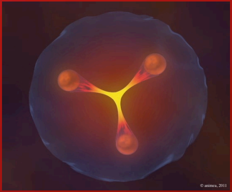
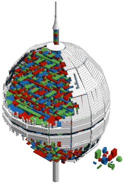
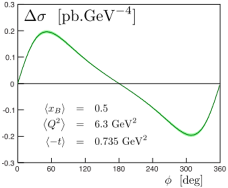
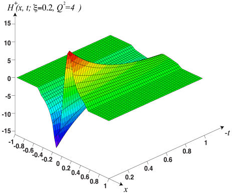
DE LA RECHERCHE À L'INDUSTRIE



Lorentz Covariance and Positivity Constraints in the Modeling of Generalized Parton Distributions







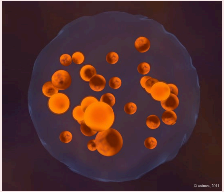
Second Sino-American Workshop and School | Hervé MOUTARDE

Nov. 20$^{\text{th}}$, 2015

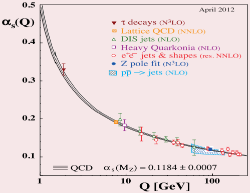
Motivation.

Study nucleon structure to shed new light on nonperturbative QCD.

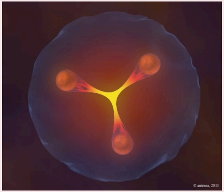
Perturbative QCD



Asymptotic freedom



Nonperturbative QCD



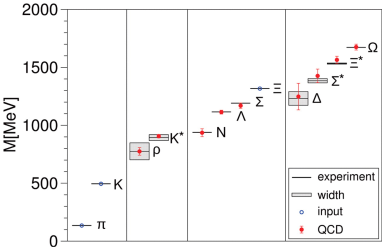
Perturbative AND nonperturbative QCD at work

- Define **universal** objects describing 3D nucleon structure: **Generalized Parton Distributions** (GPD).
- Relate GPDs to measurements using **factorization**: **Virtual Compton Scattering** (DVCS, TCS), **Deeply Virtual Meson production** (DVMP).
- Get **experimental knowledge** of nucleon structure.

Motivation.

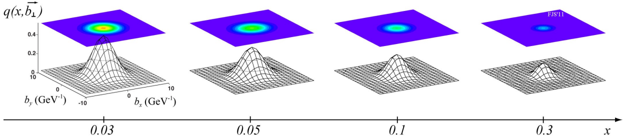
QCD large distance dynamics from the hadron structure viewpoint.

- Lattice QCD clearly shows that the mass of hadrons is generated by the **interaction**, not by the quark masses.

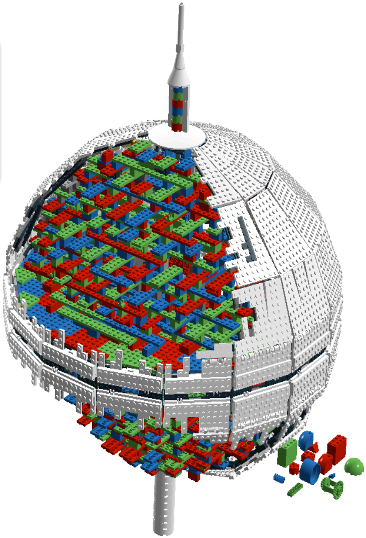


Durr *et al.*, Science **322**, 1224 (2008)

- Can we **map** the *location of mass* inside a hadron?

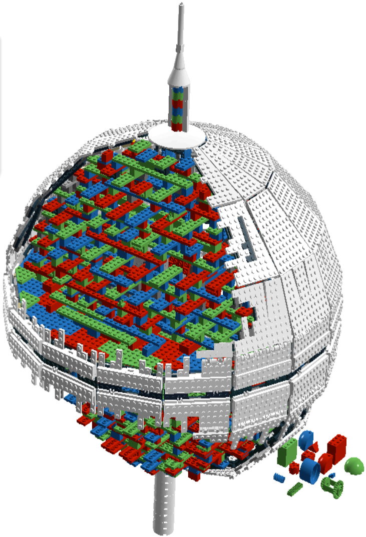


How can we recover the well-known characteristics of the nucleon from the properties of its **colored building blocks**?



How can we recover the well-known characteristics of the nucleon from the properties of its **colored building blocks**?

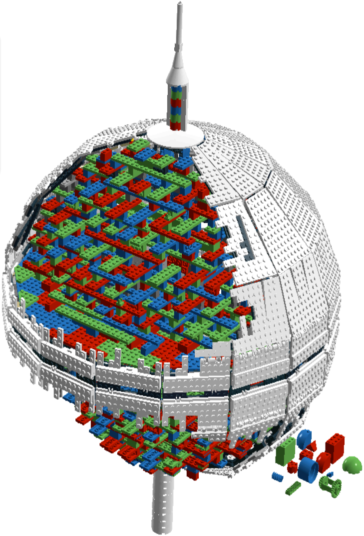
Mass?



How can we recover the well-known characteristics of the nucleon from the properties of its **colored building blocks**?

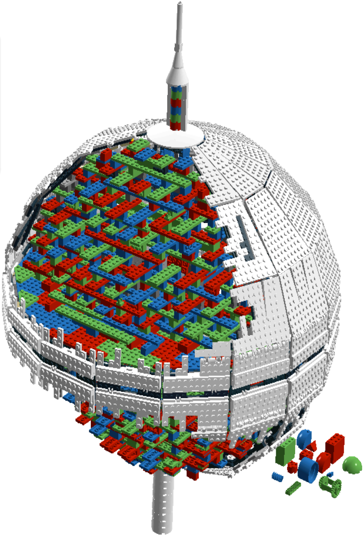
Mass?

Spin?



How can we recover the well-known characterics of the nucleon from the properties of its **colored building blocks**?

Mass?
Spin?
Charge?



