Calculation of diffusive particle acceleration by shock waves using asymmetric skew stochastic processes

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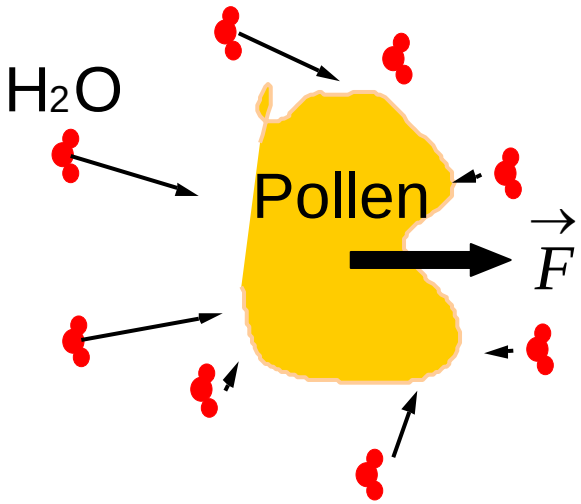
Cosmic ray or energetic particle transport in interstellar magnetic and interplanetary magnetic field is most likely a diffusion process in phase space. The distribution function of particles can be described by a Fokker-Planck equation. The applicable problems include:

1. Solar modulation of cosmic rays
2. Cosmic ray propagation through interstellar medium with nuclear interaction network
3. Diffusive shock acceleration
4. Solar energetic particles
5. Second order Fermi

Einstein Theory of Brownian Motion



for steady state

$$-k\vec{V}+\vec{F}=m\vec{a}=0$$

$$\vec{x}=\int\frac{\vec{F}}{k}dt=\int(White\ Noise)dt$$

Langevin Equation: $$\frac{dx(t)}{dt}=V(x,t)+\sigma(x,t)*"noise"$$

Stochastic differential equation:

$$dx(t)=V(x,t)dt+\sigma(x,t)dW(t) \qquad \text{(Ito type)}$$

where $dW(t)="noise"*dt$ with $\left\langle (dW(t))^2\right\rangle=dt,$

a Wiener (Gaussian) processs

Fokker-Planck Equation

- For a stochastic process:

(Ito) $dx(t) = V(x,t)dt + \sigma(x,t)dW(t)$

The probability density to find the process at a given time and location follows a Fokker-Planck equation:

$$\frac{\partial P(x_0,t_0;x,t)}{\partial t} = [\frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma(x,t)^2 - \frac{\partial}{\partial x}V(x,t)]P(x_0,t_0;x,t)]$$

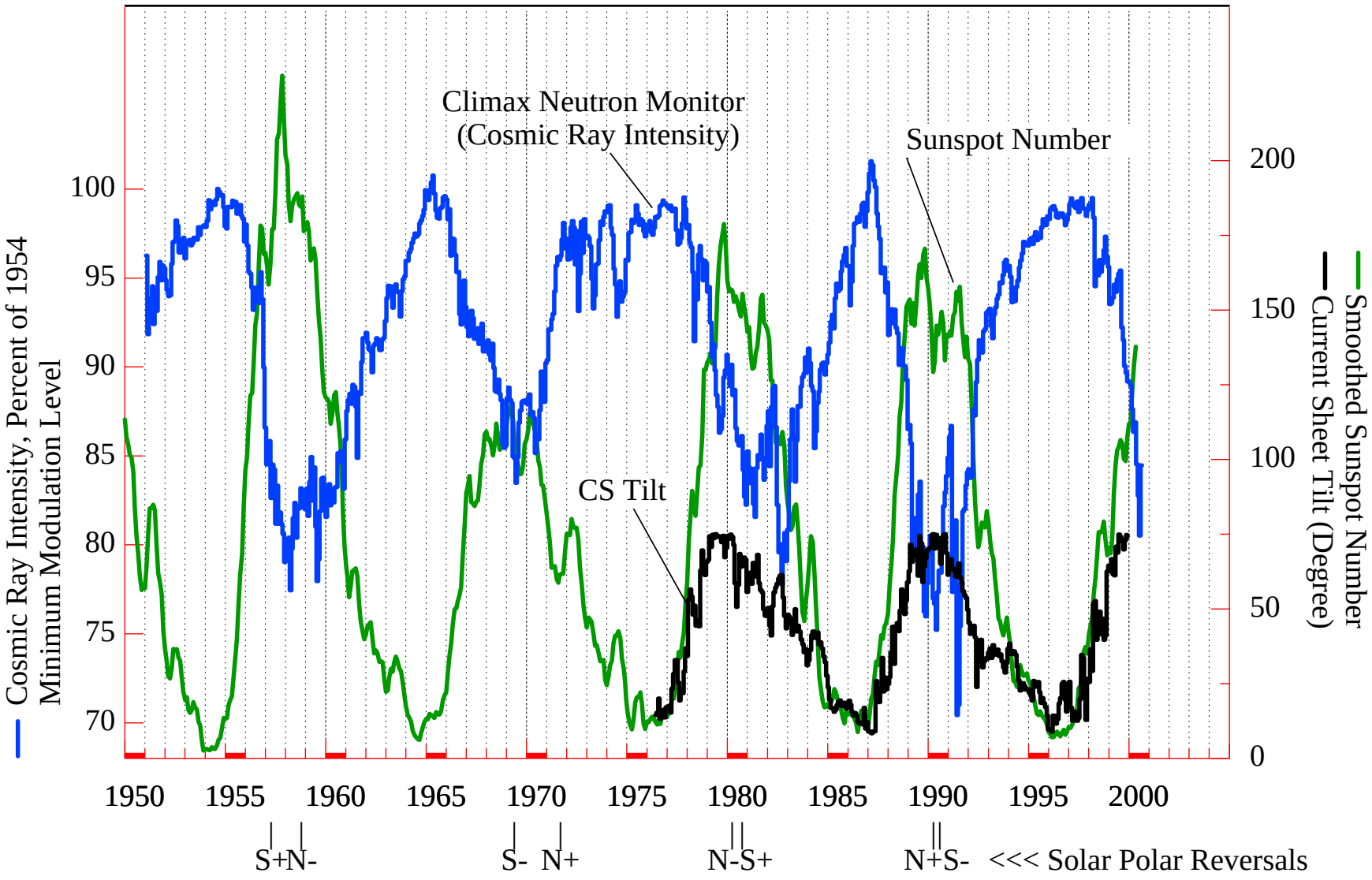
(Time forward)

$$-\frac{\partial P(x_0,t_0;x,t)}{\partial t_0} = [\frac{1}{2}\sigma(x_0,t_0)^2\frac{\partial^2}{\partial {x_0}^2} + V(x_0,t_0)\frac{\partial}{\partial x}]P(x_0,t_0;x,t)]$$

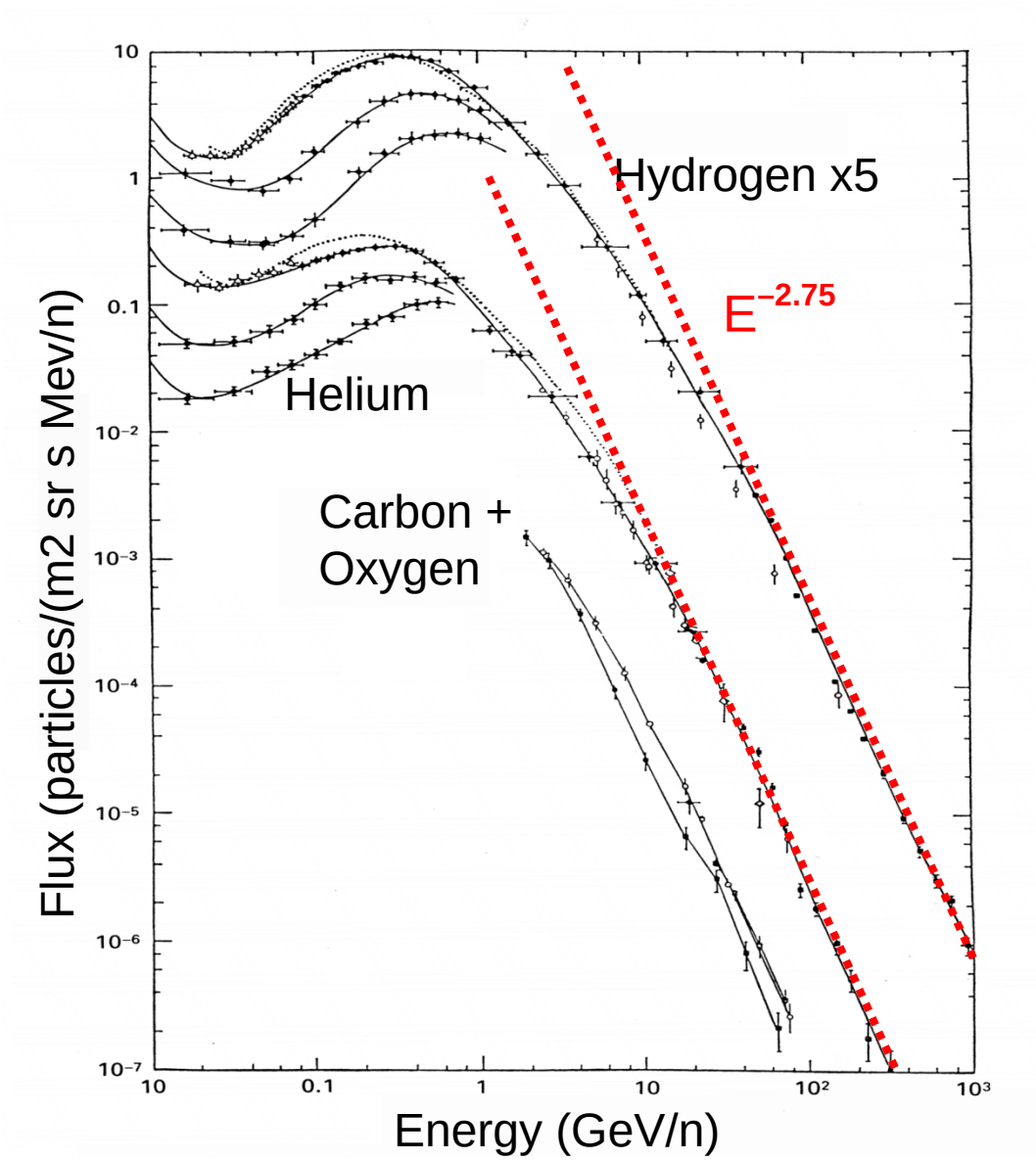
(Time backward)

