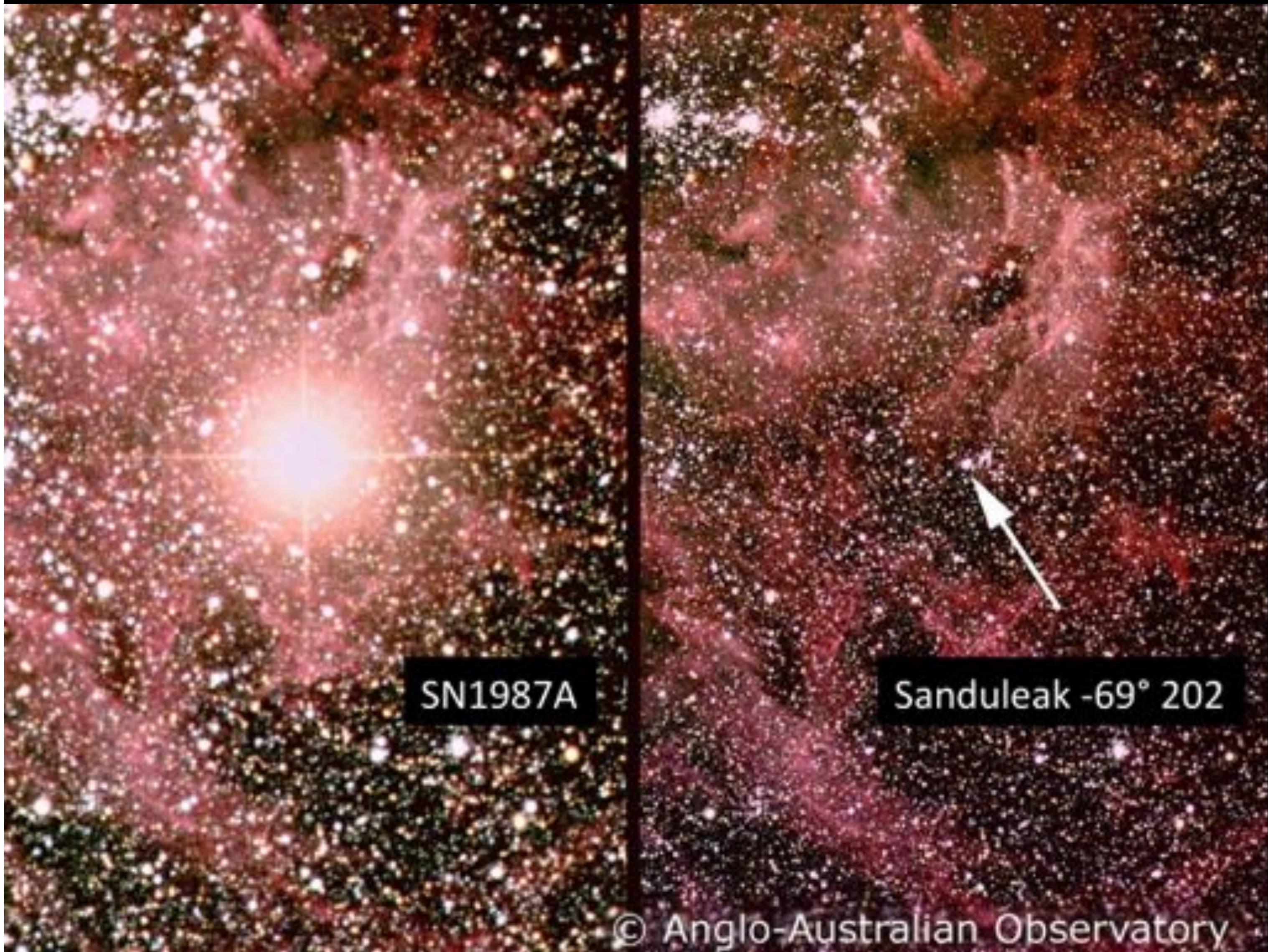
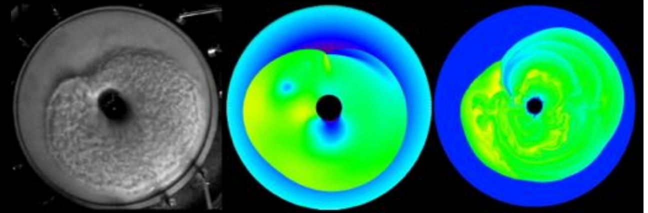




5 lectures on

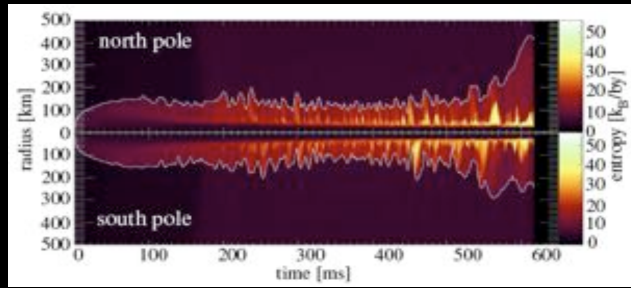
The Physics of Core-Collapse Supernovae

Outline of lecture 3

The basics of hydrodynamical instabilities

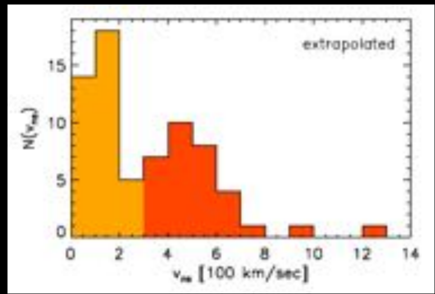
Neutrino driven convection

Why should we care about multiD instabilities



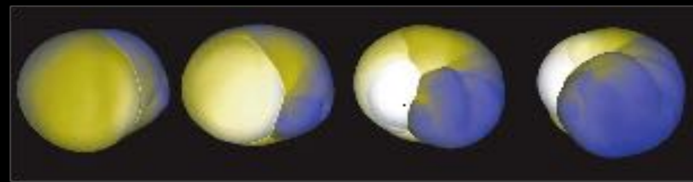
- successful explosion driven by neutrino energy

(Marek & Janka 09, Suwa+10, Müller+12, Bruenn+13, Melson+15)



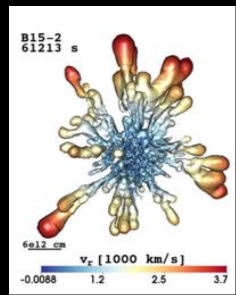
- pulsar kick

(Scheck+04, 06, Nordhaus+10, +11, Wongwathanarat+10, +13)



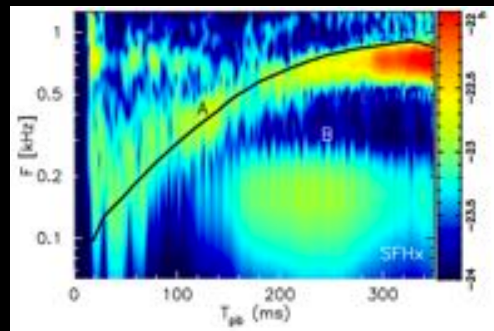
- pulsar spin

(Blondin & Mezzacappa 07, Yamasaki & Foglizzo 08, Iwakami+09, Kazeroni+16)



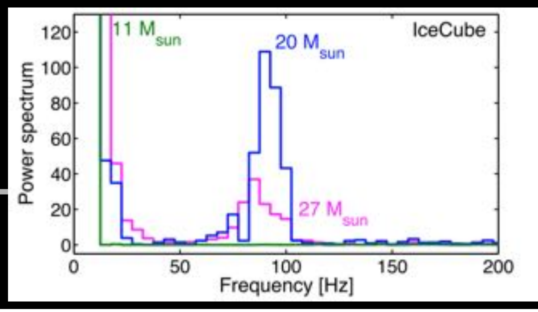
- H/He mixing and Ni clumps in SN1987A

(Kifonidis+06, Hammer+09, Utrobin+15)



- gravitational waves

(Ott+06, Kotake+07, Marek+09, Murphy+09, Kotake+11, Müller+13, Kuroda+16)



- neutrino signature

(Marek+09, Müller+12, Lund+10, 12, Tamborra+13, Müller & Janka 14)

core-collapse of a $27M_{sol}$ star in 3D
at the explosion threshold

Hanke+13, Melson+15

PRACE project 150 million hours
16.000 processors, 4.5 months/model

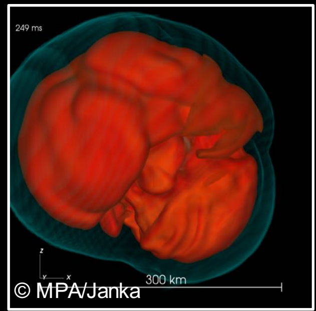
time evolution:
500ms
diameter: 300km

The many degrees of approximation of multiD core collapse

- explosion energy, neutrino signal, grav. waves, nucleosynthesis,
- pulsar kick and spin

