

Fixed point stability and decay of correlations

Ettore Vicari^{1,4} and Jean Zinn-Justin^{2,3}

¹ Dipartimento di Fisica dell, Università di Pisa and INFN, Pisa, Italy

² Dapnia, CEA/Saclay, F-91191 Gif-sur-Yvette cedex, France

³ Institut de Mathématiques de Jussieu-Chevaleret, Université de Paris 7, France

E-mail: Ettore.Vicari@df.unipi.it and zinn@dsm-mail.saclay.cea.fr

New Journal of Physics **8** (2006) 321

Received 14 November 2006

Published 14 December 2006

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/8/12/321

Abstract. In the framework of the renormalization-group (RG) theory of critical phenomena, a quantitative description of many continuous phase transitions can be obtained by considering an effective Φ^4 theories, having an N -component fundamental field Φ_i and containing up to fourth-order powers of the field components. Their RG flow is usually characterized by several fixed points (FPs). We give here strong arguments in favour of the following conjecture: the stable FP corresponds to the fastest decay of correlations, that is, is the one with the largest values of the critical exponent η describing the power-law decay of the two-point function at criticality. We prove this conjecture in the framework of the ε -expansion. Then, we discuss its validity beyond the ε -expansion. We present several lower-dimensional cases, mostly three-dimensional, which support the conjecture. We have been unable to find a counterexample.

⁴ Author to whom any correspondence should be addressed.

Contents

1. Introduction	2
2. Proof within the ϵ-expansion	4
3. Several verifications of the η conjecture	6
3.1. The $O(M) \otimes O(N)\Phi^4$ model	6
3.2. Φ^4 theory with cubic anisotropy	9
3.3. Spin-density-wave model	10
3.4. $U(N) \otimes U(N)$ -symmetric Φ^4 models	10
3.5. $SU(4)$ -symmetric Φ^4 model.	11
4. Φ^4 theories of multicritical behaviours	11
5. The GNY model	12
Appendix A. Some proofs within the ϵ-expansion	14
References	15

1. Introduction

In the framework of the renormalization-group (RG) approach to critical phenomena, a quantitative description of many continuous phase transitions can be obtained by considering effective Landau–Ginzburg–Wilson (LGW) Φ^4 theories, having an N -component fundamental field Φ_i and containing up to fourth-order powers of the field components. The fourth-degree polynomial form of the potential depends on the symmetry of the system. LGW Φ^4 theories generally present several fixed points (FPs) which are connected by RG trajectories. See, for example, figure 1, which shows the RG flow in the example of four FPs. Among them, the infrared stable FP determines the asymptotic critical behaviour of the corresponding statistical systems. FPs are determined by the zeros of β functions. The stability of the FP is related to the eigenvalues of its stability matrix: if all eigenvalues have a positive real part, then the FP is stable. An interesting question is whether a physical quantity exists such that the comparison of its values at the FPs identifies the most stable FP. In two dimensions (2D), the central charge is such a quantity: the stable FP in unitary theories is the one with the least value of the central charge [1]. But, despite several attempts and some progress, see, for example [2], no conclusive results on the extension of this theorem to higher dimensions have yet been obtained.

In this paper, we give strong arguments in favour of the following conjecture:

In general Φ^4 theories with a single quadratic invariant, the infrared stable FP is the one that corresponds to the fastest decay of correlations.

Therefore, it is the FP with the largest value of the critical exponent η which characterizes the power-law decay of the two-point correlation function $W^{(2)}(x)$ at criticality,

$$W^{(2)}(x) \propto \frac{1}{x^{d-2+\eta}}. \quad (1)$$

The exponent η is related to the RG dimension of the field, $d_\Phi = (d - 2 + \eta)/2$.

The conjecture holds in the case of the $O(N)$ -symmetric Φ^4 theory. Indeed, below 4D, the Gaussian FP, for which $\eta = 0$, is unstable against the non-trivial Wilson–Fisher FP for which

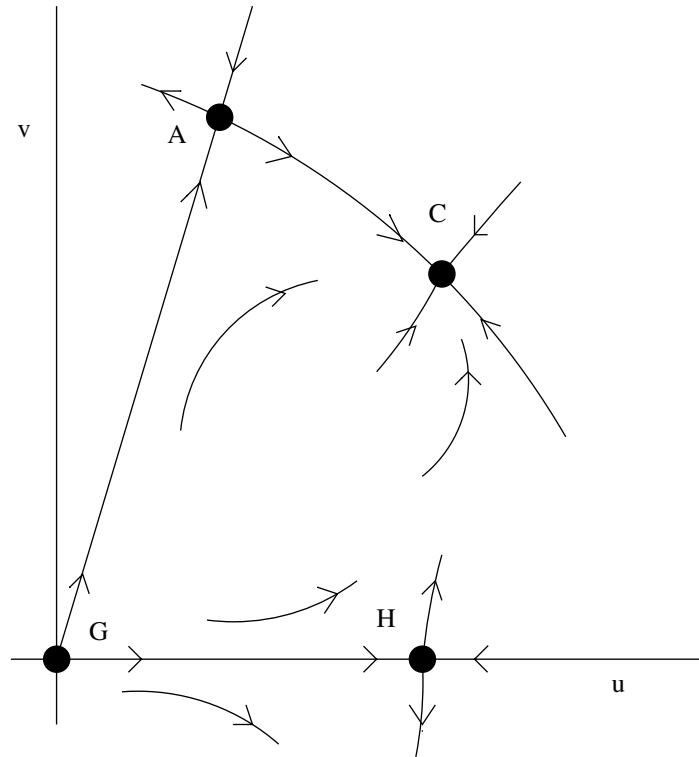


Figure 1. RG flow of the $O(M) \otimes O(N)$ model in the large- N limit.

