ONDES GRAVITATIONNELLES

- propagation des O.G. & états de polarisation
- effet d'une O.G. sur la matière
- génération des O.G.
- formules du quadrupole d'Einstein
- pulsar binaire et binaires compactes spiralantes

Gravitational waves from isolated systems

$$h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$$

small perturbation around Minkowski metric

$$\boxed{\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}}$$

ordinary flat d'Alembertian $\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$

pseudo stress-energy tensor (actually a Lorentz tensor) of matter and gravitational field in harmonic coordinates

To solve this equation we need to impose boundary conditions at infinity, saying that the source of GW is <u>isolated</u> from other sources in the Universe.

Boundary conditions are imposed at <u>past null infinity</u>

In GR one defines the spatio-temporal infinities

I^+ = future temporal infinity $\begin{pmatrix} t \longrightarrow +\infty \\ r = \text{const} \end{pmatrix}$

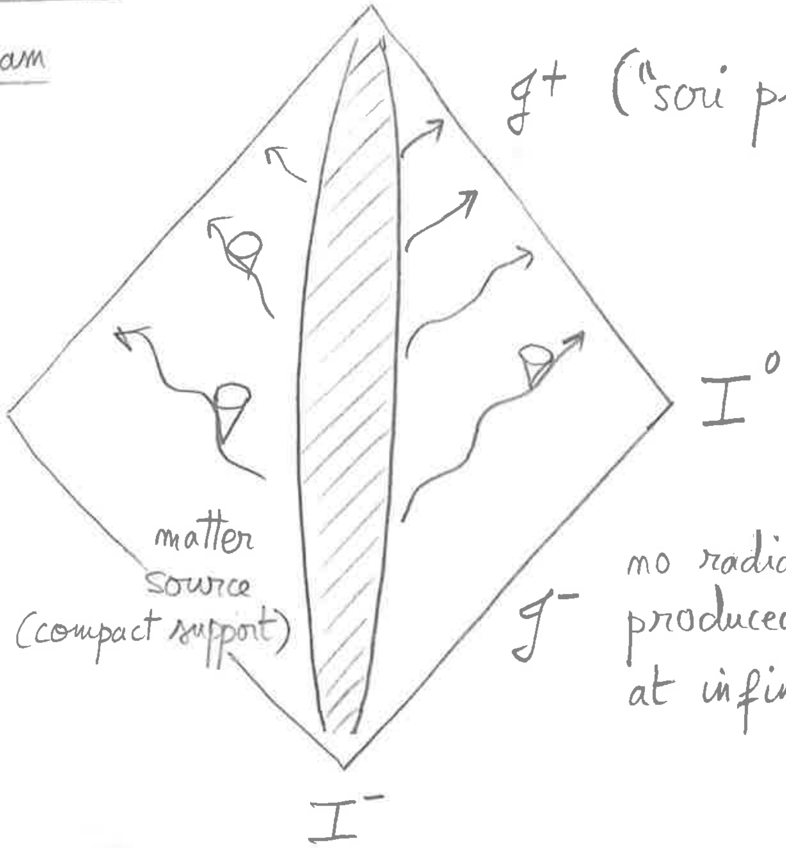
$\mathcal{J}^+$ = future null infinity $\begin{pmatrix} r \longrightarrow +\infty \\ t - r/c = \text{const} \end{pmatrix}$

I^0 = spatial infinity $\begin{pmatrix} r \longrightarrow \infty \\ t = \text{const} \end{pmatrix}$

$\mathcal{J}^-$ = past null infinity $\begin{pmatrix} r \longrightarrow +\infty \\ t + r/c = \text{const} \end{pmatrix}$

I^- = past temporal infinity $\begin{pmatrix} t \longrightarrow -\infty \\ r = \text{const} \end{pmatrix}$

Carter- Penrose diagram

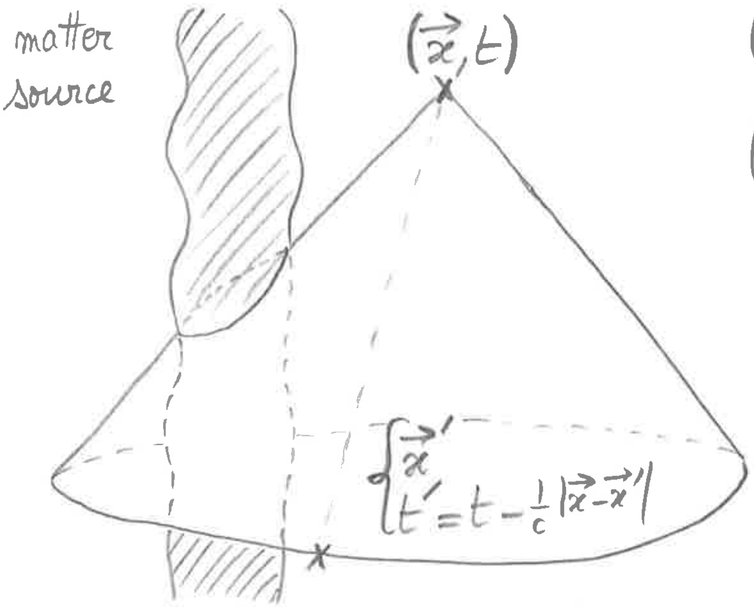


no radiation is produced by sources located at infinity

Kirchhoff's formula for the homogeneous sol. of

$$\Box\, h_{Hom} = 0$$

$$\boxed{h_{Hom}(\vec{x},t) = \lim_{|\vec{x}'|\longrightarrow\infty} \iint \frac{d\Omega'}{4\pi}\left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(r\, h_{Hom}\right)\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}$$



