Pairing from Effective Field Theory and Density Functional Theory

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June, 2010

References:

"Effective field theory for dilute fermions with pairing," rjf, H.-W. Hammer, S. Puglia, Ann. Phys. (NY) **322**, 2703 (2007)

"Toward ab initio density functional theory for nuclei," J. Drut, rjf, L. Platter, Prog. Part. Nucl. Phys. **64**, 120 (2009)

Outline

Overview: Pairing in EFT/DFT

Kohn-Sham DFT with pairing from effective action

Renormalization issues

Going forward: ideas, questions, tools

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Effective Field Theory ingredients

Simplest case: short-distance, *natural* interactions

1. Use the most general $\mathcal{L}$ with low-energy dof's consistent with global and local symmetries of underlying theory
   - $\mathcal{L}_{\rm eft} = \psi^\dagger\big[i\frac{\partial}{\partial t} + \frac{\overrightarrow{\nabla}^{2}}{2M}\big]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots$

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   - natural $a_s \sim R \Longrightarrow$ dimensional regularization and minimal subtraction (DR/MS) are most efficient but cutoff also ok

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2. Declaration of regularization and renormalization scheme
   - natural $a_s \sim R \Longrightarrow$ dimensional regularization and minimal subtraction (DR/MS) are most efficient but cutoff also ok
3. Well-defined power counting $\Longrightarrow$ small expansion parameter
   - use separation of scales $\Longrightarrow \frac{k}{\Lambda}$ with $\Lambda \sim 1/R \Longrightarrow k a_s \ll 1$, etc.
   - Recovers scattering expansion order-by-order with diagrams

$$f_0(k) \;\propto\; \frac{1}{k\cot\delta_0(k) - ik} \longrightarrow a_s\left[1 - i a_s k - (a_s^2 - a_s r_0/2)k^2 + \mathcal{O}(k^3 a_s^3)\right]$$

$$\longrightarrow R[1 - ikR - 2k^2R^2/3 + \mathcal{O}(k^3R^3)] \quad \text{[e.g., hard spheres]}$$

- with DR/MS, one power of k per diagram, *natural* coefficients
- estimate truncation error from dimensional analysis
- valid for *any* natural short-range interaction!

Effective Field Theory ingredients

Apply at finite density (no pairing) [Hammer, rjf, nucl-th/0004043]

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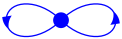
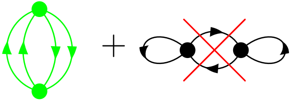
2. Declaration of regularization and renormalization scheme
   - natural $a_s \Longrightarrow$ dimensional regularization and min. subtraction
3. Well-defined power counting $\Longrightarrow$ small expansion parameters
   - use separation of scales $\Longrightarrow \frac{k_{\rm F}}{\Lambda}$ with $\Lambda \sim 1/R \Longrightarrow k_{\rm F} a_s \ll \pi$, etc.

$\mathcal{O}(k_{\rm F}^6)$:  $\mathcal{O}(k_{\rm F}^7)$: 

$$\mathcal{E} = \rho \frac{k_{\rm F}^2}{2M} \left[\frac{3}{5} + \frac{2}{3}(k_{\rm F} a_s/\pi) + \frac{4}{35}(11 - 2\ln 2)(k_{\rm F} a_s/\pi)^2 + \cdots \right]$$

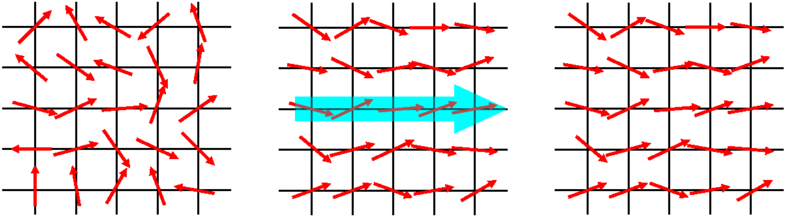
   - cleanly recovers perturbative free-space ERE and in-medium energy density (including 3-body logs), plus error estimates

Laundry list of EFT/DFT pairing issues

- Questions:
  - To what extent is an EDF unique? Kohn-Sham gap? (and so on)
  - Can we motivate/constrain ASLDA near the unitary limit?
  - For nuclei, do we want/need to include the pion explicitly?
- Ideas from other contexts:
  - Alternative (Kohn-Sham) DFT — e.g., auxiliary fields, non-local
  - Beyond quasi-local: orbital-based methods for long-range correlations (cf., RPA with OEP in Coulomb systems)
- Tools/methods:
  - Renormalized sums in finite systems (from RHA/soliton studies)
  - Operator Product Expansion (e.g., cancellation of divergences)
  - Collective coordinate methods (e.g., BRST invariance)

Effective actions and broken symmetries

- Natural framework for spontaneous symmetry breaking
  - e.g., test for zero-field magnetization M in a spin system
  - introduce an external field H to break rotational symmetry



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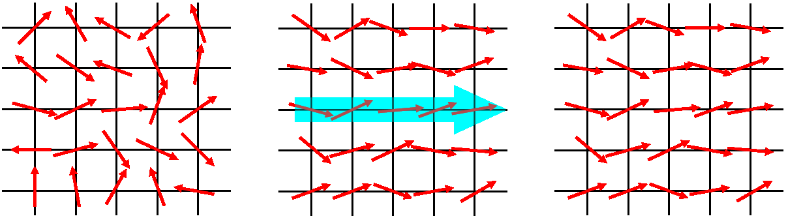
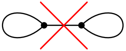


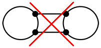
  - if $F[H]$ calculated perturbatively, $M[H=0]=0$ to all orders
  - Legendre transform Helmholtz free energy $F(H)$:

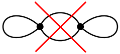
$$\text{invert}\quad M = -\partial F(H)/\partial H \quad \overset{H(M)}{\Longrightarrow} \quad \Gamma[M] = F[H(M)] + MH(M)$$

  - since $H = \partial\Gamma/\partial M \longrightarrow 0$, stationary points of $\Gamma \Longrightarrow$ ground state
- Can couple source "H" many ways (and multiple sources)

A bestiary of effective actions

- Couple source to local Lagrangian field, e.g., $J(x)\phi(x)$
  - $\Gamma[\varphi]$ where $\varphi(x) = \langle\phi(x)\rangle \Longrightarrow$ 1PI effective action 
  - Arises from fermion $\mathcal{L}$'s by introducing auxiliary fields
  - Can approximate with stationary phase $\Longrightarrow$ loop expansion

- Couple J to non-local composite op, e.g., $J(x,x')\phi(x)\phi(x')$
  - $\Gamma[G,\varphi] \Longrightarrow$ 2PI effective action [CJT] 
  - cf. Baym-Kadanoff conserving ("Φ-derivable") approximations
  - Often applied to hot, nonequilibrium QCD

- Source coupled to local composite operator, e.g., $J(x)\phi^2(x)$
  - 1.5PI effective action? Almost: 
  - Kohn-Sham DFT from inversion method
  - Careful: new divergences likely (e.g., pairing)

Partition function in $\beta \to \infty$ limit [see Zinn-Justin]

- Consider Hamiltonian with time-independent source $J(\mathbf{x})$:

$$\widehat{H}(J) = \widehat{H} + \int J\,\widehat{\phi} \quad \text{or} \quad \widehat{H}(J) = \widehat{H} + \int J\,\psi^\dagger\psi$$

- *If* ground state is isolated (and bounded from below),

$$e^{-\beta\widehat{H}(J)} = e^{-\beta E_0(J)}\left[|0\rangle\langle 0|_J + \mathcal{O}\big(e^{-\beta(E_1(J)-E_0(J))}\big)\right]$$

- As $\beta \to \infty$, $\mathcal{Z}[J] \Longrightarrow$ ground state of $\widehat{H}(J)$ with energy $E_0(J)$

$$\mathcal{Z}[J] = e^{-W[J]} \sim \mathrm{Tr}\, e^{-\beta(\widehat{H}+J\,\widehat{\rho})} \Longrightarrow E_0(J) = \lim_{\beta\to\infty} -\frac{1}{\beta}\log\mathcal{Z}[J] = \frac{1}{\beta}W[J]$$

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- $\Gamma[\rho]$: expectation value of $\widehat{H}$ in ground state generated by $J[\rho]$

$$\frac{1}{\beta}\Gamma[\rho] = E_0(J) - \int J\,\rho = \langle\widehat{H} + J\,\widehat{\rho}\rangle_J - \int J\,\rho = \langle\widehat{H}\rangle_J \xrightarrow{J\to 0} E_0$$

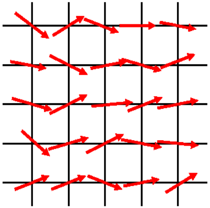
$$J(x) = -\frac{\delta\Gamma[\rho]}{\delta\rho(x)} \xrightarrow{J\to 0} \left.\frac{\delta\Gamma[\rho]}{\delta\rho(x)}\right|_{\rho_{\rm gs}(\mathbf{x})} = 0 \quad\Longrightarrow\quad \text{variational } F_{\rm HK}[\rho] \;\; (?)$$

DFT and effective actions (Fukuda et al., Polonyi, . . .)

- External field $\Longleftrightarrow$ Magnetization
- Helmholtz free energy $F[H]$
  $\Longleftrightarrow$ Gibbs free energy $\Gamma[M]$

Legendre transform $\Longrightarrow \Gamma[M] = F[H] + H\,M$

$$H = \frac{\partial \Gamma[M]}{\partial M} \xrightarrow[\text{state}]{\text{ground}} \left.\frac{\partial \Gamma[M]}{\partial M}\right|_{M_{\rm gs}} = 0$$



source magnet

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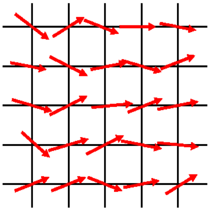

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- Partition function with sources ($J \rightarrow v$) that adjust (any) densities:

$$\mathcal{Z}[v] = e^{-W[v]} \sim \mathrm{Tr}\, e^{-\beta(\widehat{H}+v\,\widehat{\rho})} \Longrightarrow \text{e.g., path integral for } W[v]$$

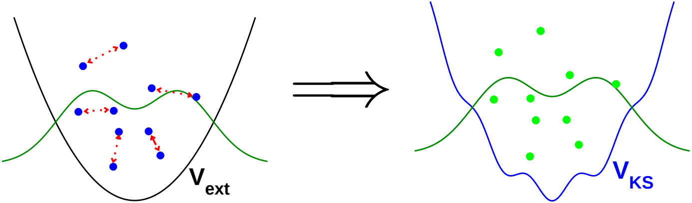
- *Invert* to find $v[\rho]$ and Legendre transform from v to ρ:

$$\rho(\mathbf{x}) = \frac{\delta W[v]}{\delta v(\mathbf{x})} \Longrightarrow \Gamma[\rho] = W[v] - \int v\,\rho \quad \text{and} \quad v(\mathbf{x}) = -\frac{\delta \Gamma[\rho]}{\delta \rho(\mathbf{x})}$$

$\Longrightarrow \Gamma[\rho] \propto$ energy functional $E[\rho]$, stationary at $\rho_{\rm gs}(\mathbf{x})$!

