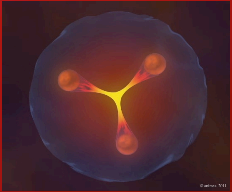
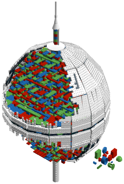
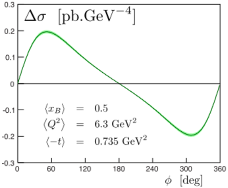
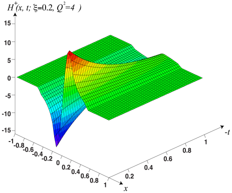
DE LA RECHERCHE À L'INDUSTRIE

The PARTONS framework: 3D hadron structure, from theory to numbers







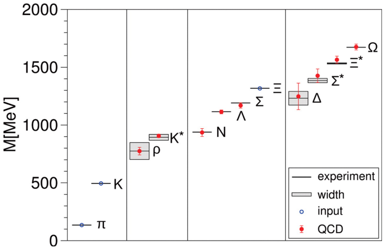


QCD Workshop | Hervé MOUTARDE

May 6$^{\text{th}}$, 2016

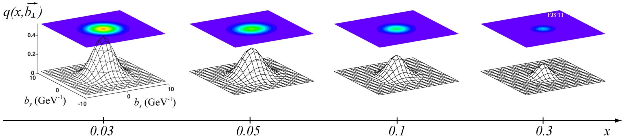
www.cea.fr

- Lattice QCD clearly shows that the mass of hadrons is generated by the **interaction**, not by the quark masses.

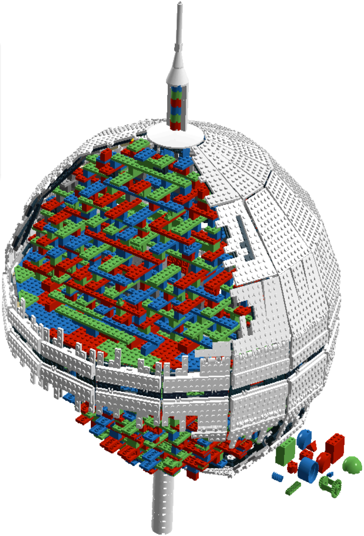


Durr *et al.*, Science **322**, 1224 (2008)

- Can we **map** the *location of mass* inside a hadron?



How can we recover the well-known characteristics of the nucleon from the properties of its **colored building blocks**?



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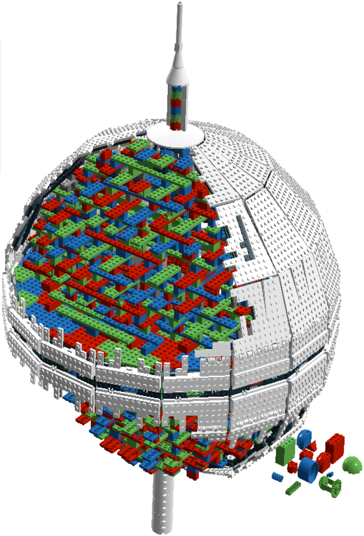
Mass?



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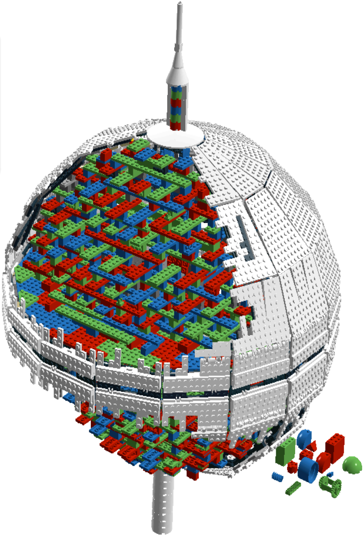
Mass?
Spin?

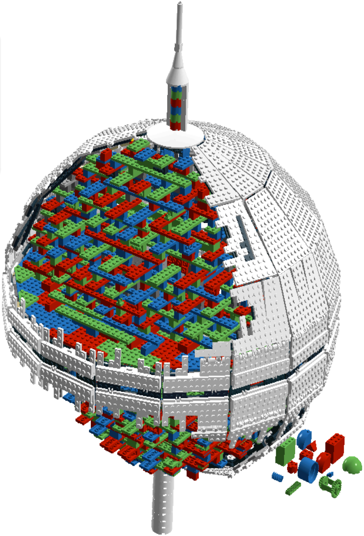


Imaging the origin of mass.

Identification of underlying mechanisms from parton distributions.

How can we recover the well-known characterics of the nucleon from the properties of its **colored building blocks**?

Mass?
Spin?
Charge?



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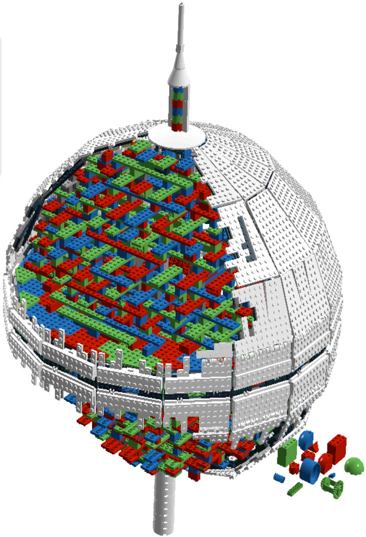
Mass?
Spin?
Charge?
...



Imaging the origin of mass.

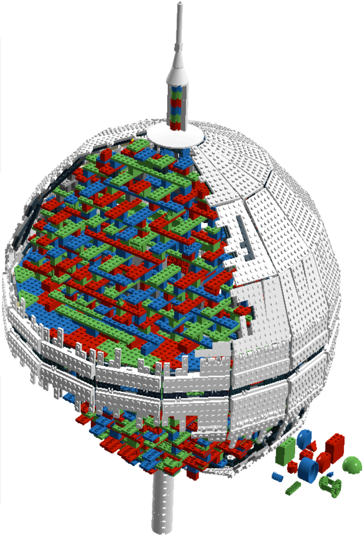
Identification of underlying mechanisms from parton distributions.

How can we recover the well-known characterics of the nucleon from the properties of its **colored building blocks**?

Mass?
Spin?
Charge?

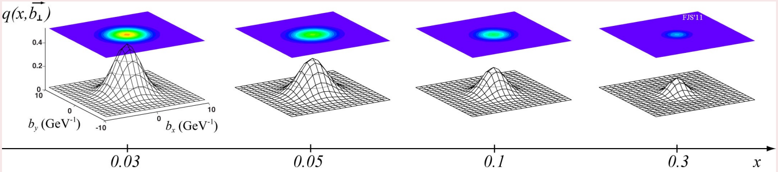
...

What are the relevant **effective degrees of freedom** and **effective interaction** at large distance?



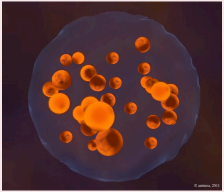
Structuring questions for the hadron physics community

- QCD **mechanisms** behind the origin of **mas** in the **visible universe**?
- **Cartography** of interactions giving its mass to the nucleon?
- **Pressure** and **density** profiles of the nucleon as a continuous medium?
- **Localization** of quarks and gluons inside the nucleon?

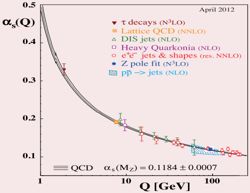


Study nucleon structure to shed new light on nonperturbative QCD.

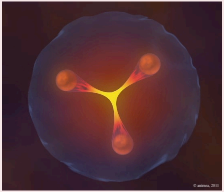
Perturbative QCD



Asymptotic freedom



Nonperturbative QCD

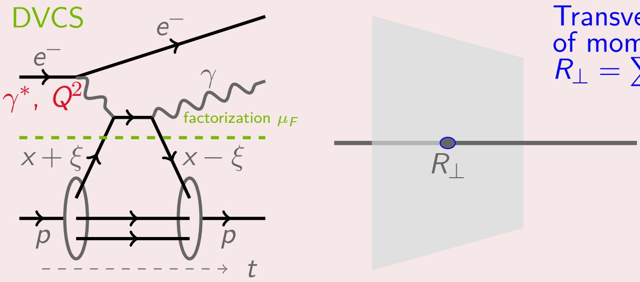


Perturbative AND nonperturbative QCD at work

- Define **universal** objects describing 3D nucleon structure: **Generalized Parton Distributions** (GPD).
- Relate GPDs to measurements using **factorization**: **Virtual Compton Scattering** (DVCS, TCS), **Deeply Virtual Meson production** (DVMP).
- Get **experimental knowledge** of nucleon structure.

