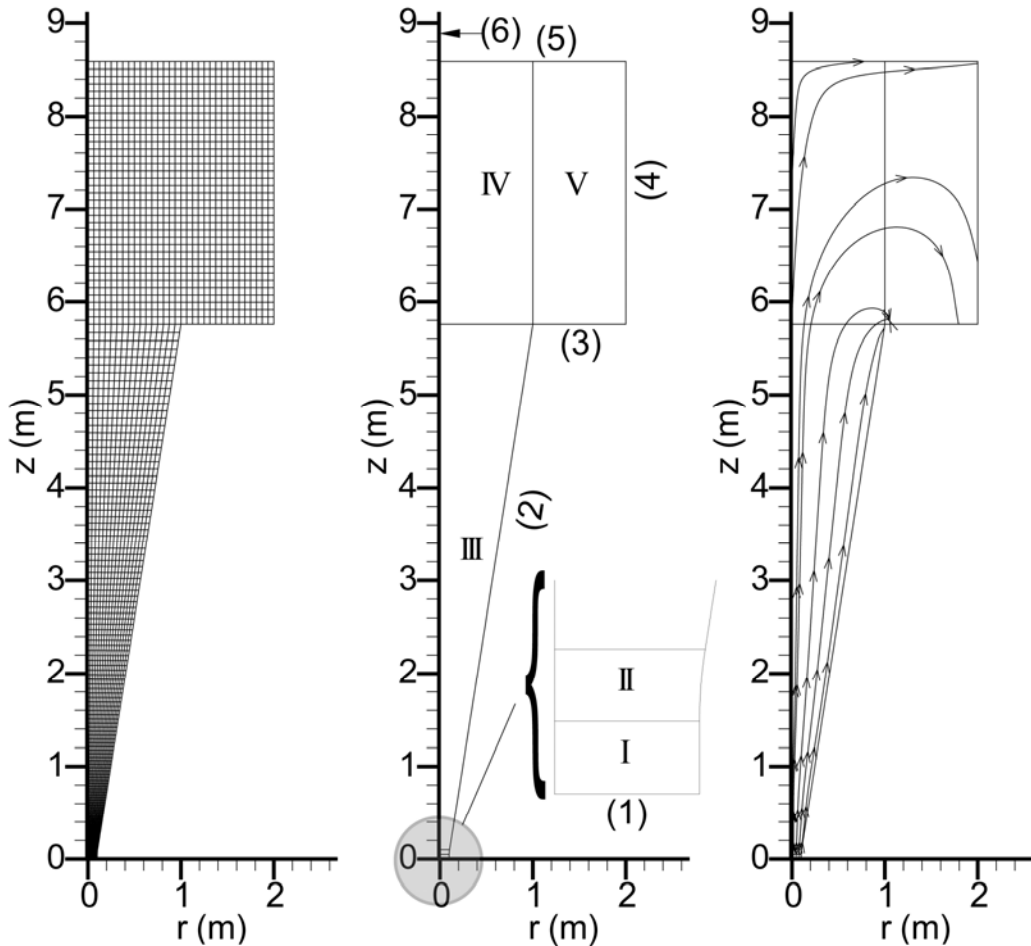
