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THE TRANSITION TEMPERATURE OF THE WEAKLY INTERACTING BOSE GAS*

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ABSTRACT

The effect of a weak repulsive two-body interaction on the transition temperature of a dilute gas Bose gas at fixed density has been controversial for a long time. We show here that the *effect is non-perturbative* in nature.

Renormalization group then allows proving that the critical temperature T_c increases linearly with the strength of the interaction, parametrized in terms of the s-wave scattering length. However, the coefficient cannot be obtained from perturbative calculations.

Recognizing that the hamiltonian of the system, which also describes the *Helium superfluid transition*, is the $N = 2$ example of the general N vector model, one generalizes the problem to arbitrary N. The coefficient of $\Delta T_c/T_c$ can then be expanded in powers of $1/N$.

The leading order result, which is *independent of N for non-trivial reasons*, is in reasonable agreement with estimates coming from numerical simulations.

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1 Introduction

The effect of a weak repulsive two-body interaction on the transition temperature of a dilute gas Bose gas at fixed density has been controversial for a long time [1,2,3,4,5,6]. It has been slowly realized that the effect was non-perturbative in nature. It has recently been argued theoretically [7] that T_c increases linearly with the strength of the interaction parametrized in terms of the scattering length a. Since the coefficient cannot be obtained from perturbation theory, in Ref. [7] a simple self-consistent approximation was used to derive an explicit estimate.

In [8], first the linear behaviour was derived more directly, using general renormalization group arguments. Then, recognizing that the hamiltonian of the system under study, which also describes the helium superfluid transition, is a particular example of the general N vector model, for N =2, the problem was generalized to arbitrary N. This generalization makes new tools available [9,10,11]; in particular, the coefficient of $\Delta T_c/T_c$ can be calculated by carrying out an expansion in $1/N$. In [8], the leading order in $1/N$ was calculated explicitly. The result happens to be independent of N, for non-trivial reasons. The calculation involves subtle technical points which are most easily dealt with by dimensional regularization [12]. Surprisingly, the result is in reasonable agreement with the most recent numerical simulations [13].

This paper is organized as follows: In section 2, we recall the basis of Bose–Einstein condensation. In section 3, we describe the field integral representation of the partition function of the dilute Bose gas (see e.g., ref. [14,15]) and lay out the basics of the problem. Then, in section 4, we derive the linear behaviour of the shift of the critical temperature for weak two-body interactions, using renormalization group arguments. Finally, in section 5, we present the general N-vector model and we calculate the leading order $1/N$ contribution to the temperature shift. After the article [8], the $1/N$ correction to the shift has been calculated [16] and other methods employed to calculate the amplitude of the shift. For a recent review see ref. [17].

2 Independent bosons: Bose–Einstein condensation

Equation of state. The equation of state of a gas relates the average number of particles $\langle \mathbf{N} \rangle$, the chemical potential μ and the inverse temperature $T = 1/\beta$. In the case of independent bosons, the equation of state takes the simple form

$$\langle \mathbf{N} \rangle = \sum_i \langle n_i \rangle , \quad \langle n_i \rangle = \frac{1}{\mathrm{e}^{\beta(\omega_i - \mu)} - 1} , \tag{2.1}$$

where $\langle n_i \rangle$ is the average occupation number of the level of energy ω_i.

The equation is defined only for $\mu < \inf_i \omega_i$.

This expression can also be expressed in terms of the one-particle hamiltonian operator $H^{(1)}$, with eigenvalues ω_i, as

$$\langle \mathbf{N} \rangle = \operatorname{tr} \frac{1}{\mathrm{e}^{\beta(H^{(1)}-\mu)} - 1} \,. \tag{2.2}$$

Bose–Einstein (BE) condensation. For a quantum hamiltonian operator of the form $H^{(1)}(\hat{p},\hat{q})$ with a discrete spectrum, the equation of state, in the high temperature limit $\beta\delta\omega \ll 1$, reduces to its semi-classical approximation. In d dimensions,

$$\langle \mathbf{N} \rangle \sim \frac{1}{(2\pi\hbar)^d} \int \frac{\mathrm{d}^d p \, \mathrm{d}^d q}{\mathrm{e}^{\beta(H^{(1)}(\mathbf{p},\mathbf{q})-\mu)} - 1} \,.$$