$\eta \geq 0$. We recall that the positivity of η in unitary theories follows rigorously from the spectral representation of the two-point function [3].

In the absence of a sufficiently large symmetry restricting the form of the Φ^4 potential, many quartic couplings must be introduced—see, for example [3]–[8]. The Hamiltonian of a general Φ^4 theory for a N -component field Φ_i can be written as

$$H = \int d^d x \left[\frac{1}{2} \sum_i (\partial_\mu \Phi_i)^2 + \frac{1}{2} \sum_i r_i \Phi_i^2 + \frac{1}{4!} \sum_{ijkl} u_{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l \right]. \quad (2)$$

The number of independent parameters r_i and u_{ijkl} depends on the symmetry group of the theory. An interesting class of models are those in which $\sum_i \Phi_i^2$ is the unique quadratic polynomial invariant under the symmetry group of the theory. In this case, all r_i are equal, $r_i = r$ and u_{ijkl} must be such not to generate other quadratic invariant terms under RG transformations, for example, it must satisfy the trace condition [7] $\sum_i u_{iikl} \propto \delta_{kl}$. In these models, criticality is driven by tuning the single parameter r , which physically may correspond to the temperature. All field components become critical simultaneously and the two-point function in the disordered phase is diagonal, that is,

$$W_{ij}^{(2)}(x-y) \equiv \langle \Phi_i(x) \Phi_j(y) \rangle = \delta_{ij} W^{(2)}(x-y). \quad (3)$$

These Φ^4 theories have generally several FPs. Our conjecture applies to such class of unitary models. We do not consider nonunitary limits, such as $N \rightarrow 0$, which are relevant to describe the critical properties of spin systems in the presence of quenched disorder.

Actually, although within the ε -expansion around $d = 4$ one can prove that only one stable FP exists, in $d < 4$ general Φ^4 theories may have more than one stable FP with separate attraction domains. The η conjecture should be then refined by comparing FP that are connected by RG trajectories starting from the Gaussian FP: among them the stable FP is the one with the largest value of η .

It was already observed in [4] that, within the ε -expansion, the stable FP of theories with four FPs, is the one with the largest value of η . Here, we extend this ε -expansion result to an arbitrary Φ^4 theory. Then, we discuss the validity of the η conjecture at fixed dimension $d < 4$, where it remains a conjecture. We present several lower-dimensional cases, mostly 3D, which support the conjecture. We have been unable to find an analytical or numerical counterexample.

Finally, we extend the conjecture to multicritical points in models with several independent correlation lengths (different r_i) that diverge simultaneously. In this situation, the exponent η is replaced by a matrix and the conjecture applies to the trace of the matrix. However, the empirical evidence, beyond the ε -expansion is more limited.

The paper is organized as follows. In section 2, we prove the η conjecture within the ε -expansion for the most general Φ^4 with a single quadratic invariant. In section 3, we discuss several lower dimensional examples, showing that in all cases the stable FP is the one with the largest value of η . In section 4, we discuss the extension of the η conjecture to Φ^4 theories describing multicritical behaviours. In section 5, we discuss the Gross–Neveu–Yukawa (GNY) model and show that also in this case the infrared stable FP is the one with the fastest decay of the critical two-point function of the boson field, within the ε -expansion and in the large- N_f limit for any dimension.

2. Proof within the ε -expansion

Expansions in powers of $\varepsilon = 4 - d$ can be most easily obtained within the minimal-subtraction scheme [9], where the RG functions are computed from the divergent part of correlation functions [3],

$$\beta_{ijkl}(g_{abcd}) \equiv \mu \frac{\partial g_{ijkl}}{\partial \mu}, \quad \eta(g_{abcd}) \equiv \mu \frac{\partial \ln Z_\Phi}{\partial \mu}, \quad (4)$$

where g_{ijkl} are the renormalized couplings corresponding to the quartic parameters u_{ijkl} .

We consider only Hamiltonians that have a symmetry such that the quadratic invariant in the field is unique and the two-point function in the disordered phase thus diagonal. As a consequence of the diagonal property of the two-point correlation function in the disordered phase, the tensor u_{ijkl} has special properties that take the form of successive constraints in the perturbative expansion. At leading order one finds [7]

$$\beta_{ijkl}(g_{abcd}) = -\varepsilon g_{ijkl} + \frac{1}{16\pi^2} \sum_{m,n} (g_{ijmn} g_{mnkl} + g_{ikmn} g_{mnjl} + g_{ilmn} g_{mnkj}). \quad (5)$$

The RG function associated with the field dimension can be inferred from the function

$$\eta(g_{ijkl}) = \frac{1}{6N(4\pi)^4} \sum_{i,j,k,l} g_{ijkl} g_{ijkl}, \quad (6)$$

this form resulting from the diagonality condition

$$N \sum_{k,l,m} g_{iklm} g_{jklm} = \delta_{ij} \sum_{k,l,m,n} g_{klmn} g_{klmn}. \quad (7)$$

One can easily verify that the general expression of the one-loop β -function derives from a potential [10]. Indeed,

$$\beta_{ijkl}(g_{abcd}) = \frac{\partial U(g_{abcd})}{\partial g_{ijkl}}, \quad (8)$$

$$U(g_{abcd}) = -\frac{\varepsilon}{2} \sum_{i,j,k,l} g_{ijkl} g_{ijkl} + \frac{1}{(4\pi)^2} \sum_{i,j,k,l,m,n} g_{ijkl} g_{klmn} g_{mnij},$$

Such a property, which has been verified to two-loop order, is shared, at leading order, by other field theories and would deserve a more systematic investigation.