diffusion coefficient is: $\kappa = \frac{1}{2}\ \sigma^2$

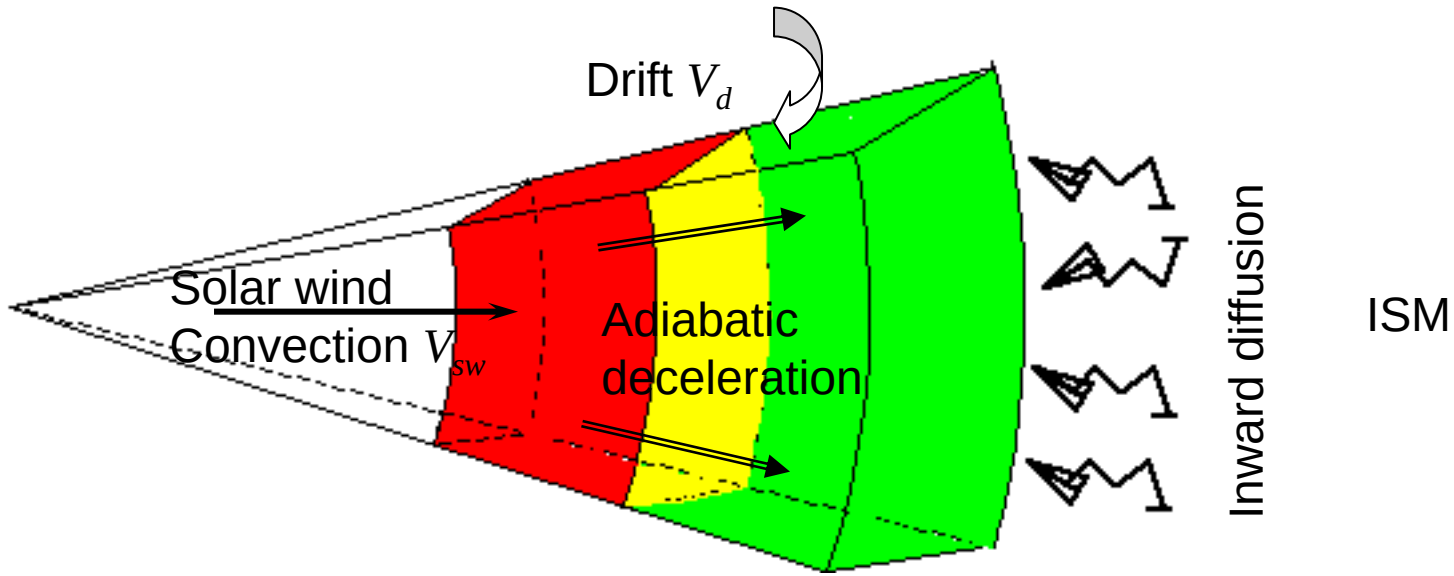
Solar Modulation of Cosmic Ray



Modulated cosmic ray spectra



Cosmic ray transport mechanisms in the heliosphere



$$\frac{\partial f(\boldsymbol{x}, p, t)}{\partial t} = \nabla \cdot (\kappa \nabla f) \quad \leftarrow \text{Diffusion}$$

$$- \boldsymbol{V}_{sw} \cdot \nabla f \quad \leftarrow \text{Convection}$$

$$- \boldsymbol{V}_d \cdot \nabla f \quad \leftarrow \text{Drift}$$

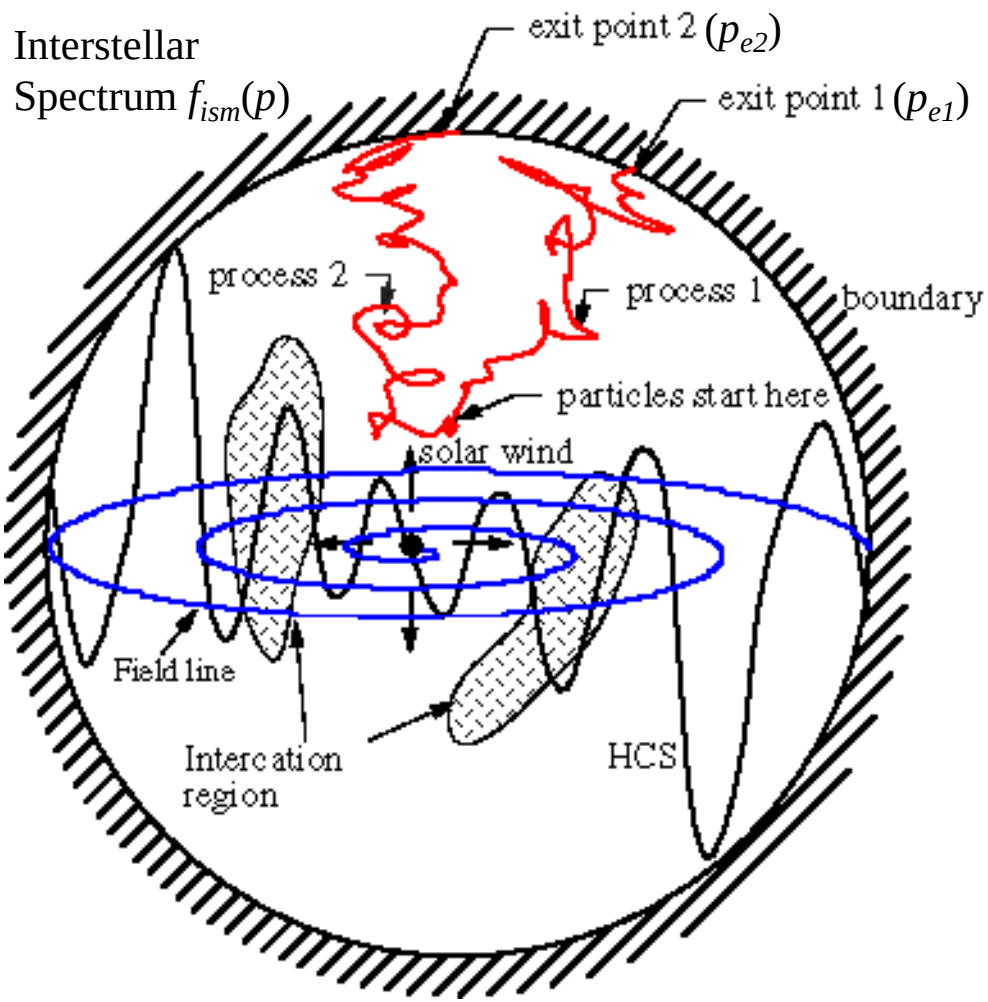
$$+ \frac{1}{3} (\nabla \cdot \mathbf{V}_{sw}) \, p \frac{\partial}{\partial p} f \quad \leftarrow \text{Adiabatic energy change}$$

$$\cancel{+ \frac{\partial}{p^2 \partial} (D_{pp} p^2 \frac{\partial f}{\partial p})} \quad \leftarrow \text{Second Fermi}$$

with $f = f_{ism}(p)$ at outer boundary

and $f = 0$ at inner boundary

Stochastic method to solve cosmic ray modulation spectra



Backward trajectory of particles

$$d\boldsymbol{x}(s) = \sum_{i=1}^{3} \sigma_i dw_i(s) + (\nabla \kappa - \boldsymbol{V} - \boldsymbol{V}_d) ds$$

$$dp(s) = \frac{1}{3}(\nabla \cdot \boldsymbol{V}) p ds$$

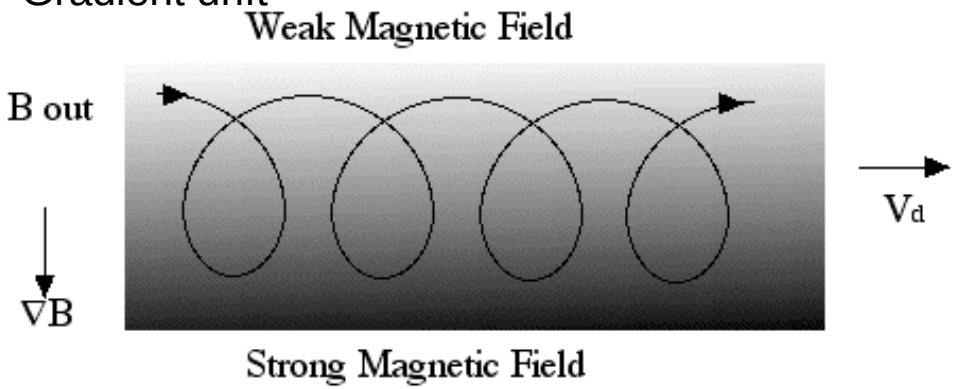
all starting at (x,p,t)

Modulated spectrum

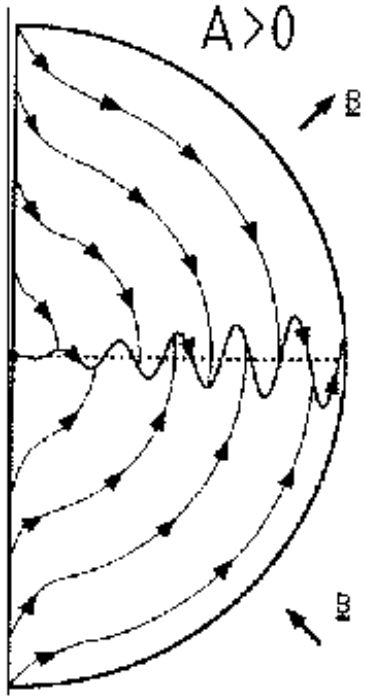
$$f(x,p,t) = \langle f_b \rangle \approx \langle f_{ism}(p_{exit}) \rangle$$