The average number of particles $\langle \mathbf{N} \rangle$ is an increasing function of $\mu \le \mu_c = \inf H^{(1)}(\mathbf{p},\mathbf{q})$. If the integral converges for $\mu = \mu_c$, one faces the apparent paradox that $\langle \mathbf{N} \rangle$ is bounded at fixed temperature.

The solution to the paradox is that the *semi-classical approximation is no longer valid for the lowest energy level*, which yields a divergent contribution for $\mu = E_{\text{ground state}}$. Returning to the exact expression, one infers that *the additional particles accumulate into the ground state*: this is the essence of BE condensation.

In the rather theoretical example of particles of mass m in a box of linear size L (a harmonic potential is closer to the experimental conditions), high temperature is equivalent to $L \to \infty$. One finds, for $d > 2$, the critical density

$$n_c = \langle \mathbf{N} \rangle \, L^{-d} \underset{L\to\infty}{\sim} \frac{1}{(2\pi\hbar)^d} \int \frac{\mathrm{d}^d p}{\mathrm{e}^{\beta \mathbf{p}^2/2m} - 1} = \zeta(d/2)/\lambda^d, \tag{2.3}$$

where $\zeta(z)$ is the Riemann function, and λ the thermal wave length:

$$\lambda = \hbar\sqrt{2\pi/mT} \,.$$

Alternatively, at density n fixed, the equation of state has no solution for temperatures $T < T_c(n) \propto (\hbar^2/m) n^{2/d}$.

In what follows, we will only consider this situation, that is, particles in a box in the infinite volume limit.

3 Statistical field theory of the weakly interacting Bose gas

We want now to investigate the effect of weak interactions between bosons.

The partition function. The partition function of a *dilute gas* of identical bosons of mass m, at temperature $T = 1/\beta$, can be expressed as an integral

$$\mathcal{Z} = \int [\mathrm{d}\psi(t,x) \mathrm{d}\psi^*(t,x)] \, \mathrm{e}^{-\mathcal{S}(\psi^*,\psi)/\hbar},$$

over fields ψ^*, ψ periodic in euclidean time,

$$\psi(0,x) = \psi(\beta,x), \quad \psi^*(0,x) = \psi^*(\beta,x),$$

associated with boson creation and annihilation.

We assume here that the gas is *sufficiently dilute* for the two-body interactions to be weak and three-body or higher interactions to be totally negligible.

Since one is interested only in long wavelength phenomena, *the two-body potential can be replaced by a delta-function* and parametrized in terms of the s-wave scattering length a (positive because the interaction is assumed repulsive).

For $d = 3$, the effective euclidean action of the system may then be written as

$$\mathcal{S}(\psi^*,\psi) = -\int_0^\beta \mathrm{d}t \int \mathrm{d}^3x \left[\psi^*(t,x)\left(\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla_x^2 + \mu\right)\psi(t,x) + \frac{2\pi\hbar^2 a}{m}\left(\psi^*(t,x)\psi(t,x)\right)^2\right],$$

where μ is the chemical potential.

The condition that the interaction is weak implies that $a \ll \lambda$, where λ is the thermal wavelength

$$\lambda = \hbar\sqrt{2\pi/mT}\,.$$

Equation of state and two-point function. Quite generally, the equation of state can be expressed in terms of the $\langle\psi^*\psi\rangle$ correlation function G. In the Fourier representation,

$$n = T\frac{\partial \ln \mathcal{Z}}{\partial \mu} = T\int \frac{\mathrm{d}^d k}{(2\pi)^d}\sum_\nu \tilde{G}(k,\omega_\nu;\mu), \tag{3.1}$$

where $\omega_\nu = 2\pi\nu T$ are Matsubara frequencies, the quantization of ω being a consequence of the periodic boundary conditions in euclidean time.