Predictions of SN and pulsars properties

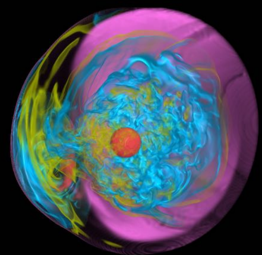
Complex comprehensive simulations



progenitor structure + nuclear EOS
+ neutrino "transport" & interactions
+ "GR" + multi-D hydro

Multi-D hydro

3D stationary accretion,
neutrino heating,
SASI+convection

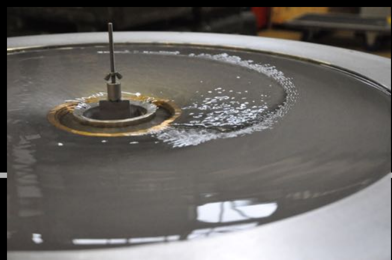


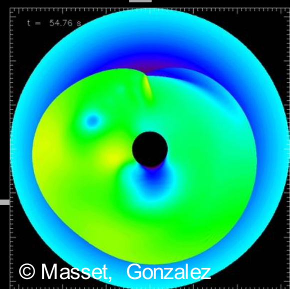
© Blondin & Mezzacappa

- 2D cylindrical SASI
  ideal gas
  neutrino cooling

SWASI experiment

Build intuition
Fast exploration the parameter space:
flow rate, Mach number,
shock radius, angular momentm



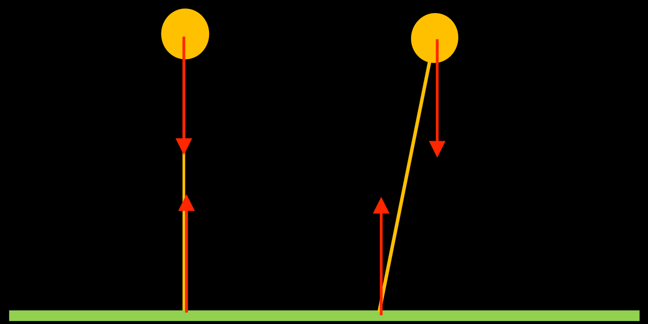


- 2D shallow water
  with viscous drag,
  then inviscid

How to characterize an instability

A linear instability is characterized by an exponential increase of small perturbation, with a rate independent of its amplitude in the linear regime.

The simplest example is the rigid pendulum:



angular momentum and torques $$ML\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = Mg\sin\theta$$

linearized equation $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \frac{g}{L}\theta = 0$$

initial perturbation $\delta\theta_0$, $(\mathrm{d}\delta\theta/\mathrm{d}t)_0$

solution
$$\begin{aligned}\delta\theta(t) &= A\exp\left(\frac{t}{\tau}\right) + B\exp\left(-\frac{t}{\tau}\right), \\ &= \delta\theta_0\cosh\left(\frac{t}{\tau}\right) + \tau\left(\frac{\mathrm{d}\delta\theta}{\mathrm{d}t}\right)_0 \sinh\left(\frac{t}{\tau}\right).\end{aligned}$$

growth rate $$\omega_i \equiv \frac{1}{\tau} \equiv \left(\frac{g}{L}\right)^{\frac{1}{2}}$$ (note: ω_i is independent of the mass)

Similarly, fluid instabilities develop on a stationary flow when the restoring forces result in an exponential amplification of the initial perturbation: e.g. a flapping flag, convective clouds...

Perturbative analysis

Example: perturbation of a uniform ideal gas with uniform velocity v_0 along the x direction

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)=0,$$
$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v+\frac{\nabla P}{\rho}=0,$$
$$\frac{\partial S}{\partial t}+v\cdot\nabla S=0.$$

$$\rho\equiv\rho_0+\delta\rho,$$
$$v\equiv v_0+\delta v,$$
$$P\equiv P_0+\delta P,$$
$$S\equiv S_0+\delta S.$$

Linearizing = keeping the first order terms

Since the unperturbed flow is stationary, a Fourier transform in time simplifies the time derivatives into multiplications by $-i\omega$ → the solution is thus a combination of exponential functions $\exp(-i\omega t)$

If the stationary flow is uniform, a Fourier transform in space simplifies the differential system into an algebraic system: $\exp(ik_x x+ik_y y)$

The relation between the eigenfrqequency ω and the wavenumber k of the perturbation is the dispersion relation.

$$\frac{\partial\delta\rho}{\partial t}+\nabla\cdot(\rho_0\delta v+v_0\delta\rho)=0,$$
$$\frac{\partial\delta v}{\partial t}+(v_0\cdot\nabla)\delta v+\frac{\nabla\delta P}{\rho_0}=0,$$
$$\frac{\partial\delta S}{\partial t}+v_0\cdot\nabla\delta S=0.$$

→

$$(\omega-k_x v_0)\frac{\delta\rho}{\rho_0}=k\cdot\delta v,$$
$$(\omega-k_x v_0)\delta v=\frac{k}{\rho_0}\delta P,$$
$$(\omega-k_x v_0)\delta S=0.$$

$$(\omega - k_x v_0)\frac{\delta\rho}{\rho_0} = k \cdot \delta v,$$
$$(\omega - k_x v_0)\delta v = \frac{k}{\rho_0}\delta P,$$
$$(\omega - k_x v_0)\delta S = 0.$$

The three types of perturbations in a gas

if $\delta S \neq 0$ then $\omega = k_x v_0$ and $\delta P = 0$: entropy perturbations are incompressible

If δS=0, then $\delta P = c_0^2 \delta\rho$ and $(\omega - k_x v_0)^2 \delta v = c_0^2 (k \cdot \delta v) k.$

If $\omega \neq k_x v_0$ the velocity perturbation δv is parallel to the wave vector k: acoustic perturbations are irrotational (kxδv=0). Their dispersion relation is $(\omega - k_x v_0)^2 = k^2 c_0^2$

Conversely, if δS=0 and $\omega = k_x v_0$ the perturbation is incompressible and corresponds to a vorticity perturbation advected with the flow.

In summary, three types of perturbations exist in a ideal uniform gas:

- -entropy perturbations
- -vorticity perturbations

} incompressible and advected with the flow, $\omega = k_x v_0$

- -acoustic waves — irrotational and adiabatic, $(\omega - k_x v_0)^2 = k^2 c_0^2$

Warning: non-uniform regions of the flow are regions of linear coupling between these 3 types of "waves"

Some examples of fluid instabilities

Gravitational potential:

Rayleigh-Taylor instability: feeds on potential energy, by carrying down dense matter exchanged with lighter matter

Sheared flow:

Kelvin-Helmholtz instability: feeds on sheared velocities, tends to smoothen the velocity gradient

Rotating flow:

Corotation instability: feeds on differential rotation and exchange angular momentum through a spiral acoustic wave

Magnetorotational instability: feeds on sheared velocities in a MHD flow, exchanging angular momentum along the field lines connecting different radial positions

Shocked flow:

Ritchmeyer Meshkov instability: similar to RT with an impulsional acceleration due to the crossing of a density interface by a shock

Standing accretion shock instability: advective-acoustic interplay of the shock surface and a downstream region of gradients

Outline of lecture 3

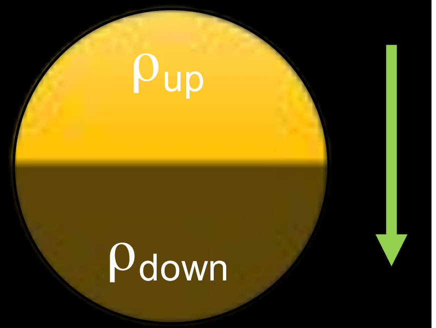
The basics of hydrodynamical instabilities

Neutrino driven convection

Instability of a top heavy disc

Denoting by $I \sim MR^2/2$ the moment of inertia of a disc with radius R and mass M
a density distribution $\rho(z)$ with a transition from ρ_{down} to ρ_{up} over a lengthscale $H=\rho/(d\rho/dz)$
z_G is the height of the center of mass above the geometric center.
The linearized variation of the angular momentum is ruled by the equation

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - Mgz_{\mathrm{G}}\theta = 0 \qquad \omega^2 = -\left(\frac{MR^2}{2I}\right)\frac{gz_{\mathrm{G}}}{R^2}.$$



If R>>H,

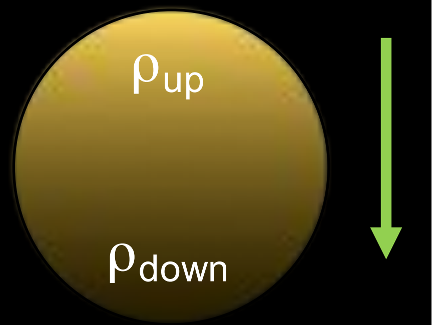
$$z_{\mathrm{G}} \equiv \frac{1}{M}\int_0^{\frac{\pi}{2}} 2\rho R^3 \sin\theta\cos^2\theta\,\mathrm{d}\theta, = \frac{4R}{3\pi}\left(\frac{\rho_{\mathrm{up}}-\rho_{\mathrm{down}}}{\rho_{\mathrm{up}}+\rho_{\mathrm{down}}}\right).$$

the growth rate (or oscillation frequency) is thus

$$\omega^2 = \frac{8g}{3\pi R}\left(\frac{MR^2}{2I}\right)\left(\frac{\rho_{\mathrm{down}}-\rho_{\mathrm{up}}}{\rho_{\mathrm{down}}+\rho_{\mathrm{up}}}\right)$$

$$\nabla\Phi = gz$$

→As for a pendulum, the smaller the disc, the shorter the time scale.



