# Gas Gain and Neff

-- Implication of Alkhazov's Theory --

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# Why We Bother?

Because we lose Neff due to gas gain fluctuation!

Some Naive Questions

(1) Why Neff < N ?

(2) Can we improve it?

# Ionization Statistics

## Ideal Readout Plane: Coordinate = Simple C.O.G.

### PDF for C.O.G. of N electrons

We assume here an ideal readout plane that can measure the x-coordinates of individual track electrons exactly. The probability distribution function for the center of gravity of N track electrons is given by

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_I(N;\bar{N}) \prod_{i=1}^{N} \left( \int dx_i \, P_D(x_i;\sigma_d) \right) \delta\left( \bar{x} - \frac{1}{N}\sum_{i=1}^{N} x_i \right)$$

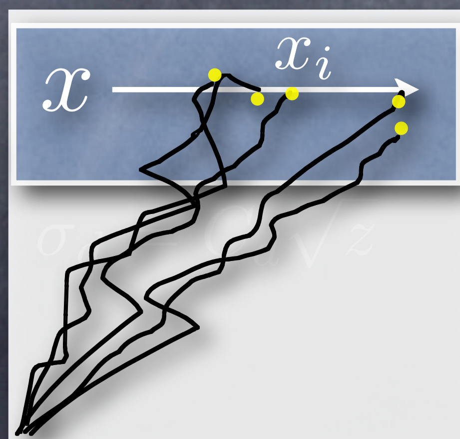
Ionization statistics

Ideal readout plane

Gaussian diffusion

$$P_D(x_i;\sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left( -\frac{x_i^2}{2\sigma_d^2} \right)$$

$$\sigma_d = C_d \sqrt{z}$$




where Cd is the diffusion coefficient and z is the drift length. The track is assumed to passed through the TPC at x=0 parallel with the readout plane and perpendicular to the pad rows.

The center of gravity of the N electrons is the best possible estimator of the incident x-coordinate of the track

$$\langle \bar{x} \rangle := \int d\bar{x} \, P(\bar{x}) \, \bar{x} = 0$$

The variance of the C.O.G. is then given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x} \, P(\bar{x}) \, \bar{x}^2 = \sigma_d^2 \left\langle \frac{1}{N} \right\rangle =: \sigma_d^2 \frac{1}{N_{\rm eff}}$$

by definition. This leads us to

$$\boxed{N_{\rm eff} := \frac{1}{\langle 1/N \rangle} < \langle N \rangle}$$

What decides the spatial resolution is not the average number of ionization electrons but the inverse of the average of its inverse.

# Gas Gain Fluctuation

## Coordinate = Gain-Weighted Mean

### PDF for gain-weighted mean

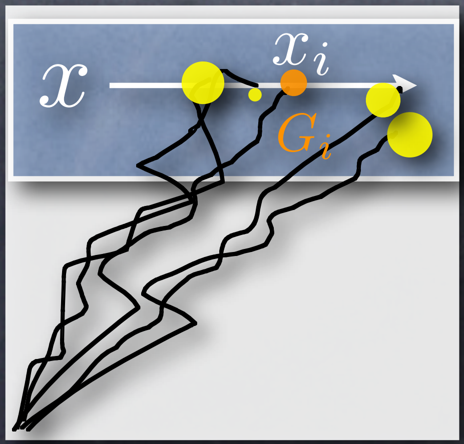
We now switch on the gas gain fluctuation and assume that the coordinate measured by the readout plane is the gain-weighted mean of the N ionization electrons.

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_I(N;\bar{N}) \prod_{i=1}^{N} \left[ \int dx_i \, P_D(x_i;\sigma_d) \right.$$
$$\left. \times \int d\left(\frac{G_i}{\bar{G}}\right) P_G\left(\frac{G_i}{\bar{G}};\theta_{\rm pol}\right) \right] \delta\left(\bar{x} - \frac{\sum_{i=1}^{N} G_i \, x_i}{\sum_{i=1}^{N} G_i}\right)$$

Gas gain fluctuation    Gain-weighted mean

We used the Polya parameter as an index even though the PG is non-Polya in general.
Notice that

$$\sum_{i=1}^{N} G_i \approx N \, \bar{G}$$




Again we assume that the charged particle passed through the TPC at x=0 parallel with the readout plane and perpendicular to the pad rows.
The average of the gain-weighted mean has then no bias

$$\langle \bar{x} \rangle := \int d\bar{x} \, P(\bar{x}) \, \bar{x} = 0$$

The variance of the C.O.G. is then given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x} \, P(\bar{x}) \, \bar{x}^2 \approx \sigma_d^2 \left\langle \frac{1}{N} \right\rangle \left\langle \left(\frac{G}{\bar{G}}\right)^2 \right\rangle =: \sigma_d^2 \frac{1}{N_{\rm eff}}$$