A $U(1)$ phase transition of superfluid Helium type occurs at a critical chemical potential μ_c given by

$$\tilde{G}^{-1}(k=0,\omega=0;\mu_c) = 0\,,$$

where the correlation length diverges.

In the limit of free bosons

$$\tilde{G}^{-1}(k,\omega;\mu) = k^2/2m - i\omega - \mu\,.$$

Then, in a box of linear size L and $d > 2$, one recovers $\mu_c = 0$ and the critical density (2.3). The general equation of state (3.1) implies that, in the limit of vanishing repulsive interactions, the phase transition of the interacting model reduces to the BE condensation of the free Bose gas. In $d = 3$ dimensions, the relation between condensation temperature and density n reads

$$T_c^0(n) \underset{d=3}{\propto} (\hbar^2/m) n^{2/3}.$$

Thus, at leading order in the interaction, the shift in the transition temperature at fixed density, $\Delta T_c = T_c - T_c^0$, can be related, to the change Δn in the density at fixed T_c by

$$\frac{\Delta T_c}{T_c} = -\frac{2}{3}\frac{\Delta n}{n}.$$

The theory of critical phenomena tells us that the variation of the critical temperature in systems with dimension $d < 4$ depends primarily on contributions from the small momenta or large distance (or IR) region. This property simplifies the problem, since *to leading order* the IR properties are only sensitive to the $\omega_\nu = 0$ Matsubara frequency. The entire calculation can thus be cast in terms of a *classical statistical field theory.* The integration over the other non-zero Matsubara frequencies yields corrections to the effective action. They do not affect the temperature shift at leading order.

4 Effective classical statistical field theory and renormalization group

After a rescaling the field ψ in order to introduce more conventional field theory normalizations, and a parametrization in terms of two real fields ϕ_1, ϕ_2:

$$\psi = \sqrt{mT}(\phi_1 + i\phi_2),$$

the partition function takes the form of the field integral

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{S}(\phi)\right]$$

with

$$\mathcal{S}(\phi) = \int \left\{ \frac{1}{2}\left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r \phi^2(x) + \frac{u}{4!}\left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x\,, \tag{4.1}$$

where $r = -2mT\mu$ and, for $d = 3$, $u = 96\pi^2 a/\lambda^2$.

The euclidean action reduces to the ordinary $O(2)$ symmetric $(\phi^2)^2$ field theory, which also describes the universal properties of the superfluid Helium transition.

A UV large momentum cutoff $\Lambda \sim 1/\lambda \propto \sqrt{mT}$ is provided by the higher frequency modes. In the action, we have kept the dimension d arbitrary in order to use dimensional regularization later.

The goal is to obtain the leading order non-trivial contribution at criticality (in the massless theory) to

$$n = 4mT\rho\,, \quad \rho = \int^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\tilde{\Gamma}^{(2)}(k)},$$

where $\tilde{\Gamma}^{(2)}(k)$ is the vertex two-point function, inverse of the two-point correlation function.

Because the interactions are weak, one may imagine calculating the change in the transition temperature by perturbation theory. However, the perturbative expansion for a critical theory does not exist for any fixed dimension $d < 4$. A discussion of this problem requires *RG arguments.*

Renormalization group equations. We introduce the dimensionless coupling constant (the large momentum cutoff Λ is proportional to $1/\lambda$)

$$g = \Lambda^{d-4} u \propto (a/\lambda)^{d-2} \ll 1\,.$$

At T_c, $\tilde{\Gamma}^{(2)}(k)$ satisfies the RG equation

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta(g)\right) \tilde{\Gamma}^{(2)}(k, \Lambda, g) = 0\,. \tag{4.2}$$

For the $O(N)$ symmetric $(\phi^2)^2$ field theory, perturbative calculations yield

$$\beta(g) = -(4-d)g + \frac{N+8}{48\pi^2} g^2 + \mathcal{O}(g^3), \quad \eta(g) = \frac{N+2}{18} \frac{1}{(4\pi)^4} g^2 + \mathcal{O}(g^3).$$

We recall here that, in the framework of the $\varepsilon = 4-d$ expansion, the β-function has, below four dimensions, a non-trivial zero [19]

$$g^* = 48\pi^2 \varepsilon/(N+8) + \mathcal{O}(\varepsilon^2),$$

which governs the large distance behaviour of connected correlation functions. Precise estimates of the fixed point values and corresponding critical exponents can be found in [20].