If R<<H, the density distribution is linearly approximated

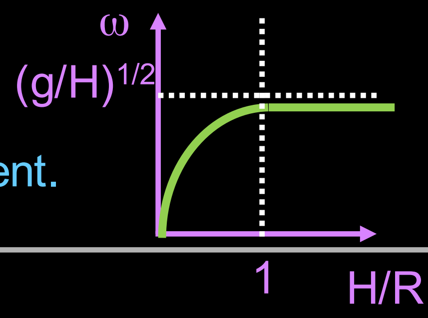
$$\rho = \rho_0\left(1+\frac{z}{H}\right)$$

$$z_{\mathrm{G}} \equiv \frac{1}{M}\int_0^{\frac{\pi}{2}} 2\rho_0\left(1+\frac{R}{H}\sin\theta\right)R^3\sin\theta\cos^2\theta\,\mathrm{d}\theta, = \frac{R^2}{4H}.$$

the growth rate is:

$$\omega^2 = -\frac{g}{2H}\left(\frac{MR^2}{2I}\right)$$

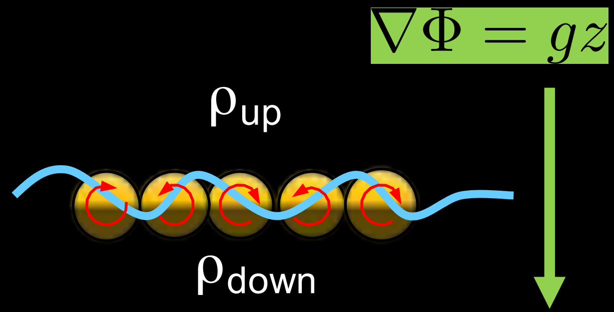
→As the radius of the disc decreases, the growth rate increases like $\sim(g/R)^{1/2}$
and reaches a maximum $\sim(g/H)^{1/2}$ as R approaches the scale H of the density gradient.



Instability of a top heavy superposition of incompressible fluids

Two incompressible fluids with uniform densities $\rho_{up} > \rho_{down}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = 0,$$
$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \frac{\nabla P}{\rho} + \nabla \Phi = 0.$$



Linearizing, + Fourier transform in time and space: $\exp(-i\omega t + ik_x x + ik_z z)$

$$ik \cdot \delta v = 0,$$
$$-i\omega \delta v + ik \frac{\delta P}{\rho} = 0.$$

$$\rightarrow \quad k^2 \frac{\delta P}{\rho} = 0 \quad \rightarrow \quad k_x^2 + k_z^2 = 0 \quad \rightarrow \quad k_z = \pm i k_x$$

$$k_x \delta v_x + k_z \delta v_z = 0,$$
$$-i\omega \delta v_x + ik_x \frac{\delta P}{\rho} = 0,$$
$$-i\omega \delta v_z + ik_z \frac{\delta P}{\rho} = 0.$$

$$\delta v_z = -i\omega \, \delta\zeta \, \mathrm{e}^{-k_x|z|} \mathrm{e}^{ik_x x},$$
$$\delta v_x = \mp \omega \, \delta\zeta \, \mathrm{e}^{-k_x|z|} \mathrm{e}^{ik_x x},$$
$$\delta P = \pm \frac{\omega^2}{k_x} \rho \, \delta\zeta \, \mathrm{e}^{-k_x|z|} \mathrm{e}^{ik_x x}.$$

Boundary condition: continuity of the interface pressure $P(\zeta) + \delta P$ at $z = \zeta$

$$\delta P_{\rm up} + \rho_{\rm up} g \delta\zeta = \delta P_{\rm down} + \rho_{\rm down} g \delta\zeta$$

$$\omega^2 = \left(\frac{\rho_{\rm down} - \rho_{\rm up}}{\rho_{\rm down} + \rho_{\rm up}} \right) k_x g$$

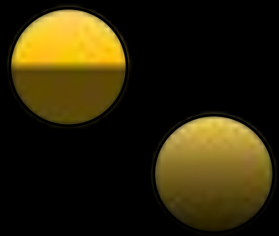
Atwood number

The Rayleigh Taylor instability from solid to fluid mechanics

A solid mechanics analogue of the RT instability is a disc of radius R with a top heavy mass distribution from ρ to $\rho+\Delta\rho$ and a transition zone extended over a distance H from the rotation axis

if H/R<<1 $\omega^2 \sim \frac{\Delta\rho}{\rho}\frac{g}{R}$

if H/R>>1 $\omega^2 \sim -\frac{g}{2H}$



The incompressible version of the RT instability is the instability of a dense fluid over a light fluid, noting k the horizontal wavelength and H the lengthscale of the density transition from ρ to $\rho+\Delta\rho$

if kH<<1, $\omega^2 \sim \frac{\Delta\rho}{\rho}kg$

if kH>>1 $\omega^2 \sim \frac{g}{H}$

In a gas in pressure equilibrium in a gravitational field, the vertical displacement of a blob of gas leads to an adiabatic change of its density to adapt to the local pressure.
The density of the blob carried upward is lighter than the surrounding gas if the entropy decreases upward: $\rho \propto P^{\frac{1}{\gamma}} \exp\left(-\frac{\gamma-1}{\gamma}S\right)$

if kH>>1 $\omega_{\rm BV}^2 \sim \frac{g}{\rho}\left(\frac{\partial\rho}{\partial z}\right)_{P={\rm cte}} = -\frac{\gamma-1}{\gamma}g\nabla S$

The Brunt Väisälä frequency ω_{BV} is the frequency of perturbations with a short horizontal wavelength compared to the stratification scale height.

The Rayleigh Taylor instability in core collapse supernovae

The oscillations driven by the buoyancy force are called internal gravity waves.

The vertical gradients of electron fraction participate in the same manner to the stability criterion.

The generalized Brunt Väisälä frequency is:

$$\omega_{\mathrm{BV}}^2 = -\frac{1}{\rho}\left[\left(\frac{\partial\rho}{\partial S}\right)_{Y_e,P}\frac{\mathrm{d}S}{\mathrm{d}r} + \left(\frac{\partial\rho}{\partial Y_e}\right)_{S,P}\frac{\mathrm{d}Y_e}{\mathrm{d}r}\right]\frac{\mathrm{d}\Phi}{\mathrm{d}r}.$$

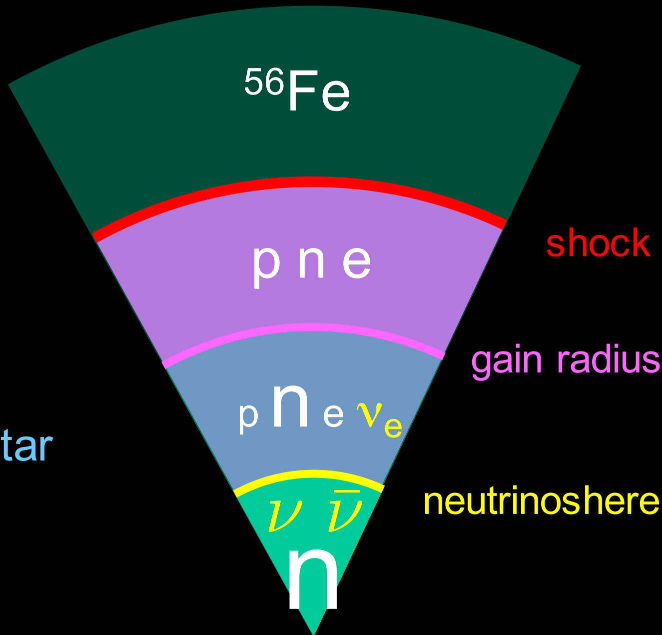
The possibility to enhance the neutrino luminosity of the proto-neutron star through lepton-driven convective instability has been proposed by Epstein (1979)