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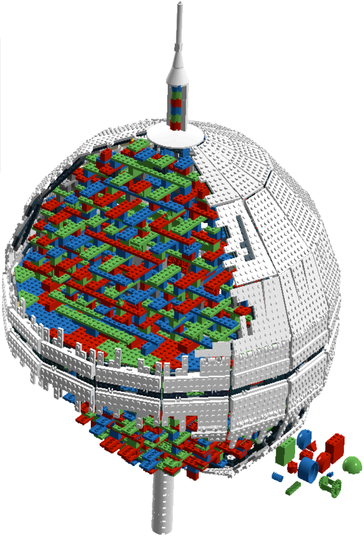
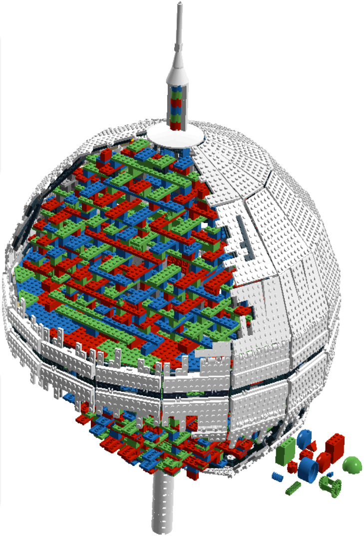
Mass?
Spin?
Charge?
...



How can we recover the well-known characterics of the nucleon from the properties of its **colored building blocks**?

Mass?
Spin?
Charge?
...

What are the relevant **effective degrees of freedom** and **effective interaction** at large distance?



Anatomy of hadrons.

GPDs, 3D hadron imaging, and beyond (1/4).

Covariant and Positive GPD Models

Introduction

Phenomenology status
- Experimental access
- DVCS Kinematics
- Universality tests
- Towards 3D images

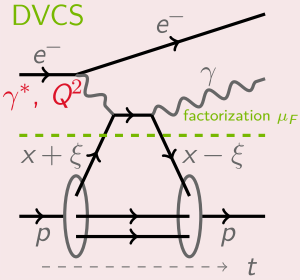
Theoretical framework
- Definition
- Polynomiality
- Double Distributions
- Positivity
- Overlap

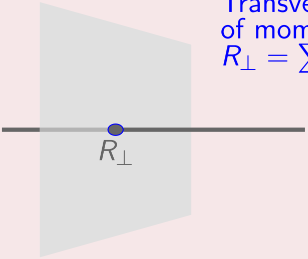
GPDs from LFWFs
- Radon transform
- Covariant extension
- Inverse Radon
- Example

Conclusion

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)

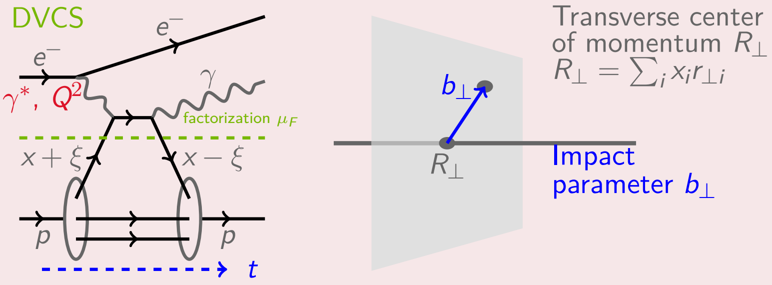




Transverse center of momentum $R_\perp$
$R_\perp = \sum_i x_i r_{\perp i}$

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)



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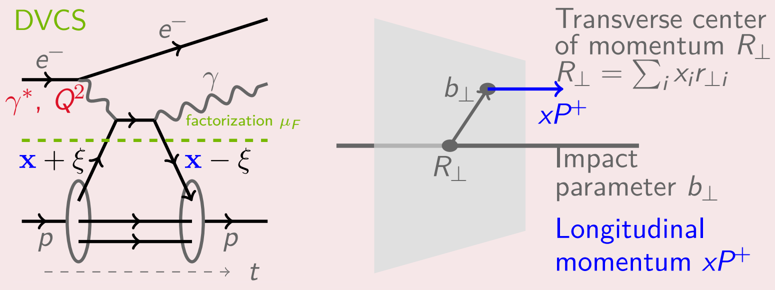

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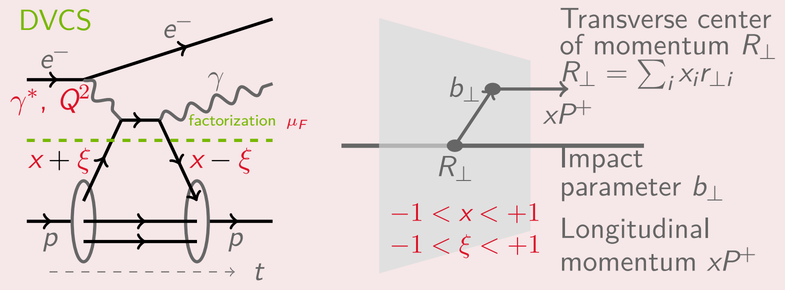
Deeply Virtual Compton Scattering (DVCS)



- **24 GPDs** $F^i(x, \xi, t, \mu_F)$ for each parton type $i = g, u, d, \ldots$ for leading and sub-leading twists.

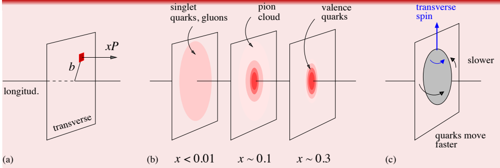
- **Probabilistic interpretation** of Fourier transform of GPD$(x, \xi = 0, t)$ in **transverse plane**.

$$\rho(x, b_\perp, \lambda, \lambda_N) = \frac{1}{2}\left[H(x,0,b_\perp^2) + \frac{b_\perp^j \epsilon_{ji} S_\perp^i}{M} \frac{\partial E}{\partial b_\perp^2}(x,0,b_\perp^2) + \lambda\lambda_N \tilde{H}(x,0,b_\perp^2)\right]$$

- Notations : quark helicity λ, nucleon longitudinal polarization λ_N and nucleon transverse spin $S_\perp$.

Burkardt, Phys. Rev. **D62**, 071503 (2000)

Can we obtain this picture from exclusive measurements?



Weiss, AIP Conf. Proc. **1149**, 150 (2009)

- Most general structure of matrix element of energy momentum tensor between nucleon states:

$$\left\langle N, P+\frac{\Delta}{2} \middle| T^{\mu\nu} \middle| N, P-\frac{\Delta}{2} \right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)\Big[A(t)\gamma^{(\mu}P^{\nu)} + B(t)P^{(\mu}i\sigma^{\nu)\lambda}\frac{\Delta_\lambda}{2M} + \frac{C(t)}{M}(\Delta^\mu\Delta^\nu - \Delta^2\eta^{\mu\nu})\Big]u\left(P-\frac{\Delta}{2}\right)$$

with $t = \Delta^2$.

