SpaceInn School - CEA Saclay Oct. 2015

Magnetic Fields in Celestial Bodies

Numerical Resolution of the Induction Equation in Mean Field Theory:

Cartesian $\alpha - \omega$ dynamo in 2-D

I: Explicit treatment of the temporal scheme

Dr. Allan Sacha Brun

The goal of this "Hands on Computer" project is to derive the induction equation in cartesian geometry and to write a 2-D code using finite differences and an explicit temporal scheme to numerically resolve the system of equation for the magneticfield.

1 Induction Equation of the Magnetic Field

In *cgs* unit system, Maxwell's equations in vacuum read:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho_0, & (1)\\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, & (2)\\
\nabla \cdot \mathbf{B} &= 0, & (3)\\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} & (4)
\end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{J}$ the electric current, ρ_0 the charge density and c the speed of light. The first equation is Gauss' law, the second Faraday's law, the 3rd states that there is no magnetic monopoles and the last is Ampère's law (with the displacement current, term later added by Maxwell)

From Faraday and Omh's laws, the latter linking electric field and current and using the definition of current in th emagnetohydrodynamic (MHD) limit (e.g. plasma motions are slow compared to the speed of light $v << c$), on can derive an equation for the time evolution of $\mathbf{B}$: the induction equation.

Start with Ampère's law, in which we neglect the displacement current:

$$\mathbf{J} = c/4\pi\,(\nabla \times \mathbf{B}),$$

In a stationnary conductor, Ohm's law is simply $\mathbf{J} = \sigma\mathbf{E}$, with σ the electric conductivity. In a conducting fluid, $\mathbf{E}$ must be evaluated in a frame of reference $\mathcal{R}'$

moving at the fluid speed $\mathbf{v}$, hence $\mathbf{E}'$ becomes: $\mathbf{E}' = \mathbf{E} + (\mathbf{v} \times \mathbf{B})/c$, and Ohm's law changes into:

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{\mathbf{c}} \right),$$

Finally, by intrucing these two equations into Faraday's law, one gets:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -c\boldsymbol{\nabla} \times \mathbf{E} = -\boldsymbol{\nabla} \times \left(\frac{c\mathbf{J}}{\sigma} - \mathbf{v} \times \mathbf{B} \right) \\ &= -\boldsymbol{\nabla} \times \left(\frac{c^2}{4\pi\sigma} \boldsymbol{\nabla} \times \mathbf{B} - \mathbf{v} \times \mathbf{B} \right) \end{aligned} \tag{5}$$

or after simple rearranging of the terms:

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \mathbf{B}), \tag{6}$$

with $\eta = c^2/4\pi\sigma$ the magnetic diffusivity. If it is constant in the media, then the induction equation simplifies to:

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \eta \Delta \mathbf{B}. \tag{7}$$

2 Induction Equation in Mean Field Theory

We are mostly interested here to understand how the magnetic field in the Sun organizes itself on large scale and varies on a 11-yr time scale. We believe this is due to the so-called fluid dynamo effect. Assuming scales separation we can write a mean field theory of magnetic field evolution (Moffatt 1978, Krause & Radler 1980). By splitting the flow and field in two parts, one representing the large scale mean field that we wish to follow and study and one representing the fluctuating components with respect to that mean we get:

$$\mathbf{B} = <\mathbf{B}> + \mathbf{b}' \text{ with } <\mathbf{b}'> = 0\,,$$

$$\mathbf{v} = <\mathbf{V}> + \mathbf{v}' \text{ with } <\mathbf{v}'> = 0$$

where $<>$ represent an ensemble average. By introducing this decomposition in the induction equation (6) and by applying the averaging operator, we are getting an equation for $<\mathbf{B}>$:

$$\frac{\partial <\mathbf{B}>}{\partial t} = \boldsymbol{\nabla} \times (<\mathbf{V}> \times <\mathbf{B}> + <\mathcal{E}> - \eta \boldsymbol{\nabla} \times <\mathbf{B}>) \tag{8}$$

where $< \mathcal{E} >=< \mathbf{v}' \times \mathbf{b}' >$ is the electromotrice mean force (emf). This equation is the induction equation for the mean field but it depends on the fluctuating field **b'**, that requires to follow the evolution of **b'** and hence to solve and extra equation. We can obtain this equation for $\mathbf{b}'$, by substracting equation (8) from equation (6), e.g.:

$$\frac{\partial \mathbf{b}'}{\partial t} = \boldsymbol{\nabla} \times (< \mathbf{V} > \times \mathbf{b}' + \mathbf{v}' \times < \mathbf{B} > + \mathbf{G} - \eta \boldsymbol{\nabla} \times \mathbf{b}') \tag{9}$$

where $\mathbf{G} = \mathbf{v}' \times \mathbf{b}' - < \mathbf{v}' \times \mathbf{b}' >$. This equation is composed of many terms implying **u'** and **b'** and requires to have an equation for **u'**. It would be most useful if we could close the system by having only to solve for the large scale field $< \mathbf{B} >$. The simplest approach to reach that goal, is called *first order smoothing approximation* (FOSA). It supposes that $\mathbf{G}$ is negligible. Then from the equation (9), we see that $\mathbf{b}' = \mathcal{F}(< \mathbf{B} >)$, where $\mathcal{F}$ is a general function depending linearly on $< \mathbf{B} >$, and likewise for $< \mathcal{E} >= \mathcal{G}(< \mathbf{B} >)$. By applying a Taylor expansion about $< \mathbf{B} >$ for $< \mathcal{E} >$, we get:

$$< \mathcal{E} >_i = \alpha_{ij} < B_j > + \beta_{ijk} \frac{\partial < B_j >}{\partial x_k} + \dots .$$