3 locations where transverse motions can feed on potential energy:

-the negative entropy gradient left by the deceleration of the shock until it stalls at 150km: "prompt convection"

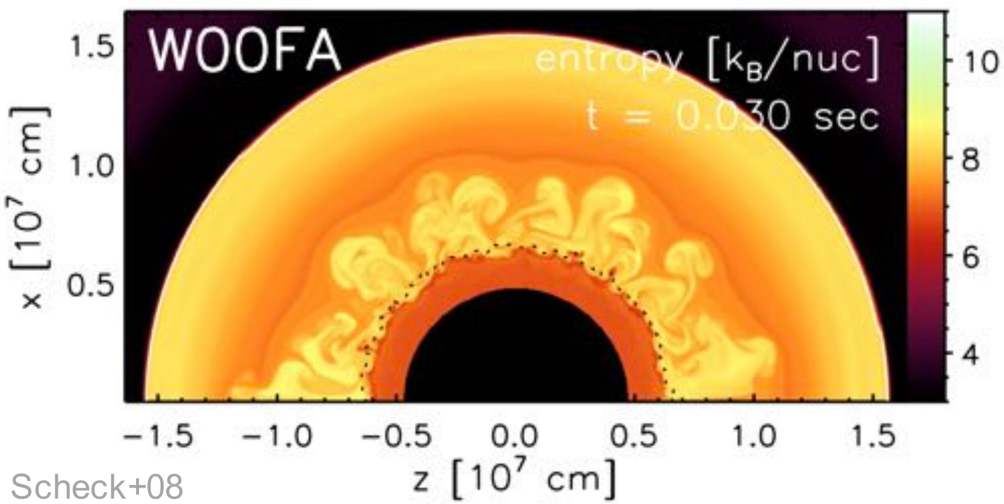
-the gradient of electronic pressure inside the proto-neutron star "thermolepton convection"

-"neutrino-driven convection" in the gain region

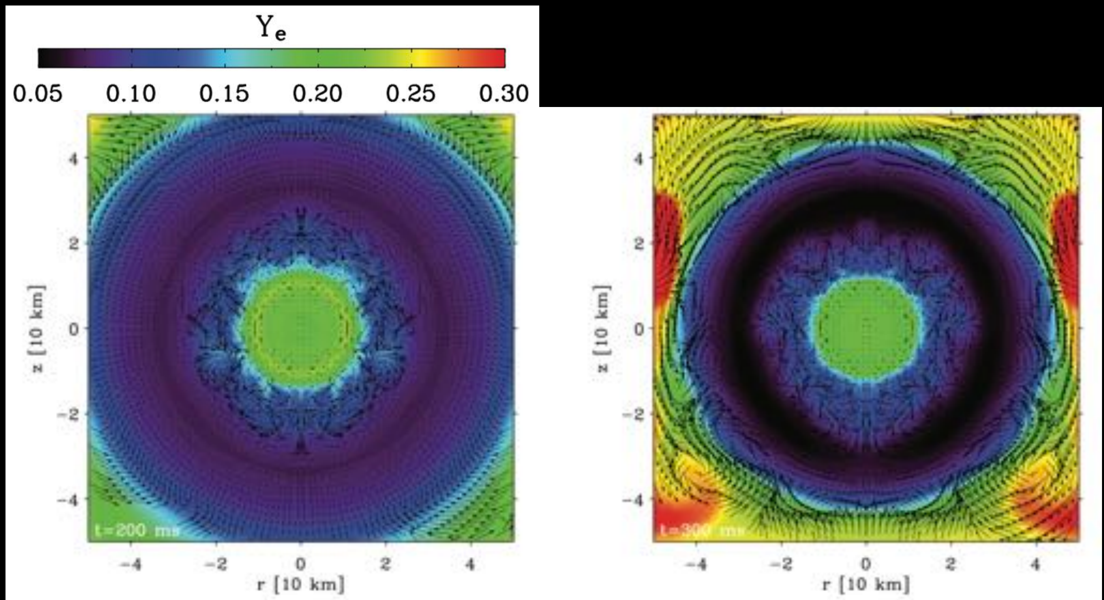


The Rayleigh Taylor instability in core collapse supernovae

Prompt convection is transient and does not affect the explosion threshold.



Scheck+08



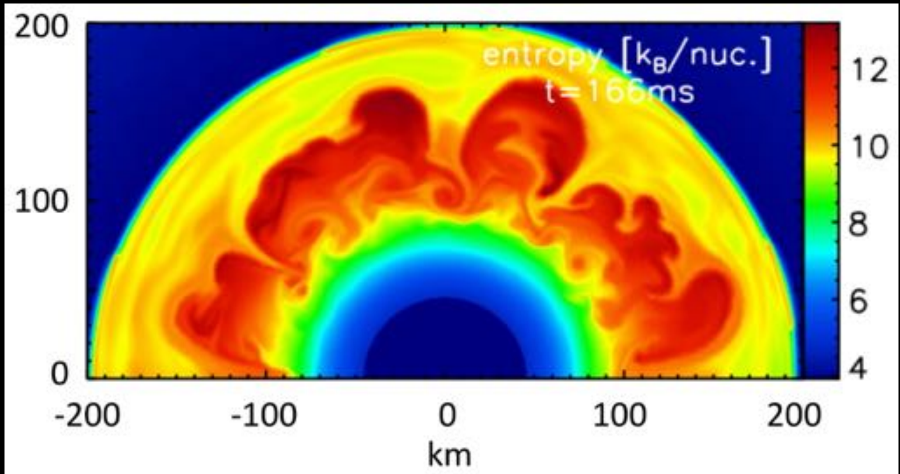
Dessart+06

The proto-neutron star convection is embedded in a stably stratified region.
It has a moderate impact on the neutrino luminosity, at a 10-20% level (Dessart+06, Buras+06, Müller & Janka 14)

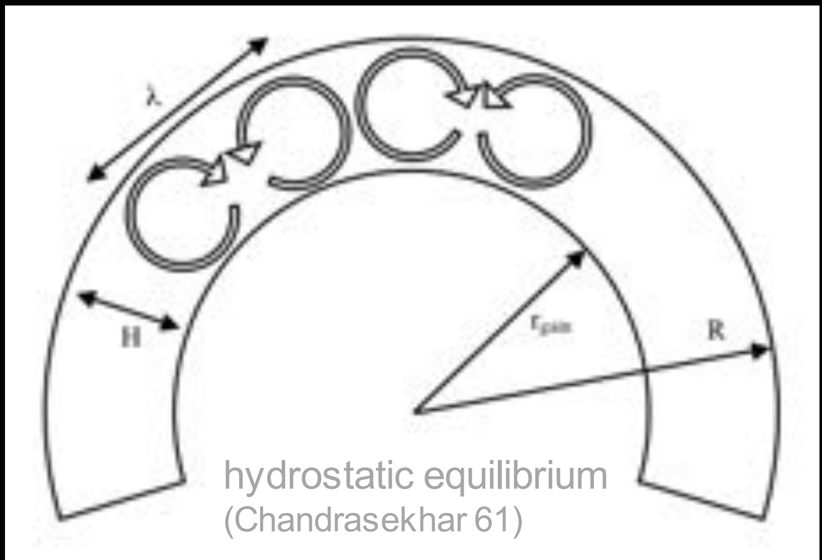
However, it may contribute to the amplification of magnetic fields (Thompson & Duncan 93).

Neutrino-driven convection in the gain region

Foglizzo +06



The negative entropy gradient is fed by the absorption in the gain region of neutrinos diffusing out of the neutrinosphere.



