Magneto-Gravito-Inertial waves in strongly stratified stellar interiors

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Abstract. Stellar radiation zones are stable strongly stratified rotating magnetic regions. The buoyancy force, the Coriolis acceleration and the Lorentz force are thus ruling the gravity waves dynamics. In this work, we examine the behaviour of these waves in stellar interiors and we show how the approximations assumed in the non-magnetic case (for gravito-inertial waves) can be generalized.

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MOTIVATION

Stellar radiation zones are stable strongly stratified rotating magnetic regions. Then, fluid dynamics in such zones is ruled by the buoyancy force, the Coriolis and the centrifugal accelerations, and the Lorentz force. Furthermore, they are the seat of transport and mixing processes that drive, with the nuclear reactions, the secular evolution of stars (cf. Zahn 1992). On the other hand, observations from the space and the ground give us now more and more refine constraints about those physical processes in stellar interiors. In this way, we thus have to go beyond the non-rotating and the non-magnetic description of stars oscillations and internal transport processes.

In this work, we focus on the waves which propagate in such regions, namely the internal (gravity) waves. These waves transport angular momentum and are a serious candidate to explain the angular velocity profile, for example in the solar radiative core (Talon & Charbonnel 2005). Since stellar radiation zones are differentially rotating and potentially magnetic regions, internal waves dynamics is then modified by the Coriolis acceleration (the centrifugal one can be neglected to the first order in rotation) and by the Lorentz force, thus becoming Magneto-Gravito-Inertial waves (MGI waves; see Kumar, Talon & Zahn 1999). They are equivalent to the MAC waves (for Magnetic-Archimedean-Coriolis force) studied in Geophysics for the liquid Earth's core, the relative importance of each force being however different with a strong domination of the stratification restoring force in stellar interiors. In this first paper, we thus focus on the study of the structure of such waves in this regime in the simplified case of a uniform rotation and of an azimuthal magnetic field such that the Alfvén frequency is uniform.

WAVES IN A ROTATING MAGNETIC STAR

First, we introduce $\vec{B}$ the macroscopic magnetic field. It is formed by the sum of the large-scale azimuthal field, $\vec{B}_0$, associated to an uniform Alfvén frequency, A_0, and of the wave's magnetic field, $\vec{b}$:

$$\vec{B}\left(\vec{r},t\right)=\vec{B}_0\left(\vec{r}\right)+\vec{b}\left(\vec{r},t\right)\ \text{with}\ \vec{B}_0=\sqrt{\mu\rho}\,r\sin\theta\,A_0\,\widehat{\mathbf{e}}_\varphi\,. \tag{1}$$

t is the time, $\vec{r}$ the position vector, and (r,θ,φ) are the usual spherical coordinates with theirs associated unit vector basis $\{\widehat{\mathbf{e}}_k\}_{k=\{r,\theta,\varphi\}}$. ρ is the density and μ is the magnetic permeability of the medium.

Next, we define $\vec{V}$ the macroscopic velocity field. It is formed by the sum of the large-scale azimuthal velocity, $\vec{V}_0$, associated to the uniform rotation (Ω_s is the angular velocity) and of the wave velocity, $\vec{u}$:

$$\vec{V}\left(\vec{r},t\right)=\vec{V}_0\left(\vec{r}\right)+\vec{u}\left(\vec{r},t\right)\ \text{with}\ \vec{V}_0=r\sin\theta\,\Omega_s\,\widehat{\mathbf{e}}_\varphi\,. \tag{2}$$

To study the dynamics of MGI waves in stellar radiation zones, the classical perfect MHD inviscid system of dynamical equations has to be solved; we linearize it around the rotating magnetic steady-state.

To achieve this aim, each scalar field X (the density, the gravitational potential Φ, and the pressure P) is then expanded as the sum of its hydrostatic value, $\overline{X}$, and of the wave's associated fluctuation, $\widetilde{X}$, as:

$$X\left(r,\theta,\varphi,t\right)=\overline{X}\left(r\right)+\widetilde{X}\left(r,\theta,\varphi,t\right). \tag{3}$$

In this work, as a first step, we neglect the non-spherical character of the hydrostatic background due to the deformations associated to the centrifugal acceleration, $\vec{\gamma}_{\rm c}(\Omega_s) = \frac{1}{2}\Omega_s^2\vec{\nabla}\left(r^2\sin^2\theta\right)$, and to the Lorentz force associated to $\vec{B}_0$, $\vec{F}_{\mathscr{L}}^{\,0}\left(\vec{B}_0\right) = \frac{1}{\mu}\left(\vec{\nabla}\times\vec{B}_0\right)\times\vec{B}_0$.