A detailed discussion of the properties of the RG flow within the ε -expansion can be found in [11]. Here, we list a number of consequences of Equation (8) within the ε -expansion. Note that none of these properties depends on the condition (3).

(i) The potential decreases along a RG trajectory and thus FPs are extrema of the potential. In particular, if two FPs are (asymptotically) connected by a RG trajectory, the stable FP corresponds to the lowest value of the potential.

(ii) The eigenvalues of the matrix of first order partial derivatives of the β functions (stability matrix) at a FP are real.

(iii) Stable FPs are local minima of the potential, that is, the matrix of second derivatives of $U(g)$ is positive.

Moreover, two additional properties depend of the special cubic form of the one-loop potential (8) (we give the proof in the appendix):

(iv) There exists at most one stable FP.

(v) The stable FP corresponds to the lowest value of the potential $U(g)$.

The latter properties are not necessarily valid beyond the ε -expansion. For example, in the physical dimensions $d = 3$, 2 LGW Φ^4 theories may have more than one stable FPs with separate attraction domains. This possibility does not occur within the ε -expansion, that is, close to 4D, but it does not contradict general RG arguments and it is found in some cases, in particular when different regions of the Hamiltonian quartic parameters are related to different symmetry breaking patterns. In the next section, we shall mention a few examples where this occurs.

In the framework of the ε -expansion, we now show that the stable FP (or at least the most stable one) corresponds to the largest value of the exponent η and thus to the case where correlation function has the fastest decay at large distance. For any FP g_{ijkl}^* , the equations

$$\beta_{ijkl}(g_{abcd}^*) = \frac{\partial U(g_{abcd}^*)}{\partial g_{ijkl}} = 0, \quad (9)$$

implies

$$\varepsilon \sum_{i,j,k,l} g_{ijkl}^* g_{ijkl}^* = \frac{3}{(4\pi)^2} \sum_{i,j,k,l,m,n} g_{ijkl}^* g_{klmn}^* g_{mnij}^*, \quad (10)$$

and, thus, at leading order,

$$U(g_{abcd}^*) = -\frac{1}{6}\varepsilon \sum_{i,j,k,l} g_{ijkl}^* g_{ijkl}^*, \quad (11)$$

which is negative, thus lower than the Gaussian FP value. At leading order, the exponent η is then given by

$$\eta = \frac{1}{6N} \frac{1}{(4\pi)^4} \sum_{i,j,k,l} g_{ijkl}^* g_{ijkl}^* = -\frac{1}{N\varepsilon} \frac{1}{(4\pi)^4} U(g_{abcd}^*). \quad (12)$$

As we have shown, the stable FP corresponds to the lowest value of U . It thus corresponds also to the largest value of the exponent η : therefore, the correlation functions corresponding to the stable FP have the fastest large distance decay.

The validity of this result beyond the ε -expansion remains a conjecture. In next section, we discuss several checks in lower dimensions, mostly in 3D.

3. Several verifications of the η conjecture

In this section, we discuss the RG flow, in $d < 4D$, of several Φ^4 theories with a single quadratic invariant, but more than one quartic term. As we shall see, in all examples considered below, the stable FP of the RG flow, within regions connected by RG trajectories starting from the Gaussian FP, is the one corresponding to the largest value of η .

3.1. The $O(M) \otimes O(N) \Phi^4$ model

We first consider the $O(M) \otimes O(N) \Phi^4$ model corresponding to the hamiltonian density

$$\mathcal{H} = \frac{1}{2} \sum_{a,i} [(\partial_\mu \Phi_{ai})^2 + r\Phi_{ai}^2] + \frac{1}{4!} u_0 \left(\sum_{a,i} \Phi_{ai}^2 \right)^2 + \frac{1}{4!} v_0 \sum_{a,i,b,j} [\Phi_{ai}\Phi_{bi}\Phi_{aj}\Phi_{bj} - \Phi_{ai}^2\Phi_{bj}^2], \quad (13)$$

where Φ_{ai} is a $M \times N$ real matrix ($a = 1, \dots, M$ and $i = 1, \dots, N$). The symmetry of this model is $O(M) \otimes O(N)$.

3.1.1. The large- N limit. The $O(M) \otimes O(N) \Phi^4$ model can be solved in the large- N limit for any fixed M [12, 13]. For any $M \geq 2$, one finds four FPs: the Gaussian FP, the Heisenberg $O(M \times N)$ FP and two new FPs which we call chiral (C) and antichiral (A). Figure 1 shows a sketch of the RG flow in the quartic-coupling space. In the large- N limit, for any $M \geq 2$ and for any $2 < d < 4$, the stable FP is the chiral one and all other FPs are unstable. The large- N critical exponent η has been calculated for all FPs

$$\eta = \frac{\eta_1 e_d}{N} + O(1/N^2), \quad (14)$$

where

$$e_d = -\frac{4\Gamma(d-2)}{\Gamma(2-d/2)\Gamma(d/2-1)\Gamma(d/2-2)\Gamma(d/2+1)}, \quad (15)$$

and

$$\eta_1 = \begin{cases} 0 & \text{Gaussian} \\ 1/M & \text{Heisenberg} \\ (M+1)/2 & \text{chiral} \\ (M-1)(M+2)/(2M) & \text{antichiral} \end{cases} \quad (16)$$

$O(1/N^2)$ calculations can be found in [13]. One easily verifies that for any M and $2 < d < 4$ the value of η at the stable chiral FP is the largest one.