Particle drift in inhomogeneous magnetic field

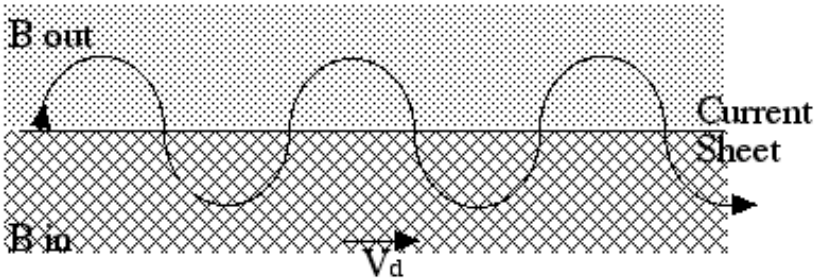
Gradient drift

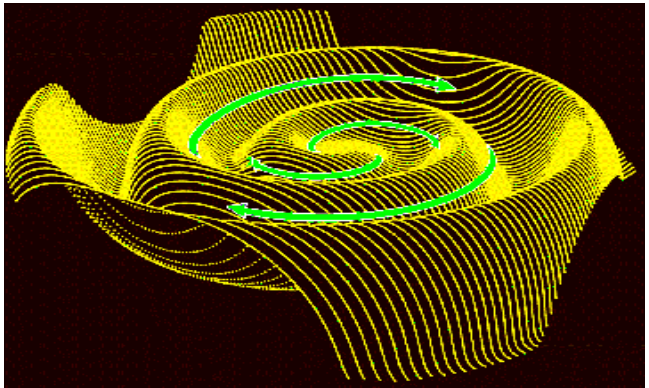


Drift path of protons



Drift in the current sheet



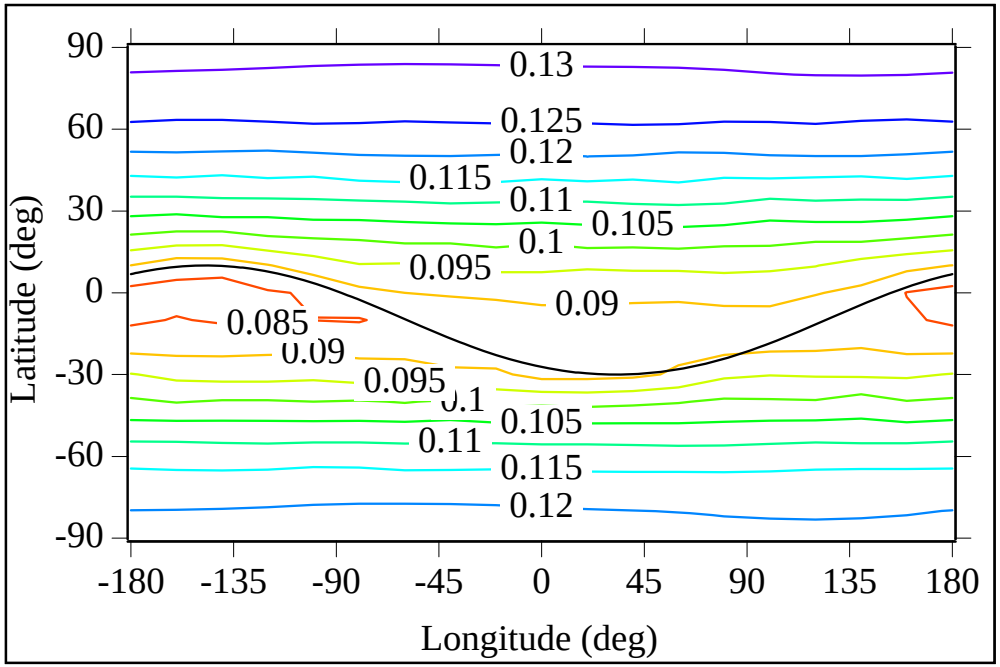


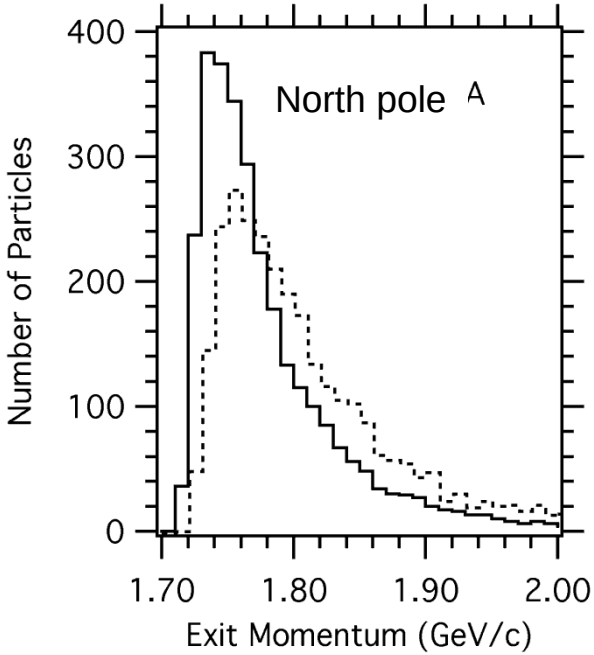
$$\vec{V}_d = \frac{pcv}{3q} \nabla \times \frac{\vec{B}}{B^2}$$

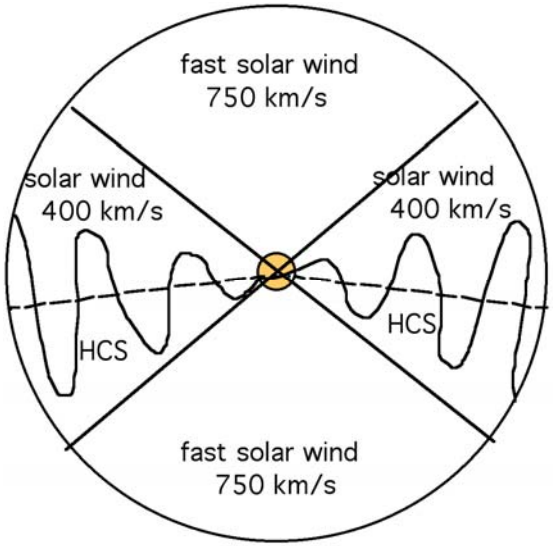
$$\vec{B} = \frac{A}{r^2}(\hat{r} - \frac{\Omega\ r \sin\theta}{V_{sw}} \hat{\varphi}) H(\theta - \theta_{cs})$$

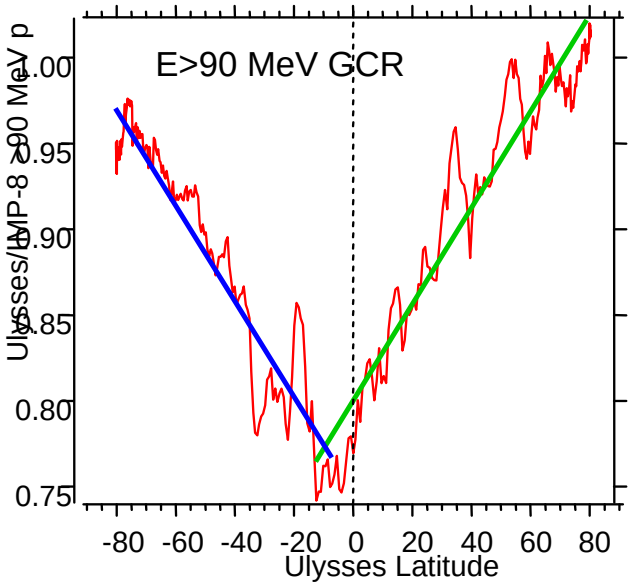
$$V_{dcs} = \frac{pcv}{3qB} \delta(\theta - \theta_{cs})$$ (singularity)

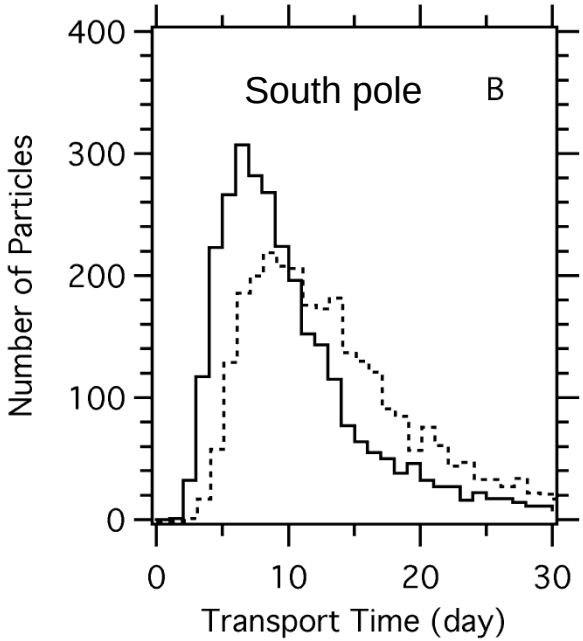
Distribution of particle (1 GeV/c) intensity on a sphere at 1 AU in an asymmetric heliospheric field