- Key observation: **link between GPDs and gravitational form factors**

$$\int \mathrm{d}x\, x H^q(x,\xi,t) = A^q(t) + 4\xi^2 C^q(t)$$

$$\int \mathrm{d}x\, x E^q(x,\xi,t) = B^q(t) - 4\xi^2 C^q(t)$$

Ji, Phys. Rev. Lett. **78**, 610 (1997)

- Spin sum rule:

$$\int \mathrm{d}x\, x\big(H^q(x,\xi,0) + E^q(x,\xi,0)\big) = A^q(0) + B^q(0) = 2J^q$$

Ji, Phys. Rev. Lett. **78**, 610 (1997)

- **Shear** and **pressure** of a hadron considered as a continuous medium:

$$\langle N | T^{ij}(\vec{r}) | N \rangle N = s(r)\left(\frac{r^i r^j}{\vec{r}^2} - \frac{1}{3}\delta^{ij}\right) + p(r)\delta^{ij}$$

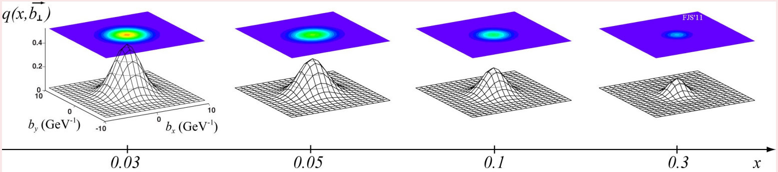
Polyakov and Shuvaev, hep-ph/0207153

Towards hadron tomography.

GPDs as a scalpel-like probe of hadron structure.

1. **Phenomenology status:** relevance and need for parameterizations.
2. **Theoretical framework:** definition and existing constraints.
3. **GPDs from Light Front Wave Functions:** a promising computing strategy.

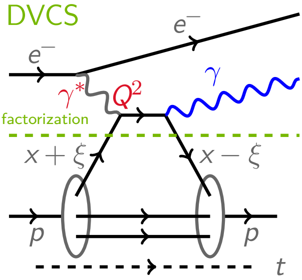
How can we make this picture? What do we learn from it?

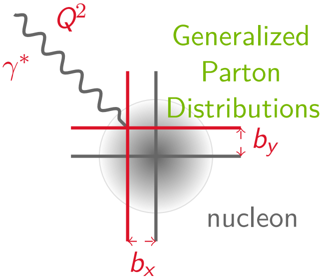


Phenomenology status

Exclusive processes of current interest (1/2).

Factorization and universality.





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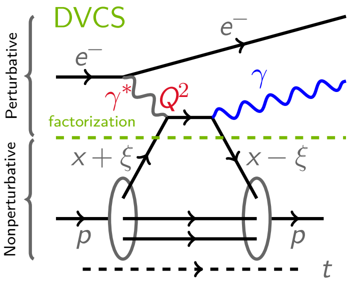
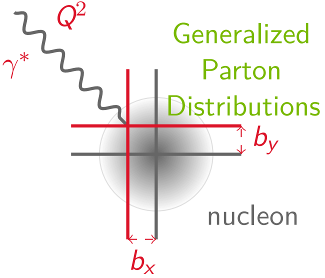
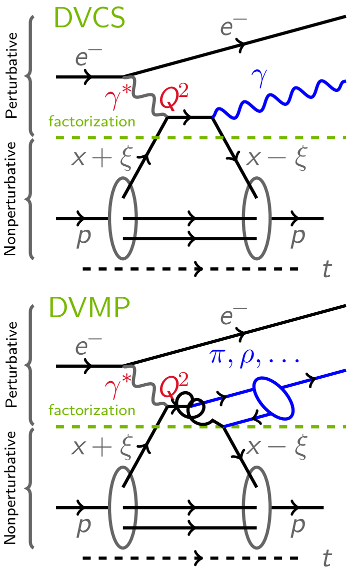
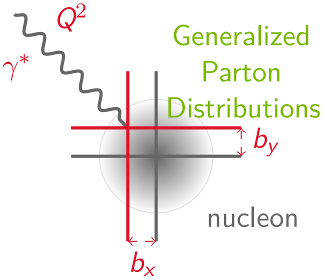


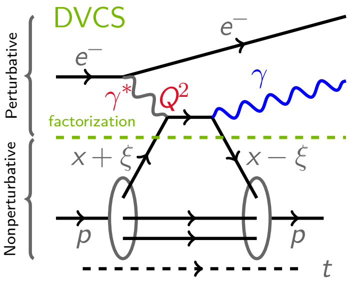


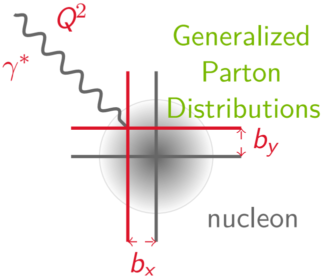
Exclusive processes of current interest (1/2).

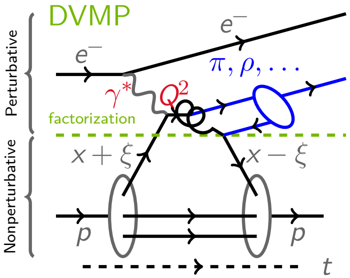
Factorization and universality.