3.1.2. $d = 3$ results from high-order FT perturbative analyses. Several results have also been obtained concerning the RG flow of the 3D $O(M) \otimes O(N)$ Φ^4 models at finite values of M, N . Below we restrict ourselves to the case $M = 2$, that is, to $O(2) \otimes O(N)$ models. The cases $N = 2, 3$ are physically interesting because they could describe transitions in non-collinear frustrated magnets, the superfluid transition in ^3He , *etc.* . . . See, for example [6, 14] and references therein. We have to distinguish the cases $v_0 > 0$ and $v_0 < 0$, because they lead to different symmetry breaking patterns:

$$O(2) \otimes O(N) \rightarrow O(2) \otimes O(N-2) \quad \text{for } v_0 > 0, \quad (17)$$

$$O(2) \otimes O(N) \rightarrow Z_2 \otimes O(N-1) \quad \text{for } v_0 < 0. \quad (18)$$

The u -axis plays the role of a separatrix and thus the RG flow corresponding to $v_0 > 0$ cannot cross the u -axis. The relevant FPs of models with the symmetry-breaking pattern (17) lie in the region $v > 0$, where v is the renormalized quartic coupling associated with v_0 , while the relevant FPs of models with the symmetry-breaking pattern (18) lie in the region $v \leq 0$.

All $O(2) \otimes O(N)$ models contain the Gaussian FP and the Heisenberg $O(2N)$ FP. They are both unstable. The relevant perturbation at the $O(2N)$ FP, which makes it unstable for any $N \geq 2$, is related to the v -term in the Hamiltonian, which is a particular combination of quartic operators transforming as the spin-0 and spin-4 representations of the $O(2N)$ group. Any spin-4 quartic perturbation is relevant at the $O(K)$ FP for $K \geq 3$, since its RG dimension $y_{4,4}$ is positive for $K \geq 3$ [15]. Therefore the $O(2N)$ FP is always unstable in the RG flow of the $O(2) \otimes O(N)$ models (actually this result extends to any $O(M) \otimes O(N)$ model with $M \geq 2$). In particular, $y_{4,4} \approx 0.11$ at the $O(4)$ FP and $y_{4,4} \approx 0.27$ at the $O(6)$ FP.

The RG flow of 3D $O(2) \otimes O(N)$ models has been investigated by computing and analysing high-order perturbative series within the massive zero-momentum (MZM) and massless $\overline{\text{MS}}$ schemes, respectively to six and five loops [16]–[19]. Some results are reported below.

RG flow for $N = 2$ and $v < 0$. One finds a stable FP in the region $v < 0$, which is in the XY universality class [6, 14]. We recall that the other relevant FPs are the Gaussian FP and the $O(4)$ FP. The best available estimates of η for $O(N)$ models are reported in table 1. They support the η conjecture, which would require $\eta_{XY} > \eta_{O(4)}$. Indeed the best available estimates are $\eta_{XY} = 0.0381(2)$ and $\eta_{O(4)} = 0.0365(10)$.

Table 1. Best theoretical estimates of the critical exponents for 3D $O(N)$ models. Concerning the methods, IHT indicates high-temperature expansion of improved lattice models with suppressed leading scaling corrections; MC indicates Monte Carlo simulations; FT indicates field-theoretical methods based on perturbative expansions; MC+IHT exploits a synergy of IHT and MC. Other results can be found in [3, 6].

N	ν	η	Method	References
1	0.63012(16)	0.0364(2)	IHT	[20]
	0.63020(12)	0.0368(2)	MC	[21]
	0.6304(13)	0.034(3)	FT	[22, 23]
2	0.6717(1)	0.0381(2)	MC+IHT	[24]
	0.6703(15)	0.035(3)	FT	[22, 23]
3	0.7112(5)	0.0375(5)	MC+IHT	[25]
	0.7073(35)	0.0355(25)	FT	[22, 23]
4	0.749(2)	0.0365(10)	MC	[26]
	0.741(6)	0.0350(45)	FT	[22]
5	0.779(3)	0.034(1)	MC	[27]
	0.764(4)	0.031(3)	FT	[28]
6	0.789(5)	0.029(3)	FT	[28]
8	0.830	0.027	FT	[29]
Large N	$1 - 32/(3\pi^2 N)$	$8/(3\pi^2 N)$	$1/N$ exp	[30]

RG flow for $N = 2$ and $v > 0$. The analysis of the high-order MZM and $\overline{\text{MS}}$ expansions provide a rather robust evidence of the existence of another stable chiral FP for $v > 0$ [16, 17, 19]. This has been confirmed by MC simulations of a lattice Φ^4 model [19]. This FP is not connected with the one found in the $v < 0$ region, because the line $v = 0$ is a separatrix. The estimates of η at this stable FP are: $\eta_{\text{ch}} = 0.09(1)$ from MZM and $\eta_{\text{ch}} = 0.09(4)$ from $\overline{\text{MS}}$. These results must be compared with the values of η at the other FPs connected by RG trajectories in the region $v > 0$, which are the Gaussian and the $O(4)$ FPs. Again the conjecture is verified because $\eta_{\text{ch}} > \eta_{O(4)} > 0$.