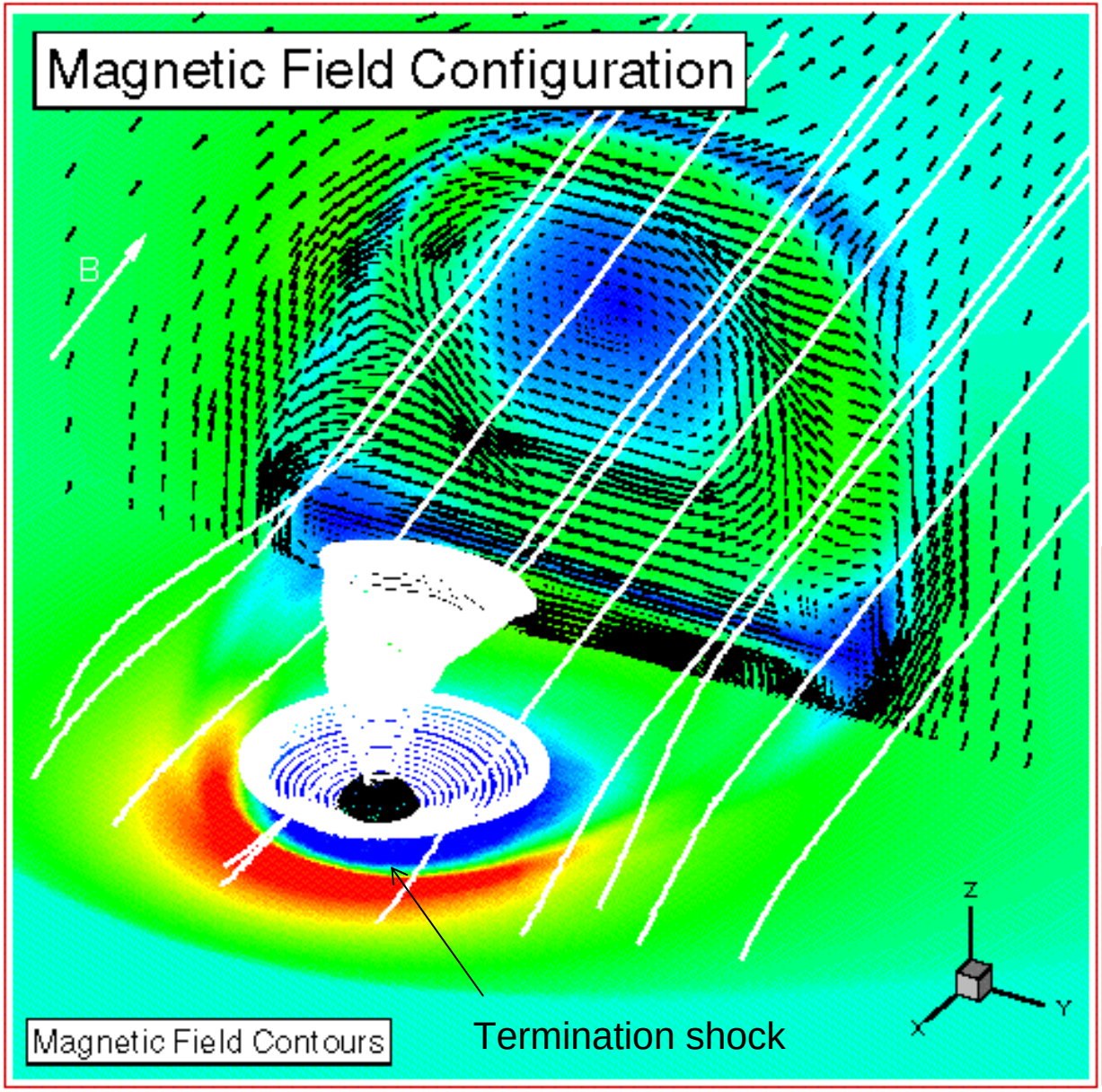


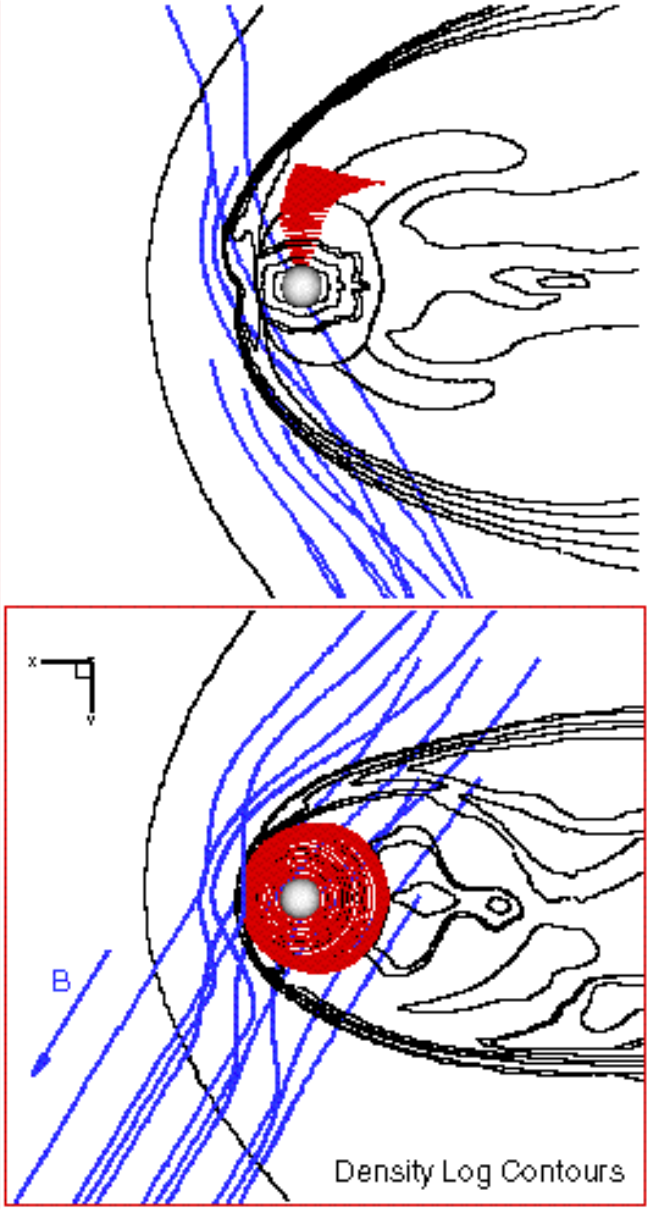






3-d MHD model heliosphere





MHD model heliosphere provided by Linde et al. (1998)

Diffusive Shock Acceleration

Nearly isotropic distribution

$$\frac{\partial f}{\partial t} = \nabla \cdot \kappa \cdot \nabla f - \mathbf{V} \cdot \nabla f + \frac{1}{3} \nabla \cdot V p \frac{\partial f}{\partial p}$$

Stochastic simulation (time forward):

$$d\mathbf{x} = \sqrt{2\kappa} \cdot d\mathbf{w} + (\nabla \cdot \kappa + \mathbf{V}) dt \qquad (3\text{-d})$$

$$dp = -\frac{p}{3} \nabla \cdot V dt$$

At shock

$$\nabla \cdot V = -(V_{n1} - V_{n2}) \delta(x - x_{sh})$$

$$\nabla \kappa = -(\kappa_1 - \kappa_2) \hat{n} \delta(x - x_{sh})$$

(singularity)

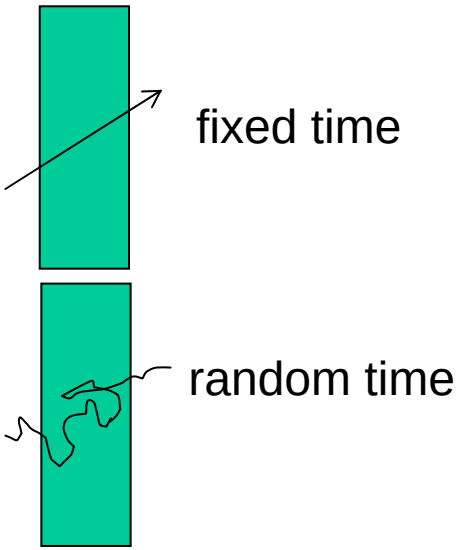
$$\text{upstream}: \quad V_{n1} = V_n(x_{sh}-), \ \kappa_1 = \kappa(x_{sh}-)$$

$$\text{downstream}: \quad V_{n2} = V_n(x_{sh}+), \ \kappa_2 = \kappa(x_{sh}-)$$

How to integrate $\delta(x - x_{sh})dt$

In normal calculus:

$$\int \delta(x - x_{sh})dt = \int \delta(x - x_{sh}) \frac{dx}{v_x dt} dt = \frac{1}{v_{xsh}}$$



In stochastic calculus:

$$\int \delta(x - x_{sh})dt = \int \delta(x - x_{sh}) \frac{dx}{\sigma dw(t) + v_x dt} dt = ?$$

Remove the singularity in stochastic differential equation

$$\xi = s(x)(x - x_{sh})$$

$$s(x) = \begin{cases} \dfrac{\kappa_2}{\kappa_1 + \kappa_2}, & \text{for } x < x_{sh} \\ \dfrac{1}{2}, & \text{for } x = x_{sh} \\ \dfrac{\kappa_1}{\kappa_1 + \kappa_2}, & \text{for } x > x_{sh} \end{cases}$$

$$d\xi = \frac{\partial \xi}{\partial x} dx + \frac{1}{2!} \frac{\partial^2 \xi}{\partial x^2} dx^2$$

$$= s(x)[\sqrt{2\kappa} dw(t) + V dt]$$

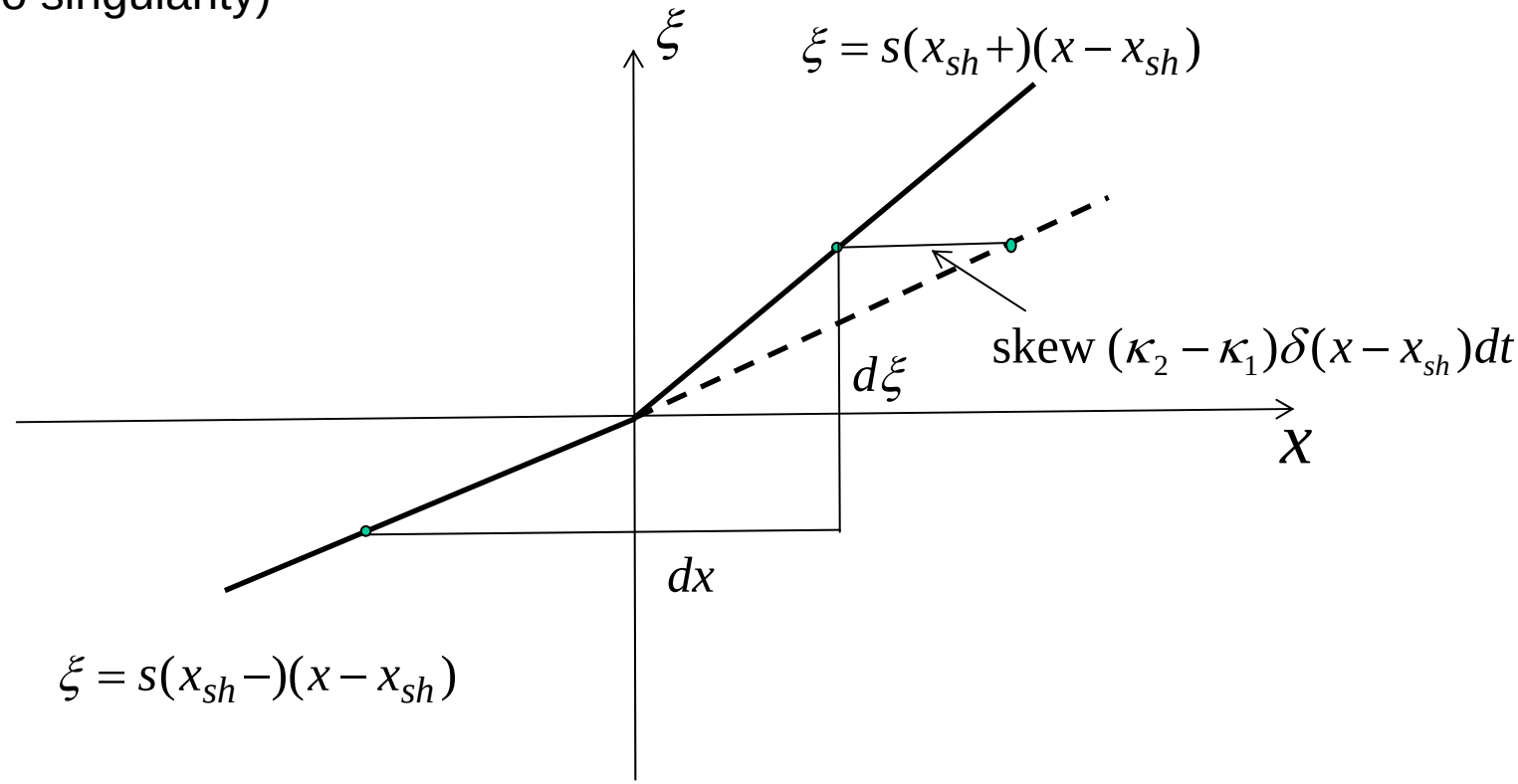
$$s(x)dx - d\xi = s(x)(\kappa_2 - \kappa_1)\delta(x - x_{sh})dt$$