Construct $W[v]$ and then $\Gamma[\rho]$ order-by-order

- Need a diagrammatic *expansion* (e.g., MBPT or EFT)
- Inversion method $\Longrightarrow$ Split source $v(\mathbf{x}) = V_{\mathrm{KS}} + v_1 + v_2 + \cdots$
  - V_{KS} *chosen* to get $\rho(\mathbf{x})$ in noninteracting (Kohn-Sham) system:



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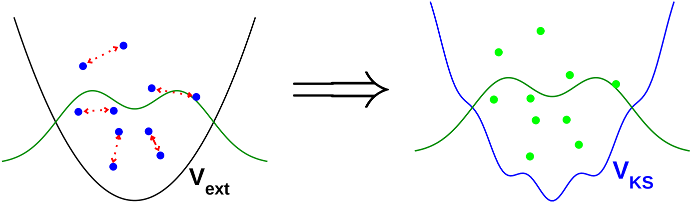

  - Orbitals $\{\psi_\alpha(\mathbf{x})\}$ in local potential $V_{\rm KS}([\rho], \mathbf{x})$

$$[-\nabla^2/2m + V_{\rm KS}(\mathbf{x})]\psi_\alpha = \varepsilon_\alpha\psi_\alpha \;\Longrightarrow\; \rho(\mathbf{x}) = \sum_{\alpha=1}^{A} |\psi_\alpha(\mathbf{x})|^2$$

  - Self-consistency from $v(\mathbf{x}) \to v_{\rm ext}(\mathbf{x}) \Longrightarrow V_{\rm KS}(\mathbf{x}) \propto \delta\Gamma_{\rm int}[\rho]/\delta\rho(\mathbf{x})$

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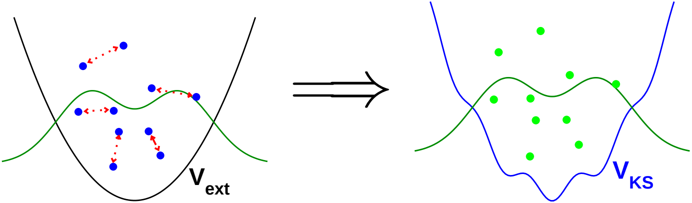

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- Alternative: Do MBPT with single particle potential $U(\mathbf{x})$ and $H = (T+U) + (V - U + v_{\rm ext})$ and choose $U = V_{\rm KS}$ (no $\Delta\rho(\mathbf{x})$)

Pairing from effective actions

- For pairing, the broken symmetry is a $U(1)$ [phase] symmetry
- Textbook effective action treatment in condensed matter
  - introduce contact interaction: $g\,\psi^\dagger\psi^\dagger\psi\psi$
  - Hubbard-Stratonovich with auxiliary pairing field $\hat{\Delta}(x)$ coupled to $\psi^\dagger\psi^\dagger \Longrightarrow$ eliminate contact interaction
  - construct 1PI $\Gamma[\Delta]$ with $\Delta = \langle\hat{\Delta}\rangle$, look for $\frac{\delta\Gamma}{\delta\Delta} = 0$
  - to leading order in the loop expansion (mean field) $\Longrightarrow$ BCS weak-coupling gap equation with gap Δ
  - Can we construct a Kohn-Sham DFT this way? ph channel?

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- Alternative: Combine an expansion (e.g., EFT) and the *inversion* method for effective actions (Fukuda et al.)
  - external current $j(x)$ coupled to pair density breaks symmetry
  - natural generalization of local Kohn-Sham DFT (Bulgac et al.)
  - cf. DFT with nonlocal source (Oliveira et al.; Kurth et al.)

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Local composite effective action with pairing

- Generating functional with sources J, j coupled to densities:

$$Z[J,j] = e^{-W[J,j]} = \int D(\psi^\dagger\psi)\, \exp\Big\{-\int d^4x\, [\mathcal{L} + J(x)\,\psi_\alpha^\dagger\psi_\alpha + j(x)(\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \psi_\downarrow\psi_\uparrow)]\Big\}$$

- Densities found by functional derivatives wrt J, j:

$$\rho(x) \equiv \langle\psi^\dagger(x)\psi(x)\rangle_{J,j} = \left.\frac{\delta W[J,j]}{\delta J(x)}\right|_j$$

$$\phi(x) \equiv \langle\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x) + \psi_\downarrow(x)\psi_\uparrow(x)\rangle_{J,j} = \left.\frac{\delta W[J,j]}{\delta j(x)}\right|_J$$

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- Effective action $\Gamma[\rho,\phi]$ by functional Legendre transformation:

$$\Gamma[\rho,\phi] = W[J,j] - \int d^4x\, J(x)\rho(x) - \int d^4x\, j(x)\phi(x)$$

Claims (hopes?) about effective action

more

- $\Gamma[\rho, \phi] \propto$ (free) energy functional $E[\rho, \phi]$
  - at finite temperature, the proportionality constant is β
- The sources are given by functional derivatives wrt ρ and ϕ

$$\frac{\delta E[\rho, \phi]}{\delta \rho(\mathbf{x})} = J(\mathbf{x}) \qquad \text{and} \qquad \frac{\delta E[\rho, \phi]}{\delta \phi(\mathbf{x})} = j(\mathbf{x})$$

  - but the sources are zero in the ground state
    $\Longrightarrow$ determine ground-state $\rho(\mathbf{x})$ and $\phi(\mathbf{x})$ by stationarity:

$$\left.\frac{\delta E[\rho, \phi]}{\delta \rho(\mathbf{x})}\right|_{\rho=\rho_{\rm gs}, \phi=\phi_{\rm gs}} = \left.\frac{\delta E[\rho, \phi]}{\delta \phi(\mathbf{x})}\right|_{\rho=\rho_{\rm gs}, \phi=\phi_{\rm gs}} = 0$$

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- This is Hohenberg-Kohn DFT extended to pairing!
- We need a method to carry out the Legendre transforms
  - to get Kohn-Sham DFT, apply inversion methods
- Can we renormalize consistently?

Kohn-Sham inversion method (general)

- Order-by-order matching in counting parameter λ

$$\begin{aligned}
\text{diagrams} &\implies W[J,j,\lambda] = W_0[J,j] + \lambda W_1[J,j] + \lambda^2 W_2[J,j] + \cdots \\
\text{assume} &\implies J[\rho,\phi,\lambda] = J_0[\rho,\phi] + \lambda J_1[\rho,\phi] + \lambda^2 J_2[\rho,\phi] + \cdots \\
\text{assume} &\implies j[\rho,\phi,\lambda] = j_0[\rho,\phi] + \lambda j_1[\rho,\phi] + \lambda^2 j_2[\rho,\phi] + \cdots \\
\text{derive} &\implies \Gamma[\rho,\phi,\lambda] = \Gamma_0[\rho,\phi] + \lambda \Gamma_1[\rho,\phi] + \lambda^2 \Gamma_2[\rho,\phi] + \cdots
\end{aligned}$$

- Start with exact expressions for Γ and ρ

$$\Gamma[\rho,\phi] = W[J,j] - \int J\,\rho - \int j\,\phi \implies \rho(x) = \frac{\delta W[J,j]}{\delta J(x)}\,, \quad \phi(x) = \frac{\delta W[J,j]}{\delta j(x)}$$

$\Longrightarrow$ plug in expansions with ρ, ϕ treated as order unity

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- 0^{th} order is Kohn-Sham system with potentials $J_0(\mathbf{x})$ and $j_0(\mathbf{x})$ $\implies$ exact densities $\rho(\mathbf{x})$ and $\phi(\mathbf{x})$ by construction

$$\Gamma_0[\rho,\phi] = W_0[J_0,j_0] - \int J_0\,\rho - \int j_0\,\phi \implies \rho(x) = \frac{\delta W_0[\,]}{\delta J_0(x)}\,, \quad \phi(x) = \frac{\delta W_0[\,]}{\delta j_0(x)}$$

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- 0^{th} order is Kohn-Sham system with potentials $J_0(\mathbf{x})$ and $j_0(\mathbf{x})$ $\Longrightarrow$ exact densities $\rho(\mathbf{x})$ and $\phi(\mathbf{x})$ by construction

$$
\Gamma_0[\rho,\phi] = W_0[J_0,j_0] - \int J_0\,\rho - \int j_0\,\phi \Longrightarrow \rho(x) = \frac{\delta W_0[\,]}{\delta J_0(x)}, \quad \phi(x) = \frac{\delta W_0[\,]}{\delta j_0(x)}
$$

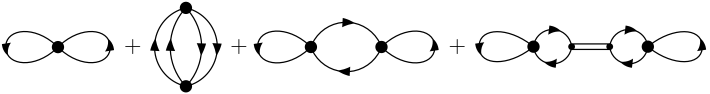
- Diagonalize with single-particle orbitals (cf. HFB with gap j_0)

$$
\begin{pmatrix} h_0(\mathbf{x}) - \mu_0 & j_0(\mathbf{x}) \\ j_0(\mathbf{x}) & -h_0(\mathbf{x}) + \mu_0 \end{pmatrix} \begin{pmatrix} u_i(\mathbf{x}) \\ v_i(\mathbf{x}) \end{pmatrix} = E_i \begin{pmatrix} u_i(\mathbf{x}) \\ v_i(\mathbf{x}) \end{pmatrix}
$$

$$
\text{where} \qquad h_0(\mathbf{x}) \equiv -\frac{\boldsymbol{\nabla}^2}{2M} + V_{\text{trap}}(\mathbf{x}) - J_0(\mathbf{x})
$$

Diagrammatic expansion of W_i

- Lines in diagrams are KS Nambu-Gor'kov Green's functions

$\Gamma_{\rm int} =$  $+ \cdots$

$$\mathbf{G} = \begin{pmatrix} \langle T_\tau \psi_\uparrow(x)\psi_\uparrow^\dagger(x')\rangle_0 & \langle T_\tau \psi_\uparrow(x)\psi_\downarrow(x')\rangle_0 \\ \langle T_\tau \psi_\downarrow^\dagger(x)\psi_\uparrow^\dagger(x')\rangle_0 & \langle T_\tau \psi_\downarrow^\dagger(x)\psi_\downarrow(x')\rangle_0 \end{pmatrix} \equiv \begin{pmatrix} G^0_{\rm ks} & F^0_{\rm ks} \\ F^{0\,\dagger}_{\rm ks} & -\widetilde{G}^0_{\rm ks} \end{pmatrix}$$

  - Extra diagrams enforce inversion (here removes anomalous)

- In frequency space, the Kohn-Sham Green's functions are

$$G^0_{\rm ks}(\mathbf{x},\mathbf{x}';\omega) = \sum_j \left[\frac{u_j(\mathbf{x})\, u_j^*(\mathbf{x}')}{i\omega - E_j} + \frac{v_j(\mathbf{x}')\, v_j^*(\mathbf{x})}{i\omega + E_j} \right]$$

$$F^0_{\rm ks}(\mathbf{x},\mathbf{x}';\omega) = -\sum_j \left[\frac{u_j(\mathbf{x})\, v_j^*(\mathbf{x}')}{i\omega - E_j} - \frac{u_j(\mathbf{x}')\, v_j^*(\mathbf{x})}{i\omega + E_j} \right]$$

Kohn-Sham self-consistency procedure

- Same iteration procedure as in Skyrme or RMF with pairing
- In terms of the orbitals, the fermion density is

$$\rho(\mathbf{x}) = 2\sum_i |v_i(\mathbf{x})|^2$$

and the pair density is

$$\phi(\mathbf{x}) = \sum_i [u_i^*(\mathbf{x})v_i(\mathbf{x}) + u_i(\mathbf{x})v_i^*(\mathbf{x})]$$

- The chemical potential μ_0 is fixed by $\int \rho(\mathbf{x}) = A$
- Diagrams for $\Gamma[\rho,\phi] \propto E_0[\rho,\phi] + E_{\rm int}[\rho,\phi]$ yields KS potentials

$$J_0(\mathbf{x})\Big|_{\rho=\rho_{\rm gs}} = \left.\frac{\delta E_{\rm int}[\rho,\phi]}{\delta\rho(\mathbf{x})}\right|_{\rho=\rho_{\rm gs}} \quad \text{and} \quad j_0(\mathbf{x})\Big|_{\phi=\phi_{\rm gs}} = \left.\frac{\delta E_{\rm int}[\rho,\phi]}{\delta\phi(\mathbf{x})}\right|_{\phi=\phi_{\rm gs}}$$

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Outline

Overview: Pairing in EFT/DFT

Kohn-Sham DFT with pairing from effective action

Renormalization issues

Going forward: ideas, questions, tools

UV divergences in nonrelativistic and relativistic effective actions

- *All* low-energy effective theories have incorrect UV behavior
- Sensitivity to short-distance physics signalled by divergences
  but finiteness (e.g., with cutoff) doesn't mean not sensitive!
  $\Longrightarrow$ must absorb (and correct) sensitivity by renormalization
- Instances of UV divergences

nonrelativistic	covariant
scattering	scattering
pairing	pairing
	anti-particles

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- Can we consistently renormalize within inversion method?
- Strategy: Verify renormalization using scale parameter Λ

Divergences: uniform dilute Fermi system

- Generating functional with constant sources μ and j:

$$e^{-W[\mu,j]} = \int D(\psi^\dagger\psi)\,\exp\Big\{-\int d^4x\,\Big[\psi^\dagger_\alpha\Big(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2M} - \mu\Big)\psi_\alpha + \frac{C_0}{2}\psi^\dagger_\uparrow\psi^\dagger_\downarrow\psi_\downarrow\psi_\uparrow + j(\psi_\uparrow\psi_\downarrow + \psi^\dagger_\downarrow\psi^\dagger_\uparrow)\Big]\Big\}$$

  - cf. adding integration over auxiliary field $\int D(\Delta^*, \Delta)\; e^{-\frac{1}{|C_0|}\int|\Delta|^2}$
    $\Longrightarrow$ shift variables to eliminate $\psi^\dagger_\uparrow\psi^\dagger_\downarrow\psi_\downarrow\psi_\uparrow$ for $\Delta^*\psi_\uparrow\psi_\downarrow$

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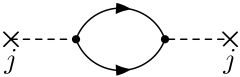
$W[\mu,j] = \cdots +$  $+ \cdots$

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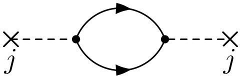
$W[\mu, j] = \cdots +$  $+ \cdots$

  - Same linear divergence as in 2-to-2 scattering
- Renormalization: Add counterterm $\frac{1}{2}\zeta|j|^2$ to $\mathcal{L}$ (cf. Zinn-Justin)
  - Additive to W (cf. $|\Delta|^2$) $\Longrightarrow$ no effect on scattering
  - How to determine ζ? Energy interpretation of Γ?