where use has been made of

$$\sum_{i=1}^{N} G_i \approx N \, \bar{G}$$

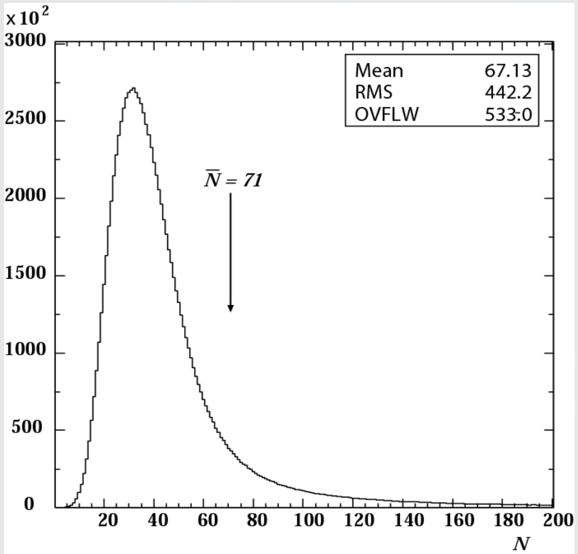
We hence have

$$N_{\rm eff} := \left[ \left\langle \frac{1}{N} \right\rangle \left\langle \left(\frac{G}{\bar{G}}\right)^2 \right\rangle \right]^{-1} = \frac{1}{\langle 1/N \rangle} \left( \frac{1+\theta_{\rm pol}}{2+\theta{\rm pol}} \right) < \langle N \rangle$$

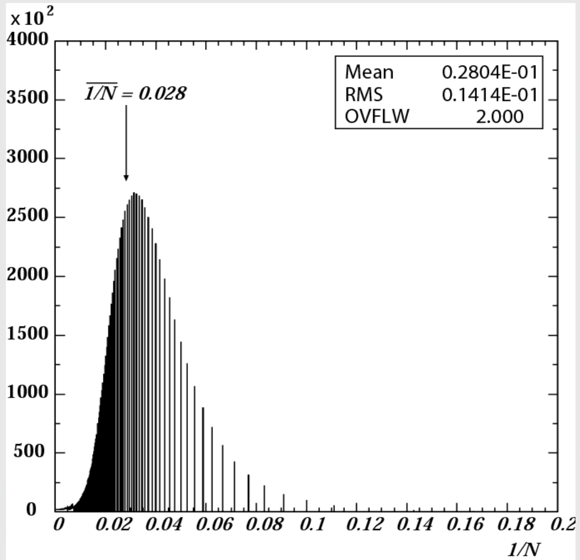
The gas gain fluctuation therefore further reduces the effective number of electrons.

# Sample Calc. for Neff

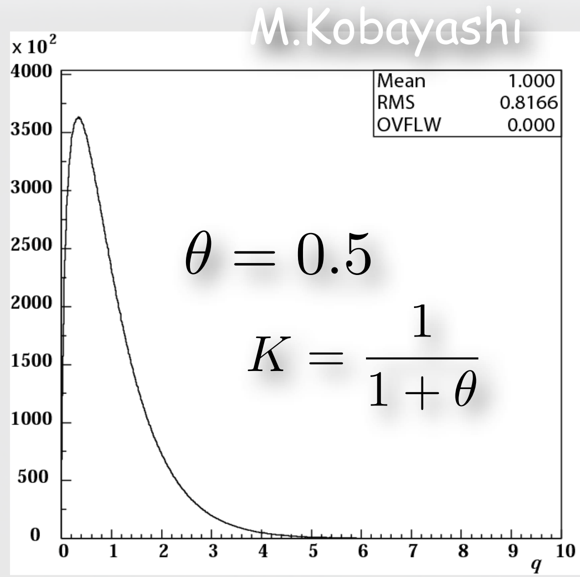
For 4 GeV pion and pad row pitch of 6mm in pure Ar




Distribution of *N*
(<*N*> = 71)




Distribution of 1/*N*
(<1/*N*> = 0.028)




Distribution of *Q*
(*K* = 0.67)

In the case of Snyder's model, gain fluctuation is exponential and K=1 (theta=0) and the Neff is reduced by a factor of 2 by it. In the case of Legler's model, theta>0 and the reduction is less sever. If we assume theta=0.5, for instance, we have a factor of 1.5 reduction:

$$N_{eff} = \left[ \left\langle \frac{1}{N} \right\rangle \left\langle \left( \frac{G}{\bar{G}} \right)^2 \right\rangle \right]^{-1} = 21 < \langle N \rangle = 71$$

$$\left\langle \left( \frac{G}{\bar{G}} \right)^2 \right\rangle = 1 + \left( \frac{\sigma_G}{\bar{G}} \right)^2 \equiv 1 + K$$

# Alkhazov's Theory of Gas Gain Fluctuation

This is a rather old work but I find it interesting to see that a set of simplistic assumptions may give at least qualitative explanations and suggestions.

# Gas Amplification

## Average Gas Gain

### Twonsend Coefficient

The probability per unit length for a seed electron in a strong E-field producing an additional ionization electron is called the first Townsend coefficient ( $\alpha$ ). We can write the average increase of electrons ($dN$) over a path ($ds$) to be

$$dN = N\,\alpha\,ds$$

The Townsend coefficient is determined by the cross sections for ionizing collisions or excitation collisions leading to secondary ionizations through Penning effect or Jesse effect. These cross sections are a function of the electron's speed or equivalently its energy, which is in turn a function of two scaling variables: "E/(gas density)" and "B/(gas density)", as far as the t-and x-derivatives of the electron state density function on the R.H.S. of Boltzmann eq. can be ignored.

Then the Townsend coefficient, having the dimension of inverse length, must scale with the mean free path inverse and hence should be proportional to the gas density:

$$\alpha = \alpha_0\left(\frac{\boldsymbol{E}}{\rho}, \frac{\boldsymbol{B}}{\rho}\right) \cdot \frac{\rho}{\rho_0}$$

unless E-field variation is so quick that the f(v;x) changes significantly over a few mean free paths.