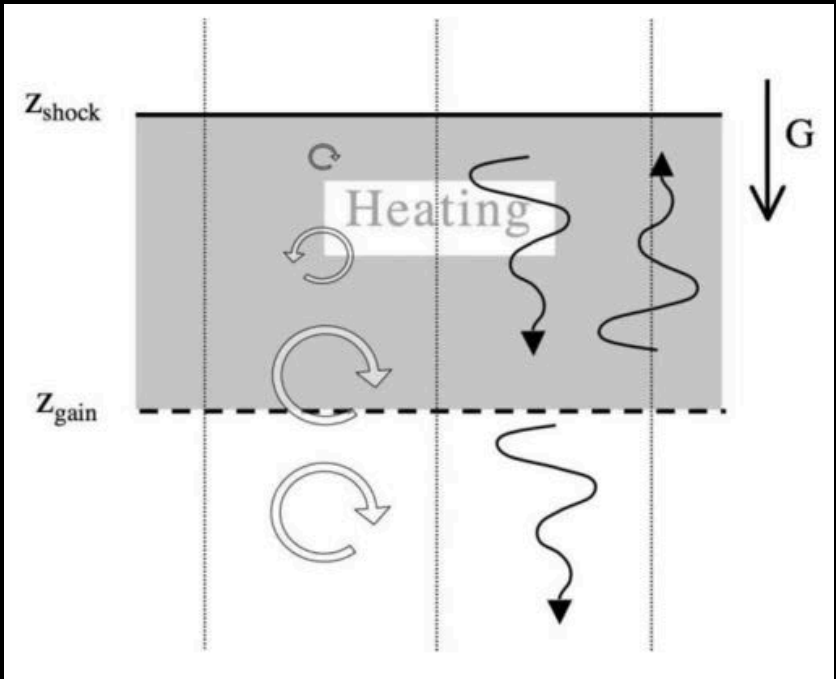
$$\omega_{\rm buoy} \equiv G^{1/2}\left|\frac{\nabla P}{\gamma P} - \frac{\nabla \rho}{\rho}\right|^{1/2} = \left(\frac{\gamma-1}{\gamma} G\nabla S\right)^{1/2},$$
$$\sim \left(\frac{G}{H}\right)^{1/2}.$$

The size of the largest unstable convective cells is comparable to the size of the gain region

$$l \sim \frac{\pi}{2}\frac{R + r_{\rm gain}}{H}$$

Neutrino-driven convection in the gain region

Foglizzo +06

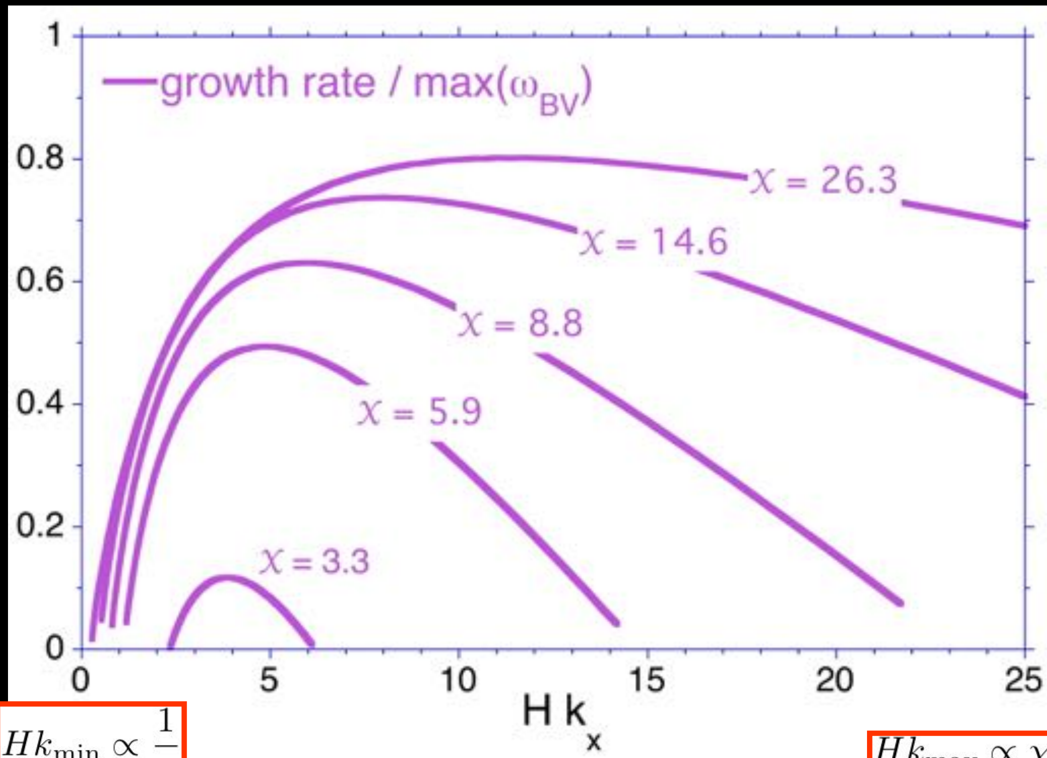


A planar toy model to study the RT instability below a stationary shock

The local timescale of convection must be compared to the timescale of advection through the gain region

$$\frac{H\omega_{\rm buoy}}{v} \sim \left(\frac{\mathcal{G}M}{r_{\rm sh}v_2^2}\right)^{\frac{1}{2}}\left(\frac{H}{r_{\rm sh}}\right)^{\frac{1}{2}}$$

$$\sim 3.1\left(\frac{v_1}{7v_2}\right)\left(\frac{H}{0.4r_{\rm sh}}\right)^{\frac{1}{2}}$$



$$Hk_{\rm min} \propto \frac{1}{\chi}$$

$$Hk_{\rm max} \propto \chi$$

horizontal wavenumber

Despite the negative entropy gradient, the flow is linearly stable if $\chi<\chi_{crit}\sim3$

$$\chi \equiv \int_{\rm gain}^{\rm shock} \omega_{\rm BV}\frac{{\rm d}r}{v_r} \sim \frac{\tau_{\rm adv}}{\tau_{\rm buoy}}$$

Neutrino-driven convection in the gain region

Foglizzo +06

The perturbative calculation requires to write

-the upper boundary conditions on a perturbed shock including photodissociation,

-the lower boundary condition of a outgoing acoustic wave.

$$\frac{\partial}{\partial z}(\rho v) = 0,$$

$$\frac{\partial}{\partial z}\left(\frac{v^2}{2} + \frac{c^2}{\gamma - 1} + Gz\right) = \frac{\mathcal{L}}{\rho v},$$

$$\frac{\partial S}{\partial z} = \frac{\mathcal{L}}{Pv},$$

Stationary flow including non adiabtic heating and cooling

$$\mathcal{L} \equiv \frac{\bar{\mathcal{L}}\rho}{1-\beta}\left[1 - \beta\left(\frac{T}{T_{\rm sh}}\right)^6\right],$$

$$\bar{\mathcal{L}} \sim 2.2\left(\frac{r_{\rm sh}}{150\ \rm km}\right)^{-2} \times 10^{20}\ \rm ergs\ g^{-1}\ s^{-1},$$

$$\frac{v_1^2}{2} + \frac{c_1^2}{\gamma - 1} = \frac{v_{\rm sh}^2}{2} + \frac{c_{\rm sh}^2}{\gamma - 1} + E.$$

$$\frac{E}{v_1^2/2 + c_1^2/(\gamma - 1)} = \left(1 - \frac{\mathcal{M}_{\rm sh}^2}{\mathcal{M}_{\rm ad}^2}\right)\left(1 - \frac{\mathcal{M}_{\rm sh}^2}{\mathcal{M}_1^2}\right),$$

$$\rho_1 v_1 = \rho_{\rm sh} v_{\rm sh},$$

$$\rho_1 \frac{c_1^2}{\gamma} + \rho_1 v_1^2 = \rho_{\rm sh}\frac{c_{\rm sh}^2}{\gamma} + \rho_{\rm sh} v_{\rm sh}^2.$$

$$\frac{v_{\rm sh}}{v_1} = \frac{\mathcal{M}_{\rm sh}^2}{\mathcal{M}_1^2}\frac{1 + \gamma\mathcal{M}_1^2}{1 + \gamma\mathcal{M}_{\rm sh}^2},$$

$$\frac{c_{\rm sh}}{c_1} = \frac{\mathcal{M}_{\rm sh}}{\mathcal{M}_1}\frac{1 + \gamma\mathcal{M}_1^2}{1 + \gamma\mathcal{M}_{\rm sh}^2}.$$

the energy losses can be parametrized by the postshock Mach number $0.1 < M_{sh} < 0.3$

Perturbations of the stationary flow

Foglizzo +06

$$f \equiv v\delta v_z + \frac{2}{\gamma-1}c\delta c,$$
$$h \equiv \frac{\delta v_z}{v} + \frac{\delta\rho}{\rho},$$
$$\delta K \equiv iv k_x \delta w_y + \frac{k_x^2 c^2}{\gamma}\delta S,$$

-f is the perturbation of the energy density

-h is the perturbation of the mass flux

-entropy perturbtions δS are simply advected in an adiabatic flow

-the combination of entropy and vorticity δK is conserved in an adiabatic flow. The validity of is conservation law seems to be limited to the linear regime.

$$\frac{\partial f}{\partial z} = \frac{i\omega v}{1-\mathcal{M}^2}\left\{h - \frac{f}{c^2} + \left[\gamma - 1 + \frac{1}{\mathcal{M}^2}\right]\frac{\delta S}{\gamma}\right\} + \delta\left(\frac{\mathcal{L}}{\rho v}\right),$$

$$\frac{\partial h}{\partial z} = \frac{i\omega}{v(1-\mathcal{M}^2)}\left\{\frac{\mu^2}{c^2}f - \mathcal{M}^2 h - \delta S\right\} + \frac{i\delta K}{\omega v},$$
$$\frac{\partial \delta S}{\partial z} = \frac{i\omega}{v}\delta S + \delta\left(\frac{\mathcal{L}}{pv}\right),$$
$$\frac{\partial \delta K}{\partial z} = \frac{i\omega}{v}\delta K + k_x^2\delta\left(\frac{\mathcal{L}}{\rho v}\right),$$