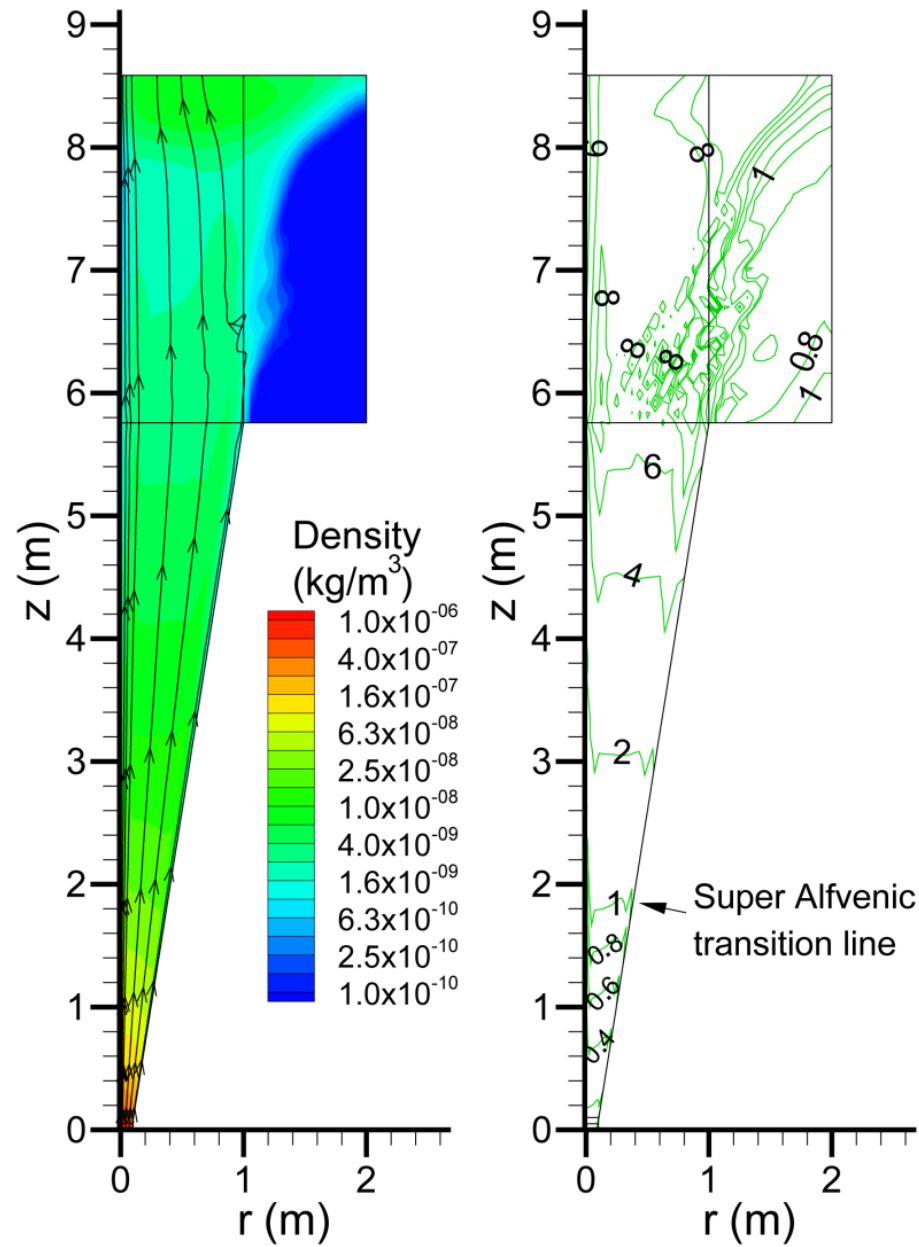