Skew stochastic process

$$d\xi = s(x)[\sqrt{2\kappa}dw(t) + Vdt]$$

$$x - x_{sh} = \frac{\xi}{s(\xi)}$$

(no singularity)



Local time $\quad L_t = \delta(x - x_{sh})dt = \dfrac{2[s(x)dx - d\xi]}{\kappa_2 - \kappa_1}$

Euler scheme $\quad \delta(x - x_{sh})\Delta t_n = \dfrac{1}{2}\left(\dfrac{1}{\kappa_1} + \dfrac{1}{\kappa_2}\right)[sign(x_{n+1} - x_{sh}) - sign(x_n - x_{sh})]\xi_{n+1}$

Full code for time-dependent diffusive shock acceleration

```
      implicit real*8 (a-h,o-z)
      implicit integer (i-n)
      dimension xp0(2),xpb(2)
      character ttime*24,cmon(12)*3
      integer iyr,imon(12),idom,idoy,ihr,imn,isc
      data
imon/0,31,59,90,120,151,181,212,243,273,304,334/
      data cmon/"Jan","Feb","Mar","Apr","May","Jun",
     +          "Jul","Aug","Sep","Oct","Nov","Dec"/
      common
/param1/v1,v2,rb,g0m,g0p,gc0,b,iseed,nz,nm
      data epx/1e-10/

      iseed=-10002882

c   start the injection

      ntp=500000
      do i=1,ntp
        xp0(1)=-epx
        xp0(2)=1.0
        xpb(1)=rb
        te=100.
        call walk(xp0,xpb,te,ns)
        write(26,960) xpb,te,ns
960         format(3e12.4,i10)
      enddo

      stop
      end

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      subroutine walk(xp0,xpb,te,ns)
c  random walk in the shock frame (x in Cartisian)

      implicit real*8 (a-h,o-z)
      implicit integer (i-n)
c  xp = (x, p)
      dimension xp0(2),xpb(2)
      common
/param1/v1,v2,rb,g0m,g0p,gc0,b,iseed,nz,nm
      data ts1/1e-2/,ts2/1.e-4/

      dt1=2*gc0/v1/v1*ts1
      dt2=2*gc0/v1/v1*ts2
      srdt1=sqrt(dt1)
      srdt2=sqrt(dt2)

      x=xp0(1)
      p=xp0(2)
      t=0.0
      n=0


c  find the convection speed and sign of x, scaling
factor, and y
      if(x.lt.0) then
        v=v1
        nsgnx0=-1
        sf=g0p/(g0p+g0m)
        y=sf*x
      else
        if(x.eq.0) then
          v=(v1+v2)/2.
          nsgnx0=0
          sf=0.5
          y=sf*x
        else
          v=v2
          nsgnx0=1
          sf=g0m/(g0p+g0m)
          y=sf*x
        endif
      endif


c  find the diffusion coeficients
1      call getdc(x,p,gc,dgc)
      g=gc+0.5*(g0p-g0m)*nsgnx0
      dg=dgc

c
c  find time step side
c    to increase accrucy near the shock
      dmax=3*sqrt(2*g)*srdt1
      dmin=3*sqrt(2*g)*srdt2
      if(nsgnx0*x.gt.dmax) then
        dt=dt1
        srdt=srdt1
      else
        if(nsgnx0*x.lt.dmin) then
          dt=dt2
          srdt=srdt2
        else
          dt=dt2+(dt1-dt2)/(dmax*dmax-
dmin*dmin)*(x*x-dmin*dmin)
          srdt=sqrt(dt)
        endif
      endif
      t=t+dt
      n=n+1

c  step forward
      dw=srdt*gasdev(iseed)
      dy=sf*((v+dg)*dt+sqrt(2*g)*dw)

      y=y+dy

c  find the convection sign of x and convert y to x
c      and speed, scaling factor for next step,
      if(y.lt.0) then
        v=v1
        nsgnx=-1
        sf=g0p/(g0p+g0m)
        x=y/sf
      else
        if(y.eq.0) then
          v=(v1+v2)/2.
          nsgnx=0
          sf=0.5
          x=y/sf
        else
          v=v2
          nsgnx=1
          sf=g0m/(g0p+g0m)
          x=y/sf
        endif
      endif

c  local time
      dlt=(0.5/g0m+0.5/g0p)*(nsgnx-nsgnx0)*y
      dp=1./3.*(v1-v2)*p*dlt
      p=p+dp

c  write the trajectory to file 28
c      write(28,980) x,p,t
980      format(3e12.4)

c      if(nsgnx*x.lt.xpb(1)) then
      if(t.lt.te) then
        nsgnx0=nsgnx
        goto 1
      endif
c  exit from the boundary
      xpb(1)=x
      xpb(2)=p
      ns=n
      te=t
      return
      end


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      subroutine getdc(x,p,gc,dgc)
c  subroutine to calculate diffusion coeffient at
(x,p) and its gradient

      implicit real*8 (a-h,o-z)
      implicit integer (i-n)
      common
/param1/v1,v2,rb,g0m,g0p,gc0,b,iseed,nz,nm

      gc=gc0*p**b*f(x)
      dgc=gc0*p**b*df(x)

      return
      end

      function f(x)
c  position dependence of the diffusion coefficient
      implicit real*8 (a-h,o-z)

      f=1.

      return
      end

      function df(x)
c  derivative of f(x)
      implicit real*8 (a-h,o-z)

      df=0.

      return
      end
```
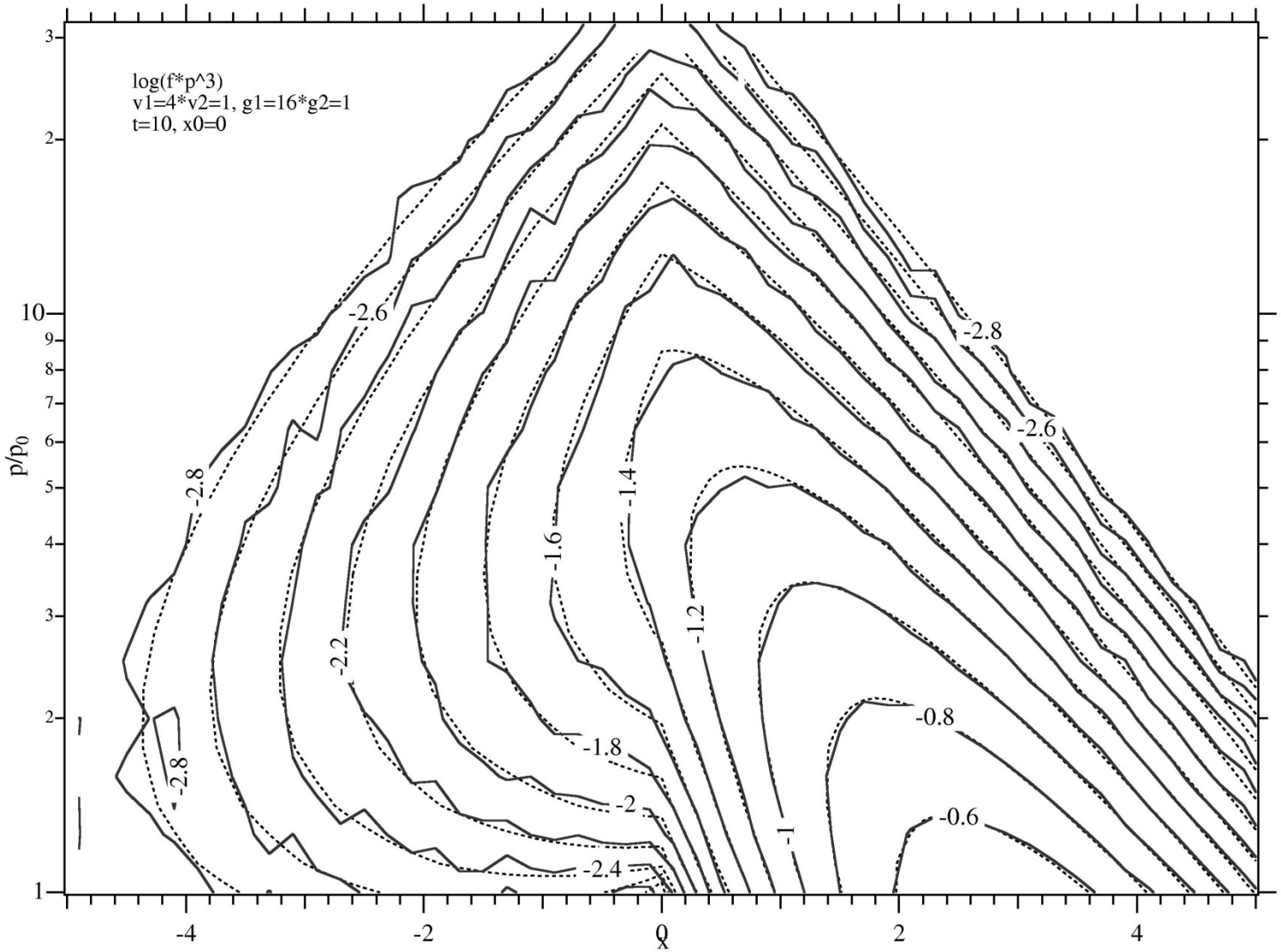
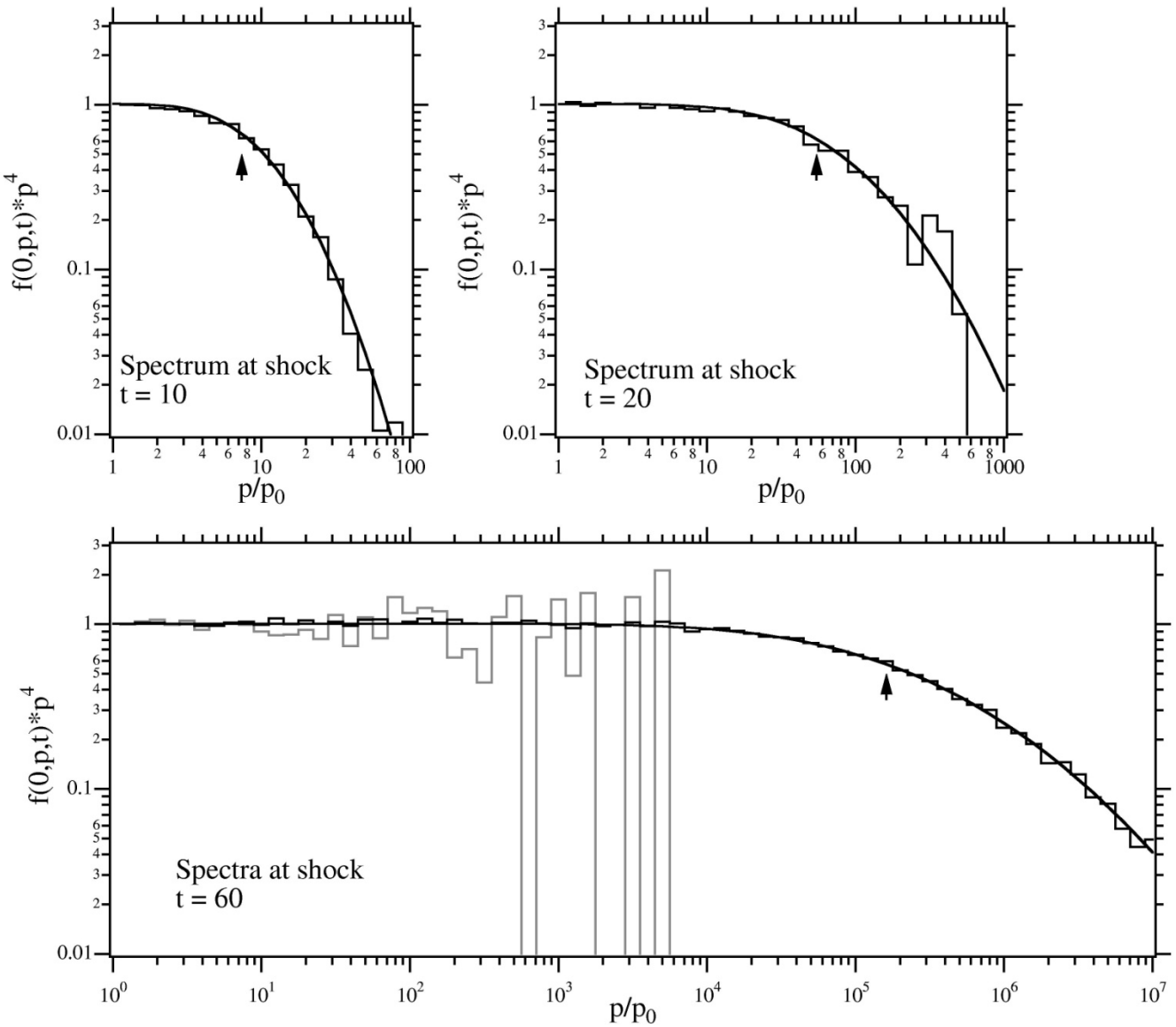