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)



Transverse center of momentum $R_\perp$

$R_\perp = \sum_i x_i r_{\perp i}$

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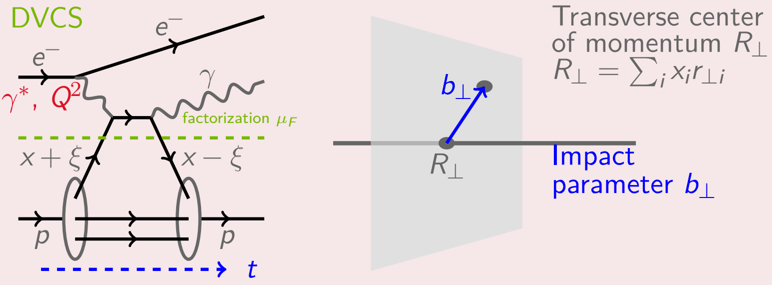

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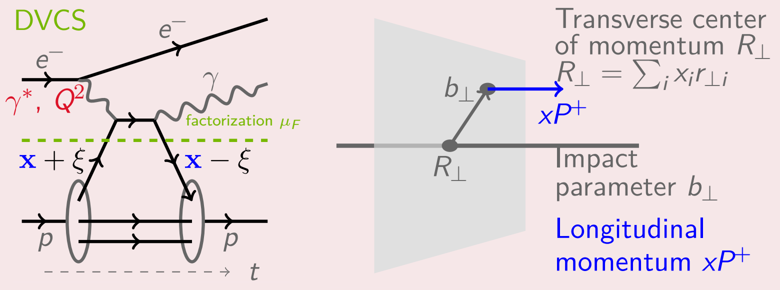

Anatomy of hadrons.

GPDs, 3D hadron imaging, and beyond (1/4).

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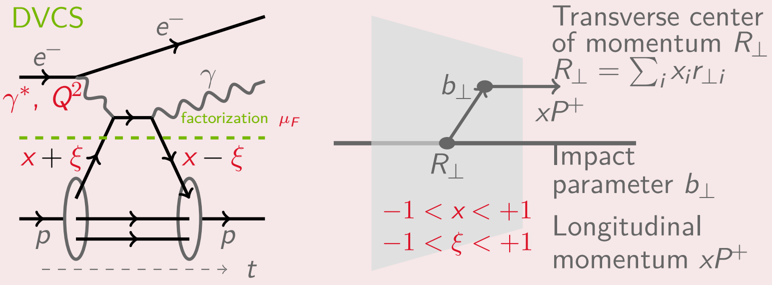
Deeply Virtual Compton Scattering (DVCS)



- **24 GPDs** $F^i(x, \xi, t, \mu_F)$ for each parton type $i = g, u, d, \ldots$ for leading and sub-leading twists.

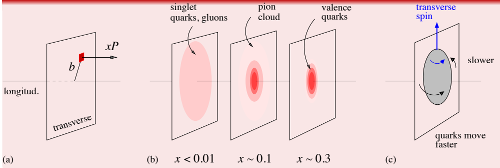
- **Probabilistic interpretation** of Fourier transform of $\mathrm{GPD}(x, \xi = 0, t)$ in **transverse plane**.

$$\rho(x, b_\perp, \lambda, \lambda_N) = \frac{1}{2}\left[H(x, 0, b_\perp^2) + \frac{b_\perp^j \epsilon_{ji} S_\perp^i}{M} \frac{\partial E}{\partial b_\perp^2}(x, 0, b_\perp^2) + \lambda \lambda_N \tilde{H}(x, 0, b_\perp^2) \right]$$

- Notations : quark helicity λ, nucleon longitudinal polarization λ_N and nucleon transverse spin $S_\perp$.

Burkardt, Phys. Rev. **D62**, 071503 (2000)

Can we obtain this picture from exclusive measurements?



Weiss, AIP Conf. Proc. **1149**, 150 (2009)

- Most general structure of matrix element of energy momentum tensor between nucleon states:

$$\left\langle N, P+\frac{\Delta}{2}\right| T^{\mu\nu} \left| N, P-\frac{\Delta}{2}\right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)\left[A(t)\gamma^{(\mu}P^{\nu)} + B(t)P^{(\mu}i\sigma^{\nu)\lambda}\frac{\Delta_\lambda}{2M} + \frac{C(t)}{M}(\Delta^\mu\Delta^\nu - \Delta^2\eta^{\mu\nu})\right]u\left(P-\frac{\Delta}{2}\right)$$

with $t = \Delta^2$.

- Key observation: **link between GPDs and gravitational form factors**

$$\int \mathrm{d}x\, x H^q(x,\xi,t) = A^q(t) + 4\xi^2 C^q(t)$$

$$\int \mathrm{d}x\, x E^q(x,\xi,t) = B^q(t) - 4\xi^2 C^q(t)$$

Ji, Phys. Rev. Lett. **78**, 610 (1997)

- Spin sum rule:

$$\int \mathrm{d}x\, x\big(H^q(x,\xi,0) + E^q(x,\xi,0)\big) = A^q(0) + B^q(0) = 2J^q$$

Ji, Phys. Rev. Lett. **78**, 610 (1997)

- **Shear** and **pressure** of a hadron considered as a continuous medium:

$$\langle N \big| T^{ij}(\vec{r}) \big| N\rangle N = s(r)\left(\frac{r^i r^j}{\vec{r}^2} - \frac{1}{3}\delta^{ij}\right) + p(r)\delta^{ij}$$

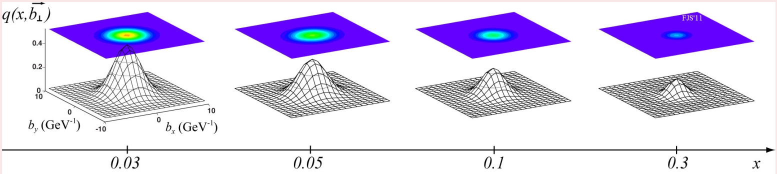
Polyakov and Shuvaev, hep-ph/0207153

1. **Status of 3D imaging:** phenomenological relevance of the field.

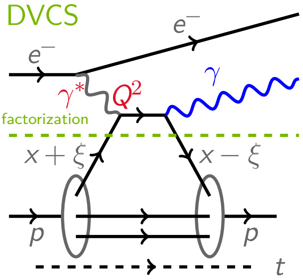
2. **Building the tools:** preparing for the high precision era.

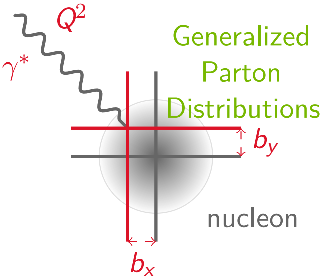
3. **Learning from GPDs:** steps towards new GPD models.

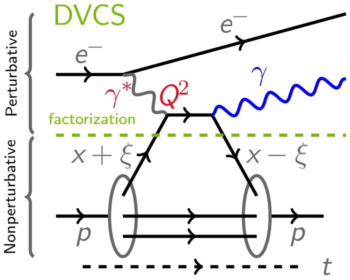
How can we make this picture? What do we learn from it?