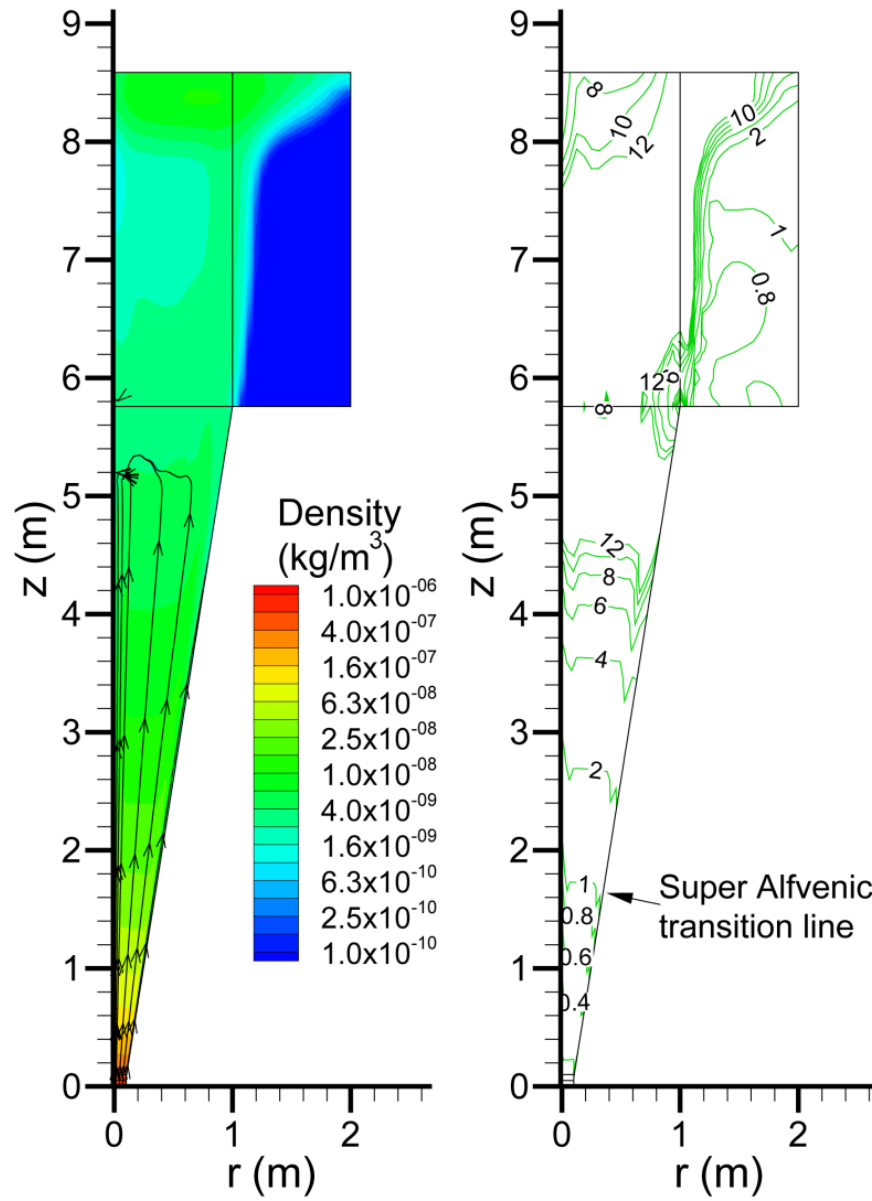
Fig. 1. Magnetic nozzle geometry showing (a) computational mesh, (b) computational block structure with blocks and boundaries labeled, and (c) initial magnetic field topology. Boundary (1), (2), (3-5), and (6) are the inlet, nozzle wall, exit, and axis of symmetry, respectively. Plasma is fed through boundary (1) of Block I. Blocks II and III are the expansion region of the nozzle. Blocks IV and V enable modeling plasma plume expansion downstream of the nozzle.

Results

Our real goal is to make an assessment of whether or not the implementation of the computational boundary conditions is successful. Numerical results indicate that the computed boundary conditions (Cases 1 and 2) are stable to small perturbations, as the model persists until the plasma plume reaches the nozzle exit, around 4.5×10^{-4} s. Case 3, with all parameters specified, is unstable and crashes at 2.6×10^{-5} s. In summary, we have noted that the computational boundary conditions (Case 1) gave results consistent with the extrapolation method (Case 2). Results from Cases 1 and 2 gave qualitative results as expected. The computational boundary conditions made the numerical model marginally more stable than those using extrapolation. Hard coding the boundary conditions caused the numerical model to crash, verifying that some flow properties must be computed based on interior conditions when some of the characteristic waves point out of the domain.



Case 1



Case 2

V. Concluding Remarks

We derive the compatibility equations using the method of projected characteristics for 3D spherical coordinate system for solar atmospheric dynamics study and cylindrical axisymmetric MHD nozzle flows. Some interesting findings can be summarized as follows:

1. The characteristic boundary conditions have shown more stable than first order extrapolation for sub-Alfvénic MHD nozzle inlet flows.
2. To assure the self-consistent evolutionary solar plasma, characteristic boundary conditions are necessary.