Comparison with analytical solution
1-d shock, monoenergetic particle injection at shock at t=0



Evolution of particle spectrum at the shock



Transport equation for particles with large anisotropy

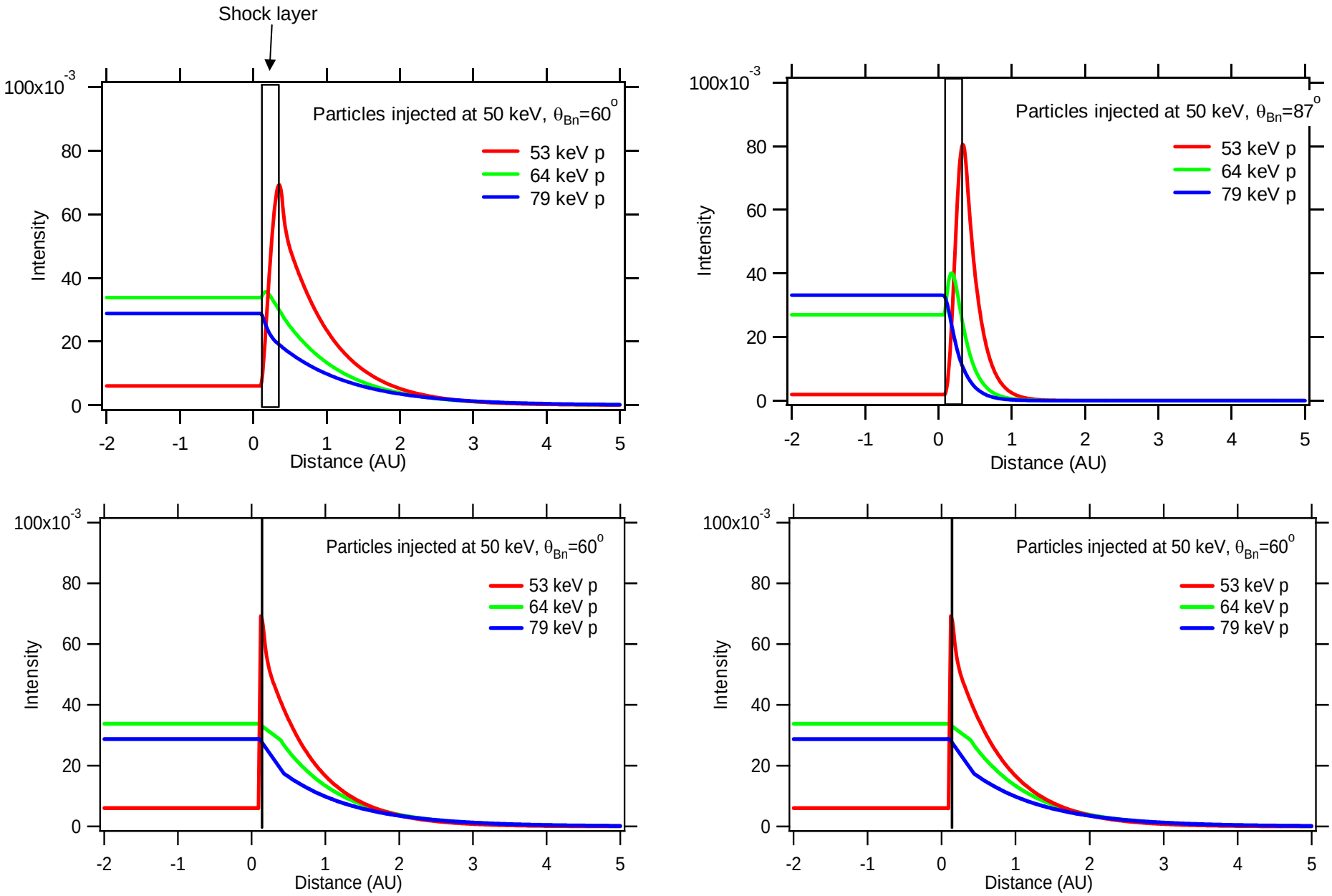
$$
\begin{aligned}
\frac{\partial f(t,\boldsymbol{r},p,\mu)}{\partial t} &= \nabla\cdot\boldsymbol{\kappa}_{\perp}\cdot\nabla f && \leftarrow \text{cross-field diffusion}\\
&-(\boldsymbol{V}_{sw}+\boldsymbol{V}_{d}+v\mu\hat{\boldsymbol{b}})\cdot\nabla f && \leftarrow \text{convection, drift and streaming}\\
&+\frac{\partial}{\partial\mu}D_{\mu\mu}\frac{\partial f}{\partial\mu} && \leftarrow \text{pitch angle diffusion}\\
&+\left[\frac{(1-\mu^2)v}{L_B}-\frac{\mu(1-\mu^2)}{2}(\nabla\cdot\boldsymbol{V}_{sw}-3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}:\nabla\boldsymbol{V}_{sw})\right]\frac{\partial f}{\partial\mu} && \leftarrow \text{focusing}\\
&+\left[qE_{\|}\mu+p\left(\frac{1-\mu^2}{2}(\nabla\cdot\boldsymbol{V}_{sw}-\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}:\nabla\boldsymbol{V}_{sw})+\mu^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}:\nabla\boldsymbol{V}_{sw}\right)\right]\frac{\partial f}{\partial p} && \leftarrow \text{energy gain}
\end{aligned}
$$