Following Braginsky (1967) and Braginsky & Roberts (1975), the wave's lagrangian displacement $\vec{\eta}$ is introduced:

$$\vec{u} = \partial_t\left[\vec{\eta}\left(\vec{r},t\right)\right] - \vec{\nabla}\times\left[\vec{V}_0\times\vec{\eta}\left(\vec{r},t\right)\right]. \quad (4)$$

Using Eq. (2), it becomes:

$$\vec{u} = \left(\partial_t + \Omega_s\partial_\varphi\right)\vec{\eta}\,. \quad (5)$$

Then, inserting Eq. (4) in the induction equation, we get

$$\vec{b} = \vec{\nabla}\times\left(\vec{\eta}\times\vec{B}_0\right), \quad (6)$$

which leads using the definition of $\vec{B}_0$ given in Eq. (1) to

$$\vec{b} = \sqrt{\mu\overline{\rho}}A_0\partial_\varphi\vec{\eta}\,. \quad (7)$$

Then, the momentum equation is given by

$$\begin{aligned}&\left(\partial_t + \Omega_s\,\partial_\varphi\right)\left[\left(\partial_t + \Omega_s\,\partial_\varphi\right)\vec{\eta} + 2\,\Omega_s\,\widehat{\mathbf{e}}_z\times\vec{\eta}\right]\\ &= -\frac{1}{\overline{\rho}}\vec{\nabla}\Pi\left(\vec{r},t\right) - \vec{\nabla}\widetilde{\Phi} + \frac{\widetilde{\rho}}{\overline{\rho}^2}\vec{\nabla}\overline{P} + \vec{F}_{\mathscr{L}}^{\,T}\left(\vec{\eta}\right). \quad (8)\end{aligned}$$

$\widehat{\mathbf{e}}_z = \cos\theta\,\widehat{\mathbf{e}}_r - \sin\theta\,\widehat{\mathbf{e}}_\theta$ is the unit vector along the rotation axis. The wave total pressure fluctuation (Π) is given by the sum of the gas pressure one ($\widetilde{P}$) and of the wave's magnetic pressure:

$$\Pi = \widetilde{P} + \frac{\vec{B}_0\cdot\vec{b}}{\mu}. \quad (9)$$

Afterwards, the volumetric wave's magnetic tension Lorentz force is

$$\begin{aligned}\vec{F}_{\mathscr{L}}^{\,T}\left(\vec{\eta}\right) &= \frac{1}{\overline{\rho}}\frac{1}{\mu}\left[\left(\vec{B}_0\cdot\vec{\nabla}\right)\vec{b} + \left(\vec{b}\cdot\vec{\nabla}\right)\vec{B}_0\right]\\ &= A_0^2\left[\partial_{\varphi^2}\vec{\eta} + 2\,\widehat{\mathbf{e}}_z\times\partial_\varphi\vec{\eta}\right]. \quad (10)\end{aligned}$$

Finally, the continuity, the energy transport, and the Poisson's equations are respectively given by:

$$\widetilde{\rho} + \vec{\nabla}\cdot\left(\overline{\rho}\,\vec{\eta}\,\right) = 0\,, \quad (11)$$

$$\left(\frac{\widetilde{P}}{\Gamma_1\overline{P}} - \frac{\widetilde{\rho}}{\overline{\rho}}\right) + \frac{N^2}{\overline{g}}\eta_r = 0\,, \quad (12)$$

where the gravity and the Brunt-Väisälä frequency are $\overline{g} = {\rm d}\overline{\Phi}/{\rm d}r$ and $N^2 = -\overline{g}\left({\rm d}\ln\overline{\rho}/{\rm d}r - 1/\Gamma_1\cdot{\rm d}\ln\overline{P}/{\rm d}r\right)$ with $\Gamma_1 = \left(\partial\ln P/\partial\ln\rho\right)_S$, and

$$\nabla^2\widetilde{\Phi} = 4\pi G\widetilde{\rho}\,, \quad (13)$$

G being the universal gravitational constant.

Next, scalar and vectorial fields are expanded in Fourier's series in φ and t:

$$\widetilde{X} = \sum_{\sigma,m}\left\{X'\left(r,\theta\right)\exp\left(im\varphi\right)\exp\left(i\sigma t\right)\right\}, \quad (14)$$

$$\vec{x} = \sum_{\sigma,m}\left\{\vec{x}^{\,\prime}\left(r,\theta\right)\exp\left(im\varphi\right)\exp\left(i\sigma t\right)\right\}, \quad (15)$$

σ being the wave's angular velocity in an inertial frame.

Then, we get for the velocity field

$$\vec{u}^{\,\prime} = i\,\sigma_s\vec{\eta}^{\,\prime} \text{ with } \sigma_s = \sigma + m\,\Omega_s\,. \quad (16)$$

In the considered rotating stellar radiation zone, waves are thus Doppler-shifted by Ω_s. Thus, the local wave angular velocity σ_s, that corresponds to the operator $\left(\partial_t + \Omega_s\partial_\varphi\right)$, appears.