Use dimensional regularization (DR)

- Generalize Papenbrock & Bertsch DR/MS calculation
- DR/PDS $\Longrightarrow$ generate explicit Λ to "check" renormalization
  - Basic free-space integral in D spatial dimensions

$$\left(\frac{\Lambda}{2}\right)^{3-D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{p^2 - k^2 + i\epsilon} \xrightarrow{\text{PDS}} -\frac{1}{4\pi}(\Lambda + ip) \qquad \left[\text{note: } \int \frac{1}{\epsilon_k^0} \to \frac{M\Lambda}{2\pi}\right]$$

  - Renormalizing free-space scattering yields:

$$C_0(\Lambda) = \frac{4\pi a_s}{M} + \frac{4\pi a_s^2}{M}\Lambda + \mathcal{O}(\Lambda^2) = C_0^{(1)} + C_0^{(2)} + \cdots \longrightarrow \frac{4\pi a_s}{M} \frac{1}{1 - a_s\Lambda}$$

  - Recover DR/MS with $\Lambda = 0$

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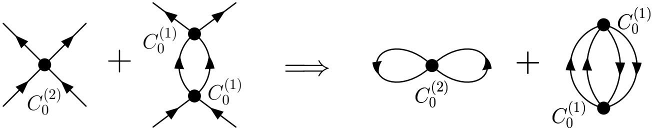
- E.g., verify NLO renormalization $\Longrightarrow$ independent of Λ



Kohn-Sham non-interacting system

- Bare density ρ:

$$\rho = -\frac{1}{\beta V}\frac{\partial W_0[\,]}{\partial \mu_0} = \frac{2}{V}\sum_{\mathbf{k}} v_k^2$$

$$= \int \frac{d^3k}{(2\pi)^3}\left(1 - \frac{\epsilon_k^0 - \mu_0}{E_k}\right)$$

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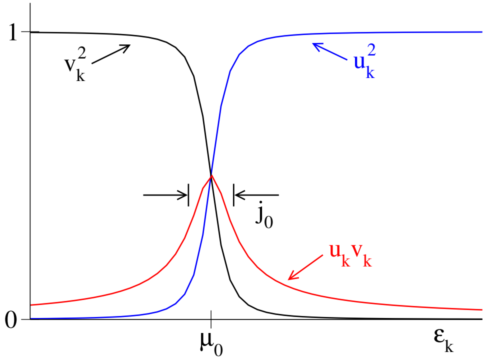

- j_0 plays role of constant gap

$$E_k = \sqrt{(\epsilon_k^0 - \mu_0)^2 + j_0^2}\,, \qquad \epsilon_k^0 = \frac{k^2}{2M}$$

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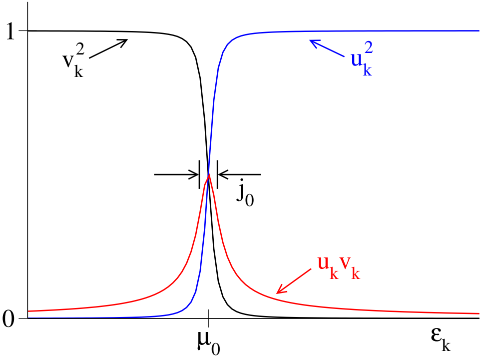

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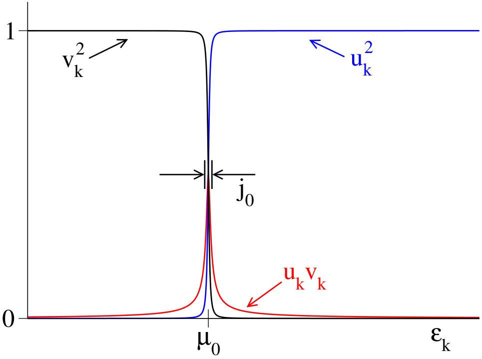

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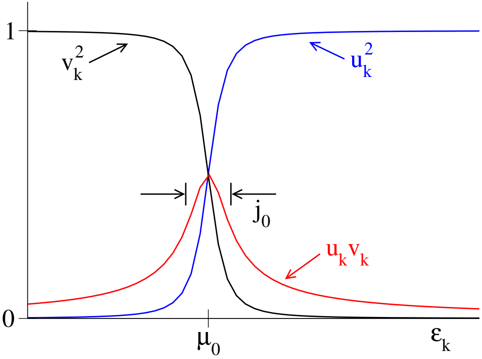

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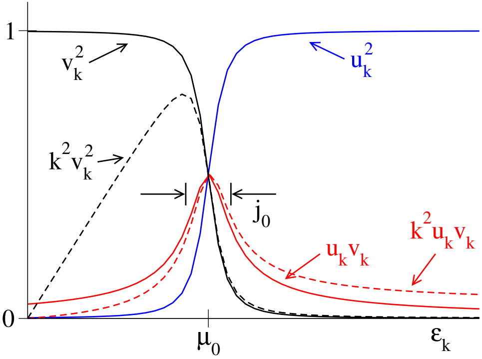

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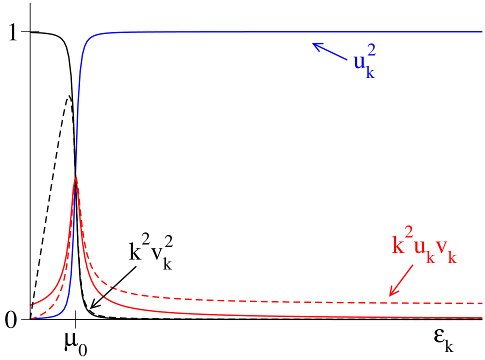

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- The basic DR/PDS integral in D dimensions, with $x \equiv j_0/\mu_0$, is

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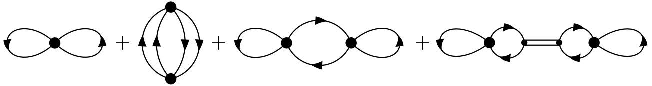
- The KS equation for the pair density ϕ fixes $\zeta^{(0)}$:

$$\phi = \frac{1}{\beta V}\frac{\partial W_0[\,]}{\partial j_0} = -\int \frac{d^3k}{(2\pi)^3}\frac{j_0}{E_k} + \zeta^{(0)} j_0 \longrightarrow -j_0\, I(0) + \zeta^{(0)} j_0 \Longrightarrow \zeta^{(0)} = \frac{M\Lambda}{2\pi}$$

Calculating to n^{th} Order

- Find $\Gamma_{1\leq i\leq n}[\rho,\phi]$ from $W_{1\leq i\leq n}[\mu_0(\rho,\phi), j_0(\rho,\phi)]$
  - including additional Feynman rules

$\Gamma_{\text{int}} =$  $+ \cdots$

- Calculate μ_i, j_i from Γ_i, then use $\sum_{i=0}^{n} j_i = j \to 0$ to find j_0
- Renormalization conditions:
  - No freedom in choosing $C_0(\Lambda) \Longrightarrow \Lambda$'s must cancel!
  - Choose $\delta Z_j^{(n)}$ and $\zeta^{(n)}$ to convert ϕ_B to renormalized ϕ

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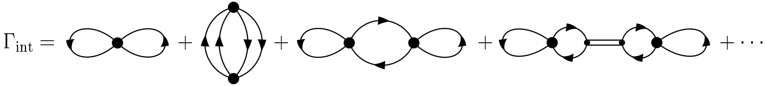

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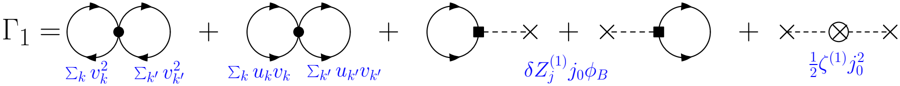
- Leading order: Diagrams for $\Gamma_1[\rho,\phi] = W_1[\mu_0(\rho,\phi), j_0(\rho,\phi)]$



$$\Longrightarrow \quad \frac{1}{\beta V}\Gamma_1[\rho,\phi] = \frac{1}{4}C_0^{(1)}\rho^2 + \frac{1}{4}C_0^{(1)}\phi^2 \quad \text{with } C_0^{(1)} = \frac{4\pi a_s}{M}$$

The "gap" equation at leading order (LO)

- Γ_1 dependence on ρ and ϕ explicit $\Longrightarrow$ easy to find μ_1 and j_1:

$$\mu_1 = \frac{1}{\beta V}\frac{\partial\Gamma_1}{\partial\rho} = \frac{1}{2}C_0^{(1)}\rho \qquad \text{and} \qquad j_1 = -\frac{1}{\beta V}\frac{\partial\Gamma_1}{\partial\phi} = -\frac{1}{2}C_0^{(1)}\phi$$

- "Gap" equation from $j = j_0 + j_1 = 0$

$$j_0 = -j_1 = -\frac{1}{2}|C_0^{(1)}|\phi = \frac{1}{2}|C_0^{(1)}|\, j_0 \left(\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{(\epsilon_k^0 - \mu_0)^2 + j_0^2}} - \zeta^{(0)} \right)$$

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$$j_0 = -j_1 = -\frac{1}{2}|C_0^{(1)}|\phi = \frac{1}{2}|C_0^{(1)}|\, j_0 \left(\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{(\epsilon_k^0 - \mu_0)^2 + j_0^2}} - \zeta^{(0)} \right)$$

- DR/PDS reproduces Papenbrock/Bertsch (with $x \equiv |j_0/\mu_0|$)

$$1 = \sqrt{2M\mu_0}\, a_s (1+x^2)^{1/4} P^0_{1/2}\left(\frac{-1}{\sqrt{1+x^2}}\right) \xrightarrow{x\to 0} k_{\rm F} a_s \left[\frac{4 - 6\log 2}{\pi} + \frac{2}{\pi}\log x\right]$$

$$\Longrightarrow \text{if } k_{\rm F} a_s < 1,\ \frac{j_0}{\mu_0} = \frac{8}{e^2} e^{-\pi/2k_{\rm F}|a_s|} \text{ holds}$$

Renormalized energy density at LO

- Renormalized effective action $\Gamma = \Gamma_0 + \Gamma_1$:

$$\frac{1}{\beta V}\Gamma = \int(\epsilon_k^0 - \mu_0 - E_k) + \frac{1}{2}\zeta^{(0)} j_0^2 + \mu_0\rho - j_0\phi + \frac{1}{4}C_0^{(1)}\rho^2 + \frac{1}{4}C_0^{(1)}\phi^2$$

- Check for Λ dependence:

$$\frac{1}{\beta V}\Gamma = 0 - I(2) + 2\mu_0 I(1) - (\mu_0^2 + j_0^2)I(0) + \frac{1}{2}\frac{M\Lambda}{2\pi} j_0^2 + \cdots$$

$$\longrightarrow \frac{M\Lambda}{2\pi}\Big(-\mu_0^2(1 - j_0^2/2\mu_0^2) + 2\mu_0^2 - \mu_0^2 - j_0^2 + \frac{1}{2}j_0^2\Big) = 0$$

- To find the energy density, evaluate Γ at the stationary point $j_0 = -\frac{1}{2}|C_0^{(1)}|\phi$ with μ_0 fixed by the equation for ρ
  $\Longrightarrow$ same results as Papenbrock/Bertsch (plus HF term)

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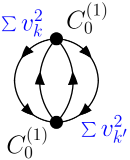
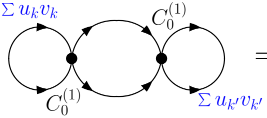
- To find the energy density, evaluate Γ at the stationary point $j_0 = -\frac{1}{2}|C_0^{(1)}|\phi$ with μ_0 fixed by the equation for ρ
  $\Longrightarrow$ same results as Papenbrock/Bertsch (plus HF term)
- Life gets more complicated at NLO
  - dependence of Γ_2 on ρ, ϕ is no longer explicit
  - analytic formulas for DR integrals not available