Taking this condition for granted we can write the average gas gain as a line integral:

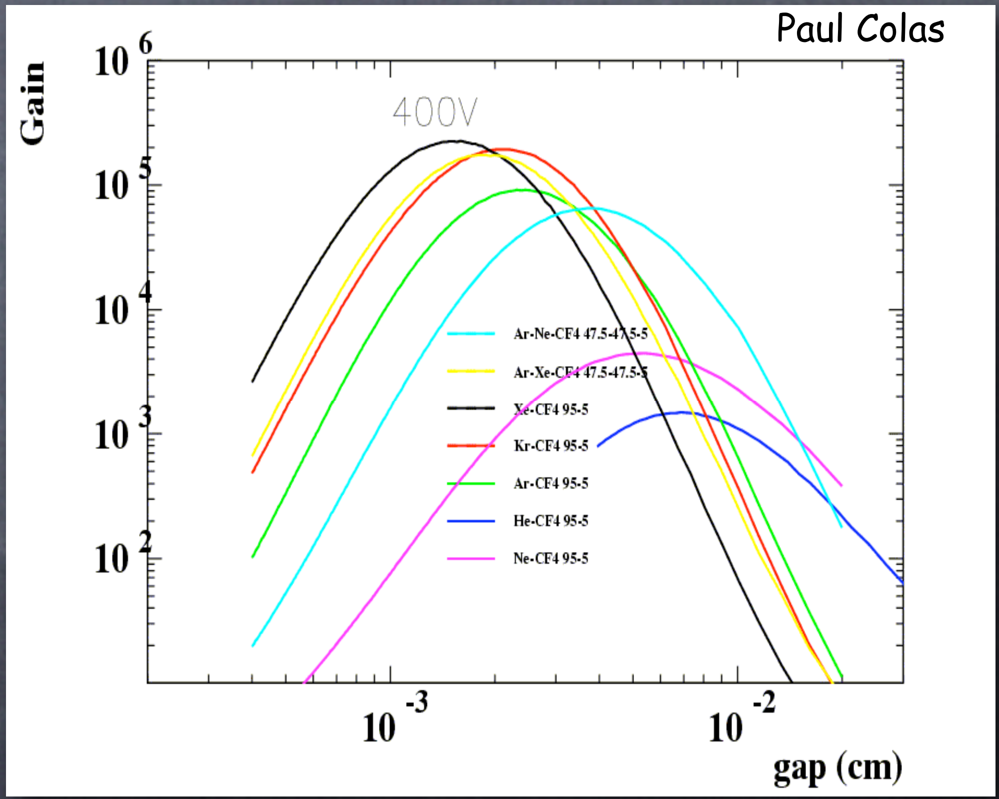
$$\bar{G} := \frac{N}{N_0} = \exp\left[\int_A^B ds\,\alpha(E(s))\right]$$

which in general depends on the possible path along which the avalanche develops.

The formula allows one to calculate the average gas gain once the 1st Townsend coefficient is given as a function of the E-field. Strictly speaking, the scaling holds only when we change both the E- and B-fields simultaneously. As far as I know there is no analytic treatment of general E and B configurations. When the E- and B-fields are parallel, however, the longitudinal motion will not be affected by the B-field and hence we can ignore the B-field effect on the Townsend coefficient (recall that the electron energy is characterized by eD/mu which is unaffected).
In the case of uniform E//B, we have

$$\bar{G}(\Delta) = \exp\left[\alpha(V/\Delta)\,\Delta\right]$$

where $\Delta$ is the amplification gap and $V$ is the high voltage across it.
This should be a good approximation for a GEM or micromegas in particular. Notice that the Townsend coefficient increases with the E-field. If the E-field is constant, the gas gain increases with the gap. The E-field, however, decreases when the gap is increased. This suggests that the gas gain must attain a maximum for an appropriate gap value, around which the gas gain is stable against gap variation. This is the operation principle of the micromegas.



# Gas Amplification

## Statistics of Avalanche Fluctuation

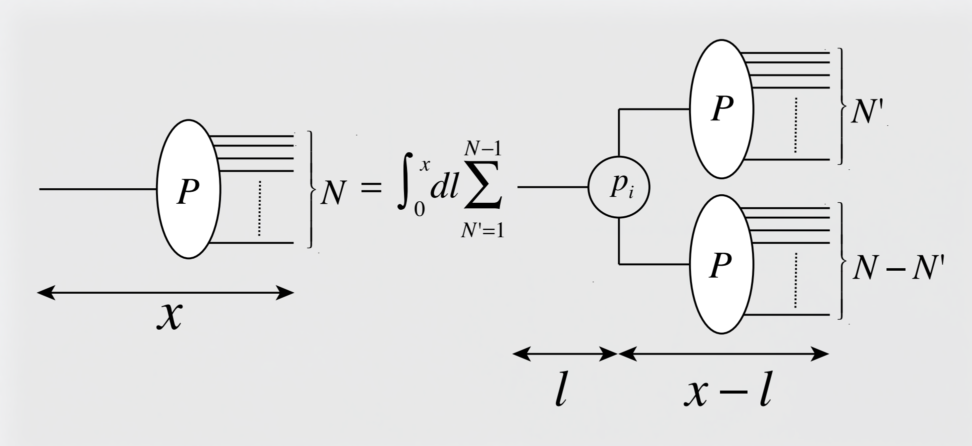
### Alkhazov's Theory (1970)

The avalanche formation involves various mechanisms: impact ionization, Penning and Jesse processes. We consider here the case where the impact ionization dominates. We further assume a uniform E-field in the amplification region. A B-field, if there is any, should be parallel to the E-field. Now let the probability of getting N electrons at the point x from the beginning of the amplification region be P(N; x), then P(N; x) must satisfy the following self-consistency equation:

$$P(N;x) = \int_0^x dl\, p_i(l) \sum_{N'=1}^{N-1} P(N';x-l)P(N-N';x-l)$$

where pi(l) is the probability of 1st ionizing collision taking place at the distance l from the origin of the seed electron.