We see that $< \mathcal{E} >$ is proportional to the mean field times a pseudo-tensor α_{ij} that can be associated to the α-effect of helicoidal turbulence (Krause & Radler 1980), and an effective (turbulent) diffusivity.We finally get the following mean field equation for $< \mathbf{B} >$:

$$\frac{\partial < \mathbf{B} >}{\partial t} = \boldsymbol{\nabla} \times (< \mathbf{V} > \times < \mathbf{B} > + \alpha < \mathbf{B} > - (\eta + \beta) \boldsymbol{\nabla} \times < \mathbf{B} >) \tag{10}$$

assuming an isotropic and homogeneous case for the coefficients $\alpha_{ij} = \alpha \delta_{ij}$ and $\beta_{ijk} = \beta \epsilon_{ijk}$. First term corresponds to the advection and shearing of $< \mathbf{B} >$ by large scale motions (meridional circulation and ω effect), the 2nd term to α-effect and the last one to magnetic diffusion enhanced by β (often luch larger than η such as $\beta + \eta \sim \beta$). Then depending on which term dominate for the regeneration of the toroidal field either α or ω, we name the dynamo[1] α^2, $\alpha - \omega$ ou $\alpha^2 - \omega$.

3 A special case: $\alpha - \omega$ solar dynamo

From equation 11, in the axisymmetric case ($\partial / \partial z = 0$), we will derive 2 coupled partial derivative equations that we will solve by writing a numerical program using finite differences. Let's start from a 2-D Cartesian domain (x,y) $\in [-1,1]$

1. note that there is no such thing as an ω^2 dynamo

$$\frac{\partial < \mathbf{B} >}{\partial t} = \mathbf{\nabla} \times (< \mathbf{V} > \times < \mathbf{B} > + \alpha < \mathbf{B} > - \beta \mathbf{\nabla} \times < \mathbf{B} >) \tag{11}$$

To treat the case of an $\alpha - \omega$ dynamo (we will thus neglect the α in the equation for the toroidal field B_z), we chose the following expression for the α and ω effects:

$$< \mathbf{V}(x,y) >= V_0 x \sin(\pi(y+1)/2)\hat{\mathbf{e}}_{\boldsymbol{z}},$$

$$\alpha(x,y) = \alpha_0 \cos(\pi(y+1)/2),$$

with V_0 (always > 0) and α_0 to be determine such as to find the dynamo threshold and the sense of dynamo wave propagation.

As clearly seen in the expression for $\mathbf{V}$ we further neglect the meridional circulation (e.g. $V_x = V_y = 0$), such that the only large scale flow is a differential rotation along $\hat{\mathbf{e}}_{\boldsymbol{z}}$.

In order to solve 2 scalar equations instead of 3 (one by magnetic field components), we use a poloidal-toroidal decomposition of equation 11:

A) Knowing that $< \mathbf{B}(x,y) >= \mathbf{\nabla}\times(A(x,y)\hat{\mathbf{e}}_{\boldsymbol{z}}) + B_z(x,y)\hat{\mathbf{e}}_{\boldsymbol{z}}$, derive the evolution equation in cartesian coordinates (x,y) for the toroidal field $B_z(x,y)$ and potential vector $A(x,y)$[1]. For simplicity we will consider a constant effective diffusivity β and neglect the α-term for B_z:

B) Using finite difference approximation of derivatives, write the algorithm for 1st and 2nd order derivates along x and y (we will consider a regular grid $x_{i+1} = \Delta x + x_i$ and $\Delta x = 2/N_x$ with N_x the number of mesh points in x direction, likewise for y).

C) From an explicit temporal scheme (such as euler or runge-kutta4) write the discretize equation for $A_{i,j}^{n+1}$ and $B_{z\,i,j}^{\,n+1}$ which will be solved by the numerical program where n is a temporal index and (i,j) the spatial index. Be aware than in explicit schemes the maximum time step must satisfy the Courant-Fredrich-Lefy (CFL) condition, e.g. $\Delta t \leq Min(\Delta x,\Delta y)/V_{max})$ to remain stable. Usually we take 50% of that maximal value.

1. hint:note that for A, the $\mathbf{\nabla}\times$ operator can be factored out

D) Write subroutines that compute 1st and 2nd order derivatives in x_i and y_j.

E) Write the main program and temporal loop (on index $n+1$, e.g. t^{n+1}) by resolving the equations in (x_i,y_j) space. For the time step Δt do consider the various time scales in the system (viscous, magnetic, Alfvèn, etc....). Then think of a simple initialization of the field $A(x_i,y_j)$ and $B_z(x_i,y_j)$ at time $t^0 = 0$ and of the key parameters of the problem. We will consider simple boundary conditions by setting $A(x_{min},y_j) = A(x_{max},y_j) = A(x_i,y_{min}) = A(x_i,y_{max}) = 0$ (likewise for B_z) for every t^n.

F) Output files with $A(x_i,y_j,t^n)$ et $B_z(x_i,y_j,t^n)$ in order to visualize them:
a) both field at a given time
b) energy as a function of time, e.g. $A(t)^2$ and $B_z(t)^2$ for a given mesh point (x_i,y_j)
c) Make a Butterfly diagram $B_z(x = cst,y,t)$ showing the dynamo wave

G) Try several cases, by changing the amplitude and sign of α_0 and the amplitude of U_0 and comment the different observed behavior.