Phenomenological status of nucleon 3D imaging


DVCS
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t


Q^2
γ^*
Generalized
Parton
Distributions
b_y
nucleon
b_x



DVCS
Perturbative
Nonperturbative
e^-
e^-
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Q^2
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$x+\xi$
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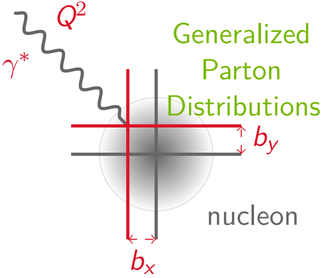
p
p
t



Q^2
γ^*
Generalized
Parton
Distributions
b_y
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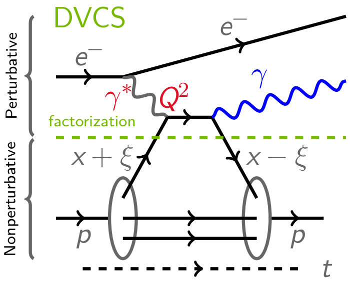
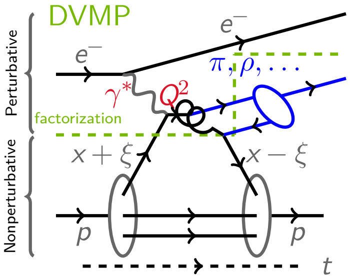
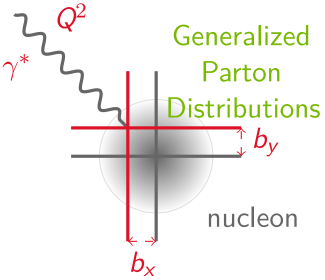
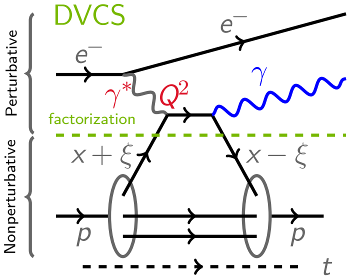
Exclusive processes of current interest.

Factorization and universality.

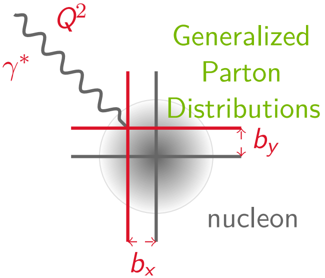




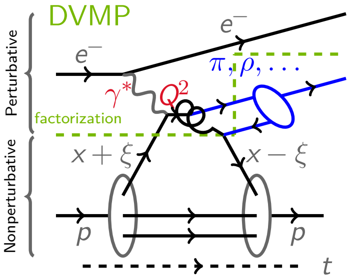




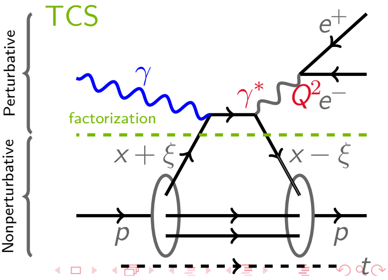
DVCS
Perturbative
Nonperturbative
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t



Q^2
γ^*
Generalized
Parton
Distributions
b_y
nucleon
b_x



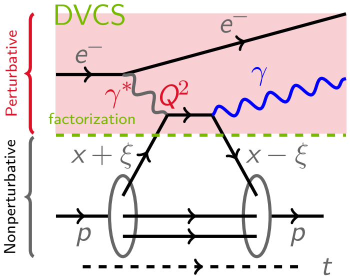
DVMP
Perturbative
Nonperturbative
e^-
e^-
$\pi, \rho, \ldots$
γ^*
Q^2
factorization
$x+\xi$
$x-\xi$
p
p
t

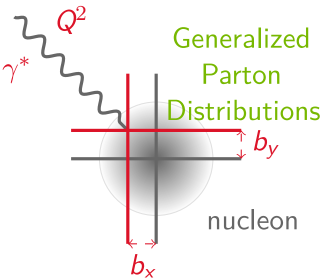


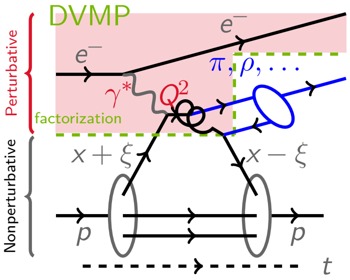
TCS
Perturbative
Nonperturbative
e^+
γ
γ^*
Q^2
e^-
factorization
$x+\xi$
$x-\xi$
p
p
t

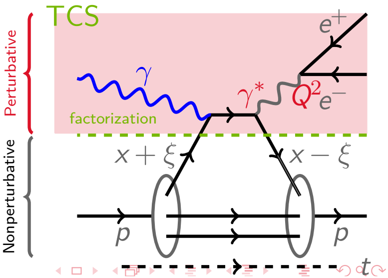
Exclusive processes of current interest.

Factorization and universality.









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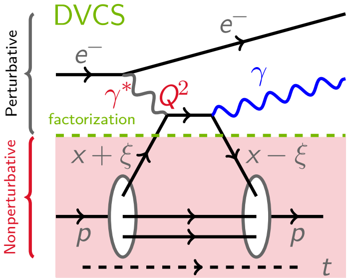
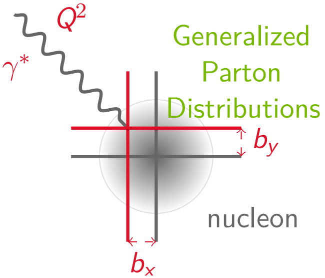
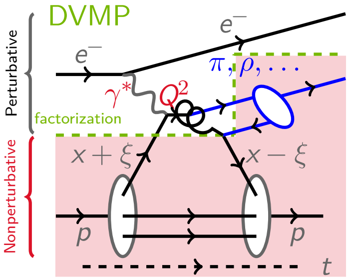
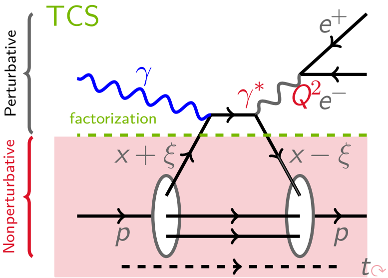








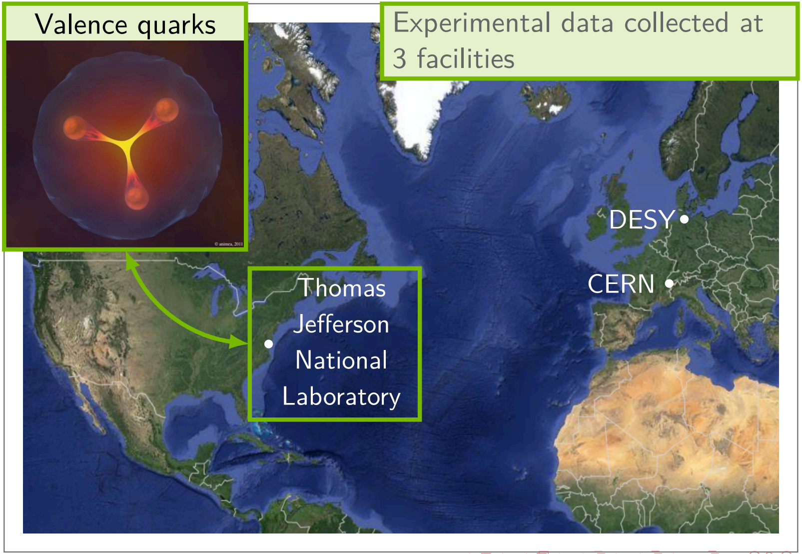
Need for global fits of world data.

Different facilities will probe different kinematic domains.

Experimental data collected at 3 facilities

Need for global fits of world data.

Different facilities will probe different kinematic domains.



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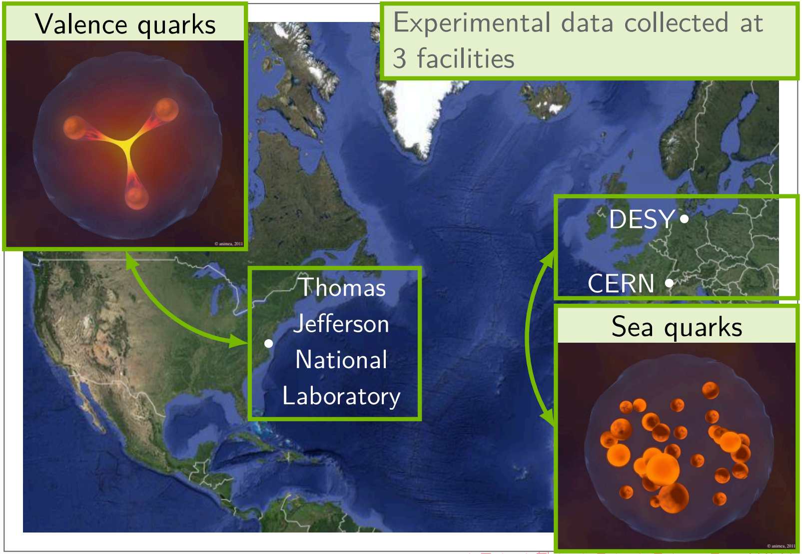

Need for global fits of world data.