RG flow for $N = 3$ and $v < 0$. There is a stable FP with attraction domain in the region $v < 0$ [18]. The corresponding estimates of η are: $\eta = 0.079(7)$ from MZM and $\eta = 0.086(24)$ from $\overline{\text{MS}}$. The other FPs are the Gaussian FP and $O(6)$ FP, which have much smaller values of η , in particular $\eta_{O(6)} = 0.029(3)$ [28].

RG flow for $N = 3$ and $v > 0$. There is a stable FP also in the region $v > 0$ [19]. The corresponding estimates of η are: $\eta = 0.10(1)$ from MZM and $\eta = 0.08(3)$ from $\overline{\text{MS}}$. This values are again much larger than the values of η of the unstable Gaussian and $O(6)$ FPs.

Note that the stable 3D FPs of $O(2) \otimes O(N)$ models with $N = 2, 3$ do not exist close to 4D, apart from the one for $N = 2$ and $v < 0$, see, for example [14]. Thus, they provide a rather non-trivial check of the η conjecture, because the 3D RG flow differs qualitatively from the RG flow close to $d = 4$, which is obtained from the ε -expansion.

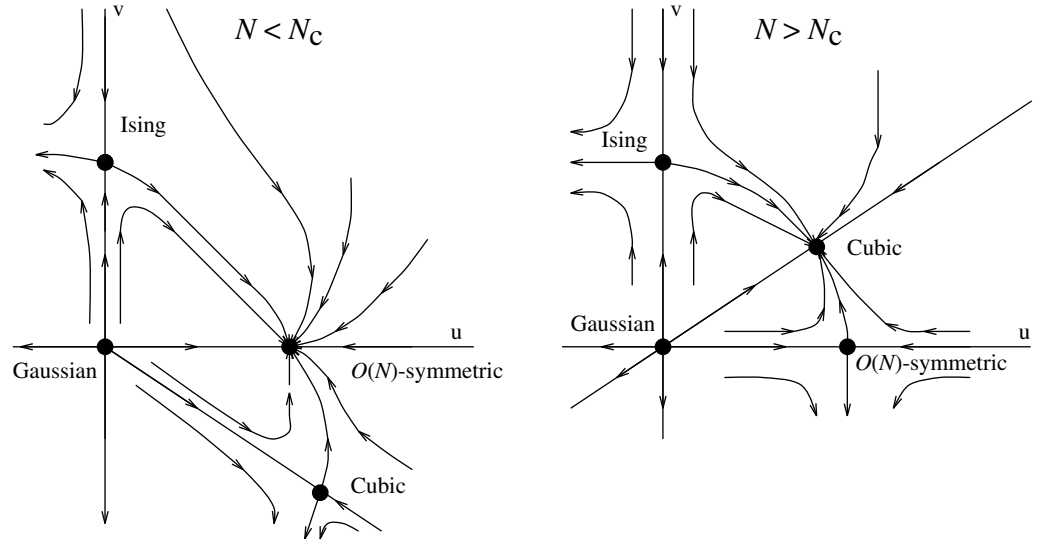


Figure 2. RG flow of the cubic model. $N_c \lesssim 3$ in $d = 3$, see, for example [31].

3.2. Φ^4 theory with cubic anisotropy

This theory is relevant for magnets with cubic anisotropy. Its Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \sum_i [(\partial_\mu \Phi_i)^2 + r\Phi_i^2] + \frac{1}{4!} u_0 \left(\sum_i \Phi_i^2 \right)^2 + \frac{1}{4!} v_0 \sum_i \Phi_i^4, \quad (19)$$

where Φ_i is a N -component field. The RG flow of this Hamiltonian has four FPs: the trivial Gaussian one, the Ising one in which the N components of the field decouple, the $O(N)$ -symmetric and the cubic FPs. The Gaussian FP is always unstable and so is the Ising FP for any number of components N . Indeed, at the Ising FP one may interpret the cubic Hamiltonian as the Hamiltonian of N Ising systems coupled by the $O(N)$ -symmetric interaction. The coupling term $\int d^d x \Phi_i^2 \Phi_j^2$ with $i \neq j$ scales as the integral of the product of two operators Φ_i^2 . Since the Φ_i^2 operator has RG dimension $1/\nu_1$ —indeed, it is associated with the temperature—the combined operator has RG dimension $2/\nu_1 - d = \alpha_1/\nu_1$ and therefore the associated crossover exponent is given by $\phi = \alpha_1$, independently of N [5]. Since $\alpha_1 > 0$, the Ising FP is unstable independently of N . On the other hand, the stability properties of the $O(N)$ -symmetric and of the cubic FPs depend on N . For sufficiently small values of N , $N < N_c$, the $O(N)$ -symmetric FP is stable and the cubic one is unstable. For $N > N_c$, the opposite is true: the RG flow is driven toward the cubic FP, which now describes the generic critical behaviour of the system. Figure 2 sketches the flow diagram in the two cases $N < N_c$ and $N > N_c$. In $d = 3$ $N_c \approx 2.9$, see for example, [6, 31] and references therein. Therefore, the $O(N)$ FP is stable only for $N = 2$. Let us now examine the various cases in more detail.

The case $N = 2$. We have four FPs: the Gaussian FP, the Ising FP along the $u = 0$ axis, the cubic FP for $v < 0$ which turns out to be equivalent to an Ising FP and the $O(2)$ FP which is the stable one. Again the η conjecture does not fail: it requires $\eta_{XY} > \eta_{\text{Ising}}$ and this is verified by the best 3D estimates reported in table 1, i.e. $\eta_{XY} = 0.0381(2)$ and $\eta_{\text{Ising}} = 0.0364(2)$.