$(\vec{x},t)$ = field point

$(\vec{x}',t')$ = source point

No-incoming rad. cond. is

$$\boxed{\lim_{\mathcal{J}^-} \left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right)(r\, h^{\mu\nu}) = 0}$$

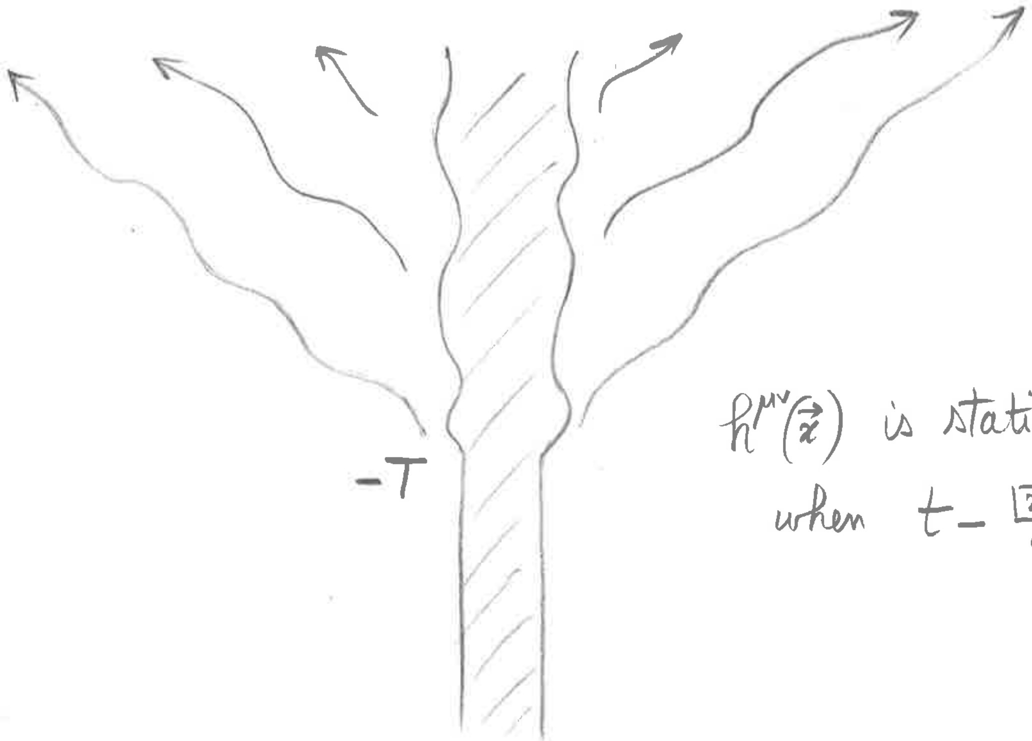
This excludes advanced waves $r\, h_{adv} \sim f(t + r/c)$ at $\mathcal{J}^-$

Einstein field eqs. can be solved (in an iterative way) by means of standard retarded integral in 3+1 dimensions

$$\boxed{h^{\mu\nu}(\vec{x},t) = -\frac{4G}{c^4} \iiint \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\, \tau^{\mu\nu}\left(\vec{x}', t - \frac{1}{c}|\vec{x}-\vec{x}'|\right)}$$

note this is infact an integro-differential equation because $\tau^{\mu\nu}$ depends on $h, \partial h, \partial^2 h$

<u>Stationarity in the past</u> (simple way to implement the no-incoming rad. condition)



$h^{\mu\nu}(\vec{x})$ is stationary (ind. of t)

when $t - \frac{|\vec{x}|}{c} \lesssim -T$

Linearized GWs in vacuum

$$\begin{cases} \Box h^{\mu\nu} = 0 \\ \partial_\nu h^{\mu\nu} = 0 \end{cases}$$ (we neglect $\mathcal{O}(h^2)$)

Gauge transformation preserving the harmonic cond. $\partial h = 0$

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu} \partial_\rho \xi^\rho$$

where $\Box \xi^\mu = 0$

Fourier decomposition $$\boxed{h^{\mu\nu}(x) = \int d^4x \, H^{\mu\nu}(k) \, e^{i k_\lambda x^\lambda}}$$

Fourier amplitude of monochromatic wave $k_\lambda = \begin{pmatrix} \text{wave} \\ \text{vector} \end{pmatrix}$

$$\boxed{\begin{aligned} & k^2 \equiv \eta_{\mu\nu} k^\mu k^\nu = 0 \\ & k_\nu H^{\mu\nu} = 0 \end{aligned}}$$

Can perform a gauge transf. with any $\xi^\mu(x) = \int d^4x \, \varepsilon^\mu(k) e^{ik\cdot x}$

TT coordinates u^μ four-vector constant (independent of x) and not orthogonal to k_μ (i.e. $u_\mu k^\mu \neq 0$) for instance

u^μ = four velocity of an observer (time-like)

There exists a gauge such that (at once)

$$\boxed{\begin{aligned} & u_\nu H^{\mu\nu} = 0 \\ & H \equiv \eta_{\mu\nu} H^{\mu\nu} = 0 \end{aligned}}$$

$\longleftarrow$ transverse (T) condition

$\longleftarrow$ traceless (T) condition

Proof: perform a gauge transf. in Fourier domain

$$H^{\mu\nu} = H_0^{\mu\nu} + i k^\mu \varepsilon^\nu + i k^\nu \varepsilon^\mu - i \eta^{\mu\nu} k_\rho \varepsilon^\rho$$