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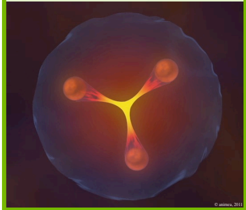
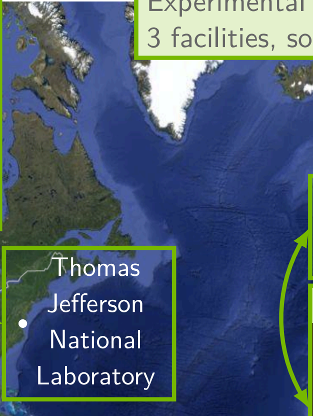
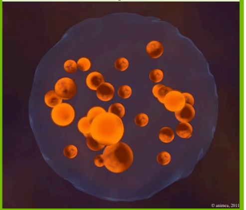
Valence quarks



Experimental data collected at 3 facilities, soon 4:EIC !



Sea quarks



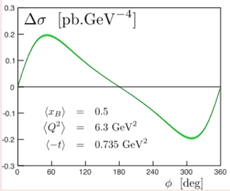
Gluons

NSAC, Long Range Plan 2015: "We recommend [...] EIC as the highest priority for new facility construction"

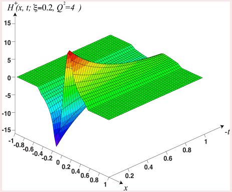
Imaging the nucleon. How?

Extracting GPDs is not enough...Need to extrapolate!

1. Experimental data fits



2. GPD extraction



3. Nucleon imaging

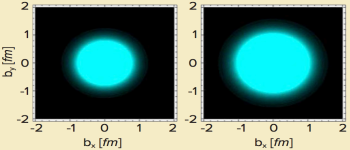
Images from Guidal et al., Rept. Prog. Phys. 76 (2013) 066202

Reaching for the Horizon

The 2015 Long Range Plan for Nuclear Science

Sidebar 2.2: The First 3D Pictures of the Nucleon

A computed tomography (CT) scan can help physicians pinpoint minute cancer tumors, diagnose tiny broken bones, and spot the early signs of osteoporosis. Now physicists are using the principles behind the procedure to peer at the inner workings of the proton. This breakthrough is made possible by a relatively new concept in nuclear physics called generalized parton distributions.



An intense beam of high-energy electrons can be used

1. **Extract** $H(x, \xi, t, \mu_F^{\mathrm{ref}})$ from experimental data.
2. **Extrapolate** to vanishing skewness $H(x, 0, t, \mu_F^{\mathrm{ref}})$.
3. **Extrapolate** $H(x, 0, t, \mu_F^{\mathrm{ref}})$ up to infinite t.
4. **Compute** 2D Fourier transform in transverse plane:

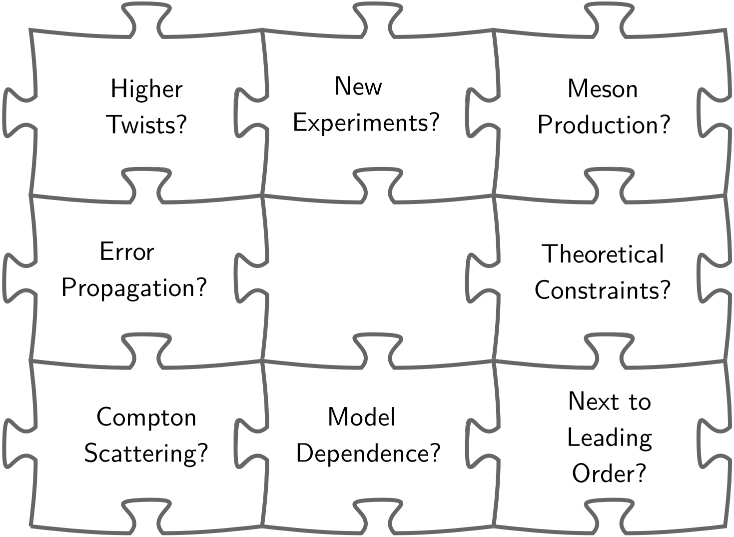
$$H(x, b_\perp) = \int_0^{+\infty} \frac{\mathrm{d}|\Delta_\perp|}{2\pi} |\Delta_\perp| J_0(|b_\perp||\Delta_\perp|) H(x, 0, -\Delta_\perp^2)$$

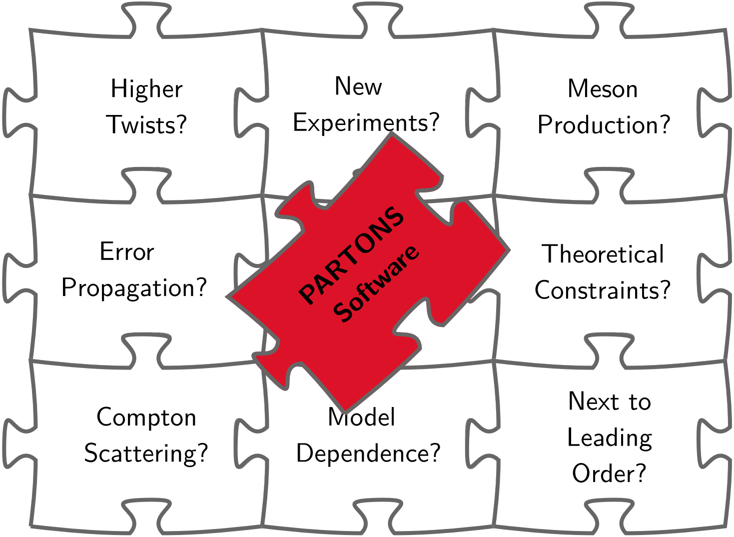
5. **Propagate** uncertainties.
6. **Control** extrapolations with an accuracy matching that of experimental data with **sound** GPD models.

- Evaluation of the impact of **higher order** effects.
- Evaluation of the impact of **target mass and finite-t** corrections.
- Evaluation of the contribution of **higher twist** GPDs.
- DVMP: sensitivity to **DA models**.
- Extrapolations with **GPD models**.

Software for the phenomenology of GPDs.

Different questions to be answered with the same tools.




Higher Twists?
New Experiments?
Meson Production?
Error Propagation?
PARTONS Software
Theoretical Constraints?
Compton Scattering?
Model Dependence?
Next to Leading Order?

Building the tools for high precision: the PARTONS project

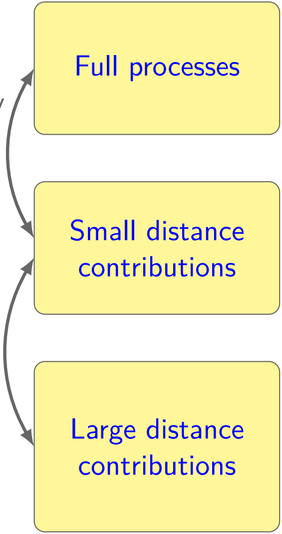
PARtonic
Tomography
Of
Nucleon
Software

Experimental data and phenomenology	Full processes
Computation of amplitudes	Small distance contributions
First principles and fundamental parameters	Large distance contributions

Experimental data and phenomenology

Computation of amplitudes

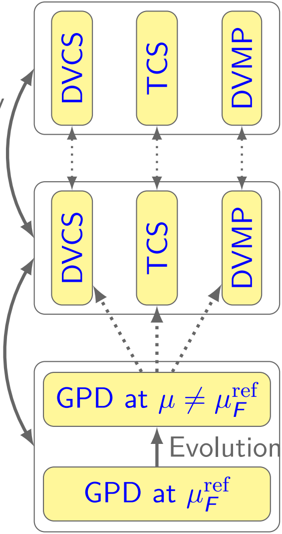
First principles and fundamental parameters



Experimental data and phenomenology

Computation of amplitudes

First principles and fundamental parameters



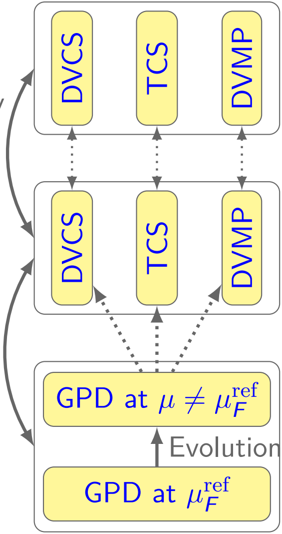
Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Computation of amplitudes

First principles and fundamental parameters



- Many observables.
- Kinematic reach.