It is also worth mentioning the 2D RG flow of this theory, where there is a line of stable FPs connecting the Ising and XY FPs [32, 33], with central charge $c = 1$ (in this case the central

charge associated with the Ising FP is $c = 1$ because it represents two decoupled Ising models). Along this line the critical exponent η does not vary, $\eta = 1/4$, but the correlation-length critical exponent goes from $\nu = 1$ to $\nu = \infty$.

The case $N = 3$. We have four FPs: the Gaussian FP, the Ising FP along the $u = 0$ axis, the $O(3)$ FP and the cubic FP for $\nu > 0$ which is the stable one. The stable cubic FP turns out to be very close to the $O(3)$ one and the critical exponents are not distinguishable with the best $O(3)$ estimates. Indeed, FT estimates of their differences give [34] $\nu_c - \nu_{O(3)} = -0.0003(3)$ and $\eta_c - \eta_{O(3)} = -0.0001(1)$. The η conjecture would require $\eta_c - \eta_{O(3)} > 0$; the above FT estimate of the difference $\eta_c - \eta_{O(3)}$, although favouring a negative sign, is not sufficiently precise to conclude that the η conjecture fails. In conclusion, also in this case the η conjecture is substantially consistent with the available results: for $N = 3$ $\eta_{\text{cubic}} \approx \eta_{O(3)} > \eta_{\text{Ising}} > 0$.

The case $N > 3$. For $N > 3$ the analysis of the six-loop series reported in [31] is consistent with the conjecture. One finds $\eta_{\text{cubic}} \approx \eta_{\text{Ising}}$, but the precision is not sufficient to determine which one is larger.

The case $N \rightarrow \infty$. For $N \rightarrow \infty$, keeping Nu and ν fixed, one can derive exact expressions for the exponents at the cubic FP. Indeed, for $N \rightarrow \infty$ the system can be reinterpreted as a constrained Ising model, leading to a Fisher renormalization [35] of the Ising critical exponents. One has

$$\eta = \eta_I + O(1/N), \quad \nu = \frac{\nu_I}{1 - \alpha_I} + O(1/N), \quad (20)$$

where η_I , ν_I and α_I are the critical exponents of the Ising model. Again the η conjecture does not fail.

3.3. Spin-density-wave model

The spin-density-wave model is a rather complicate Φ^4 model with five quartic parameters, which could describe the SDW-SC-to-SC phase transition in high- T_c superconductors (cuprates) [36]. Its Hamiltonian density is

$$\begin{aligned} \mathcal{H} = & |\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + \frac{u_{1,0}}{2}(|\Phi_1|^4 + |\Phi_2|^4) \\ & + \frac{u_{2,0}}{2}(|\Phi_1^2|^2 + |\Phi_2^2|^2) + w_{1,0}|\Phi_1|^2|\Phi_2|^2 + w_{2,0}|\Phi_1 \cdot \Phi_2|^2 + w_{3,0}|\Phi_1^* \cdot \Phi_2|^2 \end{aligned} \quad (21)$$

where Φ_a are complex N -component vectors. The RG flow of this model has been investigated in [36]. The physical interesting cases are those for $N = 2, 3$. The analysis reported in [36] suggests the existence of a stable FP in both cases, with rather large values of η : $\eta = 0.12(1)$ and $\eta = 0.18(3)$ respectively (from MZM). There are also several unstable FPs in the RG flow, but all of them have smaller values of η . Therefore, the available $d = 3$ results of this spin-density-wave model support the η conjecture.

3.4. $U(N) \otimes U(N)$ -symmetric Φ^4 models

Let us now consider the Hamiltonian density

$$\mathcal{H} = \text{tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r \text{tr} \Phi^\dagger \Phi + \frac{u_0}{4}(\text{tr} \Phi^\dagger \Phi)^2 + \frac{v_0}{4} \text{tr}(\Phi^\dagger \Phi)^2 \quad (22)$$

where Φ_{ij} is a $N \times N$ complex matrix. The symmetry is $U(N)_L \otimes U(N)_R$. In the case Φ_{ij} is also symmetric, the symmetry is $U(N)$. This model has been introduced and studied [37, 38] because it is relevant for the finite-temperature transition in QCD, that is, the theory of the strong interactions. Indeed, for $v_0 > 0$ the ground state leads to the symmetry-breaking pattern $U(N)_L \otimes U(N)_R \rightarrow U(N)_V$ (corresponding to QCD if the $U(1)_A$ anomaly is neglected).

For $N = 1$, the model reduces to the $O(2)$ -symmetric Φ^4 theory. For $N \geq 3$ no stable FP is found [38].

We focus on the case $N = 2$. One can easily identify two FPs. One is the Gaussian FP for $u = v = 0$, which is always unstable. Since for $v_0 = 0$ the Hamiltonian becomes equivalent to the one of the $O(8)$ -symmetric model, the corresponding $O(8)$ FP must exist on the $v = 0$ axis for $u > 0$. The $O(8)$ FP is also unstable because the v -term in the Hamiltonian represents a spin-4 perturbation with respect to the $O(8)$ FP and such perturbations are relevant for any $O(K)$ models with $K \geq 3$ [15]. Within one-loop ε -expansion calculations, no other FP is found [37]. By contrast, 3D analysis of both MZM and $\overline{\text{MS}}$ expansions show the presence of a stable FP in 3D [38]. The corresponding value of η is $\eta \approx 0.1$, significantly larger than $\eta_{O(8)} \lesssim 0.03$. Therefore, it supports the η conjecture.