Γ_2 at next-to-leading order (NLO)



$$\Longrightarrow \; -\big(C_0^{(1)}\big)^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{E_p + E_k + E_{p-q} + E_{k+q}} \times \Big[\, u_p^2\, u_k^2\, v_{p-q}^2\, v_{k+q}^2 - 2u_p^2\, v_k^2\, (uv)_{p-q}\, (uv)_{k+q} + (uv)_p\, (uv)_k\, (uv)_{p-q}\, (uv)_{k+q} \,\Big]$$



$$\Longrightarrow \; -\big(C_0^{(1)}\big)^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \Big[\rho (u_k v_k)^2 + \frac{1}{2}\phi_B (u_k^2 - v_k^2)\Big]^2$$

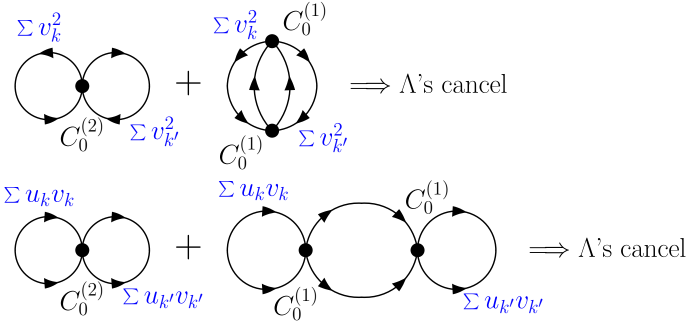
- UV divergences identified from

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right) \xrightarrow{k\to\infty} \frac{j_0^2 M^2}{k^4} \qquad u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right) \xrightarrow{k\to\infty} 1 - \frac{j_0^2 M^2}{k^4}$$

$$u_k v_k = -\frac{j_0}{2E_k} \xrightarrow{k\to\infty} -\frac{j_0 M}{k^2} \qquad \frac{1}{E_k} \xrightarrow{k\to\infty} \frac{2M}{k^2}$$

Next-to-leading-order (NLO) renormalization

- Bowtie with $C_0^{(2)} = \dfrac{4\pi a_s^2}{M}\Lambda$ vertex must precisely cancel Λ's from beachballs with $C_0^{(1)} = \dfrac{4\pi a_s}{M}$ vertices:



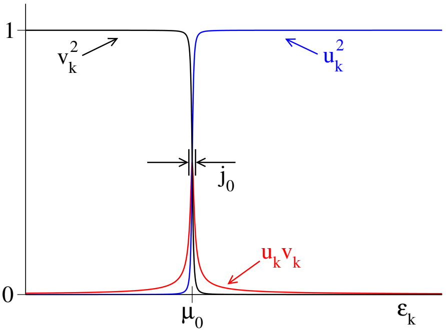
(Note that $\delta Z_j^{(1)}$ vertex takes $\phi_B \to \phi$)

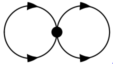
- How do we see cancellation of Λ's and evaluate renormalized results without analytic formulas? [but first . . .]

Standard induced interaction result recovered

- Look at $j_0 \Leftrightarrow \Delta$
- As $j_0 \to 0$, $u_k v_k$ peaks at μ_0
- Leading order $T = 0$:

$$\Delta_{LO}/\mu_0 = \frac{8}{e^2}\, e^{-1/N(0)|C_0|} = \frac{8}{e^2}\, e^{-\pi/2k_F|a_s|}$$



$\Gamma_1 =$  $+\, CTC + \cdots \Longrightarrow j_1 = \dfrac{\delta\Gamma_1}{\delta\phi} = \dfrac{1}{2}|C_0|\phi$

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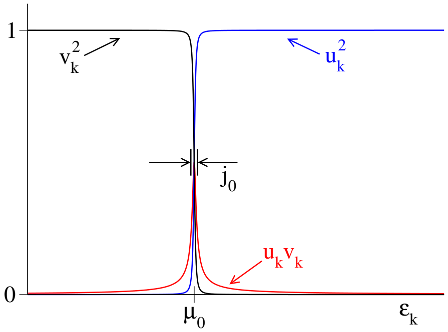
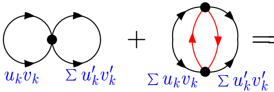
- NLO modifies exponent $\Longrightarrow$ changes prefactor
- $\Delta_{NLO} \approx \Delta_{LO}/(4e)^{1/3}$



$\Gamma_1 + \Gamma_2 =$  $\Longrightarrow j_1 + j_2 = \frac{1}{2}|C_0| \left[1 - |C_0|\langle\Pi_0\rangle_{|\mathbf{k}|=|\mathbf{k}'|=k_{\rm F}}\right]\phi$

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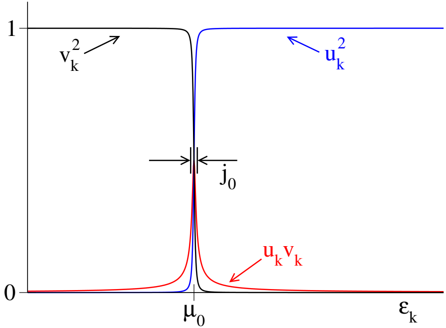
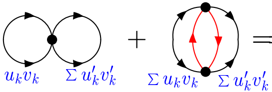


$\Gamma_1 + \Gamma_2 =$



$$\Longrightarrow j_1 + j_2 = \frac{1}{2}|C_0|\left[1 - |C_0|\langle \Pi_0 \rangle_{|\mathbf{k}|=|\mathbf{k}'|=k_F}\right]\phi$$

- How does the Kohn-Sham gap compare to “real” gap?

Renormalizing with subtractions

- NLO integrals over $E_k = \sqrt{(\epsilon_k - \mu_0)^2 + j_0^2}$ are intractable, but

$$\int \frac{1}{E_1 + E_2 + E_3 + E_4} = \int \left[\frac{1}{E_1 + E_2 + E_3 + E_4} - \frac{\mathcal{P}}{\epsilon_1^0 + \epsilon_2^0 - \epsilon_3^0 - \epsilon_4^0} \right]$$

plus a DR/PDS integral that is proportional to Λ

$\Longrightarrow$ just make the substitution in [ ]'s for renormalized result

- When applied at LO,

$$\int \frac{1}{E_k} = \int \left[\frac{1}{E_k} - \frac{\mathcal{P}}{\epsilon_k^0} \right] + \frac{M\Lambda}{2\pi}$$

- Cf. subtraction to eliminate C_0 in gap equation

$$\frac{M}{4\pi a_s} + \frac{1}{|C_0|} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_k^0} \Longrightarrow \frac{M}{4\pi a_s} = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{E_k} - \frac{1}{\epsilon_k^0} \right]$$

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  plus a DR/PDS integral that is proportional to Λ
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- Any equivalent subtraction works, e.g.,

$$\int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{P}}{\epsilon_k^0 - \mu_0} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_k^0}$$

Anomalous density in finite systems

- How do we renormalize the pair density in a finite system?

$$\phi(\mathbf{x}) = \sum_i [u_i^*(\mathbf{x}) v_i(\mathbf{x}) + u_i(\mathbf{x}) v_i^*(\mathbf{x})] \longrightarrow \infty$$

  - cf. scalar density $\rho_s = \sum_i \overline{\psi}(\mathbf{x})\psi(\mathbf{x})$ for relativistic mft

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- Usual plan: Subtracted expression for ϕ in uniform system

$$\phi = \int^{k_c} \frac{d^3k}{(2\pi)^3}\, j_0 \left(\frac{1}{\sqrt{(\epsilon_k^0 - \mu_0)^2 + j_0^2}} - \frac{1}{\epsilon_k^0} \right) \overset{k_c \to \infty}{\longrightarrow} \text{finite}$$

- Apply this in a local density approximation (Thomas-Fermi)

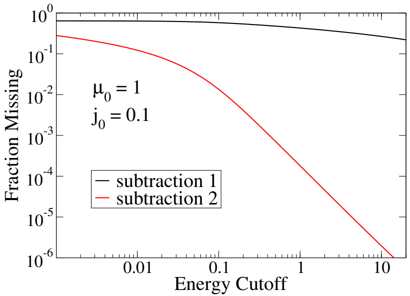
$$\phi(\mathbf{x}) = 2 \sum_i^{E_c} u_i(\mathbf{x}) v_i(\mathbf{x}) - j_0(\mathbf{x}) \frac{M\, k_c(\mathbf{x})}{2\pi^2} \quad \text{with} \quad E_c = \frac{k_c^2(\mathbf{x})}{2M} + J(\mathbf{x}) - \mu_0$$

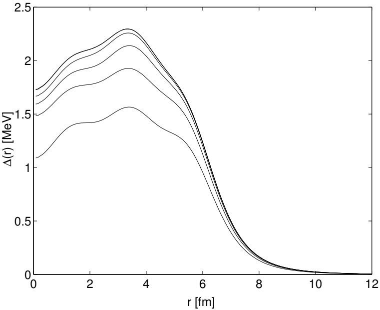
Bulgac/Yu renormalization [PRL 88 (2002) 042504]

- Convergence is very slow as the energy cutoff is increased
  $\Longrightarrow$ Bulgac/Yu: make a different subtraction

$$\phi = \int^{k_c} \frac{d^3k}{(2\pi)^3} j_0 \left(\frac{1}{\sqrt{(\epsilon_k^0 - \mu_0)^2 + j_0^2}} - \frac{\mathcal{P}}{\epsilon_k^0 - \mu_0} \right) \stackrel{k_c \to \infty}{\longrightarrow} \text{finite}$$

- Compare convergence in uniform system, in nuclei with LDA





- How do we generalize? Are RHA/soliton vacuum methods useful?

Outline

Overview: Pairing in EFT/DFT

Kohn-Sham DFT with pairing from effective action

Renormalization issues

Going forward: ideas, questions, tools

Laundry list of EFT/DFT pairing issues

- Questions:
  - To what extent is an EDF unique? Kohn-Sham gap? (and so on)
  - Can we motivate/constrain ASLDA near the unitary limit?
  - For nuclei, do we want/need to include the pion explicitly?
- Ideas from other contexts:
  - Alternative (Kohn-Sham) DFT — e.g., auxiliary fields, non-local
  - Beyond quasi-local: orbital-based methods for long-range correlations (cf., RPA with OEP in Coulomb systems)
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  - Operator Product Expansion (e.g., cancellation of divergences)
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Uniqueness of EDF's from EFT effective action

- What is the dependence of an energy density functional on the EFT representation or regularization or an RG Λ?
- Appeal to authority: Zinn-Justin on observable quantities

$$e^{-W[J]} = \int [d\phi]\, e^{-S(\phi)+J\phi} = e^{-\Gamma[\varphi]+J\varphi}\ , \quad \varphi \equiv \langle\phi\rangle = \delta W/\delta J\ , \quad J = \delta\Gamma/\delta\varphi$$

$$e^{-\Gamma[\varphi]} = \int [d\phi]\, e^{-S(\phi)+\int dx\,(\phi(x)-\varphi(x))\frac{\delta\Gamma}{\delta\varphi(x)}} \overset{\text{shift}}{\Longrightarrow} e^{-\Gamma[\varphi_c]} = \int [d\phi]\, e^{-S(\phi+\varphi_c)}$$

  - Effective action evaluated at the stationary point is independent of the representation of the fields $\phi'(x) = \sum_{k=1}^{\infty} C_k \phi^k(x)/k!$
  - These are the "on shell" points; invariance is analogous to independence of the S-matrix to interpolating field

- Use field redefinitions to illustrate this point
- Consequences for DFT from EFT effective action
  - "Universal" does not mean unique!
  - What about LDA?

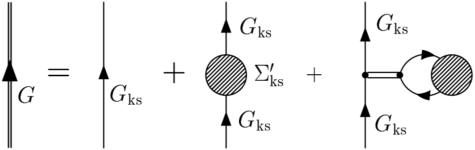
Kohn-Sham pairing questions

- How are the Kohn-Sham "gap" and gap $\Delta(\mathbf{q})$ related?
  - "Conventional" gap function from anomalous Green's function:

$$\Delta(\mathbf{q}) = -\int \frac{d^4k}{(2\pi)^4} V(\mathbf{q}-\mathbf{k})\mathcal{F}_0(k_0,\mathbf{k})e^{ik_0\eta}$$

  - Kohn-Sham Green's function vs. full Green's function more

$$G(x,x') = G_{\rm ks}(x,x') + G_{\rm ks}\left[\frac{1}{i}\frac{\delta\Gamma_{\rm int}}{\delta G_{\rm ks}} + \frac{\delta\Gamma_{\rm int}}{\delta\rho}\right] G_{\rm ks}$$



  - When do we need the "real" gap from Green's function?
  - What about the gap from odd-even mass staggering?