Computing chain design.

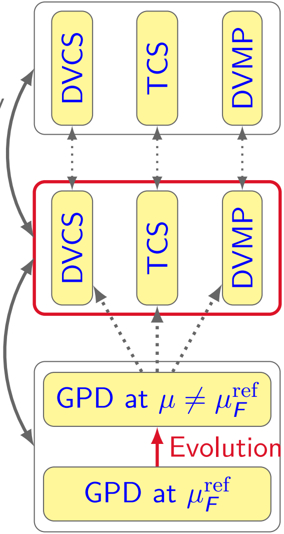
Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Need for modularity

Computation of amplitudes

First principles and fundamental parameters



- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

Computing chain design.

Differential studies: physical models and numerical methods.

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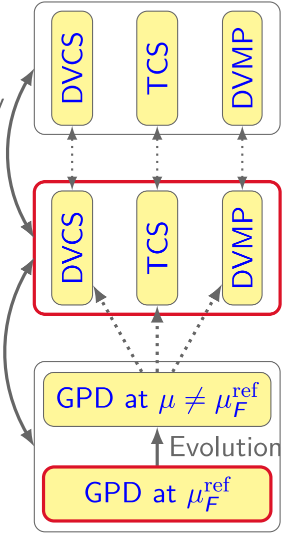
First principles and fundamental parameters



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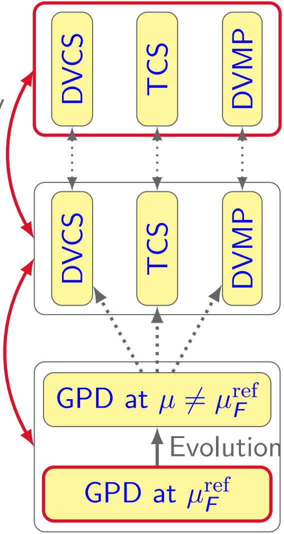

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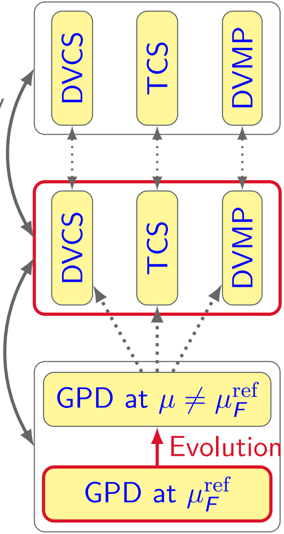

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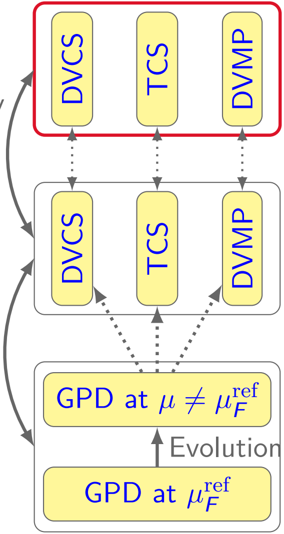


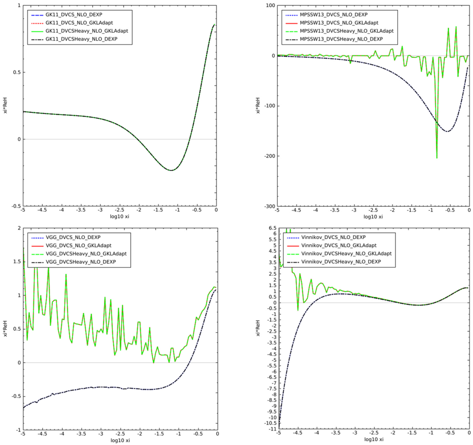
- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

- 3 stages:
  1. Design.
  2. Integration and validation.
  3. Benchmarking and production.
- Flexible software architecture.

  B. Berthou *et al.*, *PARTONS: a computing platform for the phenomenology of Generalized Parton Distributions*
  arXiv:1512.06174, *to appear in Eur. Phys. J. C.*
- 1 new physical development = 1 new module.
- *Aggregate* **knowledge** and **know-how**:
  - Models
  - Measurements
  - Numerical techniques
  - Validation
- What *can* be automated *will be* automated.

Coefficient functions, from EIC to Jefferson Lab kinematics



- *Work in progress!*
- Validation and preparation of **nonregression tools**.
- **Flexibility** at work: **physical** models and **numerical** techniques.
- $\simeq 2 \times 10^4$ GPD computed in $\lesssim 1'$.

gpdExample()

```
// Lots of includes
#include <src/Partons.h>
...
// Retrieve GPD service
GPDService* pGPDService = Partons::getInstance()->getServiceObjectRegistry
()->getGPDService();
// Load GPD module with the BaseModuleFactory
GPDModule* pGK11Model = Partons::getInstance()->getModuleObjectFactory
()->newGPDModule(GK11Model::classId);
// Create a GPDKinematic(x, xi, t, MuF, MuR) to compute
GPDKinematic gpdKinematic(0.1, 0.00050025, -0.3, 8., 8.);
// Compute data and store results
GPDResult gpdResult = pGPDService->
computeGPDModelRestrictedByGPDType(gpdKinematic, pGK11Model,
GPDType::ALL);
// Print results
std::cout << gpdResult.toString() << std::endl;

delete pGK11Model;
pGK11Model = 0;
```

GPD computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneGPD.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="01" date="" description="Example␣:␣computation␣of␣one␣GPD
␣model␣(GK11)␣without␣evolution">
    <!-- Select type of computation -->
    <task service="GPDService" method="computeGPDModel" >
        <!-- Specify kinematics -->
        <GPDKinematic>
            <param name="x" value="0.1" />
            <param name="xi" value="1.00050025" />
            <param name="t" value="-0.3" />
            <param name="MuF2" value="8" />
            <param name="MuR2" value="8" />
        </GPDKinematic>
        <!-- Choose GPD model and set parameters -->
        <GPDModule>
            <param name="id" value="GK11Model" />
        </GPDModule>
    </task>
</scenario>
```

GPD computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneGPD

```
<?xml version="1.0" encoding="UTF-8" stand
<scenario id="01" date="" description="Exam                D
␣model␣(GK11)␣without␣evolution">
     <!-- Select type of computation -->
     <task service="GPDService" method=
          <!-- Specify kinematics -->
          <GPDKinematic>
               <param name="x" valu
               <param name="xi" va
               <param name="t" valu
               <param name="MuF2"
               <param name="MuR2"
          </GPDKinematic>
          <!-- Choose GPD model and
          <GPDModule>
               <param name="id" va
          </GPDModule>
     </task>
</scenario>
```

$$H^u = 0.822557$$
$$H^{u(+)} = 0.165636$$
$$H^{u(-)} = 1.47948$$

$$H^d = 0.421431$$
$$H^{d(+)} = 0.0805182$$
$$H^{d(-)} = 0.762344$$

$$H^s = 0.00883408$$
$$H^{s(+)} = 0.0176682$$
$$H^{s(-)} = 0$$

$$H^g = 0.385611$$

and E, $\tilde{H}$, $\tilde{E}$, ...

CFF computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneCFF.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="03" date="" description="Example : computation of one convol coeff function model (DVCSCFF) with GPD model (GK11)">
	<task service="DVCSConvolCoeffFunctionService" method="computeWithGPDModel">
		<DVCSConvolCoeffFunctionKinematic>
			<param name="xi" value="0.5" />
			<param name="t" value="-0.1346" />
			<param name="Q2" value="1.5557" />
			<param name="MuF2" value="4" />
			<param name="MuR2" value="4" />
		</DVCSConvolCoeffFunctionKinematic>
		<GPDModule>
			<param name="id" value="GK11Model" />
		</GPDModule>
		<DVCSConvolCoeffFunctionModule>
			<param name="id" value="DVCSCFFModel" />
			<param name="qcd_order_type" value="LO" />
		</DVCSConvolCoeffFunctionModule>
	</task>
</scenario>
```