Solution. Equation (4.2), together with dimensional analysis, implies that the vertex two-point function has the general form

$$\tilde{\Gamma}^{(2)}(p, \Lambda, g) = p^2 Z(g) F\big(p/\Lambda(g)\big) \tag{4.3}$$

with

$$\beta(g) \frac{\partial \ln Z(g)}{\partial g} = \eta(g), \quad \beta(g) \frac{\partial \ln \Lambda(g)}{\partial g} = -1\,.$$

Therefore,

$$Z(g) = \exp \int_0^g \frac{\eta(g')}{\beta(g')} \mathrm{d}g' = 1 + \mathcal{O}(g^2)$$

and

$$\delta\rho \underset{g\to 0}{\sim} \int^{\Lambda} \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{p^2} \left(\frac{1}{F(p/\Lambda(g))} - 1 \right).$$

On dimensional grounds $\Lambda(g)$ is proportional to Λ. The function $\Lambda(g)$ is then obtained by integration:

$$\Lambda(g) = g^{1/(4-d)} \Lambda \exp \left[- \int_0^g \mathrm{d}g' \left(\frac{1}{\beta(g')} + \frac{1}{(4-d)g'} \right) \right].$$

Crossover scale. The quantity $\Lambda(g)$ is a crossover scale separating a universal long-distance regime governed by the non-trivial zero g^* of the β-function, from a universal short distance regime governed by the gaussian fixed point $g = 0$. Such a regime exists only if $\Lambda(g) \ll \Lambda$, that is, if there is an intermediate scale between the IR and the microscopic scales; otherwise only the IR behaviour can be observed. In a generic situation g is of order unity and, thus, $\Lambda(g)$ is of order Λ: the universal large momentum region is absent.

Instead, $\Lambda(g) \ll \Lambda$ implies

$$g^{1/(4-d)} \exp \left[- \int_0^g \mathrm{d}g' \left(\frac{1}{\beta(g')} + \frac{1}{(4-d)g'} \right) \right] \ll 1 .$$

Since g (equal to a/λ for $d = 3$), is $\ll 1$, this condition is satisfied in the present situation.

Then,

$$\tilde{\Gamma}^{(2)}(p) \propto p^{2-\eta} \text{ for } p \ll \Lambda(g), \quad \tilde{\Gamma}^{(2)}(p) \propto p^2 \text{ for } \Lambda(g) \ll p \ll \Lambda,$$

with $\eta \equiv \eta(g^*) \approx 0.035$ for $N = 2$.

Linear behaviour. We now show that the condition $\Lambda(g) \ll \Lambda$ implies $\Delta T_c \propto \Lambda(g)$.

First, from perturbation theory one infers that for $d = 3$, the function $F(p)$ (equation (4.3)) behaves for large p as

$$F(p) = \tilde{\Gamma}^{(2)}(p)/p^2 = 1 + \mathcal{O}(\ln p/p^2).$$

Therefore, the first correction to the density is convergent at large momentum and independent of the cutoff procedure:

$$\delta\rho = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{p^2} \left(\frac{1}{F(p/\Lambda(g))} - 1 \right).$$

Similarly, the IR behaviour implies that this integral is IR convergent.

Setting $p = \Lambda(g)k$, one then finds the general form

$$\delta\rho = \Lambda(g) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{k^2} \left(\frac{1}{F(k)} - 1 \right) .$$

The g dependence is entirely contained in $\Lambda(g)$. For g small, one concludes

$$\frac{\delta\rho}{\rho} \propto g \sim (-3/2c_0) a n^{1/3} ,$$

a linear behaviour that, however, is non-perturbative! Moreover, the amplitude $-3c_0/2$ is universal.