Then TT conditions are satisfied with gauge vector

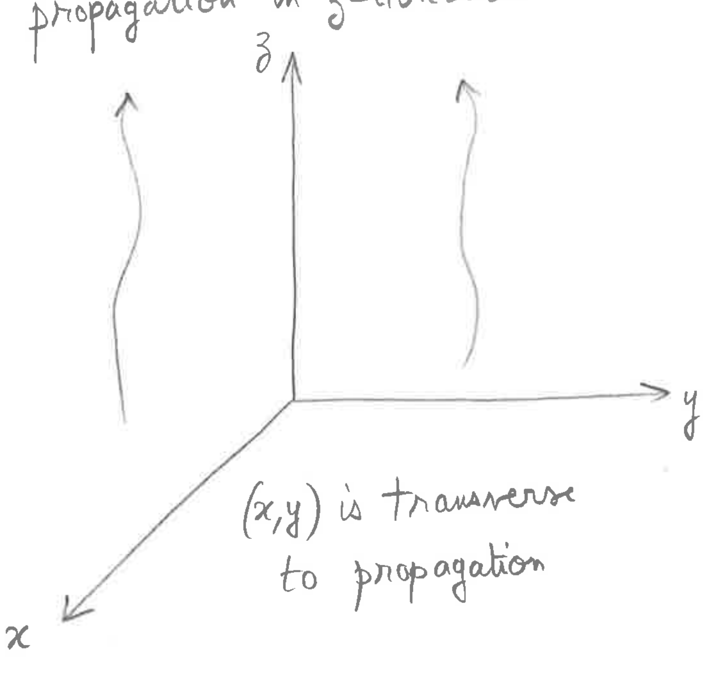
$$\varepsilon^\mu = \frac{i}{(uk)} \left[u_\nu \bar{H}_0^{\mu\nu} - \frac{k^\mu}{2(uk)} u_\rho u_\sigma \bar{H}_0^{\rho\sigma} \right]$$

where $\bar{H}_0^{\mu\nu} = H_0^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} H_0$

$$\boxed{\begin{array}{c} 10 - 4 - (4-1) - 1 = 2 \text{ independent components of } H^{\mu\nu} \\ 2 \text{ polarization states} \end{array}}$$

$u^\mu = (1, \vec{0})$ in rest frame of observer

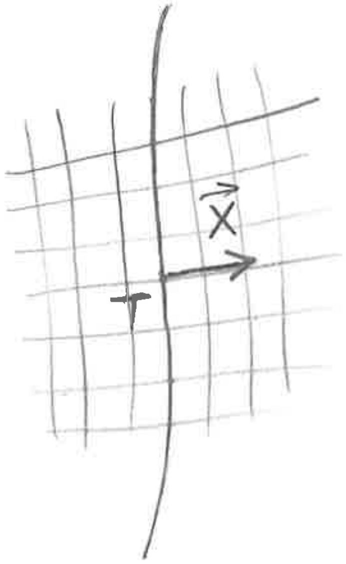
propagation in z-direction



$$h_{\mu\nu}^{TT} = \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t-z/c) & h_\times(t-z/c) & 0 \\ 0 & h_\times(t-z/c) & -h_+(t-z/c) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Action of GWs on matter

central geodesics ($X^i = 0$)



Fermi coordinates (X^i, T) in the neighborhood of central geodesics

$T =$ proper time along central geodesics

$$\boxed{g_{\mu\nu}(\vec{X}, T) = \eta_{\mu\nu} + \underbrace{F_{\mu\nu ij}(T)}_{\text{function of time } T} X^i X^j + \mathcal{O}(|\vec{X}|^3)}$$

Geodesic equ. in vicinity of central geodesic ($|\vec{X}| \ll \lambda^{GW}$)

$$\frac{d^2 X^i}{dT^2} = -c^2 \frac{\partial \Gamma^i_{00}}{\partial X^j}(T, \vec{0}) X^j = -c^2 R^i_{\cdot 0j0}(T, \vec{0}) X^j$$

(to first order in X^i)

Riemann in Fermi coord.
($-c^2 R_{i0j0}$ is a relativistic version of the tidal tensor $\partial_i \partial_j U$)

$$R^i_{\cdot 0j0} = \frac{\partial X^i}{\partial x^\lambda} \frac{\partial x^\mu}{\partial X^0} \cdots \overset{TT}{R}{}^\lambda_{,\mu\nu\rho} \approx \overset{TT}{R}{}^i_{\cdot 0j0} \approx -\frac{1}{2c^2} \frac{\partial^2 \overset{TT}{h}_{ij}}{\partial t^2}$$

Riemann in TT coordinates

$$\boxed{\frac{d^2 X^i}{dT^2} = \frac{1}{2} \frac{\partial^2 \overset{TT}{h}_{ij}}{\partial t^2}(T, \vec{0}) X^j}$$

acceleration in Fermi coord.

wave form in TT coord. evaluated on central geodesic

$$\boxed{X^i(T) = X^i(0) + \frac{1}{2} \overset{TT}{h}_{ij}(T, \vec{0}) X^j(0)}$$

(to first order in h)

position before passage of GW

Quadrupole moment formalism

Matter source is

- isolated ($T^{\mu\nu}$ has a compact support)
- post-Newtonian $\boxed{\varepsilon \approx \frac{v}{c} \ll 1}$
- self-gravitating: internal motion is due to gravitational forces