CFF computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneCFF.xml

```
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="03" date="" description="Example␣:␣computation␣of␣one␣convol␣coeff␣function␣model␣(DVCSCFF)␣with␣GPD␣model␣(GK11)">
        <task service="DVCSConvolCoeffFunctionService" method="computeWithGPDModel">
                <DVCSConvolCoeffFunctionKinematic>
                        <param name="xi" value="0.5" />
                        <param name="t" value="-0.1346" />
                        <param name="Q2" value="1.5557" />
                        <param name="MuF2" value="4" />
                        <param name="MuR2" value="4" />
                </DVCSConvolCoeffFunctionKinematic>
                <GPDModule>
                        <param name="id" value="GK11Model" />
                </GPDModule
                <DVCSConvolC
                        <para                                    >
                        <para                                    />
                </DVCSConvol
        </task>
</scenario>
```

$$\mathcal{H} = 1.47722 + 1.76698\ i$$
$$\mathcal{E} = 0.12279 + 0.512312\ i$$
$$\widetilde{\mathcal{H}} = 1.54911 + 0.953728\ i$$
$$\widetilde{\mathcal{E}} = 18.8776 + 3.75275\ i$$

Observable computing automated.

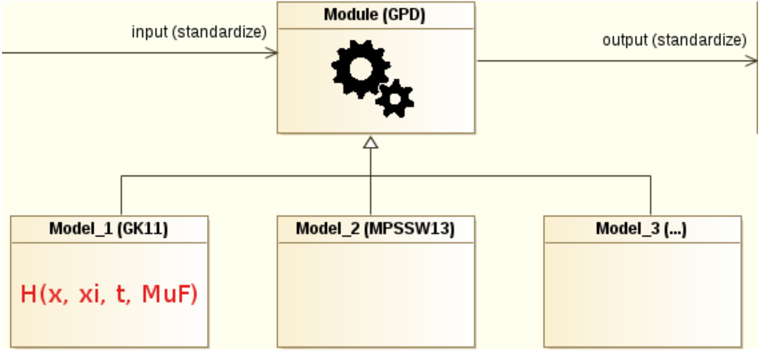
Each line of code corresponds to a physical hypothesis.

computeManyKinematicsOneModel.xml

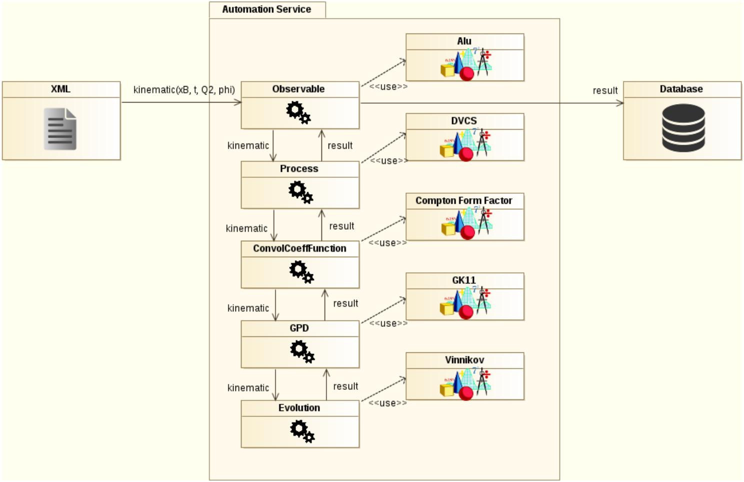
```xml
<scenario id="5" date="" description="Compute␣Alu␣E1-DVCS␣kinematics">
    <task service="ObservableService" method="
computeManyKinematicOneModel" storeInDB="0">
        <ObservableKinematic>
            <param name="file"
             value="/home/debian/workspace/PARTONS/data/E1DVCS.dat" />
        </ObservableKinematic>
        <Observable>
            <param name="id" value="Alu" />
        </Observable>
        <DVCSModule>
            <param name="id" value="BMJ2012Model" />
            <param name="beam_energy" value="5.75" />
        </DVCSModule>
        <DVCSConvolCoeffFunctionModule>
            <param name="id" value="DVCSCFFModel" />
            <param name="qcd_order_type" value="LO" />
        </DVCSConvolCoeffFunctionModule>
        <GPDModule>
            <param name="id" value="GK11Model" />
        </GPDModule>
    </task>
```

Modularity.

Inheritance, standardized inputs and outputs.



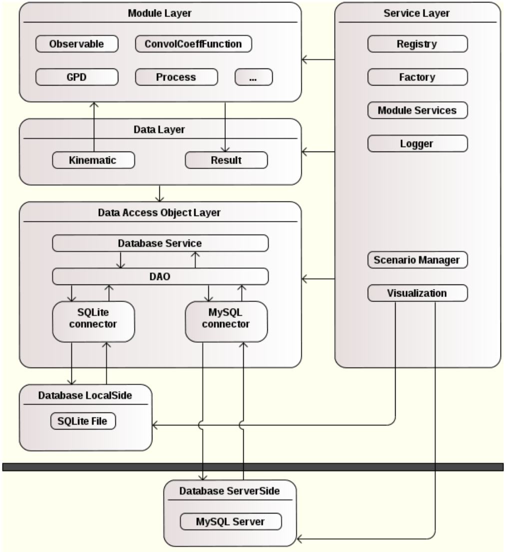
- Steps of logic sequence in parent class.
- Model description and related mathematical methods in daughter class.



Automation Service
XML
kinematic(xB, t, Q2, phi)
Observable
result
Database
Alu
<<use>>
kinematic
result
Process
DVCS
<<use>>
Compton Form Factor
kinematic
result
ConvolCoeffFunction
<<use>>
GK11
kinematic
result
GPD
<<use>>
Vinnikov
kinematic
result
Evolution
<<use>>

Modularity and layer structure.

Modifying one layer does not affect the other layers.



Pre-PARTONS times.

First mention of the PARTONS project in a conference.

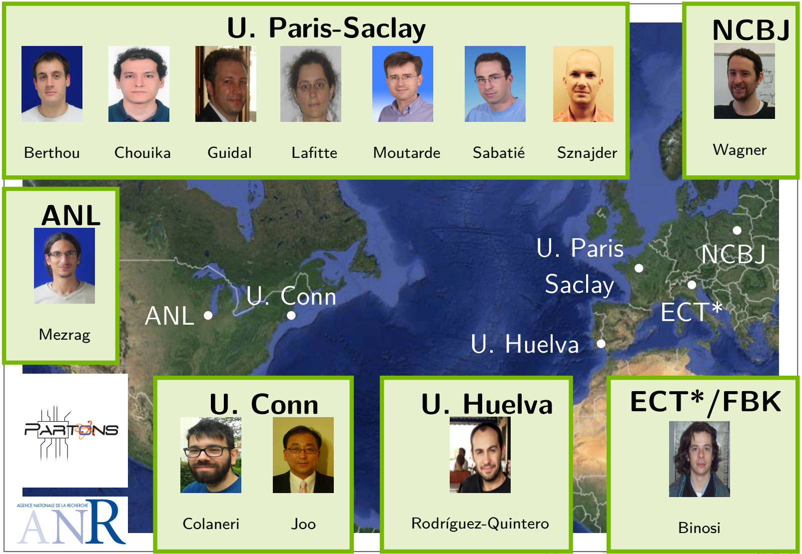
PROPHET.

Platform for **R**epresenting the **O**rganization of **P**artons inside **H**adrons and **E**xperimental **T**omographies.

1. Comprehensive **database of experimental results**.
2. Comprehensive **database of theoretical predictions**.
3. **Fitting engine**.
4. **Propagation** of statistic and systematic **uncertainties**.
5. **Visualizing software** to compare experimental results and model expectations.
6. Connection to **experimental set-up descriptions** to design new experiments.
7. **Interactive website** providing free access to model and experimental values.