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Can EFT/DFT give insight into form of ASLDA?

- Simplified but phenomenologically successful ASLDA structure motivated by Bulgac and Forbes [arXiv:0808.1436]

$$E = \int d\mathbf{x} \left\{ \mathcal{E}[n_{a,b}(\mathbf{x}), \tau_{a,b}(\mathbf{x}), \nu(\mathbf{x})] + V_{\rm ext}(\mathbf{x})[n_a(\mathbf{x}) + n_b(\mathbf{x}) + \text{sources} \ldots \right\}$$

$$\mathcal{E}[n_{a,b}, \tau_{a,b}, \nu] = \alpha_a(n_a, n_b)\frac{\tau_a}{2} + \alpha_b(n_a, n_b)\frac{\tau_b}{2} + D(n_a, n_b) + g_{\rm eff}(n_a, n_b)\nu^*\nu$$

- Cancellation of divergences? ($\alpha_a\tau_a/2 + \alpha_b\tau_b/2 - \Delta^*\nu$ is finite)
- Additional support for neglecting gradient terms?
- Power counting functionals of $[(\alpha_a\tau_a + \alpha_b\tau_b)/2 - \Delta^*\nu]/n^{5/3}$
- Decoupling of paired and normal states? (e.g., neglect of n dependence of pairing)
- Justification for semi-local form?

Effective Field Theory ingredients for EFT($\not\pi$)

Low-energy QCD (nuclear physics) but no pions ...

1. Use the most general $\mathcal{L}$ with low-energy dof's consistent with global and local symmetries of underlying theory
   - $\mathcal{L}_{\rm eft} = \psi^\dagger \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M} \right] \psi - \frac{C_0}{2}(\psi^\dagger P \psi)^2 - \frac{D_0}{6}(\psi^\dagger \psi)^3 + \ldots$
   - P is a (schematic) spin/isospin projector
2. Declaration of regularization and renormalization scheme
   - dimensional (DR/PDS) and cutoff regularization both work
   - unnatural $a_0 \gg R \Longrightarrow$ If it is to be a low-energy scale $\Longrightarrow$ nonperturbative renormalization
3. Well-defined power counting $\Longrightarrow$ small expansion parameter
   - use separation of scales $\Longrightarrow \dfrac{k}{\Lambda_b} \ll 1$ with $\Lambda_b \sim m_\pi$
   - $k|a_0| > 1 \Longrightarrow$ leading order: sum all diagrams with just C_0's
   - Is there an additional counting for finite density? $(1/N, \epsilon, \ldots)$

Same setup for an EFT for cold atoms!

EFT/DFT in unitarity limit

- Only one paper? Rupak/Schäfer, Nucl. Phys. **816** (2009)
  - Not Kohn-Sham and has normal density ρ only $\Longrightarrow$ no pairing or kinetic energy density
- Work to NLO in gradient expansion
- Match to ϵ expansion about $d = 4 - \epsilon$ spatial dimensions:

$$\mathcal{E}(x) = n(x)v_{\text{ext}}(x) + 1.364\,\frac{n(x)^{5/3}}{m} + 0.022\,\frac{[\nabla n(x)]^2}{m\,n(x)} + \mathcal{O}(\nabla^4 n)$$

- Comparison to "extended Thomas-Fermi" for free nucleons:

$$\mathcal{E}_{\text{ETF}}(x) = n(x)v_{\text{ext}}(x) + 2.871\,\frac{n(x)^{5/3}}{m} + 0.014\,\frac{[\nabla n(x)]^2}{m\,n(x)} + \mathcal{O}(\nabla^4 n)$$

- Extension to motivate/justify form of ASLDA?
  - Use ϵ or $1/N$ expansion but with sources coupled to ρ, τ, *and* pairing density
  - Apply inversion method

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Effective Field Theory Ingredients

See, e.g., Epelbaum et al., Rev. Mod. Phys. **81**, 1773 (2009)

1. Use the most general $\mathcal{L}$ with low-energy dof's consistent with the global and local symmetries of the underlying theory
2. Declaration of regularization and renormalization scheme
3. Well-defined power counting $\Longrightarrow$ expansion parameters

Effective Field Theory Ingredients: Chiral NN

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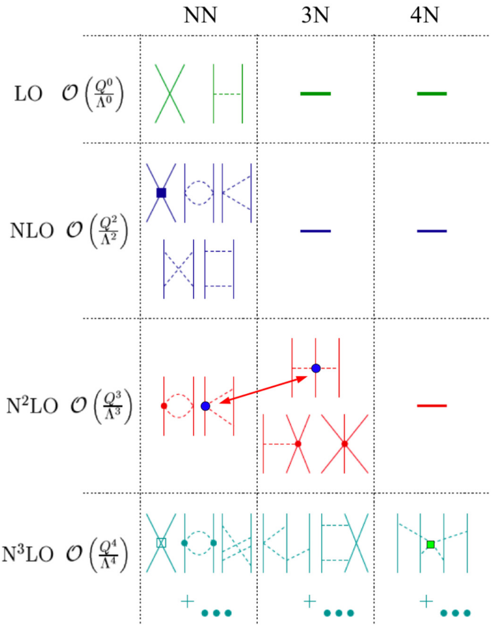
1. Use the most general $\mathcal{L}$ with low-energy dof's consistent with the global and local symmetries of the underlying theory
   - $\mathcal{L}_{\rm eft} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \mathcal{L}_{NN}$
   - chiral symmetry $\Longrightarrow$ systematic long-distance pion physics
2. Declaration of regularization and renormalization scheme
   - momentum cutoff and "Weinberg counting" (still open!) $\Longrightarrow$ define irreducible potential and sum with LS eqn
   - use cutoff sensitivity as measure of uncertainties
3. Well-defined power counting $\Longrightarrow$ expansion parameters
   - use the separation of scales $\Longrightarrow \dfrac{\{\mathbf{p}, m_\pi\}}{\Lambda_\chi}$ with $\Lambda_\chi \sim 1$ GeV
   - chiral symmetry $\Longrightarrow$ $V_{NN} = \sum_{\nu=\nu_{\rm min}}^{\infty} c_\nu Q^\nu$ with $\nu \geq 0$
   - naturalness: LEC's are $\mathcal{O}(1)$ in appropriate units

Long-range chiral EFT ⟹ enhanced Skyrme

- Add long-range (π-exchange) contributions in the density matrix expansion (DME)
    - NN/NNN through N^{2}LO [Gebremariam et al.]
- Refit Skyrme parameters for short-range parts
- Test for sensitivities and improved observables (e.g., isotope chains) [ORNL]
- Spin-orbit couplings from 2π 3NF particularly interesting
- Can we "see" the pion in medium to heavy nuclei?



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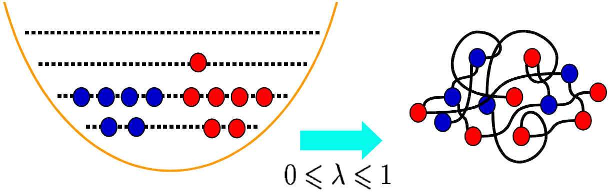
Better alternatives to local Kohn-Sham?

- Couple source to non-local pair field (Oliveira et al.):

$$\widehat{H} \longrightarrow \widehat{H} - \int dx\, dx' \, [D^*(x,x')\psi_\uparrow(x)\psi_\downarrow(x') + \text{H.c.}]$$

  - CJT 2PI effective action $\Gamma[\rho, \Delta]$ with $\Delta(x,x') = \langle \psi_\uparrow(x)\psi_\downarrow(x')\rangle$?

- DFT from Renormalization Group (J. Braun, Polonyi-Schwenk)



- Auxiliary fields: Introduce $\widehat{\Delta}^*(x)\psi(x)\psi(x) + \text{H.c.}$ via H.S.
  - 1PI effective action in $\Delta(x) = \langle \widehat{\Delta}(x) \rangle$
  - Special saddle point evaluation $\Longrightarrow$ Kohn-Sham DFT

HS decoupling in the Coulomb problem

From Atland and Simons "Condensed Matter Field Theory":

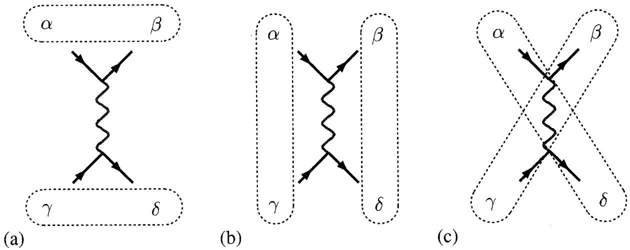


Figure 6.1 On the different channels of decoupling an interaction by Hubbard–Stratonovich transformation. (a) Decoupling in the "density" channel; (b) decoupling in the "pairing" or "Cooper" channel; and (c) decoupling in the "exchange" channel.

- May want to decouple in all three channels with $q \ll |p_i|$:

$$S_{\text{int}}[\overline{\psi},\psi] \approx \frac{1}{2}\sum_{p,p',q}\Big(\overline{\psi}_{\sigma p}\psi_{\sigma p+q}V(\mathbf{q})\overline{\psi}_{\sigma' p'}\psi_{\sigma' p'-q} - \overline{\psi}_{\sigma p}\psi_{\sigma' p+q}V(\mathbf{p}'-\mathbf{p})\overline{\psi}_{\sigma' p'+q}\psi_{\sigma' p'}$$

$$-\overline{\psi}_{\sigma p}\overline{\psi}_{\sigma' -p+q}V(\mathbf{p}'-\mathbf{p})\psi_{\sigma' p'}\psi_{\sigma' -p'+q}\Big)$$

Auxiliary Fields [Faussurier]

- Introduce scalar field φ coupled to $\psi^\dagger\psi$
- Construct $\widetilde{S}[\psi^\dagger,\psi,\varphi]$ such that $\psi,\psi^\dagger$ is only in $\psi^\dagger[G^{-1}(\varphi)]\psi$ and

$$\int \mathcal{D}\varphi\, e^{i\widetilde{S}[\psi^\dagger,\psi,\varphi]} \Longrightarrow e^{iS[\psi^\dagger,\psi]}$$

- Integrate out $\psi^\dagger\psi \Longrightarrow$ determinant $\Longrightarrow \mathrm{Tr}\ln[G^{-1}(\varphi)] + \cdots$
- Keep only leading saddle point $\phi_0(\mathbf{x}) \Longrightarrow$ Hartree
  - fluctuation corrections generate loop expansion
  - freedom to choose mean field [Kerman et al. (1983)]
    cf., $H = (T+U) + (V-U)$ for arbitrary U
- Kohn-Sham: choose special saddle-point evaluation
  - reference local potential $\phi_{\rm xc}$ such that $-\mathrm{Tr}\, G_{\rm xc}(x,x^+) = n(\mathbf{x})$
  - expand $\mathrm{Tr}\ln[G_{\rm xc}^{-1} + \delta\phi]$ in $\delta\phi = \phi - \phi_{\rm xc}$
    $\Longrightarrow \Gamma_{\rm xc}[n]$ with $\phi_{\rm xc}(\mathbf{x}) = \delta\Gamma_{\rm xc}[n]/\delta n(\mathbf{x})$
  - introduce orbitals $\{\psi_\alpha, \epsilon_\alpha\}$ to diagonalize $\mathrm{Tr}\ln[G_{\rm xc}^{-1}]$

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Orbital Dependent DFT (OEP, OPM, . . .) [J. Drut, L. Platter, rjf]

- Expansion for $E_{\rm int}[\rho, \tau, \mathbf{J}, \ldots]$; densities are sums over orbitals solving from Kohn-Sham S-eqn with $V_{\rm KS}(\mathbf{r}), \ldots$
- Self-consistency $\Longrightarrow$ $V_{\rm KS}(\mathbf{r}) = \delta E_{\rm int}[\rho, \ldots]/\delta\rho(\mathbf{r}), \ldots$
  - i.e., Kohn-Sham potential is functional derivative of interacting energy functional (or $E_{\rm xc}$) wrt (all) densities
  - How do we calculate this functional derivative?
- Approximations with explicit $\rho(\mathbf{r})$ dependence: LDA, DME, . . .