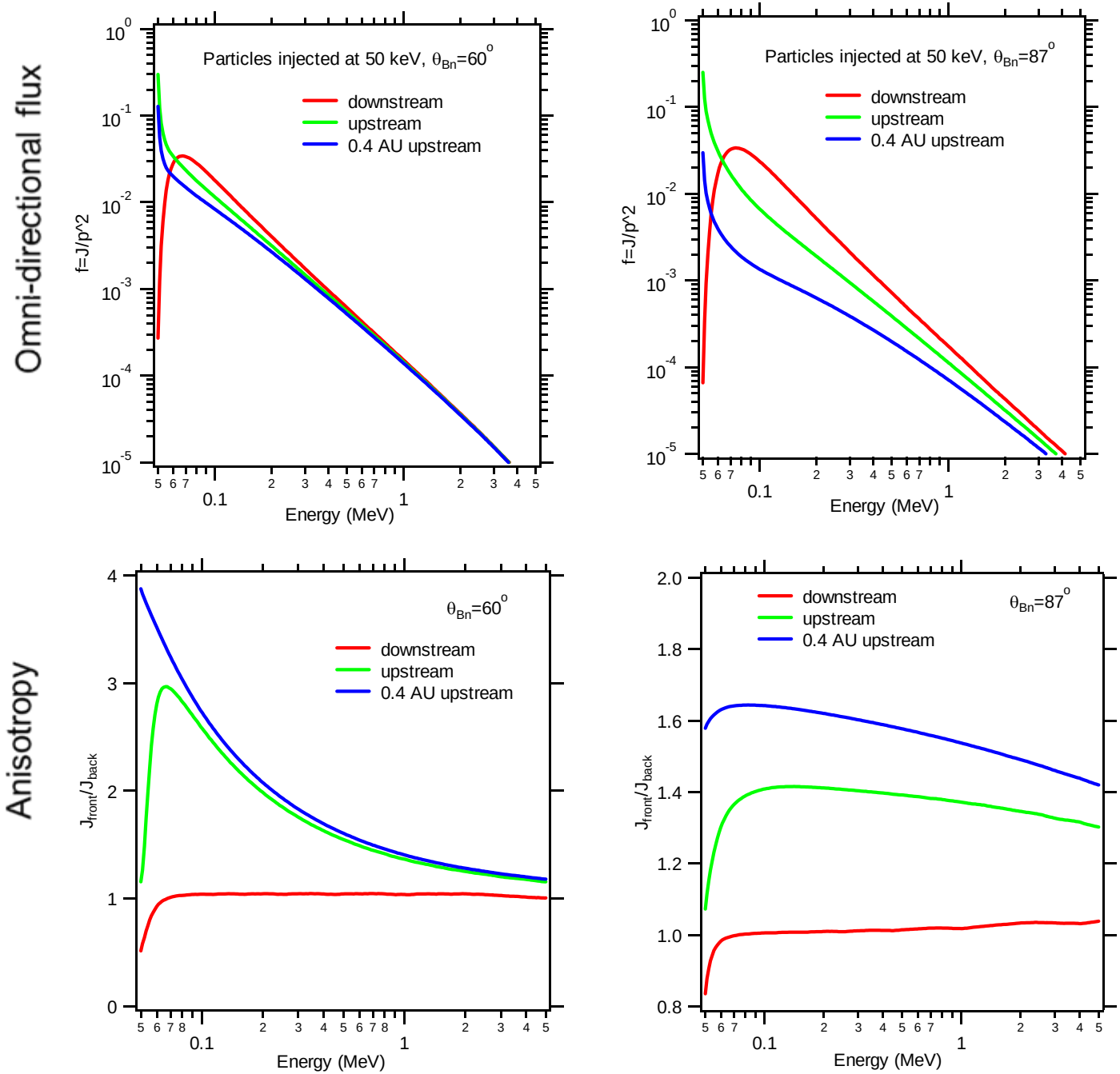
Stochastic differential equation solver

$$d\mathbf{x}=\sqrt{2\boldsymbol{\kappa}_{\perp}}\,d\mathbf{w}(s)+(\nabla\cdot\boldsymbol{\kappa}_{\perp}-\mathbf{V}_{sw}-\mathbf{V}_{d}-v\mu\hat{\mathbf{b}})ds$$

$$dp=\left[\frac{1-\mu^2}{2}(\nabla\cdot\mathbf{V}_{sw}-\hat{\mathbf{b}}\hat{\mathbf{b}}:\nabla\mathbf{V}_{sw})+\mu^2\hat{\mathbf{b}}\hat{\mathbf{b}}:\nabla\mathbf{V}_{sw}\right]pds$$

$$d\mu=\sqrt{2D_{\mu\mu}}\,dw(s)+\left[\frac{\partial D_{\mu\mu}}{\partial\mu}+\frac{(1-\mu^2)v}{2L_B}-\frac{\mu(1-\mu^2)}{2}(\nabla\cdot\mathbf{V}_{sw}-3\hat{\mathbf{b}}\hat{\mathbf{b}}:\nabla\mathbf{V}_{sw})\right]ds$$


Shock layer
Particles injected at 50 keV, θ_{Bn}=60°
53 keV p
64 keV p
79 keV p
Intensity
Distance (AU)
Particles injected at 50 keV, θ_{Bn}=87°
53 keV p
64 keV p
79 keV p
Intensity
Distance (AU)
Particles injected at 50 keV, θ_{Bn}=60°
53 keV p
64 keV p
79 keV p
Intensity
Distance (AU)
Particles injected at 50 keV, θ_{Bn}=60°
53 keV p
64 keV p
79 keV p
Intensity
Distance (AU)


Omni-directional flux
Particles injected at 50 keV, θBn=60°
downstream
upstream
0.4 AU upstream
f=J/p^2
Energy (MeV)
Particles injected at 50 keV, θBn=87°
downstream
upstream
0.4 AU upstream
f=J/p^2
Energy (MeV)
Anisotropy
θBn=60°
downstream
upstream
0.4 AU upstream
Jfront/Jback
Energy (MeV)
θBn=87°
downstream
upstream
0.4 AU upstream
Jfront/Jback
Energy (MeV)

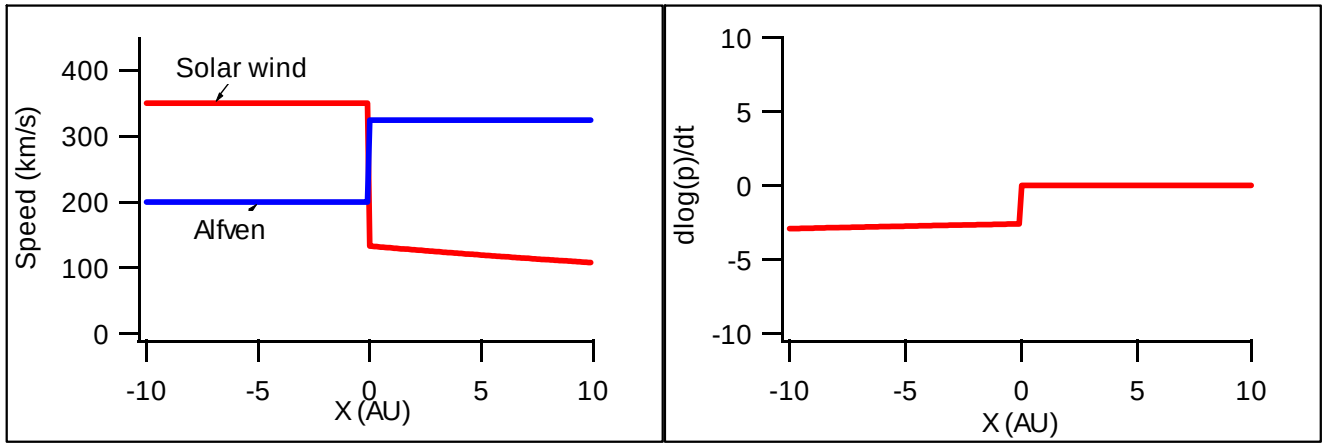
Particle acceleration by the termination shock and heliosheath

- Voyager observations show that anomalous cosmic rays are accelerated beyond the termination shock
- Acceleration mechanisms of particle acceleration in the heliosheath
  - adiabatic heating by plasma compression in the heliosheath
  - Second order Fermi acceleration with

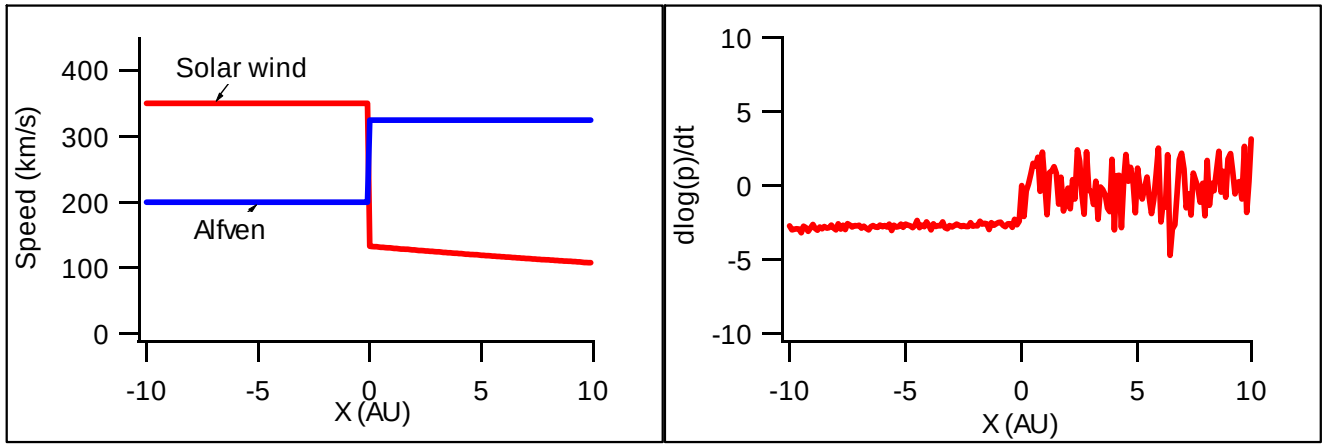
$$D_{pp} = \frac{p^2 V_A^2}{v^2} D_{\mu\mu} = A \frac{p^2 V_A^2}{\kappa_{||}} \neq 0$$