Graphically we can represent this as in the following figure:




We can define the avalanche fluctuation function as

$$p(z,x) := N(x)\, P(N(x)z;x)$$

and its n-th moment as

$$\begin{aligned} M_n &:= \int_0^\infty dz\, z^n\, p(z,x) \\ &= \sum_{N=0}^{\infty} \frac{1}{\bar{N}(x)} \left(\frac{N}{\bar{N}(x)}\right)^n \bar{N}(x)\, P(N;x) \end{aligned}$$

Because of the central limit theorem, we expect that the avalanche fluctuation fn. and hence its moments also are determined by the early stage of the avalanche growth, which implies that p(z, x) should become x-independent

$$p(z, x) \to p(z)$$

at large x where

$$\bar{N}(x) \to e^{\alpha x}$$

Keeping these in mind, we can derive from

$$P(N; x) = \int_0^x dl\, p_{\rm i}(l) \sum_{N'=1}^{N-1} P(N'; x-l) P(N-N'; x-l)$$

an equation for Mn:

$$M_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} M_k\, M_{n-k} \int_0^\infty dl\, p_{\rm i}(l)\, e^{-n\,\alpha\, l}$$

This leads us to a recurrence formula:

$$M_n = \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \frac{M_k\, M_{n-k}\, J(n)}{1 - 2J(n)}$$

with

$$J(n) := \int_0^\infty dl\, p_{\rm i}(l)\, e^{-n\,\alpha\, l}$$

determined by the probability for the 1st ionizing collision.

On the other hand, we have

$$M_0 = M_1 = 1$$

by definition. M1=1 determines the 1st Townsend coefficient:

$$2J(1) = 2 \int_0^\infty dl\, p_{\rm i}(l)\, e^{-\alpha\, l} = 1$$

Once pi(l) is given, we can hence calculate Mn recursively.

The self-consistency equation also induces an equation for p(z):

$$p(z) = \frac{1}{\alpha\, z} \int_z^\infty dz' \int_0^{z'} dz''\, p(z'')\, p(z'-z'')\, p_{\rm i}\left(\frac{1}{\alpha} \ln \frac{z'}{z}\right)$$

which can be used to get an approximate solution by iterative substitutions.

The self-consistency equation for p(z)

$$p(z) = \frac{1}{\alpha\, z} \int_z^\infty dz' \int_0^{z'} dz''\, p(z'')\, p(z'-z'')\, p_{\rm i}\left(\frac{1}{\alpha} \ln \frac{z'}{z}\right)$$

implies that the large l behavior of pi(l) controls the behavior of p(z) near z=0. Assuming the exponential shape for the large l limit:

$$p_{\rm i}(l) \to C\, e^{-a\, l} \quad \text{as} \quad l \to \infty$$

where C is a constant, we have

$$p(z) \simeq z^{\frac{a}{\alpha}-1} \int_0^\infty dz' \int_0^{z'} dz''\, p(z'') p(z'-z'')\, \frac{C}{\alpha} z'^{a/\alpha}$$

near z=0. Denoting

$$\theta := \frac{a}{\alpha} - 1$$

we hence obtain

$$p(z) \simeq C'\, z^\theta$$

where C' is a constant. In the case of Polya distribution, we have

$$\theta = \theta_{\rm pol} := \frac{1}{\sigma^2} - 1$$

## Snyder's Model

If the ionization probability is constant as given by the 1st Townsend coefficient:

$$p_{\rm i}(l) = \alpha\, e^{-\alpha\, l}$$

we have an exponential distribution

$$p(z) = e^{-z}$$

as the exact solution to the above equation. This can be easily checked by substituting this in the self-consistency equation.
In this case we have

$$M_n = n!$$

We thus have

$$M_2 = 2$$

in particular.
We will see the significance of this number later when we discuss the effective number of seed electrons (Neff). Experimentally we know that M2 is smaller than 2 for GEM and Mircomegas detectors.

# Derivations of Recurrence Formulae

$$
\begin{aligned}
M_n &= \int_0^\infty dl\, p_{\rm i}(l) \left(\frac{\bar N(x-l)}{\bar N(x)}\right)^n \sum_{N=1}^{\infty} \sum_{N'=1}^{N-1} \left(\frac{N' + (N-N')}{\bar N(x-l)}\right)^n P(N'; x-l)\, P(N-N'; x-l) \\
&= \int_0^\infty dl\, p_{\rm i}(l) \left(\frac{\bar N(x-l)}{\bar N(x)}\right)^n \sum_{N=1}^{\infty} \sum_{N'=1}^{N-1} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{N'}{\bar N(x-l)}\right)^k \left(\frac{N-N'}{\bar N(x-l)}\right)^{n-k} \\
&\qquad \times P(N'; x-l)\, P(N-N'; x-l) \\
&= \int_0^\infty dl\, p_{\rm i}(l)\, e^{-n\,\alpha\, l} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} M_k\, M_{n-k}
\end{aligned}
$$