$$\gamma \sim \frac{v^2}{a} \sim \frac{GM}{a^2}$$

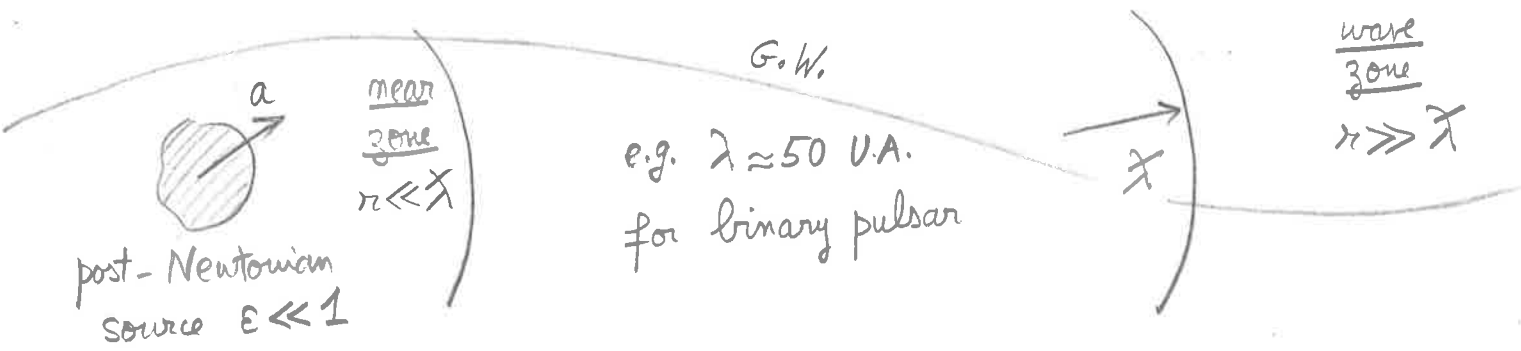
a = size of source
M = its mass

Period of motion $P \sim \dfrac{2\pi a}{v}$

Gravitational wave length $\lambda = cP$

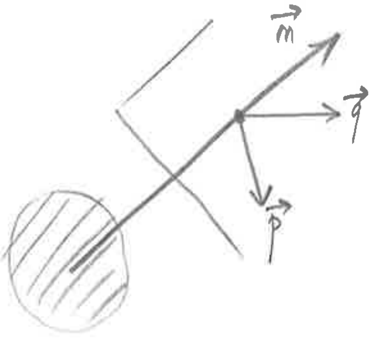
$$\lambda\!\!\!^{-} = \frac{\lambda}{2\pi}$$

$$\boxed{\frac{a}{\lambda\!\!\!^{-}} \sim \frac{v}{c} \approx \varepsilon}$$



The near zone ($r \ll \lambda\!\!\!^{-}$) covers entirely the post-Newtonian source

$$Q_{ij}(t) = \int_{\text{source}} d^3x \, \rho(\vec{x},t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^{\,2}\right)$$



$$\boxed{h_{ij}^{TT} = \frac{2G}{c^4 r} P_{ijkl}(\vec{n})\left\{\ddot{Q}_{kl}\left(t-\frac{r}{c}\right) + O(\varepsilon)\right\} + O\left(\frac{1}{r^2}\right)}$$

TT projection operator $P_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$ where $P_{ij} = \delta_{ij} - n_i n_j$

Polarization states w.r.t. $\vec{p}, \vec{q}$

$$h_+ = \frac{p_i p_j - q_i q_j}{2} h_{ij}^{TT}$$

$$h_\times = \frac{p_i q_j + p_j q_i}{2} h_{ij}^{TT}$$

$\vec{p}, \vec{q}$ polarization vectors

$$\boxed{\mathcal{F}^{GW} \equiv \left(\frac{dE}{dt}\right)^{GW} = \underbrace{\frac{G}{5c^5}}_{\text{order of magnitude of radiation reaction}} \left\{ \dddot{Q}_{ij} \dddot{Q}_{ij} + O(\varepsilon^2) \right\}}$$

Einstein quadrupole formula

order of magnitude of radiation reaction

$O(\varepsilon^5)$ called also 2.5PN

Typically $\quad Q \sim M a^2 \quad \dddot{Q} \sim M a^2 \omega^3 \quad \omega = \frac{2\pi}{P}$

Self-gravitating source $\quad \omega^2 \sim \frac{GM}{a^3}$

$$\boxed{\mathcal{F}^{GW} \sim \left(\frac{c^5}{G}\right)\left(\frac{GM\omega}{c^3}\right)^{10/3}}$$

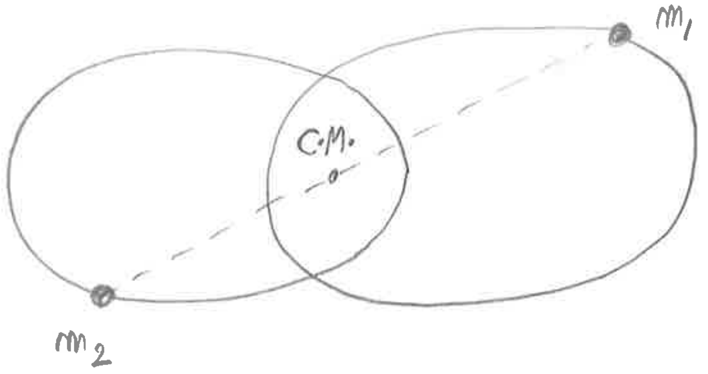
Ultra-relativistic source $\quad v \sim c \quad$ or $\quad \frac{GM\omega}{c^3} \sim 1$

$$\underbrace{\mathcal{F}^{GW}\Big|_{\substack{\text{ultra}\\\text{relativistic}}} \sim \frac{c^5}{G} = 3.63\ 10^{52}\ W}_{\text{value independent of source}}$$

GW has typically the frequency $\quad \omega \sim \frac{c^3}{GM}$

$M \sim 1\ M_\odot$	$\omega \sim 10^3\ Hz$	bandwidth of LIGO/VIRGO
$M \sim 10^6\ M_\odot$	$\omega \sim 10^{-3}\ Hz$	bandwidth of LISA