3.5. $SU(4)$ -symmetric Φ^4 model

Let us now consider the Hamiltonian density

$$\mathcal{H} = \text{tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r \text{tr} \Phi^\dagger \Phi + \frac{u_0}{4} (\text{tr} \Phi^\dagger \Phi)^2 + \frac{v_0}{4} \text{tr}(\Phi^\dagger \Phi)^2 + w_0 (\det \Phi^\dagger + \det \Phi), \quad (23)$$

where Φ is a complex and symmetric 4×4 matrix field. The symmetry of this model is $SU(4)$. For $v_0 > -\frac{3}{2}|w_0|$, the theory describes the symmetry-breaking pattern $SU(4) \rightarrow SO(4)$ [38], which is the appropriate symmetry-breaking pattern to describe transitions in a QCD-like theory with quarks in the adjoint representation.

Close to 4D there are only two FPs, the Gaussian and the $O(20)$ FPs, which are both unstable. They remain unstable even at lower dimensions and thus are of no relevance for the critical behaviour. The 3D RG flow has been investigated by field-theoretical methods based on perturbative approaches, within the MZM and $\overline{\text{MS}}$ schemes [38]. They show the presence of a stable 3D FP characterized by the symmetry-breaking pattern $SU(4) \rightarrow SO(4)$. The corresponding value of η is rather larger: $\eta \approx 0.2$, much larger than the values of η of the Gaussian and $O(20)$ FPs which is $\eta_{O(20)} \approx 0.013$.

4. Φ^4 theories of multicritical behaviours

In this section, we discuss the extension of the η conjecture to Φ^4 theories describing multicritical behaviours, characterized by more than one independent correlation lengths. In this situation, the ε -expansion indicates that the stable FP should be the one with the largest value of the trace of the η matrix.

We discuss this point within the Φ^4 theory

$$\mathcal{H} = (\partial_\mu \vec{\phi}_1)^2 + (\partial_\mu \vec{\phi}_2)^2 + r_1 \vec{\phi}_1^2 + r_2 \vec{\phi}_2^2 + u_1 (\vec{\phi}_1^2)^2 + u_2 (\vec{\phi}_2^2)^2 + w \vec{\phi}_1^2 \vec{\phi}_2^2 \quad (24)$$

where $\phi_{1,2}$ are two $O(n_1)$ and $O(n_2)$ order parameters, with n_1 and N_2 real components respectively. The symmetry is $O(n_1) \oplus O(n_2)$. This Φ^4 theory describes the multicritical

behaviour arising from the competition of orderings with symmetries $O(n_1)$ and $O(n_2)$, at the point where the corresponding transition lines meet in the phase diagram [39]. Such multicritical points arise in several physical contexts, for instance in anisotropic antiferromagnets, in high- T_c superconductors, etc See, for example [15] and references therein.

The multicritical behaviour is determined by the RG flow in the quartic-coupling space when $r_{1,2}$ are tuned to their critical values. Four FPs are found: the Gaussian FP, the isotropic $O(n_1 + n_2)$ FP (describing an effective enlargement of the symmetry), a decoupled $O(n_1)$ - $O(n_2)$ FP (which describes effectively decoupled order parameters) and a biconal FP. The main properties of the 3D RG flow are the following.

(i) For $n_1 + n_2 \geq 4$ the decoupled FP is stable. This can be inferred from non-perturbative arguments [40] that show that the RG dimension y_w of the perturbation $P_w = \phi_1^2 \phi_2^2$ that couples the two order parameters is

$$y_w = \frac{1}{\nu_1} + \frac{1}{\nu_2} - 3, \quad (25)$$

where ν_1, ν_2 are the correlation-length exponents of the $O(n_1)$ and $O(n_2)$ models. Inserting the numbers reported in table 1 one finds $y_w < 0$ for $n_1 + n_2 \geq 4$ and any $n_1, n_2 \geq 1$.

(ii) Field-theoretical methods based on perturbative expansions (six-loop in the MZM and $O(\varepsilon^5)$ in the ε -expansion) show that the isotropic $O(n_1 + n_2)$ FP is unstable for $n_1 + n_2 \geq 3$ [15]. Therefore, only in the case of two Ising order parameters can the symmetry be effectively enlarged from $Z_2 \oplus Z_2$ to $O(2)$, at the multicritical point where the Ising lines meet.

(iii) For $n_1 = 1$ (Ising), $n_2 = 2$ (XY), $O(\varepsilon^5)$ calculations show that the stable FP is the biconal FP [15]. Its critical exponents turn out to be very close to the $O(3)$ ones, in fact they are not distinguishable within the errors of the best estimates of the $O(3)$ critical exponents, see table 1.

If we compare the isotropic and the decoupled FPs, the conjecture on the trace of the η matrix should give

$$n_1 \eta_{O(n_1)} + n_2 \eta_{O(n_2)} > (n_1 + n_2) \eta_{O(n_1+n_2)}, \quad (26)$$

for $n_1 + n_2 \geq 4$ and

$$\eta_{XY} > \eta_{\text{Ising}} \quad (27)$$

Moreover, from the point (iii) above, we should have

$$\text{tr } \eta_{\text{biconal}} > \eta_{\text{Ising}} + 2\eta_{XY} \quad (28)$$

All these relations are verified by, or when the precision is not sufficient are consistent with, the best estimates of the exponent η for the $O(N)$ models, see table 1.

5. The GNY model

In this final section, we discuss the GNY model [41] and show that the infrared stable FP of its RG flow is the one characterized by the fastest decay of the critical two-point function of the boson field.