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- Approximations with explicit $\rho(\mathbf{r})$ dependence: LDA, DME, . . .
- Orbital-dependent DFT $\Longrightarrow$ full derivative via chain rule:

$$V_{\rm KS}(\mathbf{r}) = \frac{\delta E_{\rm int}[\phi_\alpha, \varepsilon_\alpha]}{\delta\rho(\mathbf{r})} = \int d\mathbf{r}' \, \frac{\delta V_{\rm KS}(\mathbf{r}')}{\delta\rho(\mathbf{r})} \sum_\alpha \biggl\{ \int d\mathbf{r}'' \biggl[\frac{\delta\phi_\alpha^\dagger(\mathbf{r}'')}{\delta V_{\rm KS}(\mathbf{r}')} \frac{\delta E_{\rm int}}{\delta\phi_\alpha^\dagger(\mathbf{r}'')} + c.c. \biggr] + \frac{\delta\varepsilon_\alpha}{\delta V_{\rm KS}(\mathbf{r}')} \frac{\partial E_{\rm int}}{\partial\varepsilon_\alpha} \biggr\}$$

  - Solve the OPM equation for $V_{\rm KS}$ using $\chi_s(\mathbf{r}, \mathbf{r}') = \delta\rho(\mathbf{r})/\delta V_{\rm KS}(\mathbf{r}')$

$$\int d^3r' \, \chi_s(\mathbf{r}, \mathbf{r}') \, V_{\rm KS}(\mathbf{r}') = \Lambda_{\rm xc}(\mathbf{r})$$

  - $\Lambda_{\rm xc}(\mathbf{r})$ is functional of the orbitals ϕ_α, eigenvalues ε_α, and $G^0_{\rm KS}$

Correlations beyond LDA: Coulomb RPA with OEP

- See Kummel and Kornik RMP article for examples/references
- RPA functional (simplified with KS dynamic response χ_{KS})

$$E_{\mathrm{xc}} = \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\,\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\int_0^1 d\lambda\left(-\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Im}\,\chi_{\mathrm{KS}}(\mathbf{r},\mathbf{r}',\omega) - n(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\right)$$

$$\chi_{\mathrm{KS}}(\mathbf{r},\mathbf{r}',\omega) = \sum_{\sigma=\uparrow,\downarrow}\sum_{j,k}(f_{k,\sigma}-f_{j,\sigma})\frac{\phi_{j,\sigma}(\mathbf{r})\phi^*_{k,\sigma}(\mathbf{r})\phi^*_{j,\sigma}(\mathbf{r}')\phi_{k,\sigma}(\mathbf{r}')}{\omega-(\epsilon_{j,\sigma}-\epsilon_{k,\sigma})+i\eta}$$

- Separate interaction into long- and short-range parts

$$\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|} = \frac{\mathrm{erf}(\mu|\mathbf{r}_i-\mathbf{r}_j|)}{|\mathbf{r}_i-\mathbf{r}_j|} + \frac{1-\mathrm{erf}(\mu|\mathbf{r}_i-\mathbf{r}_j|)}{|\mathbf{r}_i-\mathbf{r}_j|}$$

  - Treat short-range with semi-local density functional (DF) and long-range by non-local DF (or with CI)

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Are RHA or soliton vacuum methods useful?

- Casimir energy sums in finite system analogous to pairing energy sum and $\sum_\alpha \overline{\psi}(\mathbf{x})\psi(\mathbf{x})$ similar to $\sum_\alpha u^*_\alpha(\mathbf{x})v_\alpha(\mathbf{x})$

$$E_{\rm vac} \sim \sum_{\epsilon_i^0<\Lambda} (\epsilon_i - \epsilon_i^0) + E_{\rm counterterms}(\Lambda)$$

  - Sum of energy differences with and without background field
  - Many approaches to go beyond Thomas-Fermi approximation
  - E.g., Wasson and Koonin, Phys. Rev. D **43**, 3400 (1991)
- Strategies:
  - Derivative expansion with functional methods (e.g., R. Perry)
  - WKB approximations for contribution of modes with energies above cutoff $\Longrightarrow$ accelerated convergence
  - and others in the literature . . .

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Aside: Operator Product Expansion (OPE)

more

- Energy for unitary gas with separately finite terms (Tan)

$$E = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m}\left(\rho_\sigma(\mathbf{k}) - \frac{C}{k^4}\right) + \frac{C}{4\pi m a_s} + \langle V \rangle$$

  - "Contact" C appears in many relations
- OPE used by Braaten, Kang, Platter to extend Tan relations
  - In general: $\mathcal{O}_A(\mathbf{R} - \frac{1}{2}\mathbf{r})\mathcal{O}_A(\mathbf{R} + \frac{1}{2}\mathbf{r}) = \sum_C C^C_{A,B}(\mathbf{r})\mathcal{O}_C(\mathbf{R})$
- Momentum distribution can be written

$$\rho_\sigma(\mathbf{k}) = \int d^3R \int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}} \langle X|\psi^\dagger_\sigma(\mathbf{R} - \frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \frac{1}{2}\mathbf{r})|X\rangle$$

- Apply OPE standard diagrammatic techniques to get:

$$\psi^\dagger_\sigma(\mathbf{R}-\frac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R}+\frac{1}{2}\mathbf{r}) = \psi^\dagger_\sigma\psi_\sigma(\mathbf{R}) + \frac{1}{2}\mathbf{r}\cdot[\psi^\dagger_\sigma\nabla\psi_\sigma(\mathbf{R}) - \nabla\psi^\dagger_\sigma\psi_\sigma(\mathbf{R})] - \frac{r}{8\pi}g^2\psi^\dagger_1\psi^\dagger_2\psi_2\psi_1(\mathbf{R}) + \cdots$$

  - Power law in $1/k$ from function of $|\mathbf{r}|$
  - Contact is finite: $C = \int d^3R\, \langle X|g(\Lambda)^2\psi^\dagger_1\psi^\dagger_2\psi_2\psi_1(\mathbf{R})|X\rangle$
  - Use $g(\Lambda) = -\frac{\Lambda}{2\pi^2}g^2 + \frac{1}{4\pi a_s}g^2$ to split into two terms

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Collective coordinates and functional integrals

- What about dealing broken symmetries?
  - E.g., number projection for pairing
  - How to accomodate within effective action DFT framework?
    $\Longrightarrow$ consequences are zero-frequency modes (divergent perturbation expandsion)
  - Transformation to collective variables $\Longrightarrow$ work with overcomplete dof's $\Longrightarrow$ system with constraints
  - "A theory with constraints is a gauge theory in the sense that it possesses a gauge symmetry which is generated by the constraints" [Müller-Kirsten/Zhang, PRD 50]
- Possible approach: use BRST invariance
  - E.g., Bes and Kurchan, "The treatment of collective coordinates in many-body systems: An application of the BRST invariance"
  - Add *more* fermionic variables (ghosts) so more overcomplete
  - Apparent complication is actually a simplication because in gauge systems there is a supersymmetry
  - Can procedure be adapted to DFT?

Summary

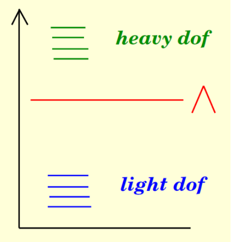
- EFT + effective action formalism generates Kohn-Sham DFT with local pairing fields $\Longrightarrow$ systematic expansion
  - Consistent treatment of renormalization through NLO
  - Energy interpretation and ambiguities?
- Some of the more general open issues for EFT-based DFT
  - Auxiliary fields or other alternatives?
  - Quasi-local functionals vs. orbital-based functionals
  - Number projection (collective coordinates)
  - Renormalization in finite systems (convergence, induced interaction)
  - Implementing low-momentum potentials $\Longrightarrow$ Power counting

Effective Field Theory Ingredients

Now QCD is the underlying theory . . .

1. Use the most general $\mathcal{L}$ with low-energy dof's consistent with global and local symmetries of underlying theory
   - What are the low-energy dof's for QCD?
   - What are the relevant symmetries?
2. Declaration of regularization and renormalization scheme
   - What choices are there?
   - Will we be able to use dimensional regularization again?
3. Well-defined power counting $\Longrightarrow$ small expansion parameter
   - Usually Q/Λ. What are the QCD scales?

- General procedures:
  - QFT: trees + loops $\rightarrow$ renormalization
  - Include long-range physics explicitly
  - Short-distance physics captured in a few LEC's (calculated from underlying or fit to data). Check naturalness.



Unraveling the magic of DFT [Kutzelnigg (2008)]

- Wavefunction-based: for anti-symmetric A-body $|\Psi\rangle$, find $E_{gs} = \min_{\Psi} \langle\Psi|\widehat{H}|\Psi\rangle$ (CI, CC use a single-particle basis for $|\Psi\rangle$)
- DFT: fermion densities as basic variables
  - Common but misleading statements:

"All information about a quantum mechanical ground state is contained in its electron density ρ."

"The energy is completely expressible in terms of the density alone."

  - At odds with kinetic and interaction energies needing $(1, 2, \cdots)$–particle density matrices!

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"The energy is completely expressible in terms of the density alone."

  - At odds with kinetic and interaction energies needing $(1, 2, \cdots)$–particle density matrices!
- Key: WF formulation deals with *single, fixed Hamiltonian*, E stationary to density matrix (or Ψ) variations, not just $\rho(\mathbf{x})$
- DFT: Consider a *family* of Hamiltonians $\widehat{H}[v] \to E[v]$, then

$$F_{\rm HK}[\rho] = \min_{v}\{E[v] - \int d\mathbf{x}\, v(\mathbf{x})\rho(\mathbf{x})\} \quad \text{and}$$

$$E[v] = \min_{\rho}\{F[\rho] + \int d\mathbf{x}\, v(\mathbf{x})\rho(\mathbf{x})\} \equiv \min_{\rho}\{E_v[\rho]\}$$

DFT as Legendre transform

- Recall ordinary thermodynamics with N particles at $T = 0$
- Use a chemical potential μ as source to change $\langle \widehat{N} \rangle$

$$\Omega(\mu) = -kT \ln Z(\mu) \qquad \text{and} \qquad N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{TV}$$

- *Invert* to find $\mu(N)$, Legendre transform to $F(N)$

$$F(N) = \Omega(\mu(N)) + \mu(N)N$$

$\Longrightarrow$ This is our energy function!

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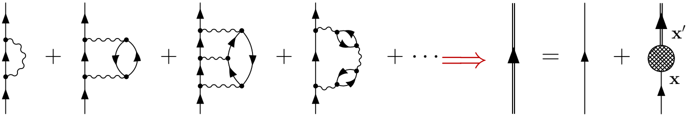
- Generalize to spatially dependent chemical potential $v(\mathbf{x})$

$$Z(\mu) \longrightarrow Z[v(\mathbf{x})] \qquad \text{and} \qquad \mu N = \mu \int \psi^\dagger \psi \longrightarrow \int v(\mathbf{x}) \psi^\dagger \psi(\mathbf{x})$$

- LT from $\ln Z[v(\mathbf{x})]$ to $\Gamma[\rho(\mathbf{x})]$, where $\rho = \langle \psi^\dagger \psi \rangle_v \Longrightarrow$ DFT!