Perturbations of the stationary flow

Foglizzo +06

the conservation of mass flux, momentum flux and energy flux are written across the shock whose poition is displaced by $\Delta\zeta$ with a velocity Δv

$$\rho_1(v_1 - \Delta v) = (\rho_{\rm sh} + \delta\rho_{\rm sh})(v_{\rm sh} + \delta v_{\rm sh} - \Delta v),$$

$$\rho_1(v_1 - \Delta v)^2 + \rho_1 \frac{c_1^2}{\gamma} = (\rho_{\rm sh} + \delta\rho_{\rm sh})(v_{\rm sh} + \delta v_{\rm sh} - \Delta v)^2 + (\rho_{\rm sh} + \delta\rho_{\rm sh})\frac{(c_{\rm sh} + \delta c_{\rm sh})^2}{\gamma},$$

$$\frac{(v_1 - \Delta v)^2}{2} + \frac{c_1^2}{\gamma - 1} = \frac{(v_{\rm sh} + \delta v_{\rm sh} - \Delta v)^2}{2} + \frac{(c_{\rm sh} + \delta c_{\rm sh})^2}{\gamma - 1} + E,$$

$$\rho_1 v_1 h_{\rm sh} - (\rho_{\rm sh} - \rho_1)\Delta v = \Delta\zeta\left[\frac{\partial}{\partial z}(\rho v)_1 - \frac{\partial}{\partial z}(\rho v)_{\rm sh}\right],$$

$$v_{\rm sh}^2\delta\rho_{\rm sh} + 2\rho_{\rm sh}v_{\rm sh}\delta v_{\rm sh} + \frac{2}{\gamma}\rho_{\rm sh}c_{\rm sh}\delta c_{\rm sh} + \delta\rho_{\rm sh}\frac{c_{\rm sh}^2}{\gamma} = \Delta\zeta\left[\frac{\partial}{\partial z}\left(\rho v^2 + \rho\frac{c^2}{\gamma}\right)_1 - \frac{\partial}{\partial z}\left(\rho v^2 + \rho\frac{c^2}{\gamma}\right)_{\rm sh}\right],$$

$$f_{\rm sh} - (v_{\rm sh} - v_1)\Delta v = \Delta\zeta\left[\frac{\partial}{\partial z}\left(\frac{v^2}{2} + \frac{c^2}{\gamma - 1}\right)_1 - \frac{\partial}{\partial z}\left(\frac{v^2}{2} + \frac{c^2}{\gamma - 1}\right)_{\rm sh}\right].$$

$$f_{\rm sh} = \Delta v(v_{\rm sh} - v_1) - \Delta\zeta\frac{c_{\rm sh}^2}{\gamma}\nabla S_{\rm sh},$$

$$h_{\rm sh} = \frac{\Delta v}{v_{\rm sh}}\left(1 - \frac{v_{\rm sh}}{v_1}\right),$$

$$\frac{\delta S_{\rm sh}}{\gamma} = -\Delta\zeta\left[\frac{\nabla S_{\rm sh}}{\gamma} + \left(1 - \frac{v_{\rm sh}}{v_1}\right)\frac{G}{c_{\rm sh}^2}\right] - \frac{v_1\Delta v}{c_{\rm sh}^2}\left(1 - \frac{v_{\rm sh}}{v_1}\right)^2,$$

$$\delta K_{\rm sh} = -k_x^2\Delta\zeta\frac{c_{\rm sh}^2}{\gamma}\nabla S_{\rm sh},$$

If the flow is adiabatic, the quantity δK is uniformly zero:

-the perturbed shock geneates $\delta K_{sh}=0$

-δK is conserved when advected

In this case, the vorticity is directly related to the perturbed entropy through

$$\delta w_y = \frac{ik_x c^2}{\gamma v}\delta S$$

Perturbations of the stationary flow

Foglizzo +06

the outgoing wave condition at the lower boundary requires the decomposition of the perturbation into four components:

- advected entropy f_S h_S associated to δS ($\delta K=0$, $\delta P=0$)
- advected entropy/vorticity f_K h_K associated to δK ($\delta S=0$, $\delta P=0$)
- downward propagating acoustic waves f_+ ($\delta S=0$, $\delta K=0$)
- upward propagating acoustic waves f_- ($\delta S=0$, $\delta K=0$)

$$f = f_S + f_K + f_+ + f_-,$$
$$h = h_S + h_K + h_+ + h_-.$$

The outgoing acoustic condition at the lower boundary is simply $f_-=0$, $h_-=0$

In a non uniform flow the identification of the acoustic waves relies on the WKB approximation.

Using the WKB approximation to identify acoustic waves

In the adiabatic approximation, the differential system can be reduced to a single differential equation of second order. A change of variable allows for a compact formulation:

$$\frac{dX}{dz} \equiv \frac{v}{1-\mathcal{M}^2},$$

$$\mu^2 \equiv 1 - \frac{k^2c^2}{\omega^2}(1-\mathcal{M}^2)$$

$$\left\{\frac{\partial^2}{\partial X^2} + \left(\frac{\omega\mu}{vc}\right)^2\right\}\left(\frac{f}{i\omega}\mathrm{e}^{\int_{\mathrm{sh}} \frac{i\omega}{c^2}\mathrm{d}X}\right) = \frac{\delta S_{\mathrm{sh}}}{\gamma}\frac{\partial}{\partial X}\left(\frac{1-\mathcal{M}^2}{\mathcal{M}^2}\mathrm{e}^{\int_{\mathrm{sh}} \frac{i\omega}{v^2}\mathrm{d}X}\right)$$

If the frequency is high enough, the acoustic wave adapts adiabatically to the radial gradients of the flow according to the WKB approximation of the homogeneous differential equation

$$f_{\mathrm{wkb}}^{\pm} \equiv f_{\mathrm{sh}}^{\pm}\frac{c}{c_{\mathrm{sh}}}\left(\frac{\mu_{\mathrm{sh}}\mathcal{M}}{\mu\mathcal{M}_{\mathrm{sh}}}\right)^{\frac{1}{2}}\mathrm{e}^{\int_{\mathrm{sh}} \frac{i\omega}{c}\frac{\mathcal{M}\mp\mu}{1-\mathcal{M}^2}\mathrm{d}z}$$

$$h_{\pm} \sim \pm\frac{\mu}{\mathcal{M}}\frac{f_{\pm}}{c^2}$$

The WKB criterion specifies the domain of validity of this approximation

$$\frac{\omega\mu}{c} \gg (1-\mathcal{M}^2)\frac{\partial\log}{\partial z}\left(\frac{\mu^2}{v^2c^2}\right)$$

reminder about the WKB approximation

The WKB approximation is a useful tool to approximate the solution of a 2nd order differential equations, written in the canonical form

$$\left\{\frac{\partial^2}{\partial X^2} + W(X)\right\} f(X) = 0$$

If W(X) were constant, the solution would be a simple combination of two exponential $\exp(\pm i W^{\frac{1}{2}} X)$

The WKB approximation of the solution accounts for the variations of W(X)

$$f_{\text{WKB}}(X) \equiv \frac{1}{W^{\frac{1}{4}}} \exp\left[\pm i \int^X W(X')^{\frac{1}{2}} \mathrm{d}X'\right]$$

The domain of validity of this approximation is set by the WKB condition $\frac{\partial W}{\partial X} \ll W^{\frac{3}{2}}$

It requires a slow enough spatial variation of the wavevector compared to the wavelength

This approximation is significantly more elaborate than a ray tracing approximation.
The factor $W^{-1/4}$ garantees that the energy of the approximate wave is correctly propagated.

The domain where the WKB criterion is not fulfilled correspond to regions of coupling between waves.

For example, the turning point x=0 in the Airy function solution of y"+xy=0

reminder about the general solution to a second order differential equation with a source term

2^{nd} order linear differential equation with a source term S(X)

$$\left\{ \frac{\partial^2}{\partial X^2} + a(X) \frac{\partial}{\partial X} + b(X) \right\} f(X) = S(X)$$

Let f_1, f_2 two independent solutions of the homogeneous equation.