$$
\begin{aligned}
p(z) &= \int_0^\infty dl\, p_{\rm i}(l) \int_0^{z e^{\alpha l}} dz''\, e^{\alpha l}\, p(z'')\, p(z e^{\alpha l} - z'') \\
&= \int_0^\infty dl\, p_{\rm i}(l)\, e^{\alpha l} \int_z^\infty dz'\, \delta(z' - z e^{\alpha l}) \int_0^{z'} dz''\, p(z'')\, p(z' - z'') \\
&= \int_z^\infty dz' \int_0^{z'} dz''\, p(z'')\, p(z' - z'') \int_0^\infty dl\, p_{\rm i}(l) \delta(z' - z e^{\alpha l})\, e^{\alpha l} \\
&= \frac{1}{\alpha\, z} \int_z^\infty dz' \int_0^{z'} dz''\, p(z'')\, p(z' - z'')\, p_{\rm i}\left(\frac{1}{\alpha} \ln \frac{z'}{z}\right)
\end{aligned}
$$

# Legler's Model

Legler assumed that any ionizing collision may take place only after the seed electron flying over a minimum distance:

$$x_0 := U_0/E$$

so as to gain enough energy for ionization from the E-field. Legler further assumed the probability of ionizing collision being constant after the seed electron having reached the threshold. The probability of the 1st ionizing collision is then given by

$$p_{\rm i}(l) = a_{\rm i}\, e^{-a_{\rm i}(l-x_0)}\theta(l-x_0)$$

As mentioned before, 2J(1) = 1 gives

$$a_{\rm i} = \frac{\alpha}{2e^{-\alpha x_0} - 1} \qquad (0 \le \alpha\, x_0 \le \ln 2)$$

Notice that in the low E-field limit, where

$$\alpha x_0 \to 0 \text{ as } E/\rho \to 0$$

and hence

$$a_{\rm i} \to \alpha \text{ as } E/\rho \to 0$$

converging to Snyder's model.

It is hence important to have a high E-field in the early stage of the avalanche growth in order to suppress gain fluctuation.

From

$$M_n = \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \frac{M_k\, M_{n-k}\, J(n)}{1 - 2J(n)}$$

with

$$J(n) := \int_0^\infty dl\, p_{\rm i}(l)\, e^{-n\,\alpha\, l}$$

$$M_0 = M_1 = 1$$

we have

$$M_2 = \frac{2\,J(2)}{1 - 2\,J(2)}$$

which leads us to

$$\sigma^2 = M_2 - 1 = \frac{(2 - e^{\alpha\, x_0})^2}{2 - (2 - e^{\alpha\, x_0})^2}$$

Denoting

$$\kappa := (2 - e^{\alpha\, x_0})^2$$

we then obtain

$$\sigma^2 = M_2 - 1 = \frac{\kappa}{2 - \kappa} \qquad (0 \le \kappa \le 1)$$

$$\theta = \frac{2(1 - \sqrt{\kappa})}{\sqrt{\kappa}} = \frac{\sqrt{\kappa}}{1 + \sqrt{\kappa}}\theta_{\rm pol} \le \theta_{\rm pol}$$

The theta parameter controls the behavior near z=0. The inequality

$$\theta = \frac{2(1-\sqrt{\kappa})}{\sqrt{\kappa}} = \frac{\sqrt{\kappa}}{1+\sqrt{\kappa}}\theta_{\rm pol} \le \theta_{\rm pol}$$

states that the turn over near z=0 is less prominent than that expected from the variance assuming a Polya distribution

$$\theta_{\rm pol} := \frac{1}{\sigma^2} - 1$$

or for the same theta, the variance is smaller than that expected for the Polya. Legrer's model thus suggests a probability distribution for the gas gain fluctuation, p(z), being non-Polya. Nevertheless, we can calculate the variance by

$$\sigma^2 = M_2 - 1 = \frac{(2-e^{\alpha\, x_0})^2}{2-(2-e^{\alpha\, x_0})^2}$$

with

$$x_0 := U_0/E$$

If we set

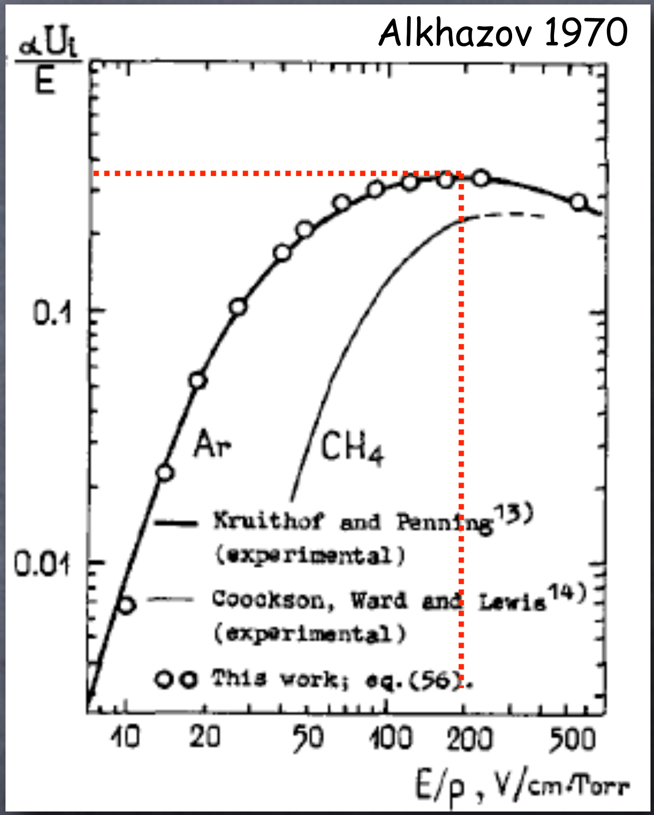
$$U_0 = U_I \,:\, \text{ionization pot.}$$