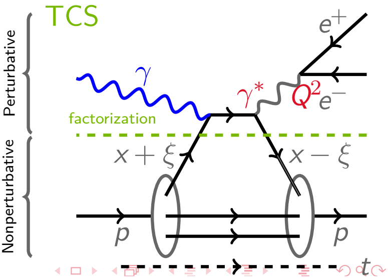





DVCS
Perturbative
Nonperturbative
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t

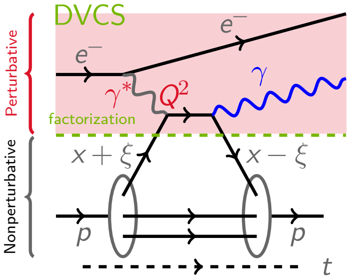

γ^*
Q^2
Generalized
Parton
Distributions
b_y
nucleon
b_x

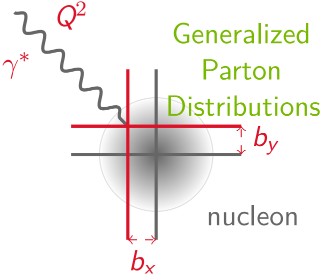

DVMP
Perturbative
Nonperturbative
e^-
e^-
$\pi, \rho, \ldots$
γ^*
Q^2
factorization
$x+\xi$
$x-\xi$
p
p
t

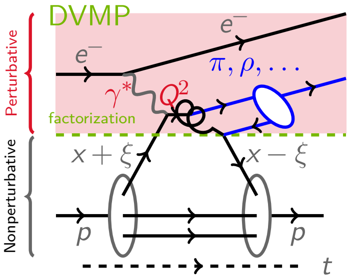

TCS
Perturbative
Nonperturbative
e^+
γ
γ^*
Q^2
e^-
factorization
$x+\xi$
$x-\xi$
p
p
t

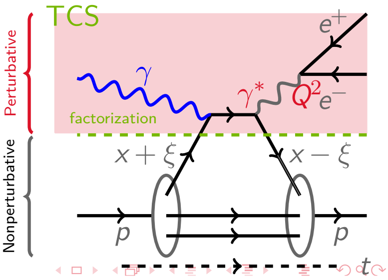
Exclusive processes of current interest (1/2).

Factorization and universality.



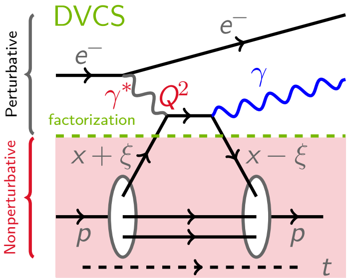


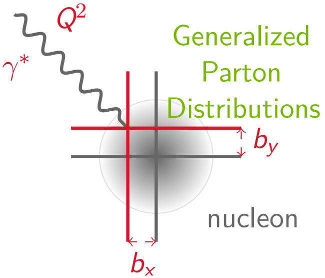


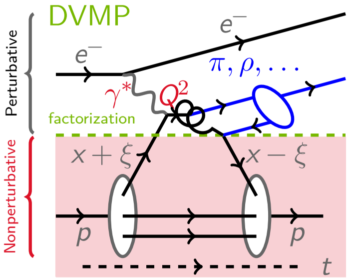


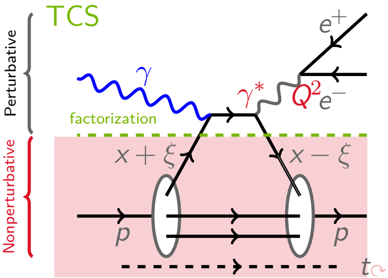
Exclusive processes of current interest (1/2).

Factorization and universality.









Exclusive processes of present interest (2/2).

Factorization and universality.

Bjorken regime : large Q^2 and fixed $xB \simeq 2\xi/(1+\xi)$

- Partonic interpretation relies on **factorization theorems**.
- All-order proofs for DVCS, TCS and some DVMP.
- GPDs depend on a (arbitrary) factorization scale μ_F.
- **Consistency** requires the study of **different channels**.

- GPDs enter DVCS through **Compton Form Factors** :

$$\mathcal{F}(\xi, t, Q^2) = \int_{-1}^{1} dx\, C\left(x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right) F(x, \xi, t, \mu_F)$$

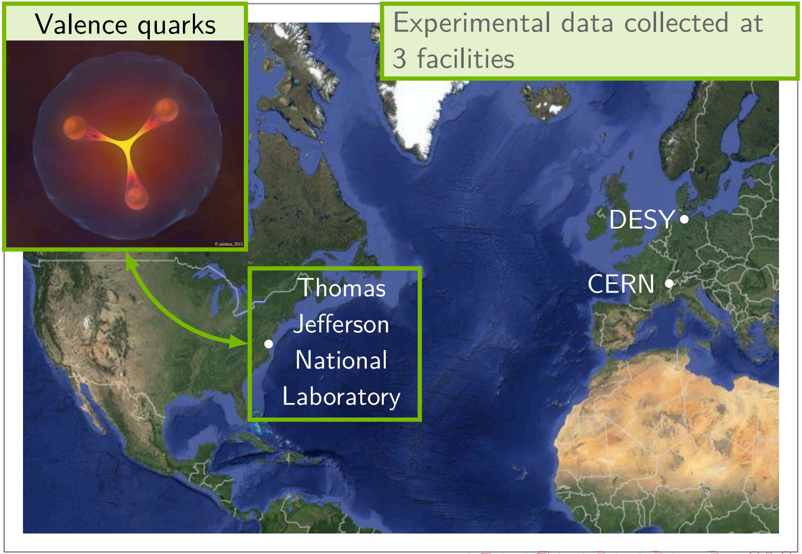
for a given GPD F.

- CFF $\mathcal{F}$ is a **complex function**.

Experimental data collected at 3 facilities

Need for global fits of world data.

Different facilities will probe different kinematic domains.



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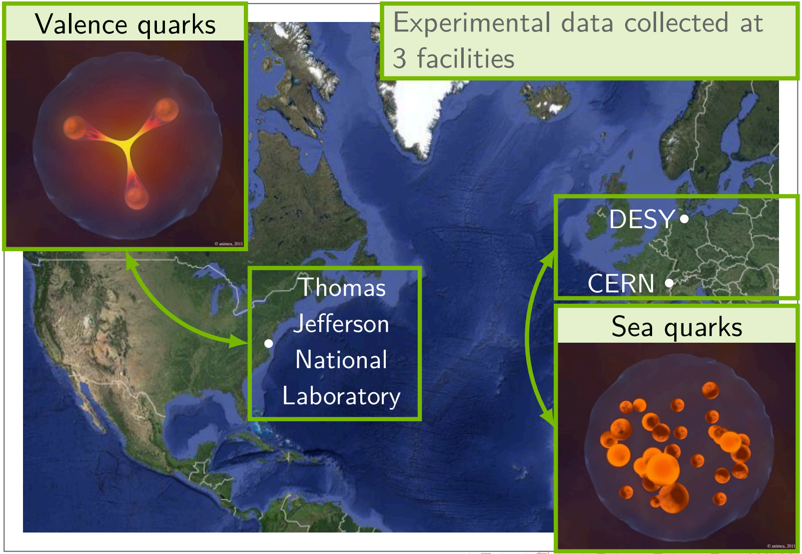