5 The N-vector model. The large N expansion at order $1/N$

Since the function $F(p)$ cannot be obtained from a perturbative calculation, we consider the $O(N)$ symmetric generalization of the model corresponding to the euclidean action (4.1): the field $\phi(x)$ then has N real components ϕ_i, $i = 1, \ldots, N$.

The advantage of such a generalization is that it provides us with a tool, the large N expansion, which allows calculating at the critical point. The large N limit is taken at Nu fixed.

The first non-trivial correction to $\tilde{\Gamma}^{(2)}(p)$ appears only at order $1/N$ [18,14,11]:

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \frac{2}{N} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(6/Nu) + B(q)} \left(\frac{1}{(p+q)^2} - \frac{1}{q^2} \right) + \mathcal{O}\left(\frac{1}{N^2} \right) ,$$

where $B(q)$ is the one-loop contribution (the bubble diagram) to the perturbative four-point function

$$B(q) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 (k+q)^2} \underset{q\to 0}{\sim} b(\varepsilon) q^{-\varepsilon} ,$$

($\varepsilon = 4 - d > 0$) which is UV finite for $d < 4$. For $d = 3$, $b(1) = 1/8$.

One evaluates

$$\delta\rho = -\frac{2}{N} \int \frac{\mathrm{d}^d p}{(2\pi)^{2d}} \frac{1}{p^4} \frac{\mathrm{d}^d q}{(6/Nu) + b(\varepsilon) q^{-\varepsilon}} \left(\frac{1}{(p+q)^2} - \frac{1}{q^2} \right)$$

by keeping the dimension d generic and using dimensional regularization. This allows exchanging the order between the p and q integrations, something which is not allowed directly in dimension 3. In the $d = 3$ limit, the two integrations yield $(1/32\pi^2)(Nu/6)$.

As expected, $\delta\rho \propto u$:

$$\delta\rho = -u/96\pi^2 = -\frac{a}{\lambda^2}.$$

One finally obtains the change in the transition temperature [8]:

$$\frac{\Delta T_c}{T_c} = \frac{8\pi}{3\zeta(3/2)}\frac{a}{\lambda} = c_0 a n^{1/3} \text{ with}$$
$$c_0 = \frac{8\pi}{3\zeta(3/2)^{4/3}} = 2.33\ldots .$$

For non-trivial reasons, the leading order result does not depend on N. The calculation involves a $1/N$ correction, but the result is proportional to u, which appears only in the combination uN.

Nevertheless, the result is valid only for N large. Taking into account, the $1/N$ correction [16], one finds $c_0 = 1.71\ldots$. A number of other methods, lattice calculations [13], summation of perturbative expansions, suggest $c_0 \approx 1.3$. These $1/N$ estimates have a precision typical of other $1/N$ calculations.

6 Conclusions

Using RG arguments, we have shown that the properties of the dilute, weakly interacting Bose gas remain dominated by the UV fixed point up to large length scales; this is why one can still refer to BE condensation when discussing the phase transition of the interacting Bose gas.

RG arguments enabled us to confirm that the relative shift of the transition temperature at fixed density is proportional to the dimensionless combination $an^{1/3}$ for weak interactions. This result is non-perturbative and the proportionality coefficient, which is *universal*, cannot be obtained from perturbation theory.

Therefore, a *non-perturbative method*, the large N expansion, has been introduced that allows a systematic, analytic, calculation of this coefficient as a power series in $1/N$, where eventually one has to set $N = 2$.

The leading order contribution is formally of order $1/N$ multiplied by a function of aN, which is kept fixed in the large N limit. Because for $d = 3$ the result is linear in a, the $1/N$ factor somewhat surprisingly cancels and the result is independent of N. Adding the 30% $1/N$ correction, one finds a value in reasonable agreement with the most recent numerical estimates.

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