Particle acceleration by the termination shock and heliosheath

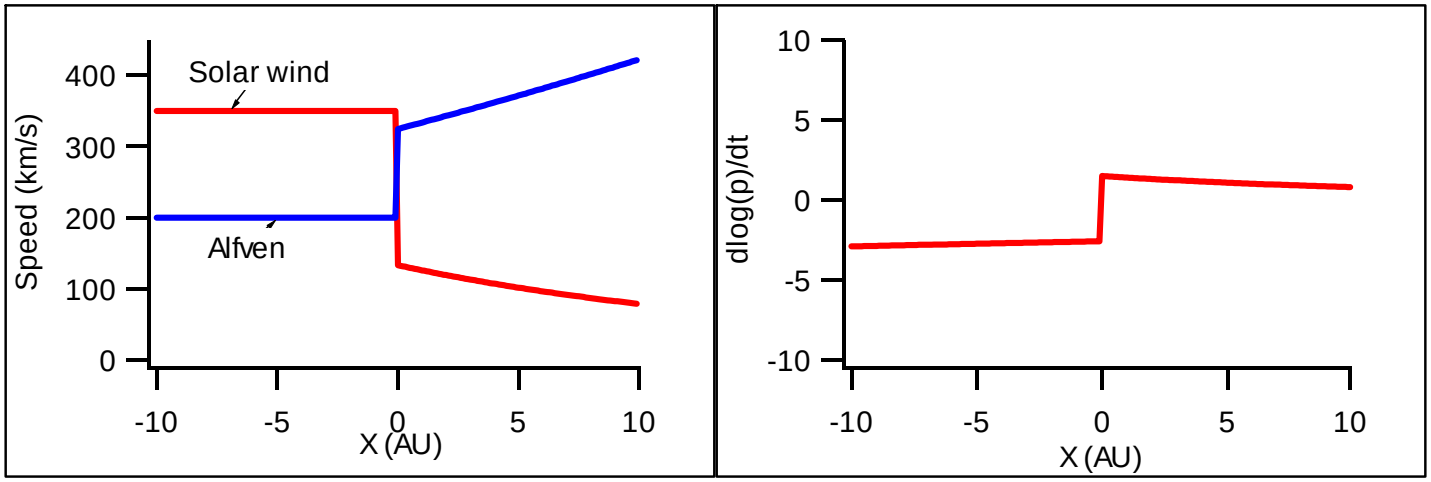
A. Plain shock acceleration



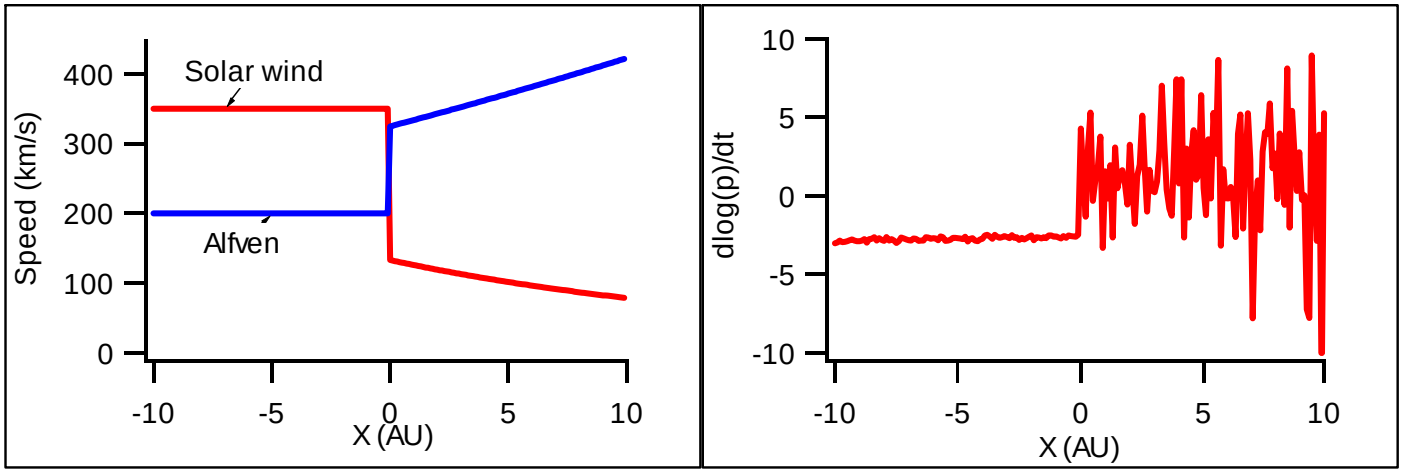
B. Shock acceleration with second-order Fermi acceleration

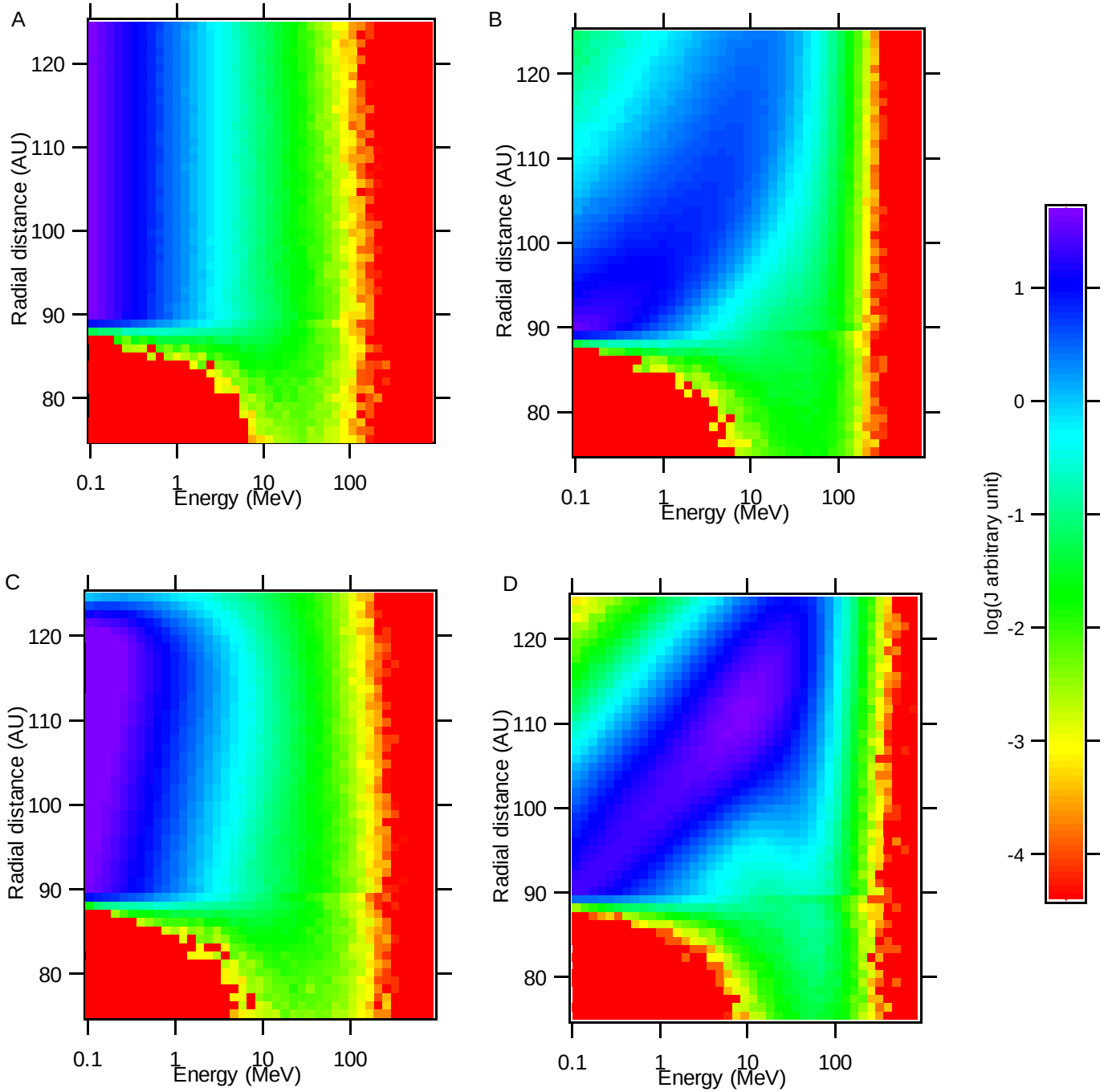


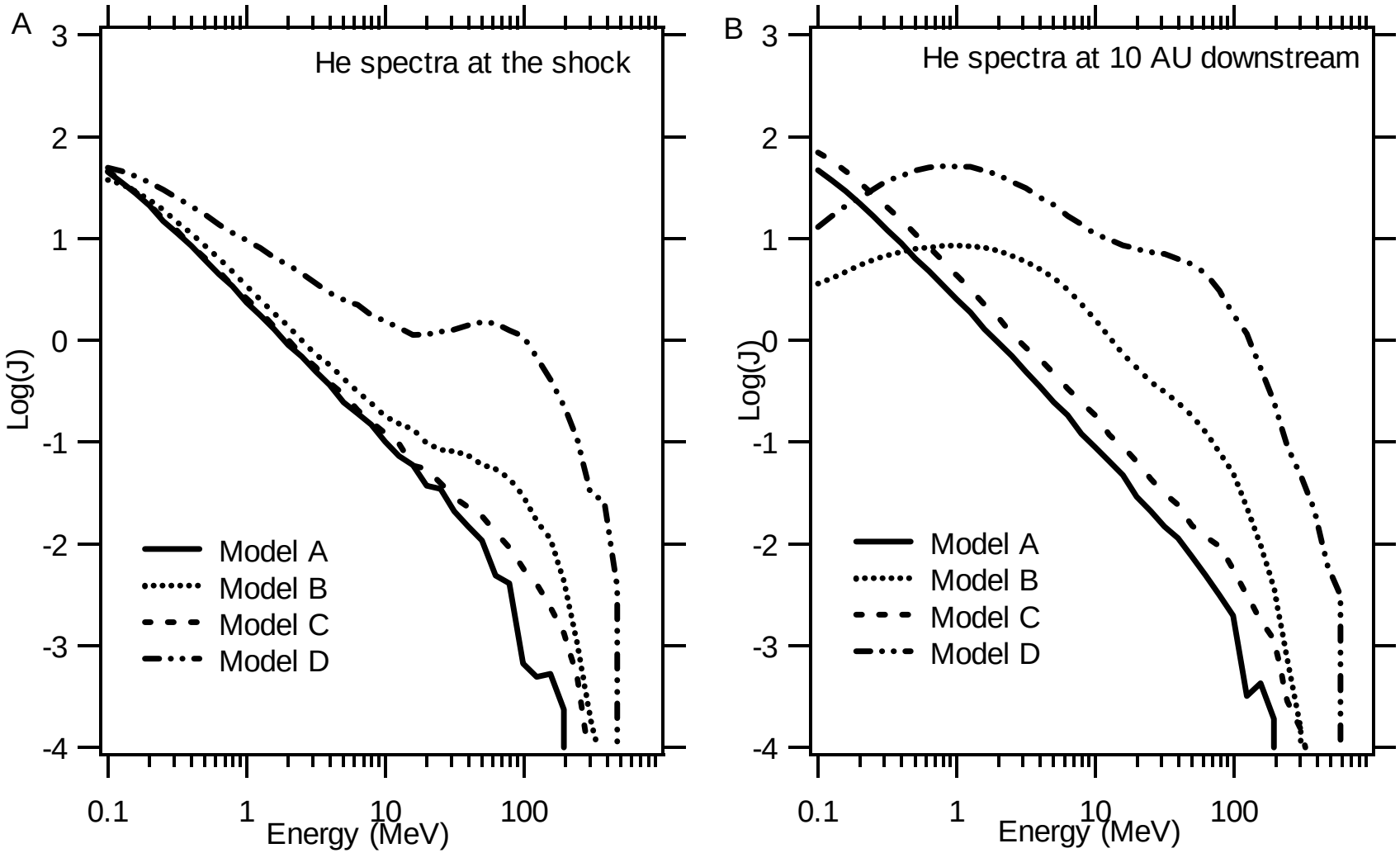
C. Shock acceleration with adiabatic post-shock compression



D. Shock acceleration with adiabatic post-shock compression and stochastic acceleration




A
B
C
D
Radial distance (AU)
Energy (MeV)
120
110
100
90
80
0.1
1
10
100
log(J arbitrary unit)
1
0
-1
-2
-3
-4


A
He spectra at the shock
Log(J)
Energy (MeV)
Model A
Model B
Model C
Model D
B
He spectra at 10 AU downstream
Log(J)
Energy (MeV)
Model A
Model B
Model C
Model D

Stochastic differential equation
vs.
Fokker-Planck equation

- ✓ Monte-Carlo simulation with SDE can be used to solve Fokker-Planck equation.
- ✓ Track stochastic trajectories to look into the details of particle transport.
- ✓ Problems in high dimensions do not necessarily increase computation demand.
- ✓ Programming is extremely simple and less numerical instability.
- ✓ Singularity in stochastic differential equations can be solved by skew Brownian motion
- ✓ Application to shock acceleration (both isotropic and focus transport)
- ✓ Application to particle drift in heliospheric current sheet