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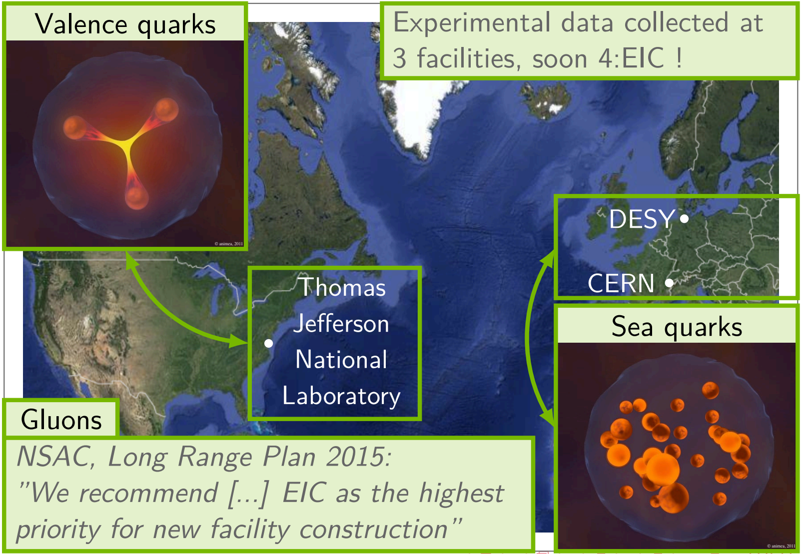
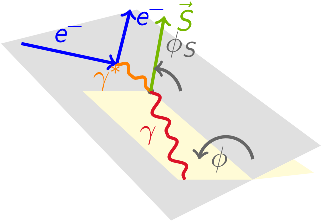


Typical DVCS kinematics.

Probing gluons, sea and valence quarks through DVCS.



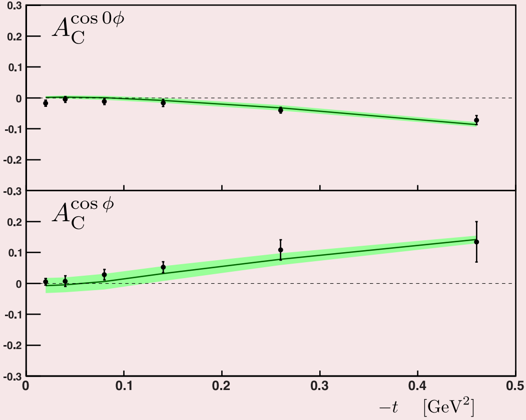
- Study the **harmonic structure** of $ep \to ep\gamma$ amplitude.

Diehl *et al.*,
Phys. Lett. **B411**, 193 (1997)

Experiment	Kinematics		
	x_B	Q^2 [GeV^2]	t [GeV^2]
HERA	0.001	8.00	-0.30
COMPASS	0.05	2.00	-0.20
HERMES	0.09	2.50	-0.12
CLAS	0.19	1.25	-0.19
HALL A	0.36	2.30	-0.23

No parameter of the GK model **was tuned** to analyse DVCS.

Beam Charge Asymmetry, HERMES

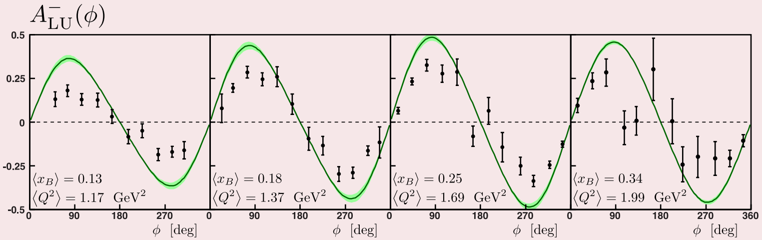


Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

Goloskokov-Kroll (GK) model on DVCS.

No parameter of the GK model **was tuned** to analyse DVCS.

Beam Spin Asymmetry, CLAS



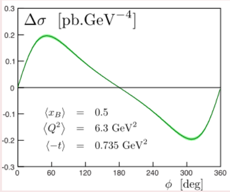
Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

- **Dominance** of twist-2 and **validity** of a GPD analysis of DVCS data.
- $Im\mathcal{H}$ **best determined**. Large uncertainties on $Re\mathcal{H}$.
- However sizable **higher twist contamination** for DVCS measurements.
- Already some indications about the **invalidity** of the H-dominance hypothesis with **unpolarized data**.

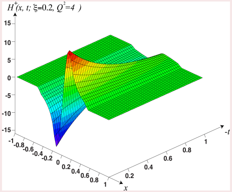
Imaging the nucleon. How?

Extracting GPDs is not enough...Need to extrapolate!

1. Experimental data fits



2. GPD extraction



3. Nucleon imaging

Images from Guidal et al., Rept. Prog. Phys. 76 (2013) 066202

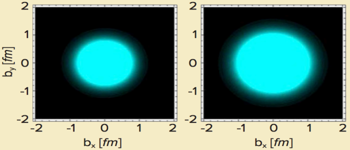
Reaching for the Horizon

The 2015 Long Range Plan for Nuclear Science

Sidebar 2.2: The First 3D Pictures of the Nucleon

A computed tomography (CT) scan can help physicians pinpoint minute cancer tumors, diagnose tiny broken bones, and spot the early signs of osteoporosis. Now physicists are using the principles behind the procedure to peer at the inner workings of the proton. This breakthrough is made possible by a relatively new concept in nuclear physics called generalized parton distributions.

An intense beam of high-energy electrons can be used



1. **Extract** $H(x, \xi, t, \mu_F^{\mathrm{ref}})$ from experimental data.
2. **Extrapolate** to vanishing skewness $H(x, 0, t, \mu_F^{\mathrm{ref}})$.
3. **Extrapolate** $H(x, 0, t, \mu_F^{\mathrm{ref}})$ up to infinite t.
4. **Compute** 2D Fourier transform in transverse plane:

$$H(x, b_\perp) = \int_0^{+\infty} \frac{\mathrm{d}\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp) H(x, 0, -\Delta_\perp^2)$$

5. **Propagate** uncertainties.
6. **Control** extrapolations with an accuracy matching that of experimental data with **sound** GPD models.