1 multidisciplinary team over 5 countries

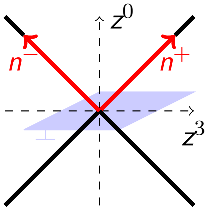
Theorists, experimentalists, 1 mathematician + 1 software engineer



Learning on the strong interaction from GPD models

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)

Ji, Phys. Rev. Lett. **78**, 610 (1997)

Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**

$$H^q(x, 0, 0) = q(x)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

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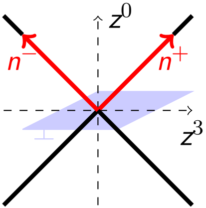

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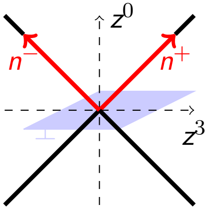
- PDF **forward limit**
- Form factor **sum rule**

$$\int_{-1}^{+1} dx\, H^q(x, \xi, t) = F_1^q(t)$$

$$H^q_\pi(x, \xi, t) \quad = \quad \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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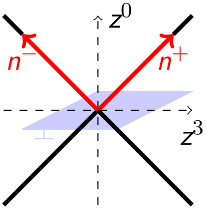
- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.

$$H^q_\pi(x, \xi, t) \quad = \quad \frac{1}{2} \int \frac{\mathrm{d}z^-}{2\pi} \, e^{ixP^+ z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.

Polynomiality

$$\int_{-1}^{+1} dx\, x^n H^q(x, \xi, t) = \text{polynomial in } \xi$$

Polynomiality

Lorentz covariance

- **Polynomiality**

Lorentz covariance

- **Positivity**

$$H^q(x, \xi, t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

Spin-0 Generalized Parton Distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

Spin-0 Generalized Parton Distribution.

Not so simple properties.

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  Relativistic quantum mechanics

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2} \phi^q_\pi \left(\frac{1 + x}{2} \right)$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

How can we implement *a priori* these theoretical constraints?

- There is no known GPD parameterization **relying only on first principles**.
- In the following, focus on **polynomiality** and **positivity**.

- Representation of GPD:

$$H^q(x,\xi,t) = \int_{\Omega_{\rm DD}} \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t)+\xi G^q(\beta,\alpha,t)\big)$$

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x,\xi,t) = x\int_{\Omega_{\rm DD}} \mathrm{d}\beta \mathrm{d}\alpha\, f^q_{\rm BMKS}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Belitsky *et al.*, Phys. Rev. **D64**, 116002 (2001)

$$H^q(x,\xi,t) = (1-x)\int_{\Omega_{\rm DD}} \mathrm{d}\beta \mathrm{d}\alpha\, f^q_{\rm P}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Pobylitsa, Phys. Rev. **D67**, 034009 (2003)

Müller, Few Body Syst. **55**, 317 (2014)

- Choose $F^q(\beta, \alpha) = 3\beta\theta(\beta)$ ad $G^q(\beta, \alpha) = 3\alpha\theta(\beta)$:

$$H^q(x, \xi) = 3x \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi)$$

- Simple analytic expressions for the GPD:

$$H(x, \xi) = \frac{6x(1-x)}{1-\xi^2} \text{ if } 0 < |\xi| < x < 1,$$

$$H(x, \xi) = \frac{3x(x+|\xi|)}{|\xi|(1+|\xi|)} \text{ if } -|\xi| < x < |\xi| < 1.$$

- Compute first Mellin moments.

n	$\int_{-\xi}^{+\xi} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{+\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{-\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$
0	$\frac{1+\xi-2\xi^2}{1+\xi}$	$\frac{2\xi^2}{1+\xi}$	1
1	$\frac{1+\xi+\xi^2-3\xi^3}{2(1+\xi)}$	$\frac{2\xi^3}{1+\xi}$	$\frac{1+\xi^2}{2}$
2	$\frac{3(1-\xi)(1+2\xi+3\xi^2+4\xi^3)}{10(1+\xi)}$	$\frac{6\xi^4}{5(1+\xi)}$	$\frac{3(1+\xi^2)}{10}$
3	$\frac{1+\xi+\xi^2+\xi^3+\xi^4-5\xi^5}{5(1+\xi)}$	$\frac{6\xi^5}{5(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{5}$
4	$\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5-6\xi^6}{7(1+\xi)}$	$\frac{6\xi^6}{7(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{7}$

- Expressions get more complicated as n increases... But they always yield polynomials!

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x \mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x}\mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \big(\psi_N^{(\beta,\lambda)}\big)^* (\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWFs.

Overlap representation.

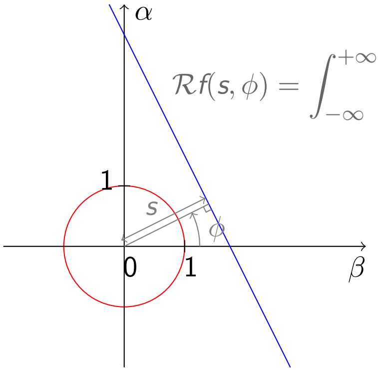
Advantages and drawbacks.

- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of** N**-body problems**.

What is not obvious anymore

What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

Diehl, Phys. Rept. **388**, 41 (2003)



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s,\phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha\, f(\beta,\alpha)\delta(s - \beta\cos\phi - \alpha\sin\phi)$$

and:

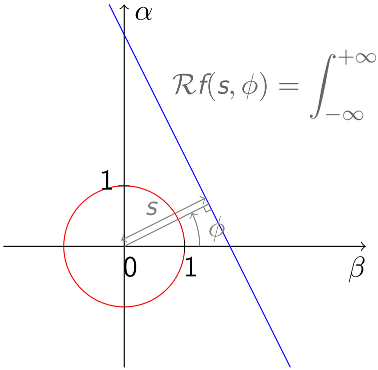
$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Belistky *et al.* gauge

$$\frac{\sqrt{1+\xi^2}}{x} H(x,\xi) = \mathcal{R}f_{\mathrm{BMKS}}(s,\phi) ,$$



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and:

$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Pobylitsa gauge

$$\frac{\sqrt{1+\xi^2}}{1-x} H(x,\xi) = \mathcal{R}f_{\mathrm{P}}(s,\phi) ,$$

The range of the Radon transform.

The polynomiality property a.k.a. the Ludwig-Helgason condition.

- The Mellin moments of a Radon transform are **homogeneous polynomials** in $\omega = (\sin\phi, \cos\phi)$.
- The converse is also true:

Theorem (Hertle, 1983)

*Let $g(s, \omega)$ an even compactly-supported distribution. Then g is itself the Radon transform of a compactly-supported distribution if and only if the **Ludwig-Helgason consistency condition** hold:*

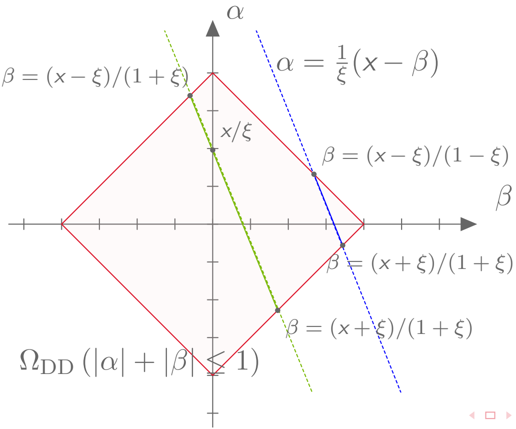
(i) *g is C^∞ in ω,*

(ii) *$\int \mathrm{d}s\, s^m g(s, \omega)$ is a homogeneous polynomial of degree m for all integer $m \geq 0$.*

- Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.

DGLAP and ERBL regions

$$(x, \xi) \in \text{DGLAP} \Leftrightarrow |s| \geq |\sin\phi| ,$$
$$(x, \xi) \in \text{ERBL} \Leftrightarrow |s| \leq |\sin\phi| .$$

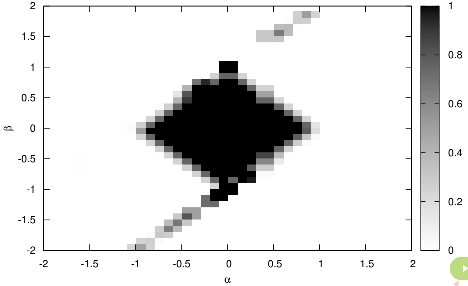


- Each point (β, α) with $\beta \neq 0$ contributes to **both** DGLAP and ERBL regions.
- Expressed in **support theorem**.