The Wronskien associated to (f_1, f_2) is defined as $\mathcal{W} \equiv f_2 \frac{\partial f_1}{\partial X} - f_1 \frac{\partial f_2}{\partial X}$

it satisfies $\frac{1}{\mathcal{W}} \frac{\partial \mathcal{W}}{\partial X} = -a$

The general solution is
$$f(X) = f_1 \int^X \frac{f_2 S}{\mathcal{W}} \mathrm{d}X' - f_2 \int^X \frac{f_1 S}{\mathcal{W}} \mathrm{d}X'$$

One can easily check its first derivative:
$$\frac{\partial f}{\partial X} = \frac{\partial f_1}{\partial X} \int^X \frac{f_2 S}{\mathcal{W}} \mathrm{d}X' - \frac{\partial f_2}{\partial X} \int^X \frac{f_1 S}{\mathcal{W}} \mathrm{d}X'$$

and its second derivative:
$$\frac{\partial^2 f}{\partial X^2} = \frac{\partial^2 f_1}{\partial X^2} \int^X \frac{f_2 S}{\mathcal{W}} \mathrm{d}X' - \frac{\partial^2 f_2}{\partial X^2} \int^X \frac{f_1 S}{\mathcal{W}} \mathrm{d}X' + S$$

The boundary of the integral are defined by the boundary conditions of the differential system.

$$\chi \equiv \int_{\mathrm{gain}}^{\mathrm{shock}} \omega_{\mathrm{BV}} \frac{\mathrm{d}r}{v_r} \sim \frac{\tau_{\mathrm{adv}}}{\tau_{\mathrm{buoy}}}$$

Convection vs advection in 2D

Test case: a planar subsonic toy model without a shock $\chi_{\mathrm{crit}}=2$

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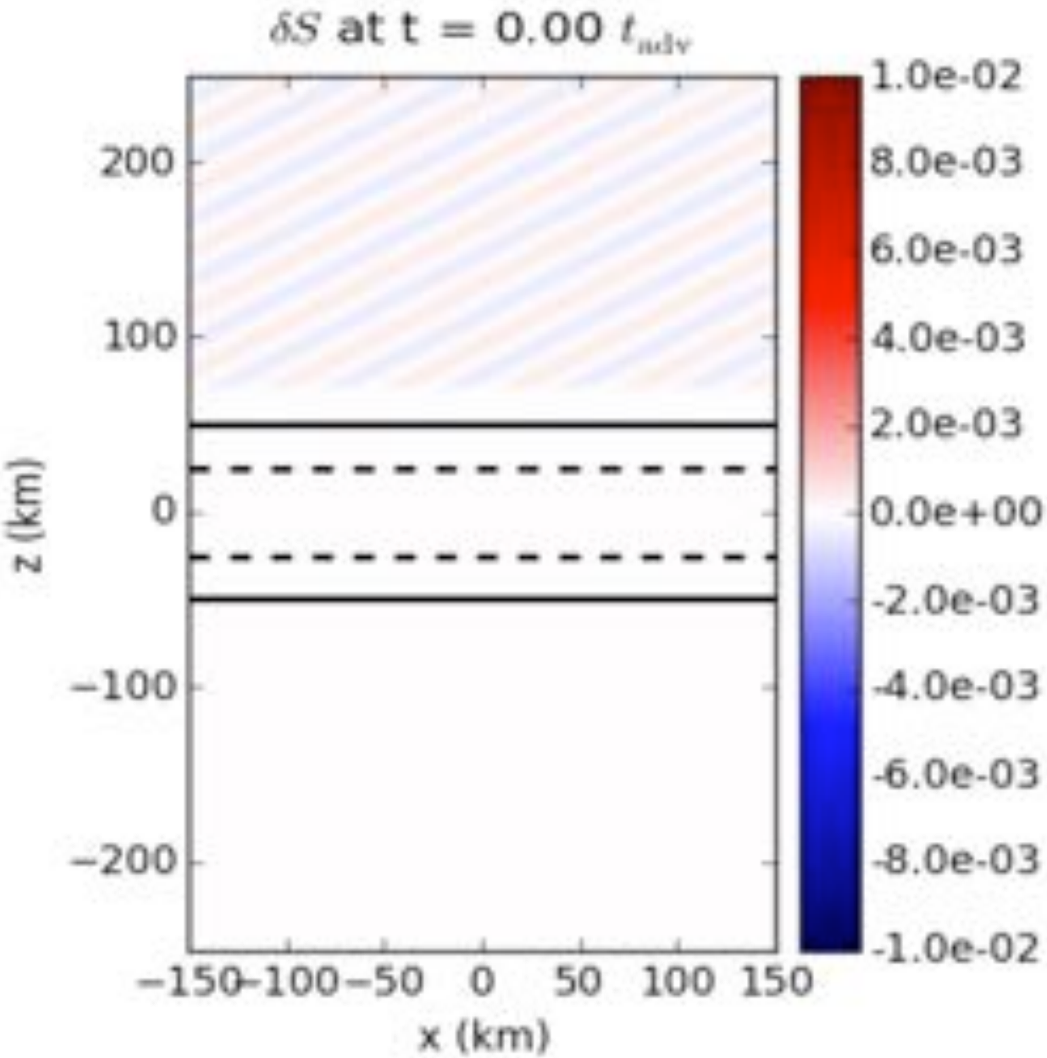
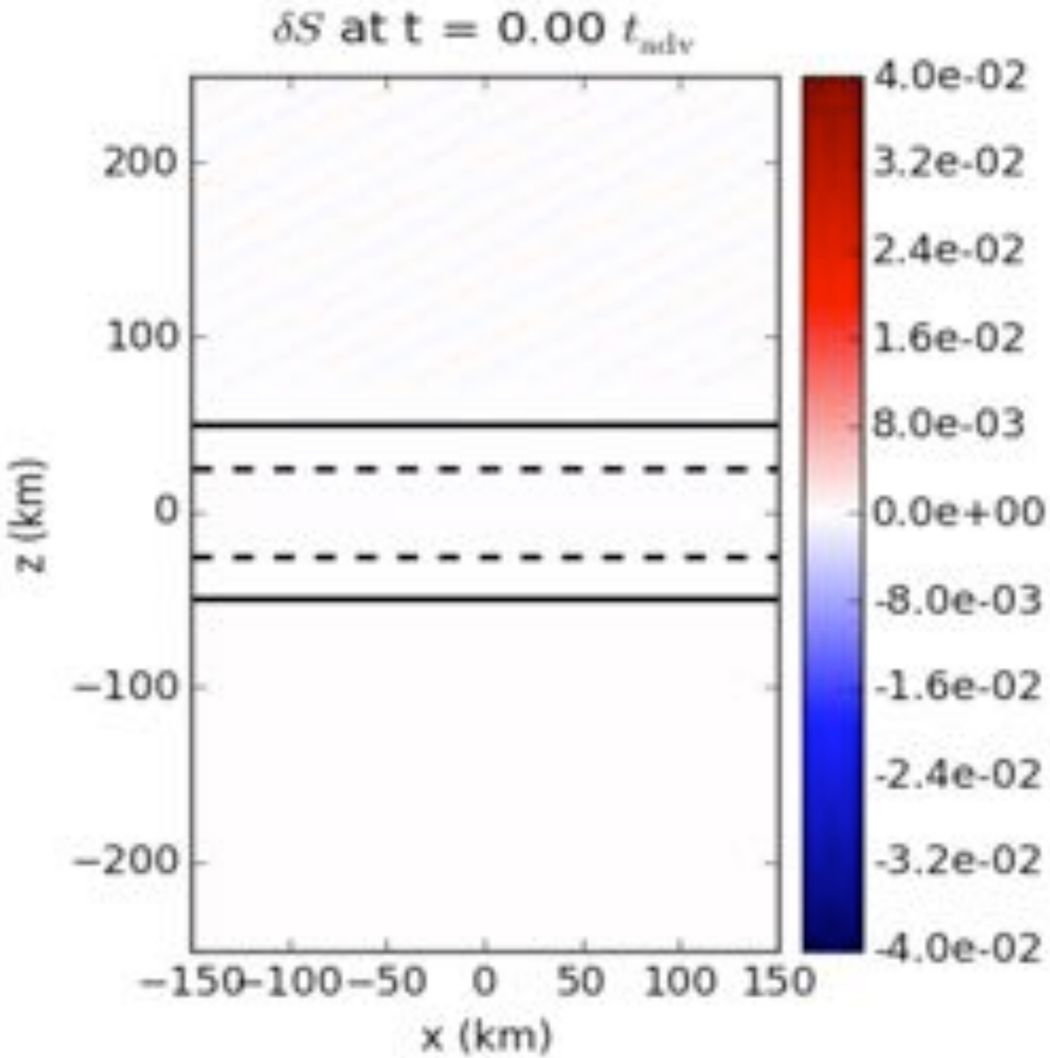
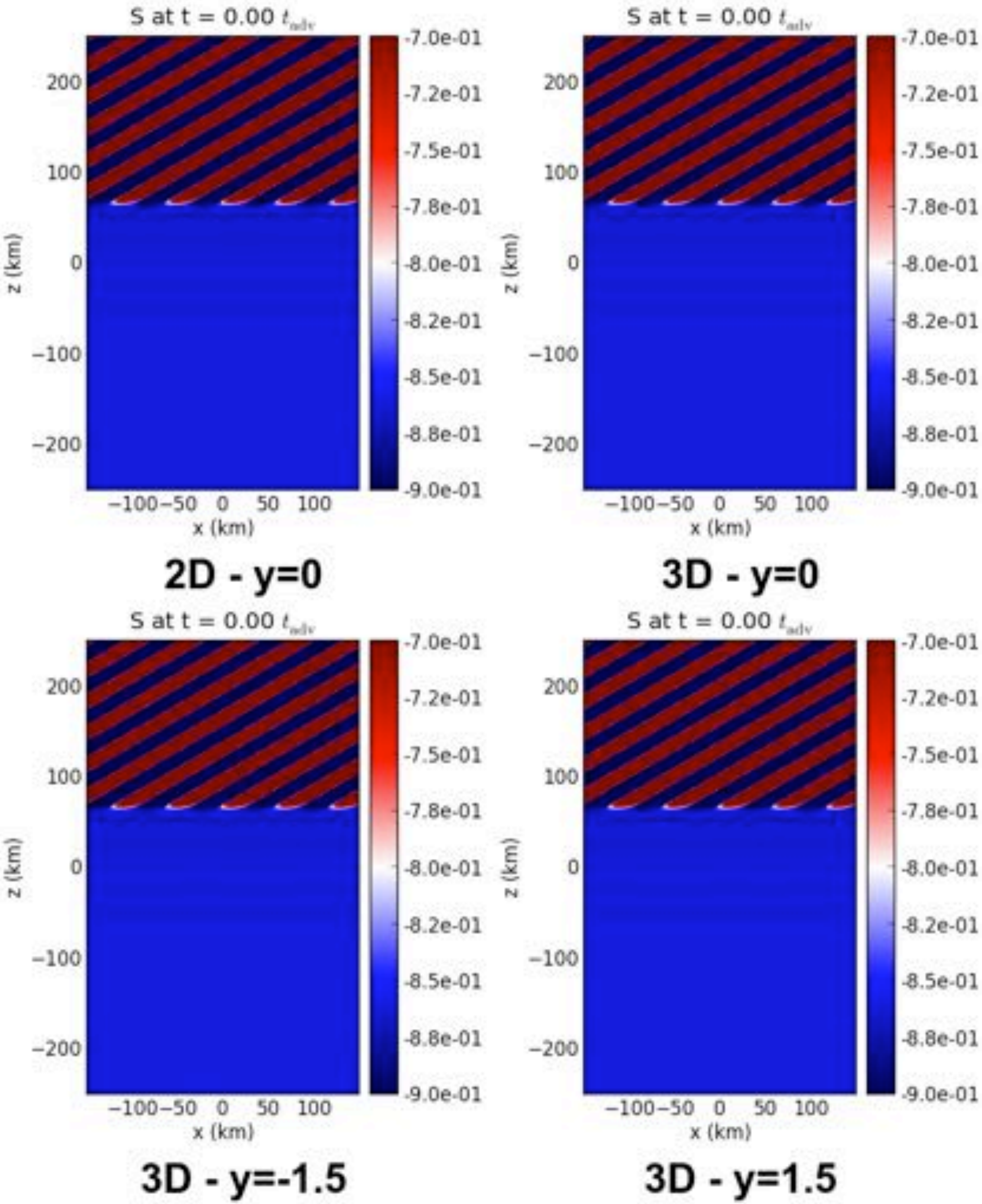
$\chi = 1.5$ $\quad \frac{\delta\rho}{\rho} = 0.01\%$ $\quad \chi = 5.0$