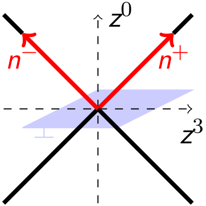
Theoretical framework

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) \quad = \quad \frac{1}{2} \int \frac{dz^-}{2\pi} \, e^{ixP^+ z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0 \\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**

$$H^q(x, 0, 0) = q(x)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) \quad = \\ \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\middle| \pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

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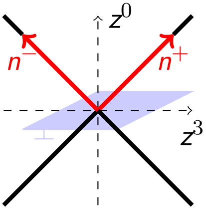

References

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Ji, Phys. Rev. Lett. **78**, 610 (1997)

Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**

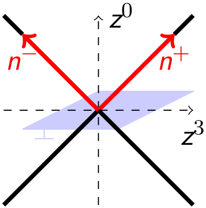
$$\int_{-1}^{+1} dx\, H^q(x, \xi, t) = F^q_1(t)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) = \frac{1}{2} \int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+ z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0 \\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



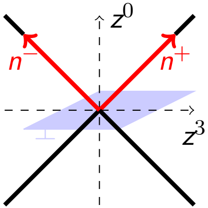
References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.

Polynomiality

$$\int_{-1}^{+1} dx\, x^n H^q(x, \xi, t) = \text{polynomial in } \xi$$

Polynomiality

Lorentz covariance

- **Polynomiality**

Lorentz covariance

- **Positivity**

$$H^q(x,\xi,t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

Spin-0 Generalized Parton Distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2}\phi_\pi^q\left(\frac{1+x}{2}\right)$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

How can we implement *a priori* these theoretical constraints?

- There is no known GPD parameterization **relying only on first principles**.
- In the following, focus on **polynomiality** and **positivity**.

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x, \xi, t)$$

$$= \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

linear combination of $(P^+)^{m+1-k}(\Delta^+)^k$ for $0 \leq k \leq m+1$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x, \xi, t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \,.$$

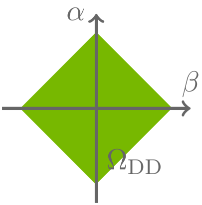
Double Distributions.

A convenient tool to encode GPD properties.

- Define Double Distributions F^q and G^q as matrix elements of **twist-2 quark operators**:

$$\left\langle P+\frac{\Delta}{2}\right| \bar{q}(0)\gamma^{\{\mu} i\overleftrightarrow{D}^{\mu_1}\dots i\overleftrightarrow{D}^{\mu_m\}} q(0)\left|P-\frac{\Delta}{2}\right\rangle = \sum_{k=0}^{m}\binom{m}{k}$$

$$\left[F^q_{mk}(t)2P^{\{\mu} - G^q_{mk}(t)\Delta^{\{\mu}\right] P^{\mu_1}\dots P^{\mu_{m-k}}\left(-\frac{\Delta}{2}\right)^{\mu_{m-k+1}}\dots\left(-\frac{\Delta}{2}\right)^{\mu_m\}}$$



with

$$F^q_{mk} = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\, \alpha^k \beta^{m-k} F^q(\beta,\alpha)$$

$$G^q_{mk} = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\, \alpha^k \beta^{m-k} G^q(\beta,\alpha)$$

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Rev. **D59**, 014030 (1999)
Radyushkin, Phys. Lett. **B449**, 81 (1999)

Double Distributions.

Relation to Generalized Parton Distributions.

- Representation of GPD:

$$H^q(x,\xi,t) = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\,\delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t)+\xi\,G^q(\beta,\alpha,t)\big)$$

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Pobylitsa gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x,\xi,t) = (1-x)\int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\, f^q(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Pobylitsa, Phys. Rev. **D67**, 034009 (2003)
Müller, Few Body Syst. **55**, 317 (2014)

- Choose $F^q(\beta, \alpha) = 3\beta\theta(\beta)$ ad $G^q(\beta, \alpha) = 3\alpha\theta(\beta)$:

$$H^q(x, \xi) = 3x \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi)$$

- Simple analytic expressions for the GPD:

$$\begin{aligned} H(x, \xi) &= \frac{6x(1-x)}{1-\xi^2} \text{ if } 0 < |\xi| < x < 1, \\ H(x, \xi) &= \frac{3x(x+|\xi|)}{|\xi|(1+|\xi|)} \text{ if } -|\xi| < x < |\xi| < 1. \end{aligned}$$

Double Distributions.

Lorentz covariance by example.

- Compute first Mellin moments.

n	$\int_{-\xi}^{+\xi} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{+\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{-\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$
0	$\frac{1+\xi-2\xi^2}{1+\xi}$	$\frac{2\xi^2}{1+\xi}$	1
1	$\frac{1+\xi+\xi^2-3\xi^3}{2(1+\xi)}$	$\frac{2\xi^3}{1+\xi}$	$\frac{1+\xi^2}{2}$
2	$\frac{3(1-\xi)(1+2\xi+3\xi^2+4\xi^3)}{10(1+\xi)}$	$\frac{6\xi^4}{5(1+\xi)}$	$\frac{3(1+\xi^2)}{10}$
3	$\frac{1+\xi+\xi^2+\xi^3+\xi^4-5\xi^5}{5(1+\xi)}$	$\frac{6\xi^5}{5(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{5}$
4	$\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5-6\xi^6}{7(1+\xi)}$	$\frac{6\xi^6}{7(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{7}$

- Expressions get more complicated as n increases... But they always yield polynomials!

Positivity.

A consequence of the positivity of the nom in a Hilbert space.

- Identify the matrix element defining a GPD as an **inner product** of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, *e.g.*:

$$|H^q(x,\xi,t)| \leq \sqrt{\frac{1}{1-\xi^2} q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- This procedures yields **infinitely many inequalities** stable under LO evolution.

Pobylitsa, Phys. Rev. **D66**, 094002 (2002)

- The **overlap representation** guarantees *a priori* the fulfillment of positivity constraints.