Ill-posedness in the sense of Hadamard.

A first glimpse at the inverse Radon transform.

- Numerical evaluation ***almost unavoidable*** (polar vs cartesian coordinates).
- Ill-posedness by **lack of continuity**.
- The **unlimited** Radon inverse problem is **mildly** ill-posed while the **limited** one is **severely** ill-posed.
- Careful selection of **algorithms** and **numerical methods**.



Mezrag
PhD dissertation

See more on inverse Radon transform.

Conclusion

- **Challenging constraints** expected from Jefferson Lab in valence region and later from EIC in gluon sector.
- **Good theoretical control** on the path between GPD models and experimental data: from theory to measurements, and conversely.
- Development of the PARTONS framework for **phenomenology** and **theory** purposes.
- **Systematic** procedure to construct GPD models from any "reasonable" Ansatz of LFWFs.
- **First release** of PARTONS in summer 2016!

Appendix

Implementing Lorentz covariance.

Extend an overlap in the DGLAP region to the whole GPD domain.

For **any model of LFWF**, one has to address the following three questions:

1. Does the extension exist?
2. If it exists, is it unique?
3. How can we compute this extension?

Work in progress!

Theorem

Let f be a compactly-supported locally summable function defined on $\mathbb{R}^2$ and $\mathcal{R}f$ its Radon transform.
Let $(s_0, \omega_0) \in \mathbb{R} \times S^1$ and U_0 an open neighborhood of ω_0 such that:

$$\text{for all } s > s_0 \text{ and } \omega \in U_0 \quad \mathcal{R}f(s, \omega) = 0 \; .$$

Then $f(\aleph) = 0$ on the half-plane $\langle \aleph | \omega_0 \rangle > s_0$ of $\mathbb{R}^2$.

Consider a GPD H being zero on the DGLAP region.

- Take ϕ_0 and s_0 *s.t.* $\cos \phi_0 \neq 0$ and $|s_0| > |\sin \phi_0|$.
- Neighborhood U_0 of ϕ_0 *s.t.* $\forall \phi \in U_0 \quad |\sin \phi| < |s_0|$.
- The underlying DD f has a zero Radon transform for all $\phi \in U_0$ and $s > s_0$ (DGLAP).
- Then $f(\beta, \alpha) = 0$ for all $(\beta, \alpha) \in \Omega_{DD}$ with $\beta \neq 0$.
- Extension **unique** up to adding a **D-term**: $\delta(\beta) D(\alpha)$.

A discretized problem

Consider $N+1$ Hilbert spaces H, H_1, ..., H_N, and a family of continuous surjective operators $R_n : H \to H_n$ for $1 \leq n \leq N$.
Being given $g_1 \in H_1$, ..., $g_n \in H_n$, we search f solving the following system of equations:

$$R_n f = g_n \quad \text{for } 1 \leq n \leq N$$

Fully discrete case

Assume f piecewise-constant with values f_m for $1 \leq m \leq M$.
For a collection of lines $(L_n)_{1 \leq n \leq N}$ crossing Ω_{DD}, the Radon transform writes:

$$g_n = \mathcal{R}f = \int_{L_n} f = \sum_{m=1}^{M} f_m \times \mathrm{Measure}(L_n \cap C_m) \quad \text{for } 1 \leq n \leq N$$

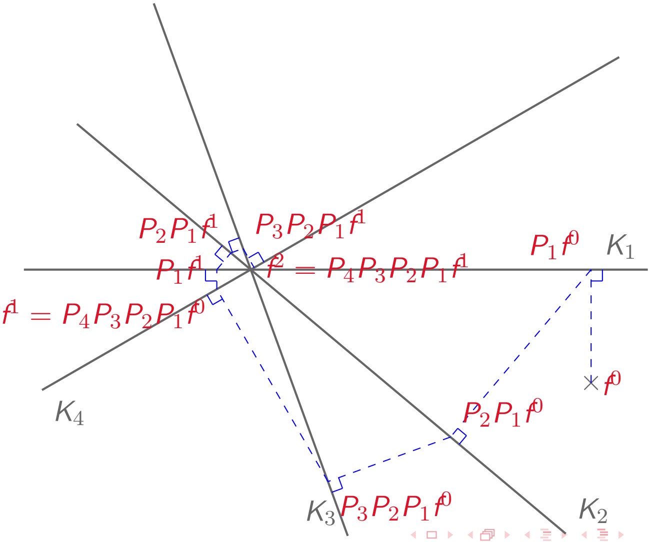
Kaczmarz algorithm

Denote P_n the orthogonal projection on the *affine* subspace $R_n f = g_n$. Starting from $f^0 \in H$, the sequence defined iteratively by:

$$f^{k+1} = P_N P_{N-1} \dots P_1 f^k$$

converges to the solution of the system.
The convergence is exponential if the projections are randomly ordered.

Strohmer and Vershynin, Jour. Four. Analysis and Appl. **15**, 437 (2009)



$P_2 P_1 f^1$
$P_3 P_2 P_1 f^1$
$P_1 f^0$
K_1
$P_1 f^1$
$f^2 = P_4 P_3 P_2 P_1 f^1$
$f^1 = P_4 P_3 P_2 P_1 f^0$
f^0
K_4
$P_2 P_1 f^0$
K_3
$P_3 P_2 P_1 f^0$
K_2

And if the input data are inconsistent?

- Instead of solving $g = \mathcal{R}f$, find f such that $\|g - \mathcal{R}f\|_2$ is **minimum**.
- The solution **always exists**.
- The input data are **inconsistent** if $\|g - \mathcal{R}f\|_2 > 0$.

Relaxed Kaczmarz algorithm

Let $\omega \in]0, 2[$ and:

$$P_n^\omega = (1 - \omega)\,\mathrm{Id}_H + \omega P_n \quad \text{for } 1 \leq n \leq N$$

Write:

$$RR^\dagger = (R_i R_j^\dagger)_{1 \leq i,j \leq N} = D + L + L^\dagger$$

where D is diagonal, and L is lower-triangular with zeros on the diagonal.

Theorem

Let $0 < \omega < 2$. For $f^0 \in \mathrm{Ran}\ R^\dagger$ (e.g. $f^0 = 0$), the Kaczmarz method with relaxation converges to the unique solution $f^\omega \in \mathrm{Ran}\ R^\dagger$ of:

$$R^\dagger (D + \omega L)^{-1} (g - R f^\omega) = 0 \ ,$$

where the matrix D and L appear in the decomposition of $RR^\dagger$. If $g = \mathcal{R} f$ has a solution, then f^ω is its solution of minimal norm. Otherwise:

$$f^\omega = f_{MP} + \mathcal{O}(\omega) \ ,$$

where f_{MP} is the minimizer in H of:

$$\langle g - \mathcal{R} f | g - \mathcal{R} f \rangle_D \ ,$$

the inner product being defined by:

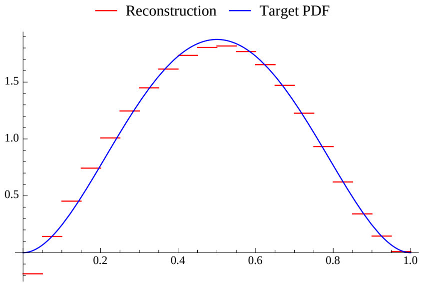
$$\langle h | k \rangle_D = \langle D^{-1} h | k \rangle \ .$$

Test on a 1D example.

Recovering a PDF from the knowledge of its Mellin moments.

A pion valence PDF-like example

Aim: reconstruct the PDF $q(x) = 30x^2(1-x)^2$ from the knowledge of its first 30 Mellin moments.



- Piecewise-constant PDF: 20 values.
- Input: 30 Mellin moments.
- Unrelaxed method $\omega = 1$.
- 10000 iterations.

- Extensive testing *in progress*
  - Various inputs: PDFs and LFWFs.
  - Numerical noise.

Back to covariant extensions.

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