Convection vs advection in 2D/3D

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$$\chi \equiv \int_{\rm gain}^{\rm shock} \omega_{\rm BV} \frac{{\rm d}r}{v_r} \sim \frac{\tau_{\rm adv}}{\tau_{\rm buoy}}$$

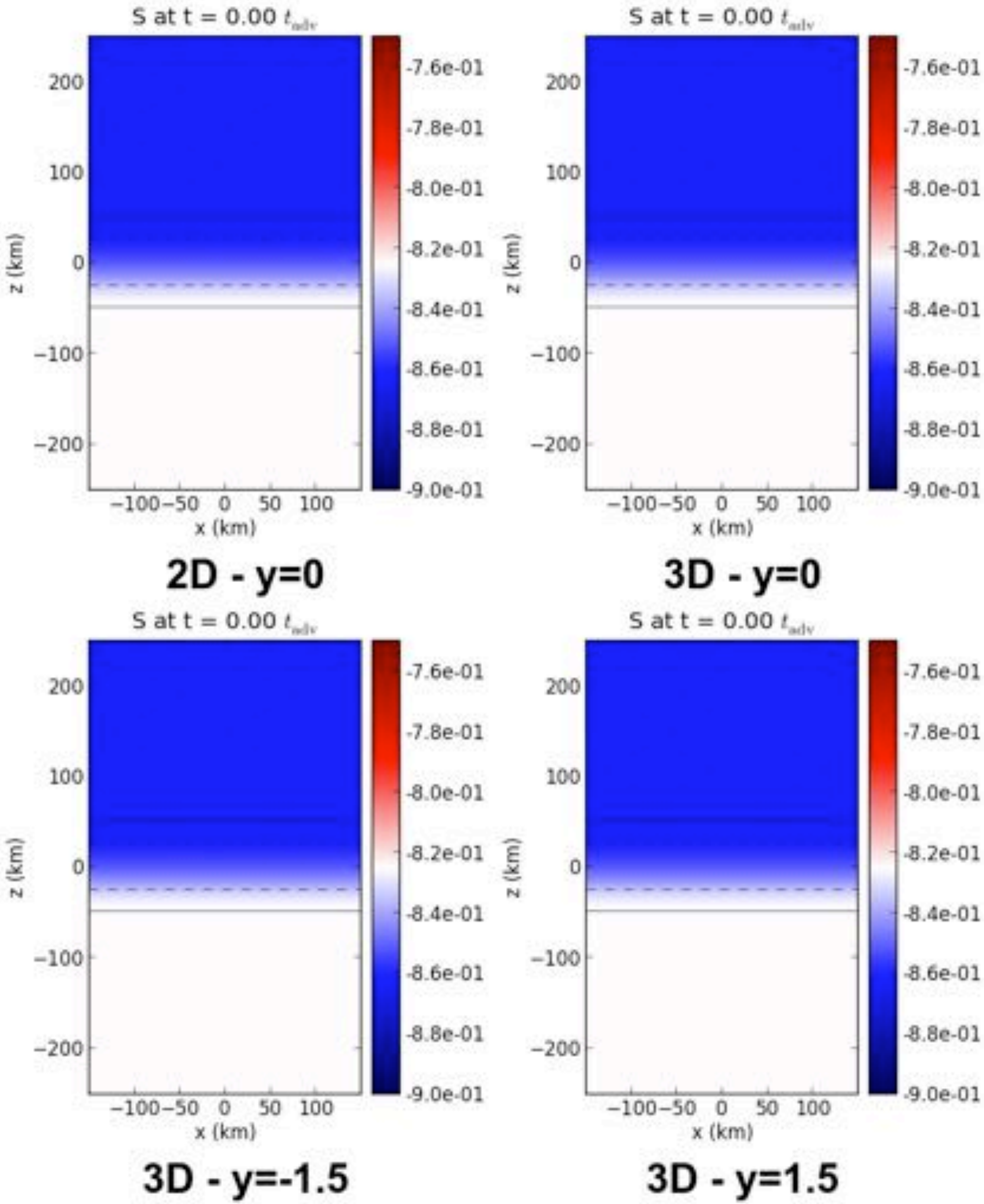
$$\chi = 1.5 < \chi_{\rm crit} = 2$$

$$\frac{\delta \rho}{\rho} = 30\%$$

Density perturbations with a very large amplitude are buoyant but ultimately washed away if $\chi<\chi_{crit}$

Self sustained convective motions last longer if χ is close to the linear stability threshold χ_{crit}

Their evacuation is faster in 2D than in 3D



Convection vs advection in 2D/3D

Kazeroni +17

$$\chi \equiv \int_{\mathrm{gain}}^{\mathrm{shock}} \omega_{\mathrm{BV}} \frac{\mathrm{d}r}{v_r} \sim \frac{\tau_{\mathrm{adv}}}{\tau_{\mathrm{buoy}}}$$

$$\chi = 5 > \chi_{\mathrm{crit}} = 2$$

$$\frac{\delta\rho}{\rho} = 0.1\%$$

Density perturbations with a small amplitude are linearly unstable if $\chi > \chi_{\mathrm{crit}}$

The linear phase of the instability is identical in 2D and 3D

Their non linear saturation is stronger in 3D than in 2D despite the stronger mixing in 3D

→favourable to 3D explosions