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x\mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x}\mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \big(\psi_N^{(\beta,\lambda)}\big)^* (\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWFs.

- Expressions for vertices and propagators:

$$
\begin{aligned}
S(p) &= \left[-i\gamma \cdot p + M \right] \Delta_M(p^2) \\
\Delta_M(s) &= \frac{1}{s + M^2} \\
\Gamma_\pi(k, p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z \, \rho_\nu(z) \; \left[\Delta_M(k_{+z}^2) \right]^\nu \\
\rho_\nu(z) &= R_\nu (1 - z^2)^\nu
\end{aligned}
$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:

Overlap representation.

Positivity by example: GPD H^q in the Dyson-Schwinger approach.

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Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.

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Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν

Overlap representation.

Positivity by example: GPD H^q in the Dyson-Schwinger approach.

- Expressions for vertices and propagators:

$$\begin{aligned} S(p) &= \left[-i\gamma \cdot p + M \right] \Delta_M(p^2) \\ \Delta_M(s) &= \frac{1}{s + M^2} \\ \Gamma_\pi(k, p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z \, \rho_\nu(z) \, \left[\Delta_M(k_{+z}^2) \right]^\nu \\ \rho_\nu(z) &= R_\nu (1 - z^2)^\nu \end{aligned}$$

with R_ν a normalization factor and $k_{+z} = k - p(1 - z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν. **Fixed to 1** to recover asymptotic pion DA.

Overlap representation.

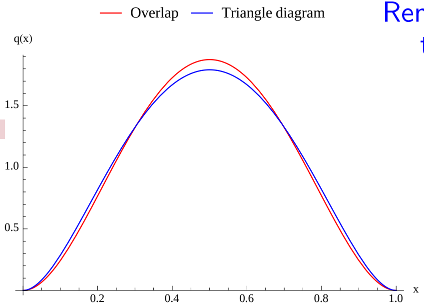
Positivity by example: GPD H^q in the Dyson-Schwinger approach.

- Evaluate LFWF in algebraic model:

$$\psi(x, \mathbf{k}_\perp) \propto \frac{x(1-x)}{[(\mathbf{k}_\perp - x\mathbf{P}_\perp)^2 + M^2]^2}$$

- Expression for the GPD at $t = 0$:

$$H(x, \xi, 0) \propto \frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}$$



Remember J. Rodríguez-Quintero's talk last Wednesday!

- Manifest 2-body symmetry.
- Expression for the PDF:

$$q(x) = 30x^2(1-x)^2$$

- Off-forward case: *in progress*.

Overlap representation.

Advantages and drawbacks.

- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of** N**-body problems**.

What is not obvious anymore

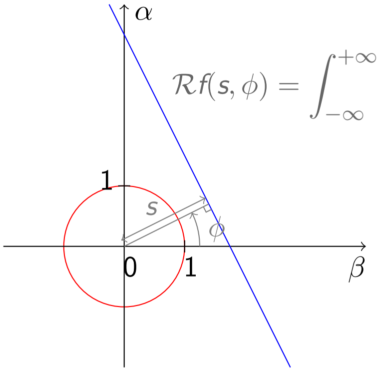
What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

Diehl, Phys. Rept. **388**, 41 (2003)

GPDs from Light Front Wave Functions

The Radon transform.

Definition and properties.



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s,\phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha \, f(\beta,\alpha)\delta(s - \beta\cos\phi - \alpha\sin\phi)$$

and:

$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Pobylitsa gauge

$$\frac{\sqrt{1+\xi^2}}{1-x} H(x,\xi) = \mathcal{R}f^{\text{Pobylitsa}}(s,\phi) \ ,$$

The range of the Radon transform.

The polynomiality property a.k.a. the Ludwig-Helgason condition.

- The Mellin moments of a Radon transform are **homogeneous polynomials** in $\omega = (\sin\phi, \cos\phi)$.
- The converse is also true:

Theorem (Hertle, 1983)

Let $g(s, \omega)$ an even compactly-supported distribution. Then g is itself the Radon transform of a compactly-supported distribution if and only if the ***Ludwig-Helgason consistency condition*** *hold:*

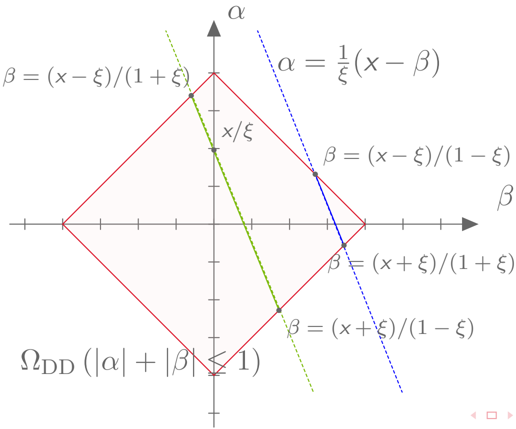
(i) *g is C^∞ in ω,*

(ii) *$\int \mathrm{d}s\, s^m g(s, \omega)$ is a homogeneous polynomial of degree m for all integer $m \geq 0$.*

- Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.

DGLAP and ERBL regions

$$(x, \xi) \in \text{DGLAP} \Leftrightarrow |s| \geq |\sin\phi| ,$$
$$(x, \xi) \in \text{ERBL} \Leftrightarrow |s| \leq |\sin\phi| .$$



Each point (β, α) with $\beta \neq 0$ contributes to **both** DGLAP and ERBL regions.

For **any model of LFWF**, one has to address the following three questions:

1. Does the extension exist?
2. If it exists, is it unique?
3. How can we compute this extension?

Theorem

Let f be a compactly-supported locally summable function defined on $\mathbb{R}^2$ and $\mathcal{R}f$ its Radon transform.
Let $(s_0, \omega_0) \in \mathbb{R} \times S^1$ and U_0 an open neighborhood of ω_0 such that:

$$\textit{for all } s > s_0 \textit{ and } \omega \in U_0 \quad \mathcal{R}f(s, \omega) = 0 .$$

Then $f(\aleph) = 0$ on the half-plane $\langle \aleph | \omega_0 \rangle > s_0$ of $\mathbb{R}^2$.

Consider a GPD H being zero on the DGLAP region.