Secular decrease of orbital period of binary pulsar



Two compact objects (without spin) on a Keplerian ellipse

$a =$ semi-major axis

$e =$ eccentricity

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \text{ such that } 0 < \nu \le \frac{1}{4}}$$

test-mass limit — equal masses

$$\boxed{\langle \mathcal{F}^{GW} \rangle \equiv \frac{1}{P}\int_0^P dt\, \mathcal{F}^{GW}(t) = \frac{32}{5}\frac{c^5}{G}\nu^2\left(\frac{GM}{ac^2}\right)^5 \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}}$$

eccentricity dependent "enhancement" factor $f(e)$ (Peters & Mathews 1964)

Energy balance argument

$$\frac{dE}{dt} = -\langle \mathcal{F}^{GW} \rangle \quad \text{with} \quad E = -\frac{GM\nu^2}{2a}$$

$$GM = \omega^2 a^3$$

$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\left(\frac{2\pi GM}{P}\right)^{5/3}\nu\, f(e) = -2.4\ 10^{-12}\ \mathrm{s/s}}$$

Binary pulsar PSR 1913+16

in agreement with observations (Taylor et al).

Actually the masses m_1 and m_2 (or equivalently M and γ) are measured by the relativistic effects themselves.

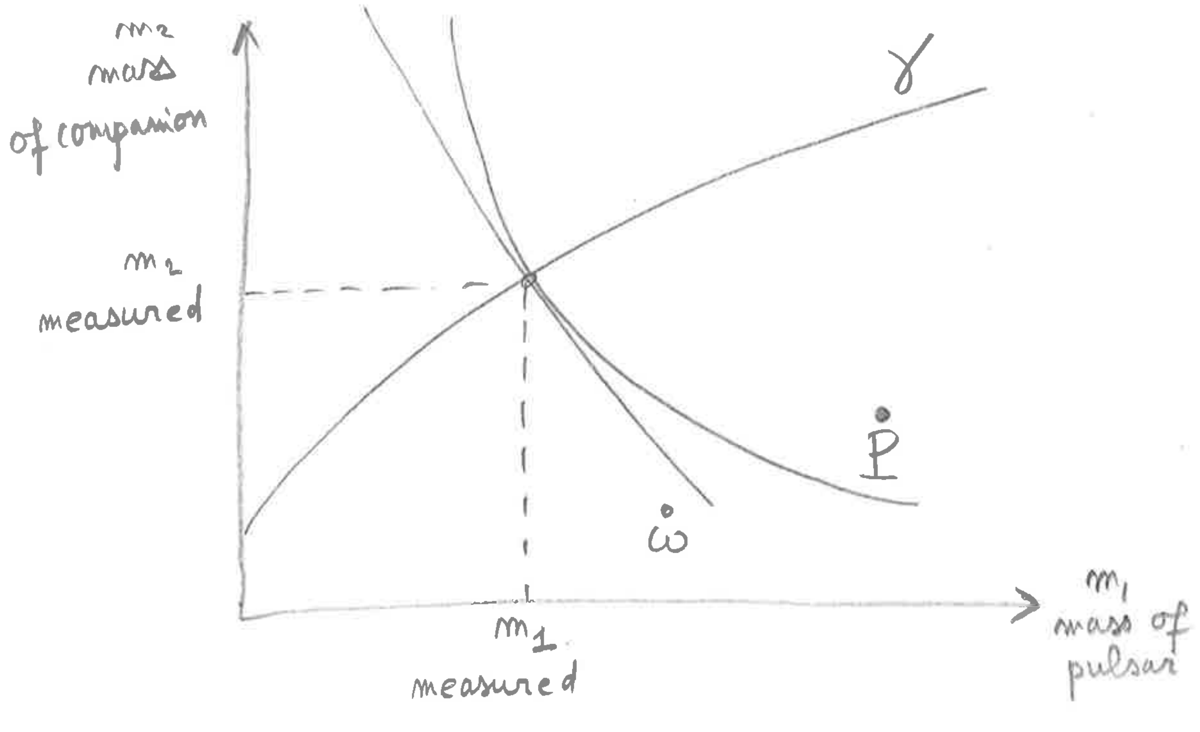
One measures

- $\dot{\omega}$ relativistic advance of periastron (like for Mercury, but here the effect is of order of 4° per year)
- γ combination between the gravitational redshift due to the companion and the second-order Doppler effect
- $\dot{P}$ due to gravitational radiation

In GR

$$\boxed{\begin{aligned} \dot{\omega} &= \frac{3G^{2/3}}{c^2}\left(\frac{P}{2\pi}\right)^{-5/3}\frac{M^{2/3}}{1-e} \\ \gamma &= \frac{G^{2/3}}{c^2}\, e \left(\frac{P}{2\pi}\right)^{1/3} m_2\,(m_1+2m_2)\, M^{-4/3} \end{aligned}}$$

and we draw the mass plane



Inspiralling compact binaries

Evolution of eccentricity $e(t)$

Orbit's energy and angular momentum

$$\boxed{\begin{aligned} \frac{E}{\nu} &= -\frac{GM^2}{2a} \\ \frac{J}{\nu} &= \sqrt{GM^3 a(1-e^2)} \end{aligned}}$$

$$\nu \equiv \frac{\mu}{M}$$

Apply quadrupole formulas for both E and J

$$\dot{E} = -\left\langle \frac{G}{5c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} \right\rangle$$

$$\dot{J}^i = -\left\langle \frac{2G}{5c^5} \varepsilon_{ijk} \ddot{Q}_{jl} \dddot{Q}_{kl} \right\rangle$$

$$\boxed{\frac{e^2}{(1-e^2)^{19/6}} \left(1 + \frac{121}{304} e^2\right)^{\frac{145}{121}} = \left(\frac{\omega}{\omega_0}\right)^{-\frac{19}{9}}}$$

gives $e(t)$ as a function of $\omega(t)$ during the inspiral ($e^2 \sim P^{19/9}$ for small e)
(ω_0 is determined from initial conditions)