and define

$$\chi := \frac{\alpha\, U_I}{E}$$

we have

$$\sigma^2 = M_2 - 1 = \frac{(2-e^{\chi})^2}{2-(2-e^{\chi})^2}$$



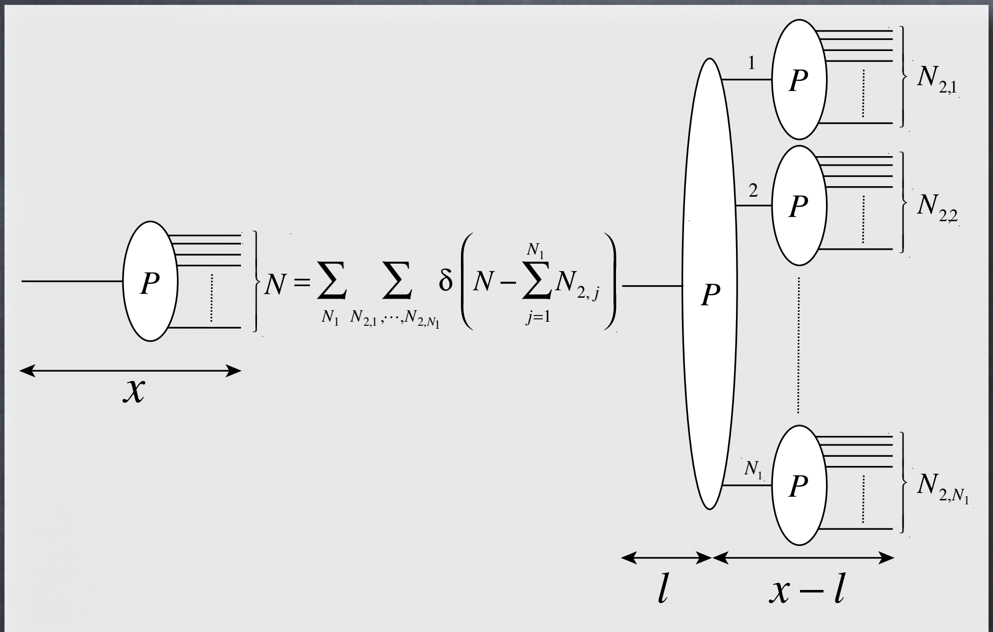

The variance depends on the E-field. The data suggest

$$\sigma^2 = M_2 - 1 \gtrsim 0.2$$

for Ar.

# Extension to a nonuniform E field

Consider first the avalanche development in a uniform E field. Dividing the amplification region (0,x) into two parts (0,l), (l,x).




The self-consistency equation for this division reads

$$P(N;x) = \sum_{N_1} \sum_{N_{2,1},\cdots,N_{2,N_1}} \delta\left(N - \sum_{j=1}^{N_1} N_{2,j}\right) \times P(N_1;l) \left(\prod_{j=1}^{N_1} P(N_{2,j};x-l)\right)$$

Here we have assumed that N1 2nd stage avalanches develop independently.
The average of N is then given by

$$\begin{aligned}\bar{N}_{12} &= \sum_{N_{12}} P(N_{12};x)\, N_{12} \\ &= \sum_{N_1} \sum_{N_{2,1},\cdots,N_{2,N_1}} N_1\, N_{2,j} P(N_1;l) \left(\prod_{j=1}^{N_1} P(N_{2,j};x-l)\right) \\ &= \bar{N}_1\, \bar{N}_2\end{aligned}$$

which leads us to a functional equation

$$\bar{N}(x) = \bar{N}(l)\,\bar{N}(x-l)$$

Noting that Nbar(0)=1, we have from this

$$\begin{aligned}\frac{d\bar{N}}{dx}(x) &= \lim_{l\to 0} \frac{\bar{N}(x) - \bar{N}(x-l)}{l} \\ &= \lim_{l\to 0} \bar{N}(x-l)\,\frac{\bar{N}(l) - \bar{N}(0)}{l} \\ &= \bar{N}(x)\,\left.\frac{d\bar{N}}{dx}\right|_{x=0}\end{aligned}$$

We find again the familiar equation

$$\frac{d\bar{N}}{dx} = \alpha\,\bar{N} \quad \text{with} \quad \alpha := \left.\frac{d\bar{N}}{dx}\right|_{x=0}$$

where $\alpha$ is the 1st Townsend coefficient.

This eq. allows us to extend our uniform E-field result to a nonuniform case

$$\bar{G}(x) := \bar{N}(x) = \exp\left[\int_0^x dl\, \alpha(l)\right]$$

This is none other than the average gas gain formula we have derived before.