The Lagrangian of the GNY model is

$$\mathcal{L} = - \sum_i \bar{\psi}_i (\gamma_\mu \partial_\mu + g_0 \sigma) \psi_i + \frac{1}{2} [(\partial_\mu \sigma)^2 + m^2 \sigma^2] + \frac{1}{4!} u_0 \sigma^4, \quad (29)$$

where ψ_i are N_f fermionic fields and σ is a real scalar field. The relation between the GNY and the standard Gross–Neveu model is discussed in [41]. Its RG flow can be investigated within the ε -expansion. The RG functions at one-loop order are [41]

$$\beta_u = -\varepsilon u + \frac{1}{8\pi^2} \left(\frac{3}{2} u^2 + Nu g^2 - 6Ng^4 \right), \quad (30)$$

$$\beta_{g^2} = -\varepsilon g^2 + \frac{N+6}{16\pi^2} g^4,$$

where $N = N_f \text{tr} I$ (the trace is in the γ -matrix space) is the total number of fermion components. In 4D $\text{tr} I = 4$ and thus $N = 4N_f$ in equation (30). u and g are the MS renormalized couplings associated with u_0 and g_0 , respectively. Note that at one-loop the RG β function for a completely general Yukawa coupling [42] derives also for a potential (see also [3]).

The β functions of the GNY model have three FPs. Beside the unstable Gaussian FP, there is an Ising FP at

$$u_* = \frac{16\pi^2}{3} \varepsilon, \quad g_*^2 = 0, \quad (31)$$

which is also unstable. The infrared stable FP of the theory is the Gross–Neveu FP at

$$u_* = \frac{384N\pi^2}{(N+6)[(N-6) + (N^2 + 132N + 36)^{1/2}]} \varepsilon, \quad g_*^2 = \frac{16\pi^2}{N+6} \varepsilon. \quad (32)$$

We can now compare the corresponding RG dimensions of the scalar field and check if the infrared stable FP is the one with the largest value of η_σ . Calculations of the scalar field RG functions show that the critical exponent η_σ is maximum at the GN FP

$$\eta_\sigma = \frac{N}{N+6} \varepsilon, \quad (33)$$

while $\eta_\sigma = O(\varepsilon^2)$ at the Ising FP.

The GNY model is soluble in the large- N_f limit for any d [41], for a review see [30]. In this limit one finds

$$\eta_\sigma = 4 - d + O(1/N_f), \quad (34)$$

and therefore $\eta_\sigma = 1 + O(1/N_f)$ in 3D. The values of η_σ at the Ising FP are definitely smaller, for example $\eta_\sigma \approx 0.036$ in 3D, see table 1, and $\eta_\sigma = 1/4$ in 2D.

In conclusions, the above analytical results suggest that in the GNY model the infrared stable FP is the one that corresponds to the fastest decay of correlations of the scalar field, as in the Φ^4 theories.

Appendix A. Some proofs within the ε -expansion

In the framework of the ε -expansion, we prove two consequences of the property of gradient flow discussed in section 2: (i) there exists at most one stable FP; (ii) the stable FP corresponds to the lowest value of the potential. Indeed, let us assume the existence of two FPs corresponding to the parameters g^* and g'^* . We then consider the parameters g of the form

$$g(s) = sg^* + (1-s)g'^*, \quad 0 \leq s \leq 1, \quad (\text{A.1})$$

and the corresponding potential $u(s) = U(g(s))$. As the explicit form (8) shows, at leading order $u(s)$ is a third degree polynomial in s . The derivative

$$u'(s) = \sum_a g'_a(s) \frac{\partial U}{\partial g_a} = \sum_a (g_a^* - g_a'^*) \frac{\partial U}{\partial g_a} = \sum_a (g_a^* - g_a'^*) \beta_a(g(s)) \quad (\text{A.2})$$

vanishes due to the fixed point conditions at $s = 0$ and $s = 1$: $u'(0) = u'(1) = 0$. Since $u'(s)$ is a second degree polynomial, it then has necessarily the form

$$u'(s) = As(1-s). \quad (\text{A.3})$$

The second derivative $u''(s)$ is given in terms of the matrix of second partial derivatives of U and, thus, the partial derivatives of the β -functions, by

$$u''(s) = \sum_{a,b} (g_a^* - g_a'^*) \frac{\partial^2 U(g(s))}{\partial g_a \partial g_b} (g_b^* - g_b'^*) = A(1-2s) \quad (\text{A.4})$$

In particular, for $s = 0$ and $s = 1$

$$A = \sum_{a,b} (g_a^* - g_a'^*) \frac{\partial^2 U(g'^*)}{\partial g_a \partial g_b} (g_b^* - g_b'^*), \quad (\text{A.5})$$

$$-A = \sum_{a,b} (g_a^* - g_a'^*) \frac{\partial^2 U(g^*)}{\partial g_a \partial g_b} (g_b^* - g_b'^*). \quad (\text{A.6})$$

At a stable FP, the matrix \mathbf{U}'' of partial second derivatives of U is positive. Thus, if g^* and g'^* are stable FPs, A and $-A$ are both given by the expectation value of a positive matrix and thus are both positive, which is contradictory: the two FPs cannot both be stable.

More generally, the sign of A characterizes, in some sense, the relative stability of these two FPs. Let us assume, for example, $A < 0$ which is consistent with the assumption that g^* is stable. Then $u'(s) < 0$ in $[0, 1]$ and $U(g(s))$ is a decreasing function. Thus,

$$U(g^*) < U(g'^*). \quad (\text{A.7})$$

In particular, if g^* is a stable FP, it corresponds, among all FPs, to the lowest value of the potential.

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