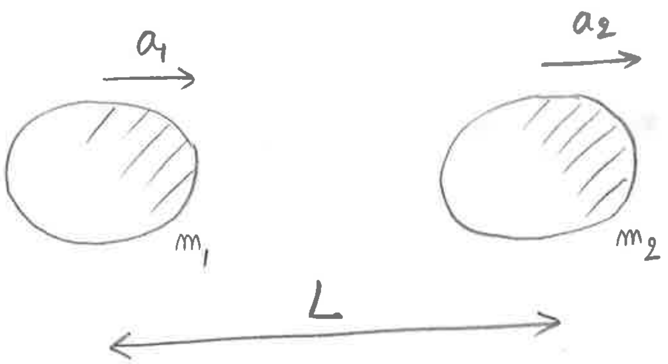
For the binary pulsar $\quad e_{now} = 0.617$
$\qquad\qquad\qquad\qquad\quad \omega_{now} = 2.24 \; 10^{-4}\, Hz$

hence GWs are visible by VIRGO/LIGO when

$$\boxed{\omega \sim 30\ Hz \quad \Rightarrow \quad e \sim 5\ 10^{-6}}$$

eccentricity is negligible in general.

Finite size effects



Look for influence of quadrupole moments Q_1 and Q_2 induced by tidal interactions between non-spinning compact objects

$$\boxed{Q_1 = k_1 m_2 \frac{a_1^5}{L^3} \qquad Q_2 = k_2 m_1 \frac{a_2^5}{L^3}}$$

$k_{1,2}$ = Love numbers (depend on internal structure)

$Q_{1,2}$ scale like L^{-3} because of tidal field $\partial_{ij} U \sim \frac{1}{L^3}$

Introduce the compacity parameters

$$K_1 = \frac{2Gm_1}{a_1 c^2} \qquad K_2 = \frac{2Gm_2}{a_2 c^2}$$

The quadrupoles modify the energy and GW flux and the orbital frequency ω and phase $\phi = \int \omega dt$

$$\dot{E} = -\mathcal{F}^{GW} \quad \Rightarrow \quad \phi = -\int \frac{\omega \, dE}{\mathcal{F}^{GW}}$$

Effect of quadrupoles is

depends on internal structure

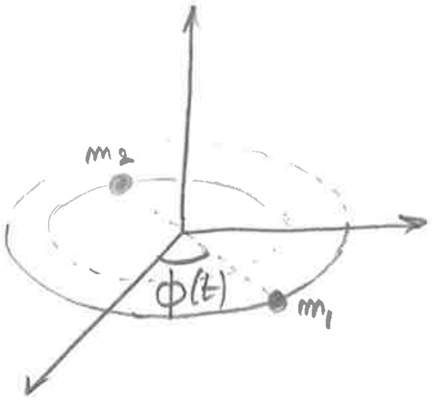
$$\boxed{\phi^{\text{finite-size}} = \underbrace{\phi_0 - \frac{1}{8x^{5/2}}\Big\{ 1}_{\text{point-mass result}} + (\text{const}) \left(\frac{x}{K}\right)^5 \Big\}}$$

$x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3}$ Since $K \sim 1$ for compact objects the formal order of magnitude of the <u>finite-size effect</u> is 5PN (namely $x^5 \sim \frac{1}{c^{10}}$)

Orbital phase evolution $\phi(t)$

(same as for binary pulsar, i.e. based on

$$\frac{dE}{dt} = -\mathcal{F}^{GW})$$



where $\frac{E}{M} = -\frac{c^2}{2}\nu x$

$$\mathcal{F}^{GW} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5$$

$$\boxed{x = \left(\frac{GM\omega}{c^3}\right)^{2/3}} = \text{PN parameter } \mathcal{O}(\varepsilon^2)$$

$$\dot{E} = -\mathcal{F}^{GW} \Rightarrow \dot{x} = \frac{64}{5}\frac{c^3}{G}\frac{\nu}{M}x^5 \Rightarrow x(t) = \left[\frac{256}{5}\frac{c^3}{G}\frac{\nu}{M}(t_c - t)\right]^{-1/4}$$

t_c = instant of coalescence

$$\phi(t) = \int \omega\, dt = \frac{5}{64\nu}\int x^{-7/2}\,dx \Rightarrow \boxed{\phi(t) = \phi_c - \frac{x(t)^{-5/2}}{32\nu}}$$

Number of orbital cycles left till coalescence from time t

$$\mathcal{N} = \frac{\phi_0 - \phi(t)}{\pi} = \frac{1}{32\pi\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} = \underbrace{\mathcal{O}(\varepsilon^{-5})}_{\text{inverse of order of radiation reaction}}$$