Let us now consider the variance of the avalanche fluctuations:

$$\overline{(N_{12})^2} - \left(\bar{N}_{12}\right)^2 := \sum_{N_1 2} P(N_{12}; x)\, (N_{12})^2 - \left(\bar{N}_{12}\right)^2$$

Recalling the self-consistency equation

$$P(N; x) = \sum_{N_1} \sum_{N_{2,1},\cdots,N_{2,N_1}} \delta\left(N - \sum_{j=1}^{N_1} N_{2,j}\right) \times P(N_1; l) \left(\prod_{j=1}^{N_1} P(N_{2,j}; x-l)\right)$$

we have

$$\overline{(N_{12})^2} = \sum_{N_1} \sum_{N_{2,1},\cdots,N_{2,N_1}} P(N_1; l) \left(\prod_{j=1}^{N_1} P(N_{2,j}; x-l)\right) \left(\sum_{j=1}^{N_1} N_{2,j}\right)^2$$

which leads us to

$$\begin{aligned} \overline{(N_{12})^2} &= \sum_{N_1} P(N_1; l) \left[N_1\, \overline{(N_2)^2} + \left((N_1)^2 - N_1\right)(\bar{N}_2)^2\right] \\ &= \bar{N}_1 \left(\overline{(N_2)^2} - (\bar{N}_2)^2\right) + \overline{(N_1)^2}(\bar{N}_2)^2 \end{aligned}$$

Denoting

$$\overline{N^2} - \bar{N}^2 := \bar{N}^2 f(\bar{N})$$

we arrive at

$$f(\bar{N}_1 \bar{N}_2) = f(\bar{N}_1) + (\bar{N}_1)^{-1} f(\bar{N}_2)$$

or

$$f(\bar{N}(x)) = f(\bar{N}(l)) + (\bar{N}(l))^{-1} f(\bar{N}(x-l))$$

If the gain of the 1st stage is large, the fluctuation in the 2nd stage is negligible, being consistent with naive expectation.

Differentiating both sides with respect to x and then taking l-to-x limit, we get

$$\frac{df}{d\bar{N}} \frac{d\bar{N}}{dx} = \frac{1}{\bar{N}} \left(\frac{df}{d\bar{N}}\right)_{x=0} \left(\frac{d\bar{N}}{dx}\right)_{x=0}$$

Recalling that

$$\frac{dN}{dx} = \alpha \bar{N} \quad \text{and} \quad \bar{N}(0) = 1$$

we obtain

$$\frac{df}{d\bar{N}} = \frac{1}{\bar{N}^2} \left(\frac{df}{d\bar{N}}\right)_{x=0} =: \frac{C}{\bar{N}^2}$$

General solution to this equation is

$$f(\bar{N}) = C' - \frac{C}{\bar{N}}$$

We need to impose the boundary condition

$$f(N(0)) = f(1) = 0$$

since

$$P(N;0) = \delta(N-1)$$

which requires

$$C' = C$$

Denoting

$$f_0 := f(\infty) = C$$

we can rewrite the equation for f in the following form:

$$\frac{df}{d\bar{N}} = \frac{f_0}{\bar{N}^2}$$

This equation allows us to extend our uniform field results to a nonuniform field.

Recalling

$$\bar{G}(x) := \bar{N}(x) = \exp\left[\int_0^x dl\,\alpha(l)\right]$$

we arrive at

$$f\left(\bar{G}(x)\right) = \int_0^x dx'\,\alpha(x')\,\frac{f_0(x')}{[\bar{G}(x')]^2}$$

From this and

$$f \equiv \sigma^2 =: \frac{1}{\theta_{\rm pol} + 1}$$

we can calculate the Polya parameter if the Townsend coefficient and f0 are known. Notice that the avalanche fluctuation is in general non-Polya. Nevertheless we may use the Polya parameter as an index.

In the case of Legler's model, we have

$$f_0(x) = \frac{(2 - e^{\chi(x)})^2}{2 - (2 - e^{\chi(x)})^2}$$

with

$$\chi(x) := \alpha(x)\,x_0(x) = \frac{\alpha(x)\,U_0}{E(x)}$$

# Summary

# What Theory Predicts

- Some implications of Alkahzov's theory
  - Theory predicts that at low E-field, the gas gain fluctuation becomes exponential (Snyder's model).
  - Comparing Snyder's model to Legler's, we find the importance of the non negligible minimum distance for ionization. Penning and Jesse effects do not have one. They may create a secondary electron at the upstream of the parent. When they set in, we hence expect degradation of resolution. The space charge effect may suppress the higher tail though.
  - For uniform E-fields, Legler's model predicts a turn over towards low gain, and a smaller gas gain fluctuation (shorter tail) than that expected from the theta parameter as determined from the turn over near the zero gain.
  - Theta depends on the E field configuration. For smaller gas gain fluctuation, higher E-field in the beginning is essential! This is probably the reason why the micromegas energy resolution is so good. We need a large chi value.

# Implication on Neff

- To increase Neff, one needs
  - high 1/(<1/N>)
  - low gas gain fluctuation (large theta)
  - (high transmission gating GEM)
- If we are to lower the gain in order to avoid a gating GEM, we have to make sure that Neff does not decrease too much. If it does, there will be no strong reason to avoid the gating GEM. We need to consider the optimization of Neff considering both the transmission coefficient for the gating GEM and a possible gas gain dependence of Neff. High enough gas gain by increasing the E-field may benefit.
- Of course, the theory has to be tested by experiments. Fortunately French groups are measuring single photon spectra and Polya parameters. I am looking forward to seeing their results!