- Take ϕ_0 and s_0 *s.t.* $\cos\phi_0 \neq 0$ and $|s_0| > |\sin\phi_0|$.
- Neighborhood U_0 of ϕ_0 *s.t.* $\forall \phi \in U_0 \quad |\sin \phi| < |s_0|$.
- The underlying DD f has a zero Radon transform for all $\phi \in U_0$ and $s > s_0$ (DGLAP).
- Then $f(\beta, \alpha) = 0$ for all $(\beta, \alpha) \in \Omega_{\text{DD}}$ with $\beta \neq 0$.
- Extension **unique** up to adding a **D-term**: $\delta(\beta) D(\alpha)$.

A discretized problem

Consider $N+1$ Hilbert spaces H, H_1, ..., H_N, and a family of continuous surjective operators $R_n : H \to H_n$ for $1 \leq n \leq N$. Being given $g_1 \in H_1$, ..., $g_n \in H_n$, we search f solving the following system of equations:

$$R_n f = g_n \quad \text{for } 1 \leq n \leq N$$

Fully discrete case

Assume f piecewise-constant with values f_m for $1 \leq m \leq M$. For a collection of lines $(L_n)_{1 \leq n \leq N}$ crossing Ω_{DD}, the Radon transform writes:

$$g_n = \mathcal{R}f = \int_{L_n} f = \sum_{m=1}^{M} f_m \times \mathrm{Measure}(L_n \cap C_m) \quad \text{for } 1 \leq n \leq N$$

Kaczmarz algorithm

Denote P_n the orthogonal projection on the *affine* subspace $R_n f = g_n$. Starting from $f^0 \in H$, the sequence defined iteratively by:

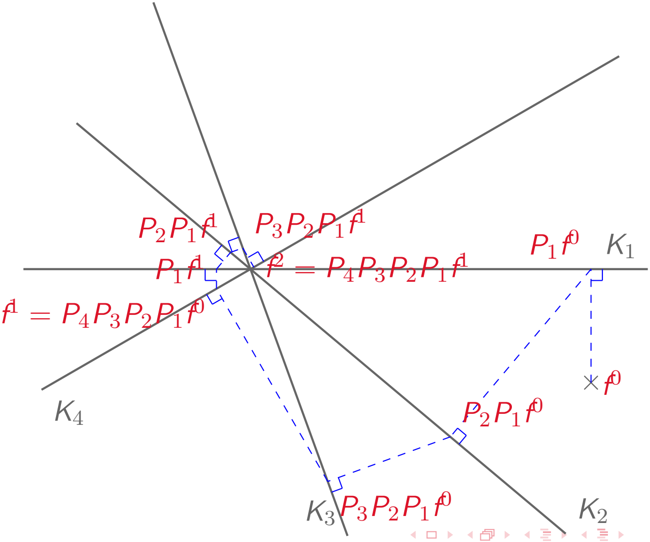
$$f^{k+1} = P_N P_{N-1} \ldots P_1 f^k$$

converges to the solution of the system.

The convergence is exponential if the projections are randomly ordered.

Strohmer and Vershynin, Jour. Four. Analysis and Appl. **15**, 437 (2009)

Computation of the extension.
Numerical evaluation of the inverse Radon transform (2/3).



And if the input data are inconsistent?

- Instead of solving $g = \mathcal{R}f$, find f such that $\|g - \mathcal{R}f\|_2$ is **minimum**.
- The solution **always exists**.
- The input data are **inconsistent** if $\|g - \mathcal{R}f\|_2 > 0$.

Relaxed Kaczmarz algorithm

Let $\omega \in]0, 2[$ and:

$$P_n^\omega = (1-\omega)\,\mathrm{Id}_H + \omega P_n \quad \text{for } 1 \leq n \leq N$$

Write:

$$RR^\dagger = (R_i R_j^\dagger)_{1 \leq i,j \leq N} = D + L + L^\dagger$$

where D is diagonal, and L is lower-triangular with zeros on the diagonal.

Theorem

Let $0 < \omega < 2$. For $f^0 \in \mathrm{Ran}\ R^\dagger$ (e.g. $f^0 = 0$), the Kaczmarz method with relaxation converges to the unique solution $f^\omega \in \mathrm{Ran}\ R^\dagger$ of:

$$R^\dagger (D + \omega L)^{-1} (g - R f^\omega) = 0 \ ,$$

where the matrix D and L appear in the decomposition of $RR^\dagger$. If $g = \mathcal{R} f$ has a solution, then f^ω is its solution of minimal norm. Otherwise:

$$f^\omega = f_{MP} + \mathcal{O}(\omega) \ ,$$

where f_{MP} is the minimizer in H of:

$$\langle g - \mathcal{R} f | g - \mathcal{R} f \rangle_D \ ,$$

the inner product being defined by:

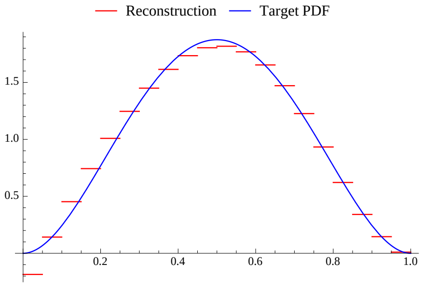
$$\langle h | k \rangle_D = \langle D^{-1} h | k \rangle \ .$$

Test on a 1D example.

Recovering a PDF from the knowledge of its Mellin moments.

A pion valence PDF-like example

Aim: reconstruct the PDF $q(x) = 30x^2(1-x)^2$ from the knowledge of its first 30 Mellin moments.



- Piecewise-constant PDF: 20 values.
- Input: 30 Mellin moments.
- Unrelaxed method $\omega = 1$.
- 10000 iterations.

- Extensive testing *in progress*
  - Various inputs: PDFs and LFWFs.
  - Numerical noise.

Conclusion

Conclusions and prospects.

Positivity and polynomiality constraints consistently implemented.

- Last decade demonstrated **maturity of GPD phenomenology**.
- **Good theoretical control** on the path between GPD models and experimental data.
- **Challenging constraints** expected from Jefferson Lab in the valence region.
- Building of **QCD-inspired models** to make progress.
- **Systematic** procedure to construct GPD models from any "reasonable" Ansatz of LFWFs.
- **Characterization** of the **existence** and **unicity** of the extension from the DGLAP to the ERBL region.
- Numerical tests *in progress*. Stay tuned!

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