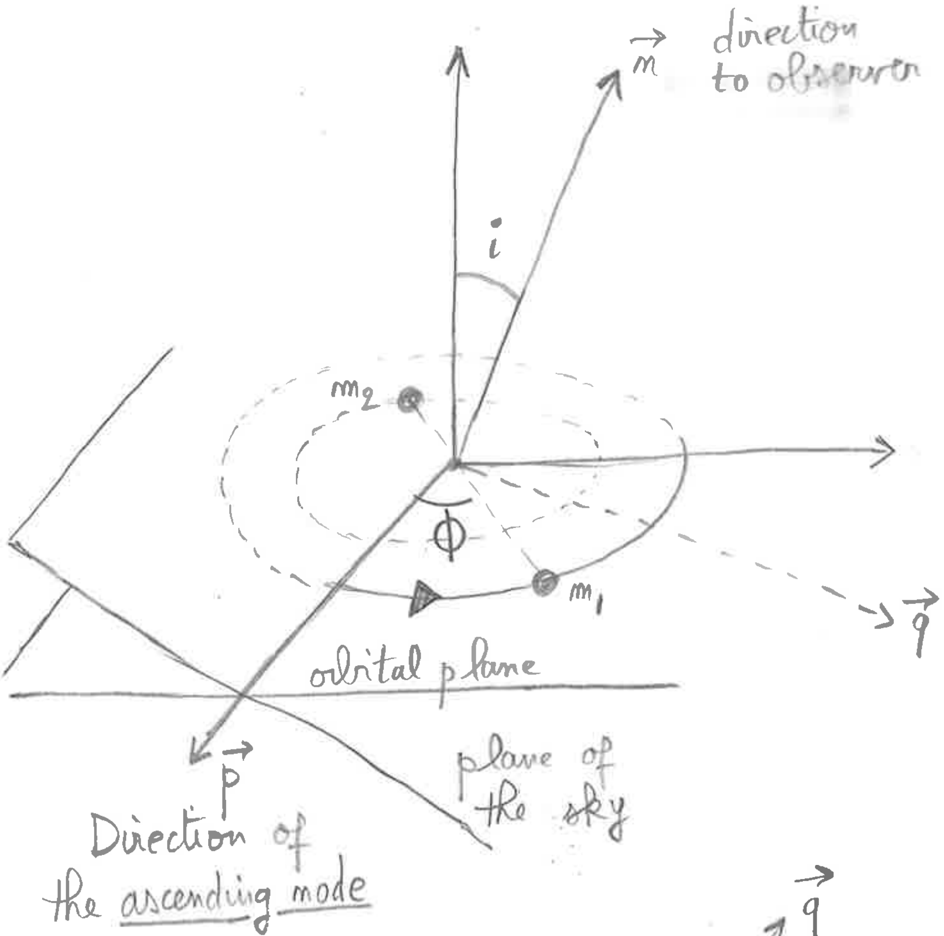
$$\varepsilon^{-5} \sim \left(\frac{c}{v}\right)^5$$

But $\mathcal{N}$ should be monitored in LIGO/VIRGO with precision

$$\delta\mathcal{N} \sim 1$$

so it is evident that PN corrections in the phase will play a crucial role up to at least the 2.5PN order. Detailed analysis show that good templates for inspiralling compact binaries should have 3PN accuracy. Current theoretical prediction is 3.5PN.

Wave form of inspiralling compact binaries (ICBs)

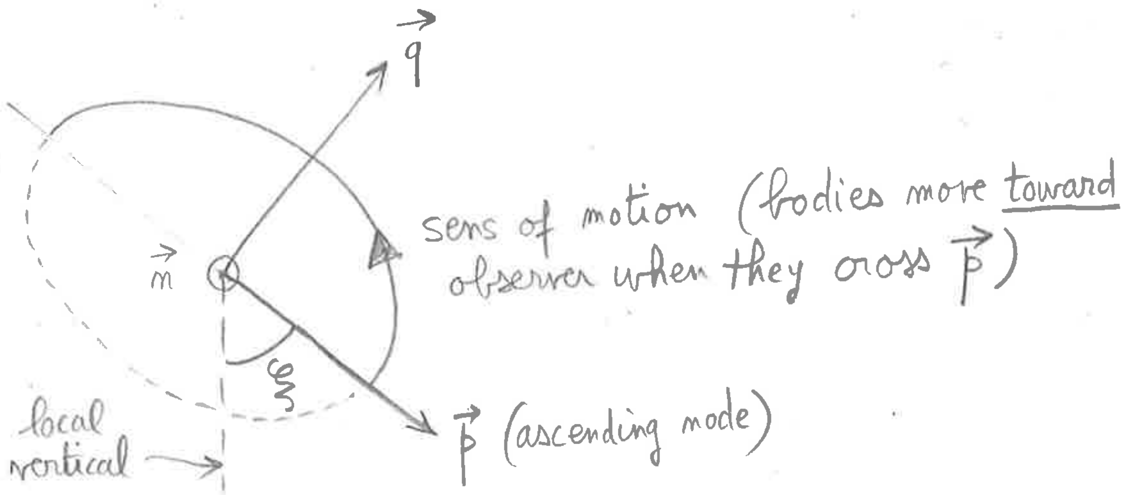


$\vec{p}, \vec{q}$ = polarization vectors (in the plane of sky)

i = inclination angle

$\phi(t)$ = orbital phase

As seen from observer:



ξ = polarization angle (between $\vec{p}$ and local vertical of observer)

Response of detector $\boxed{h \equiv \frac{2\delta L}{L} = F_+ h_+ + F_\times h_\times}$