How is the Full G Related to $G_{\rm ks}$? [nucl-th/0410105]

back



$\Longrightarrow \Sigma^*(\mathbf{x},\mathbf{x}';\omega)$

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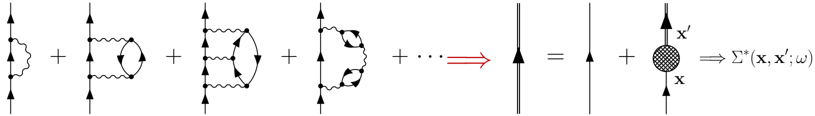
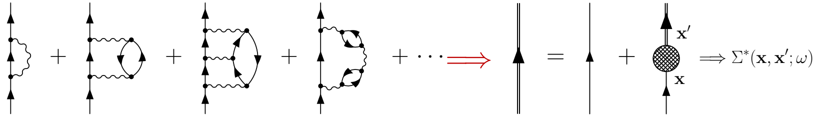


- Add a non-local source $\xi(x', x)$ coupled to $\psi(x)\psi^\dagger(x')$:

$$Z[J,\xi] = e^{iW[J,\xi]} = \int D\psi D\psi^\dagger \; e^{i\int d^4x\;[\mathcal{L} + J(x)\psi^\dagger(x)\psi(x) + \int d^4x'\;\psi(x)\xi(x,x')\psi^\dagger(x')]}$$

How is the Full G Related to $G_{\rm ks}$? [nucl-th/0410105] back



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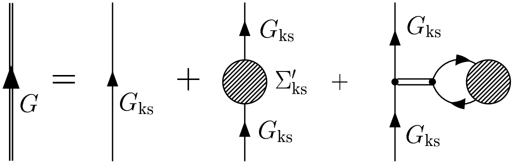
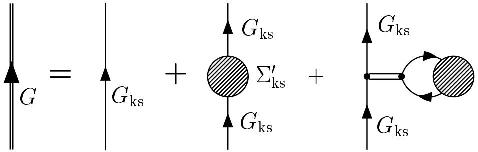
- With $\Gamma[\rho,\xi] = \Gamma_0[\rho,\xi] + \Gamma_{\rm int}[\rho,\xi]$,

$$G(x,x') = \left.\frac{\delta W}{\delta\xi}\right|_J = \left.\frac{\delta\Gamma}{\delta\xi}\right|_\rho = G_{\rm ks}(x,x') + G_{\rm ks}\left[\frac{1}{i}\frac{\delta\Gamma_{\rm int}}{\delta G_{\rm ks}} + \frac{\delta\Gamma_{\rm int}}{\delta\rho}\right]G_{\rm ks}$$



G and $G_{\rm ks}$ Yield the Same Density by *Construction*

- Claim: $\rho_{\rm ks}(\mathbf{x}) = -i\nu G^0_{\rm KS}(x,x^+)$ equals $\rho(\mathbf{x}) = -i\nu G(x,x^+)$
- Start with



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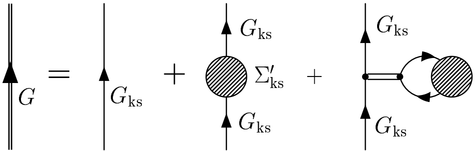
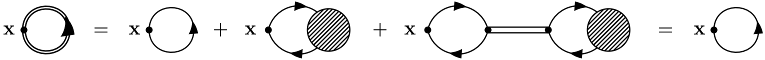


- Simple diagrammatic demonstration:



- Densities agree by construction!

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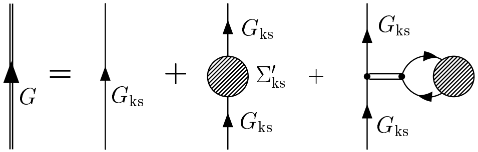

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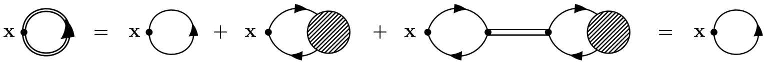

- Densities agree by construction!
- Is the Kohn-Sham basis a useful one for G?
- Better single-particle energies with more sources

Add Sources [Bhattacharyya, rjf, nucl-th/0408014]

- Add additional sources to Lagrangian, e.g., $\eta(\mathbf{x})\, \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi$

$$\Gamma[\rho,\tau] = W[J,\eta] - \int J(x)\rho(x) - \int \eta(x)\tau(x)$$

- Two Kohn-Sham potentials: $[\rho \equiv \langle\psi^\dagger\psi\rangle, \quad \tau \equiv \langle\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi\rangle]$

$$J_0(\mathbf{x}) = \frac{\delta\Gamma_{\rm int}[\rho,\tau]}{\delta\rho(\mathbf{x})} \quad \text{and} \quad \eta_0(\mathbf{x}) = \frac{\delta\Gamma_{\rm int}[\rho,\tau]}{\delta\tau(\mathbf{x})}$$

- Kohn-Sham equation $\Longrightarrow$ defines $\frac{1}{2M^*(\mathbf{x})} \equiv \frac{1}{2M} - \eta_0(\mathbf{x})$:

$$\left(-\nabla \cdot \frac{1}{2M^*(\mathbf{x})}\nabla - J_0(\mathbf{x})\right)\phi_\alpha(\mathbf{x}) = \epsilon_\alpha\, \phi_\alpha(\mathbf{x})$$

- HF dilute energy density with ρ only vs. ρ and τ (for $\nu = 2$) :

$$\frac{C_2}{8}\left[\frac{3}{5}\left(\frac{6\pi^2}{\nu}\right)^{2/3}\rho^{8/3}\right] + \ldots \quad \Longrightarrow \quad \frac{C_2}{8}\left[\rho\tau + \frac{3}{4}(\boldsymbol{\nabla}\rho)^2\right] + \ldots$$

Beyond Kohn-Sham LDA: Kinetic energy density

- Skyrme E is functional of ρ *and* $\tau \equiv \langle \nabla\psi^\dagger \cdot \nabla\psi \rangle$

$$E[\rho,\tau,\mathbf{J}] = \int d^3x \left\{ \frac{1}{2M}\tau + \frac{3}{8}t_0\rho^2 + \frac{1}{16}t_3\rho^{2+\alpha} + \frac{1}{16}(3t_1+5t_2)\rho\tau \right.$$
$$\left. + \frac{1}{64}(9t_1-5t_2)(\nabla\rho)^2 - \frac{3}{4}W_0\rho\nabla\cdot\mathbf{J} + \frac{1}{32}(t_1-t_2)\mathbf{J}^2 \right\}$$

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$$\left. + \frac{1}{64}(9t_1-5t_2)(\nabla\rho)^2 - \frac{3}{4}W_0\rho\nabla\cdot\mathbf{J} + \frac{1}{32}(t_1-t_2)\mathbf{J}^2 \right\}$$

- To do this in DFT/EFT, add to Lagrangian $+\,\eta(\mathbf{x})\,\nabla\psi^\dagger\nabla\psi$

$$\Gamma[\rho,\tau] = W[J,\eta] - \int J(x)\rho(x) - \int \eta(x)\tau(x)$$

Beyond Kohn-Sham LDA: Kinetic energy density

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- Two Kohn-Sham potentials:

$$J_0(\mathbf{x}) = \frac{\delta\Gamma_{\rm int}[\rho,\tau]}{\delta\rho(\mathbf{x})} \quad \text{and} \quad \eta_0(\mathbf{x}) = \frac{\delta\Gamma_{\rm int}[\rho,\tau]}{\delta\tau(\mathbf{x})}$$

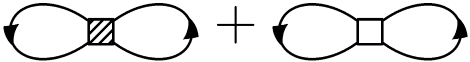
- Kohn-Sham equation $\Longrightarrow$ defines $\frac{1}{2M^*(\mathbf{x})} \equiv \frac{1}{2M} - \eta_0(\mathbf{x})$:

$$\left(-\nabla\cdot\frac{1}{2M^*(\mathbf{x})}\nabla + v_{\rm ext}(\mathbf{x}) - J_0(\mathbf{x})\right)\phi_\alpha(\mathbf{x}) = \epsilon_\alpha\,\phi_\alpha(\mathbf{x})$$

First step: HF diagrams with ∇'s [nucl-th/0408014]

- Consider bowtie diagrams from vertices with derivatives:

$$\mathcal{L}_{\rm eft} = \ldots + \frac{C_2}{16}\left[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{ h.c.}\right] + \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) + \ldots$$



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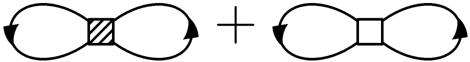

- Energy density in Kohn-Sham LDA ($\nu = 2$):

$$\mathcal{E}_{\rm int}[\rho] = \ldots + \frac{C_2}{8}\left[\frac{3}{5}\left(\frac{6\pi^2}{\nu}\right)^{2/3}\rho^{8/3}\right] + \frac{3C_2'}{8}\left[\frac{3}{5}\left(\frac{6\pi^2}{\nu}\right)^{2/3}\rho^{8/3}\right] + \ldots$$

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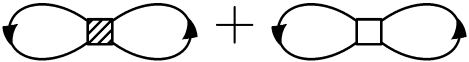

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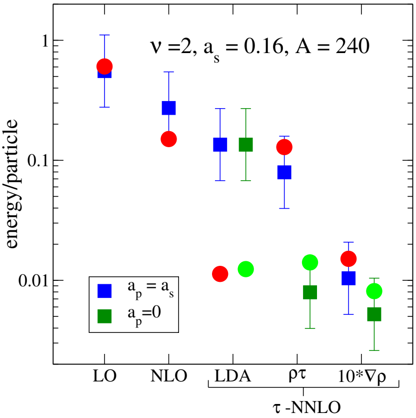
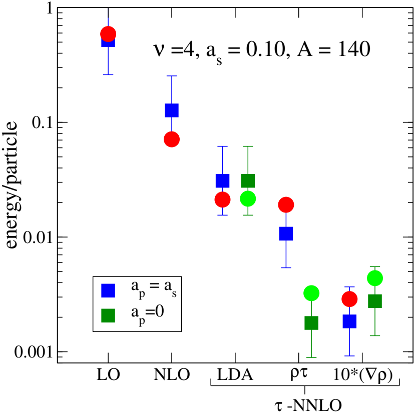
- Energy density in Kohn-Sham with τ ($\nu = 2$):

$$\mathcal{E}_{\rm int}[\rho,\tau] = \ldots + \frac{C_2}{8}\left[\rho\tau + \frac{3}{4}(\boldsymbol{\nabla}\rho)^2\right] + \frac{3C_2'}{8}\left[\rho\tau - \frac{1}{4}(\boldsymbol{\nabla}\rho)^2\right] + \ldots$$

Power counting estimates work for gradients!





Comparing Skyrme and dilute functionals

- Skyrme energy density functional (for $N = Z$)

$$E[\rho,\tau,\mathbf{J}] = \int d^3x \left\{ \frac{\tau}{2M} + \frac{3}{8} t_0 \rho^2 + \frac{1}{16}(3t_1 + 5t_2)\rho\tau + \frac{1}{64}(9t_1 - 5t_2)(\nabla\rho)^2 - \frac{3}{4} W_0 \rho \nabla \cdot \mathbf{J} + \frac{1}{16} t_3 \rho^{2+\alpha} + \cdots \right\}$$

- Dilute $\rho\tau\mathbf{J}$ energy density functional for $\nu = 4$ ($V_{\text{external}} = 0$)

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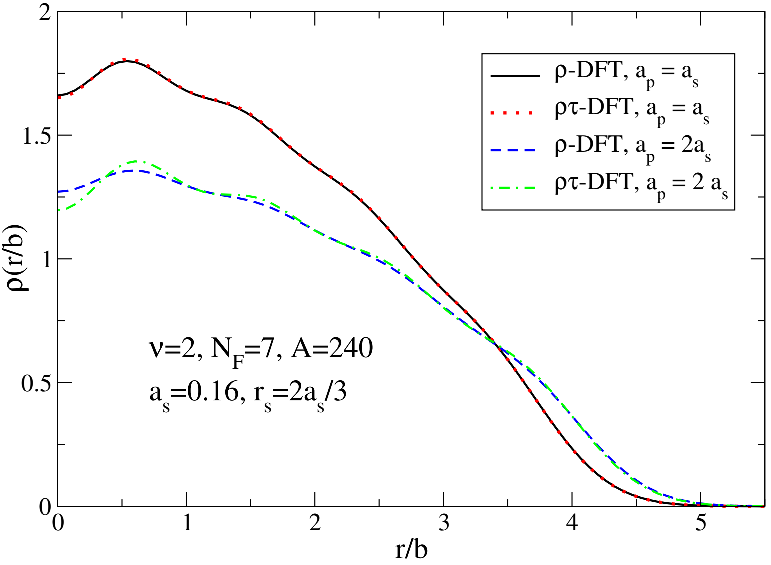
- Same functional as dilute Fermi gas with $t_i \leftrightarrow C_i$
  - equivalent $a_s \approx -2$–3 fm but $|k_{\mathrm{F}} a_p|$, $|k_{\mathrm{F}} r_0| < 1$ (with $a_p < 0$)
  - missing non-analytic terms, NNN, …

Kohn-Sham LDA ρ vs. $\rho\tau$ [Anirban Bhattacharyya]