# Appendix

# Central Limit Theorem

## Sketch of Its Proof

### Characteristic Function

The characteristic function of a probability distribution function P(x) is defined by

$$\phi(s) := \int dx\, e^{isx}\, P(x)$$

which is essentially the Fourier transform of the p.d.f. and hence uniquely specifies it. The characteristic function comes in handy for calculations of moments:

$$M_n := \int dx\, x^n\, P(x)$$

By definition, we have

$$\begin{aligned} M_0 &= 1 \\ M_1 &= \bar{x} \\ M_2 &= \sigma^2 + \bar{x}^2 \end{aligned}$$

Once a characteristic function is given, we can calculate these moments as

$$M_n = (-i)^n \left.\frac{d^n}{ds^n}\phi(s)\right|_{s=0}$$

### Examples

For instance, the characteristic function of a Gaussian distribution is

$$\begin{aligned} \phi_G(s) &= \int_{-\infty}^{+\infty} dx\, e^{isx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \\ &= e^{-\frac{1}{2}\sigma^2 s^2 + i\bar{x}s} \end{aligned}$$

It is easy to make sure that the first three moments obtained from this characteristic function indeed coincide the above.

$$\begin{aligned} M_1 &= \bar{x} \\ M_2 &= \sigma^2 + \bar{x}^2 \end{aligned}$$

For an exponential distribution, we have

$$\begin{aligned}\phi_E(s) &= \int_0^{+\infty} dx\, e^{isx}\frac{1}{\lambda}e^{-x/\lambda}\\ &= (1-is\lambda)^{-1}\end{aligned}$$

and hence

$$\begin{aligned}M_1 &= \bar{x} = \lambda\\ M_2 &= \sigma^2+\bar{x}^2 = 2\lambda^2\end{aligned}$$

For a Polya distribution

$$P_P(x) = \frac{(\theta+1)^{\theta+1}}{\Gamma(\theta+1)}x^\theta e^{-(\theta+1)x}$$

we have

$$\begin{aligned}\phi_P(s) &= \int_0^{\infty} dx\, e^{isx}\frac{(\theta+1)^{\theta+1}}{\Gamma(\theta+1)}x^\theta e^{-(\theta+1)x}\\ &= \left(\frac{\theta+1}{\theta+1-is}\right)^{\theta+1}\end{aligned}$$

The 1st and the 2nd moments obtained from the characteristic function are

$$\begin{aligned}M_1 &= \bar{x} = 1\\ M_2 &= \sigma^2+\bar{x}^2 = \frac{2+\theta}{1+\theta}\end{aligned}$$

from which we have

$$\begin{aligned}\bar{x} &= 1\\ \sigma^2 &= \frac{1}{1+\theta}\end{aligned}$$

If theta=0, the Polya distribution becomes an exponential one with lambda=1 as is clearly seen either from the definition or from its characteristic function.

The Polya distribution becomes a delta-function in the limit of theta going to infinity as is easily seen from its characteristic function:

$$\begin{aligned}\phi_P(s) &= \left(\frac{\theta+1}{\theta+1-is}\right)^{\theta+1}\\ &\rightarrow e^{is} \quad \text{as } \theta\rightarrow\infty\end{aligned}$$

The asymptotic form coincides with the characteristic function for a Gaussian with a mean value of unity and with a zero width.

## Composition Rules

A p.d.f. for a random variable x induces a p.d.f. for a variable (ax). The characteristic function for (ax) is then given by

$$\phi_{ax}(s) = \int d(ax)\, e^{is(ax)}\, \frac{1}{a} P(x) = \phi_x(as)$$

The characteristic function for (x+a) is

$$\phi_{x+a} = \int d(x+a)\, e^{is(x+a)}\, P(x) = e^{ias}\, \phi_x(s)$$

A p.d.f. for a variable x1 and another p.d.f. for a variable x2 induce a p.d.f. for their sum (x1+x2). The characteristic function for this reads

$$\begin{aligned} \phi_{1+2}(s) &= \int dx\, e^{isx} \int dx_1 \int dx_2\, P_1(x_1)\, P_2(x_2) \\ &\qquad \times \delta\left(x - (x_1 + x_2)\right) \\ &= \phi_1(s) \cdot \phi_2(s) \end{aligned}$$

For N variables with the same p.d.f., we get

$$\phi_N(s) = [\phi(s)]^N$$

## Proof of Central Limit Theorem

We consider N variables x1, ..., xN, obeying the same p.d.f.: P(x), and consider the distribution of

$$z := \frac{1}{\sqrt{N}\sigma} \sum_{i=1}^{N} (x_i - \bar{x})$$

The characteristic function for this is

$$\phi_z(s) = [\phi_{x-\bar{x}}(s/\sqrt{N}\sigma)]^N$$

Recall now that we can expand phi in terms of moments as follows

$$\begin{aligned} \phi_{x-\bar{x}}(s/\sqrt{N}\sigma) &= \sum_{k=0}^{\infty} \frac{(is/\sqrt{N}\sigma)^k}{k!} M_k \\ &= 1 - \frac{s^2}{2N} + O\left(\frac{1}{N^{3/2}}\right) \end{aligned}$$

In the large N limit, we hence have

$$\begin{aligned} \phi_z(s) &= [\phi_{x-\bar{x}}(s/\sqrt{N}\sigma)]^N \\ &\rightarrow \lim_{N\to\infty} \left[1 - \frac{s^2}{2N}\right]^N = e^{-\frac{1}{2}s^2} \end{aligned}$$

implying that the p.d.f. for z is a Gaussian centered at zero with a variance of 1.