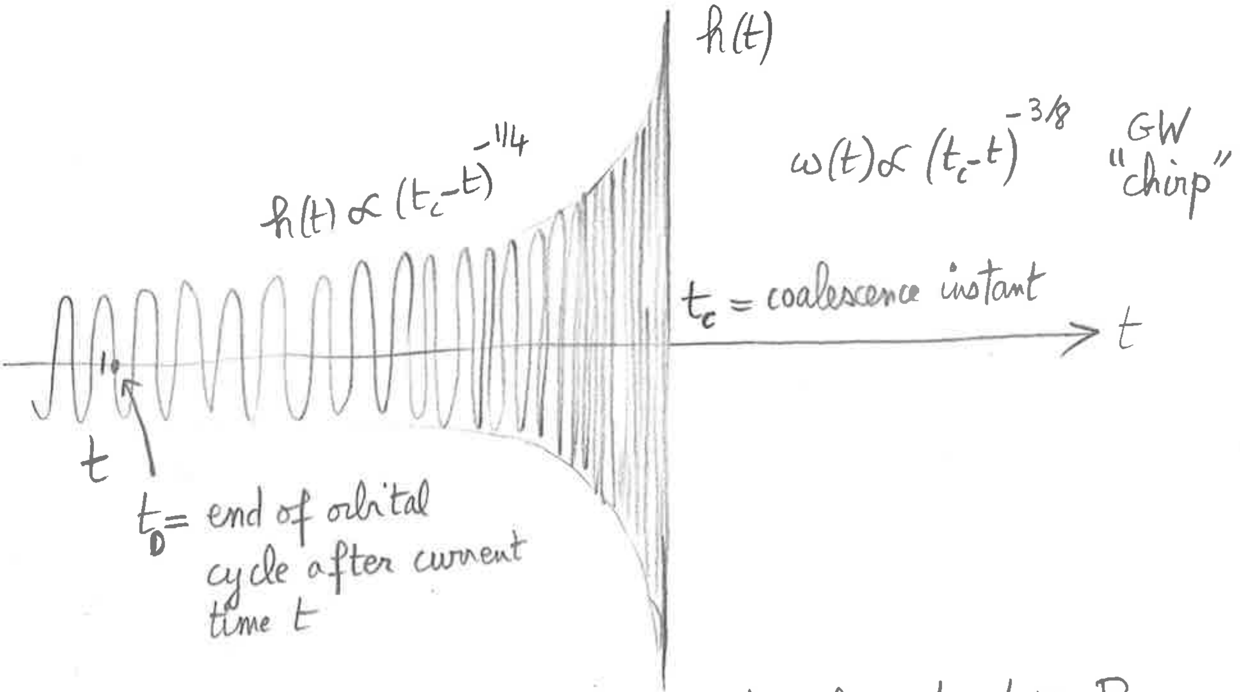
$F_{+,\times}$ = detector's pattern functions depend on $-\vec{m}$ (direction of source) and ξ

In quadrupole approximation

$$\boxed{\begin{aligned} h_+ &= \frac{2G\mu}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{2/3}\left(1+\cos^2 i\right)\cos(2\phi) \\ h_\times &= \underline{\qquad\qquad\qquad}\ (2\cos i)\sin(2\phi) \end{aligned}}$$

D = distance of source
= luminosity distance in cosmology

where
$$\begin{aligned} \phi(t) &= \phi_c - \frac{1}{\nu}\left(\frac{\nu c^3}{5GM}(t_c-t)\right)^{5/8} \\ \omega(t) &= \frac{c^3}{8GM}\left(\frac{\nu c^3}{5GM}(t_c-t)\right)^{-3/8} \end{aligned}$$



Suppose current time t is such that $t_c - t \gg P$
(non-relativistic limit, two bodies are well-separated)

$$t_c - t = (t_c - t_0)\left[1 + \frac{t_0-t}{t_c-t_0}\right] \qquad \text{with} \quad \frac{t_0-t}{t_c-t_0} \ll 1$$

$$\begin{aligned} \phi(t) &\approx \phi_c - \frac{1}{\nu}\left(\frac{\nu c^3}{5GM}(t_c-t_0)\right)^{5/8}\left[1 + \frac{5}{8}\frac{t_0-t}{t_c-t_0} + \cdots\right] \\ &\approx \phi_0 + \frac{5}{8\nu}\left(\frac{\nu c^3}{5GM}\right)^{5/8}(t_c-t_0)^{-3/8}\, t + \cdots \end{aligned}$$

thus $\boxed{\phi(t) \approx \phi_0 + \omega_0 t + \cdots}$ constant orbital motion in the non relativistic limit

Orders of magnitude

$$h \sim \frac{GM\nu}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{2/3}$$

Number of cycles around frequency ω

$$n = \frac{\omega^2}{\dot{\omega}} \sim \frac{1}{\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} = \mathcal{O}(\varepsilon^{-5})$$ inverse of rad. reaction order

Effective amplitude after matched filtering

$$\boxed{h_{eff} = h\sqrt{n} \sim \frac{GM\sqrt{\nu}}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{-1/6}}$$

Example: coalescence of two supermassive BHs in LISA

Characteristic frequency $\omega_c \sim \omega_{I.C.O.}$

innermost circular orbit (defined by the minimum of the energy function)

$$\frac{GM\omega_c}{c^3} \sim 0.1 \quad \Rightarrow \quad f_c \sim 10^4 \, Hz \left(\frac{M_\odot}{M}\right)$$

(from 3PN theory) For LISA $f_c \in [10^{-4} \, Hz, 10^{-1} \, Hz]$

Hence LISA should observe

$$\boxed{10^5 \, M_\odot \lesssim M \lesssim 10^8 \, M_\odot}$$

$$\boxed{h_{eff} \sim 10^{-14} \left(\frac{1\,\mathrm{Gpc}}{D}\right)\left(\frac{\nu}{0.25}\right)^{1/2}\left(\frac{M}{10^7 M_\odot}\right)^{-5/6}\left(\frac{f}{10^{-4}\,\mathrm{Hz}}\right)^{-1/6}}$$

Separation of BHs ($M \sim 10^7 M_\odot$) at entry frequency of LISA

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3} \sim 1 \text{ A.U.}$$

Time left till coalescence

$$T = \frac{5GM}{\nu c^3}\left(\frac{8GM\omega}{c^3}\right)^{-8/3} \sim 10 \text{ days}$$

The signal-to-noise of the supermassive BH coalescence in LISA is enormous

$$\boxed{\frac{S}{N} = \left(\int_{-\infty}^{+\infty} d\omega \, \frac{|\tilde{h}(\omega)|^2}{S_m(\omega)}\right)^{1/2} \sim \frac{h_{eff}}{\sqrt{\omega S_m(\omega)}} \sim 10^4}$$

$S_m(\omega) \sim 10^{-34}\,\mathrm{Hz}^{-1}$ for LISA