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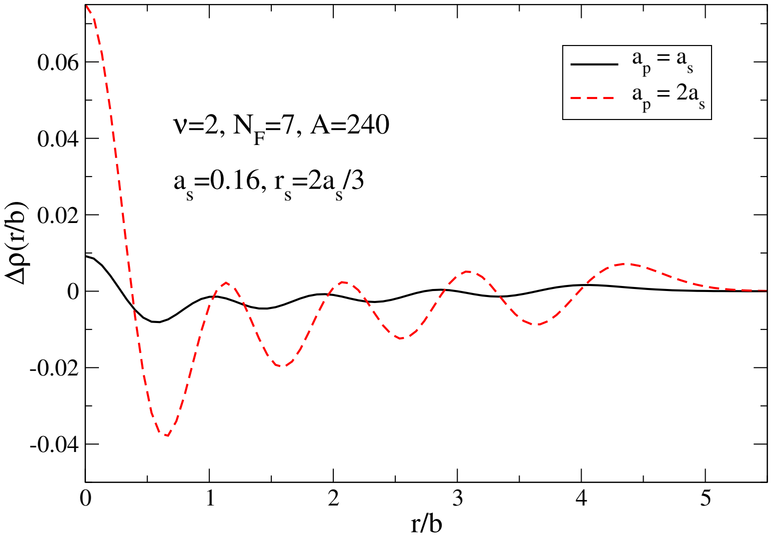
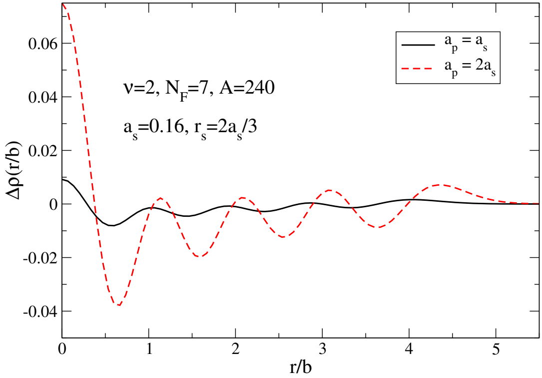


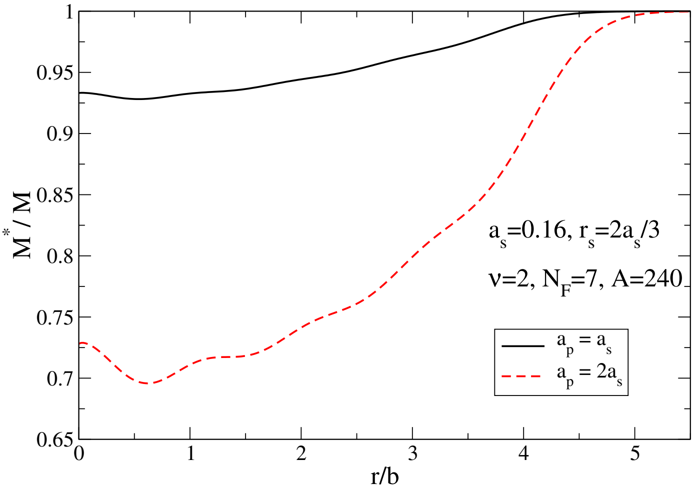
Kohn-Sham LDA ρ vs. $\rho\tau$: Differences

$a_p = a_s$	E/A	$\sqrt{\langle r^2 \rangle}$
ρ	7.66	2.87
$\rho\tau$	7.65	2.87

$a_p = 2a_s$	E/A	$\sqrt{\langle r^2 \rangle}$
ρ	8.33	3.10
$\rho\tau$	8.30	3.09

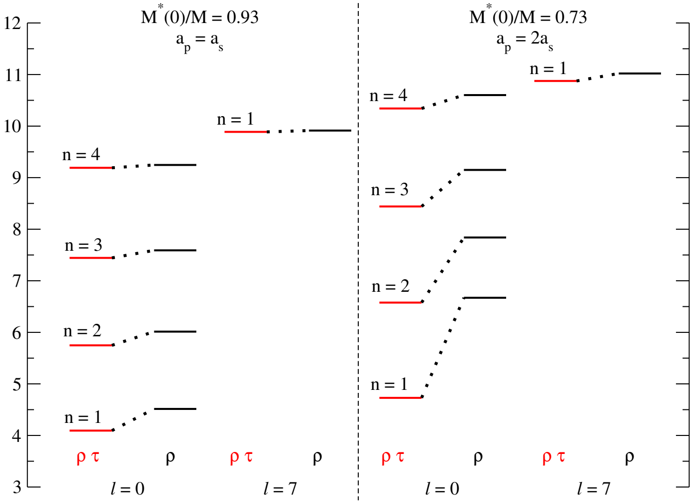


Effective mass and the single-particle spectrum



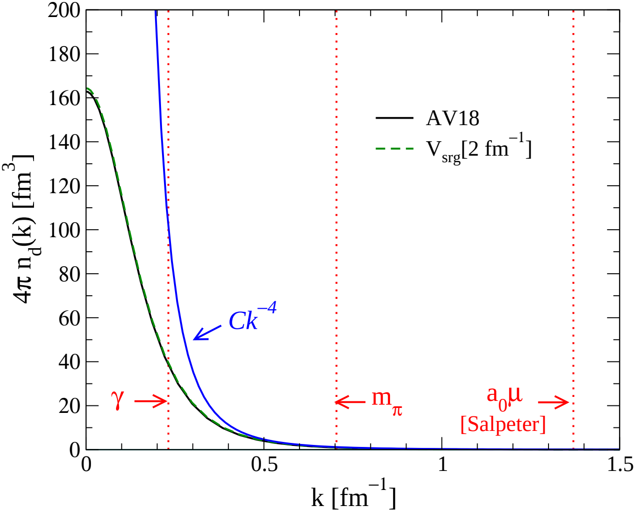
- Effective mass M^* related to single-particle levels

Effective mass and the single-particle spectrum



- Uniform system: $\varepsilon^{\rho}_{\mathbf{k}} - \varepsilon^{\rho\tau}_{\mathbf{k}} = \frac{\pi}{\nu}[(\nu-1)a_s^2 r_s + 2(\nu+1)a_p^3]\,\frac{k_F^2 - \mathbf{k}^2}{2M}\,\rho$

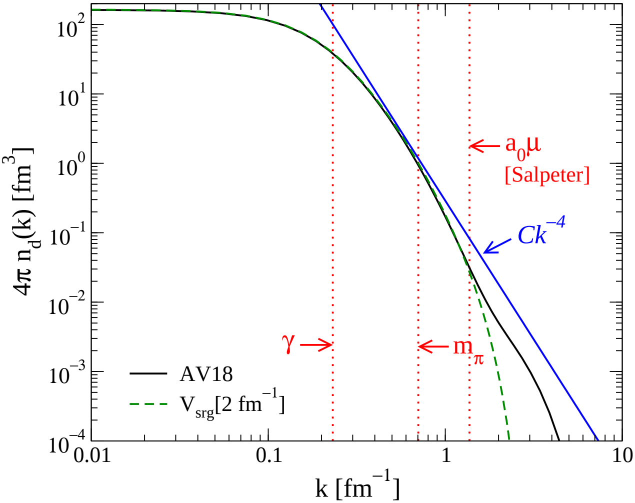
$1/k^4$ in deuteron momentum distribution? back



Salpeter wave function: $\phi(\mathbf{k}) \propto \dfrac{1}{(k^2+\gamma^2)(k^2+(a_0\mu)^2)}$

$1/k^4$ in deuteron momentum distribution?

back



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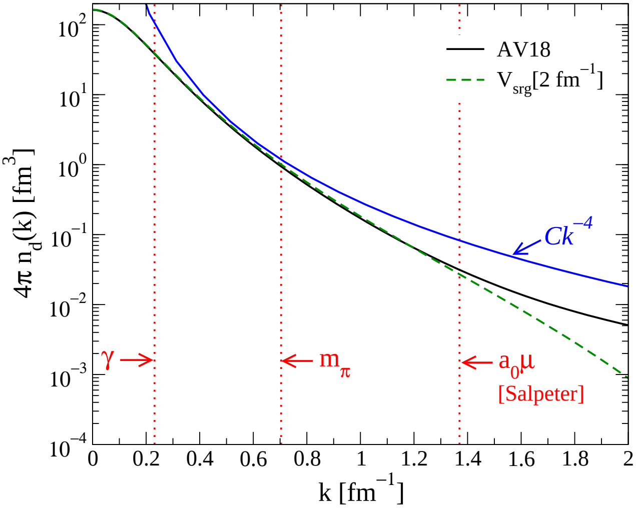

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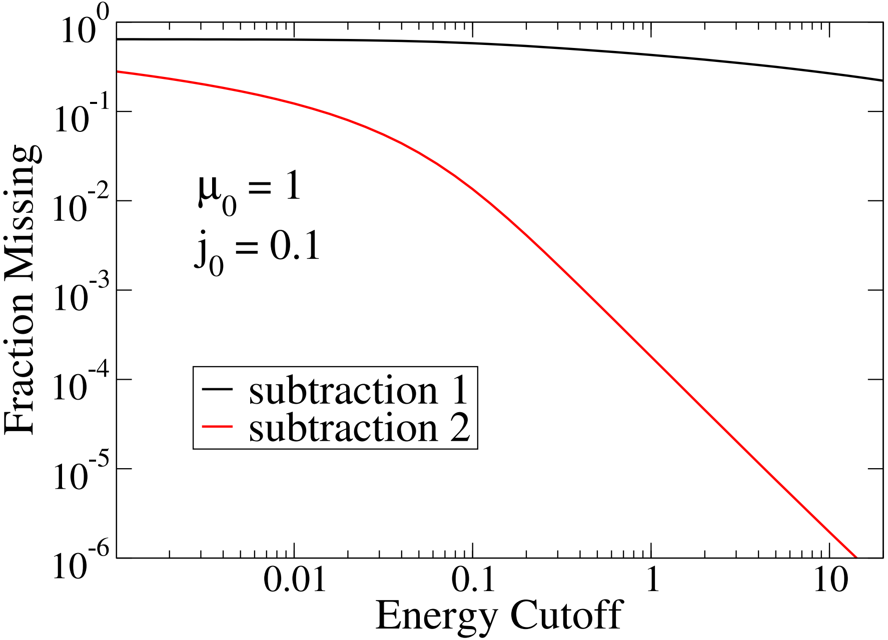
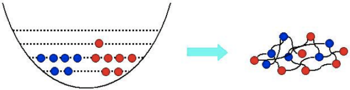


Fraction Missing
Energy Cutoff
$\mu_0 = 1$
$j_0 = 0.1$
subtraction 1
subtraction 2
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0.01
0.1
1
10

Density Functional RG for Nuclei

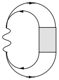
Density Functional: $\Gamma_\lambda[\rho] = \ln \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \mathrm{e}^{-S_\lambda[\psi^\dagger,\psi] + \int \frac{\delta \Gamma_\lambda}{\delta \rho} \cdot (\psi^\dagger \psi)}$



$\lambda = 0$ $\lambda = 1$

start from mean-field (background potential U_λ) and include many-body correlations

$$\partial_\lambda \Gamma_\lambda[\rho] = \frac{1}{2} V_{\mathrm{low\,k}}$$

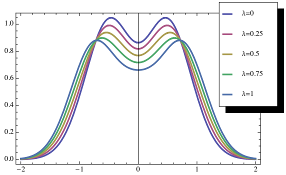


$\Gamma_\lambda[\rho]$ is the density functional

$$S_\lambda[\psi^\dagger, \psi] = \int \psi^\dagger \Big[\partial_t - \frac{1}{2m}\Delta + (1-\lambda)U_\lambda\Big]\psi + \frac{1}{2}\int \psi^\dagger \psi \lambda V_{\mathrm{low\,k}} \psi^\dagger \psi$$

+ ($\lambda V_{3\mathrm{N}}$ will be included later)

- density basis expansion scales favorably to heavy nuclei
- allows for a calculation of ground-state energy and density from nuclear forces
- in production: results for ^{16}O



Example: RG flow of the ground-state density for 1d model ("smeared-out" δ-function interaction)

Energy Interpretation

back

- Effective actions of local composite operators 30 years ago
  - "Sentenced to death" by Banks and Raby
  - Underlying problems from new UV divergences
- Connection between effective action and variational energy
  - Euclidean space (see Zinn-Justin)

$$\frac{1}{\beta}\Gamma[\rho] = \langle \widehat{H}(J) \rangle_J - \int J\,\rho = \langle \widehat{H} \rangle_J$$

  - Minkowski space constrained minimization (see Weinberg)
    - source terms serve as Lagrange multipliers
  - Are these properties invalidated by nonlinear source terms?

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- Potential ambiguities in the renormalization
  - Arbitrary finite part of added counterterms $\Longrightarrow$ shift minima
  - Verschelde et al. claim not arbitrary
- Are the stationary points valid in any case?