On the other hand, Eq. (7) leads to

$$\vec{b}^{\,\prime} = im\sqrt{\mu\overline{\rho}}A_0\vec{\eta}^{\,\prime}\,. \quad (17)$$

The momentum equation then becomes

$$-\mathscr{A}\,\vec{\eta} + i\mathscr{B}\,\widehat{\mathbf{e}}_z\times\vec{\eta} = -\vec{\nabla}W' + \frac{\rho'}{\overline{\rho}^2}\vec{\nabla}\overline{P} - \Pi'\frac{\vec{\nabla}\overline{\rho}}{\overline{\rho}^2}, \quad (18)$$

where $\mathscr{A} = \sigma_{\rm M}^2 = \sigma_s^2 - m^2A_0^2$ and $\mathscr{B} = 2\left(\Omega_s\sigma_s - mA_0^2\right)$. $\sigma_{\rm M} = \sqrt{\mathscr{A}}$ can be seen as a modified local wave's angular velocity that corresponds to the modification of σ_s due to the presence of the magnetic field. In the case where $\sigma_{\rm M}^2 < 0$, waves become trapped and do not propagate (Schatzman 1993; Barnes, MacGregor & Charbonneau 1998). On the other hand, we have defined

$$W\left(\vec{r},t\right) = \frac{\Pi}{\overline{\rho}} + \widetilde{\Phi}, \quad (19)$$

which is the sum of the total wave dynamical pressure fluctuation[1] and of the gravific potential fluctuation.

From now on, we adopt the Cowling's approximation where the wave's gravific potential fluctuation is neglected. Therefore, we get $W = \Pi/\overline{\rho}$.

LOW-FREQUENCY WAVES IN A ROTATING MAGNETIC STELLAR RADIATION ZONE

The Traditional approximation

In the general case, the operator which governs the spatial structure of the waves, the Poincaré operator, is

[1] The dynamical pressure is defined by P/ρ.

of mixed type (elliptic and hyperbolic) and not separable (for a detailed discussion we refer the reader to Friedlander & Siegman 1982 and to Dintrans & Rieutord 2000 in the hydrodynamical and in the MHD cases). This leads to the appearance of detached shear layers associated with the underlying singularities of the adiabatic problem that could be crucial for transport and mixing processes in stellar radiation zones, since they are the seat of strong dissipation (cf. Dintrans & Rieutord 2000 and references therein).

Let us first focus on the hydrodynamical case (namely on the gravito-inertial waves). In the largest part of stellar radiation zones, we are in a regime where $2\Omega_s \ll N$. Since we are interested here in low-frequency waves ($\sigma \ll N$), the Traditional approximation, which consists in neglecting the latitudinal component of the rotation vector ($\Omega_s \widehat{\mathbf{e}}_z$), $-\Omega_s \sin\theta\,\widehat{\mathbf{e}}_\theta$, in the Coriolis acceleration, can be adopted in the super-inertial regime where $2\Omega_s < \sigma \ll N$ if a uniform rotation (Ω_s) is assumed (see Mathis et al. 2008). Then, variables separation in radial and horizontal eigenfunctions remains possible that corresponds to the ergodic (regular) elliptic gravito-inertial modes family (the E_1 modes in Dintrans & Rieutord 2000). This approximation has to be carefully used , as it changes the nature of the Poincaré operator, and removes the singularities and associated shear layers that appear. In the sub-inertial regime, where $\sigma \leq 2\Omega_s$, that corresponds to the equatorially trapped hyperbolic modes (the H_2 modes in Dintrans & Rieutord 2000), the Traditional approximation fails to reproduce the waves behaviour and the complete momentum equation has to be solved.

In the MHD case, our purpose is to generalize the Traditional Approximation to $i\mathscr{B}\widehat{\mathbf{e}}_z \times \vec{\eta}$, thus neglecting the latitudinal component of $\widehat{\mathbf{e}}_z$ in the case where $\sigma \ll N$, $2\Omega_s \ll N$ and $A_0 \ll N$.

Therefore, we restrict here ourselves to the regular low-frequency waves for which the Traditional approximation is usable. Its application domain will be discussed in the last section.

Dynamical equations

Assuming the Traditional Approximation for low-frequency Magneto-Gravito-Inertial waves, we respectively get for the linearized momentum equation components:

$$-\mathscr{A}\,\eta_r^{'} = -\partial_r W^{'} - \frac{\rho^{'}}{\overline{\rho}}\overline{g} - \Pi^{'}\frac{\partial_r\overline{p}}{\overline{\rho}^2}\,, \qquad (20)$$

$$-\mathscr{A}\,\eta_\theta^{'} - i\mathscr{B}\cos\theta\,\eta_\varphi^{'} = -\frac{1}{r}\partial_\theta W^{'}\,, \qquad (21)$$

$$-\mathscr{A}\,\eta_\varphi^{'} + i\mathscr{B}\cos\theta\,\eta_\theta^{'} = -\frac{imW^{'}}{r\sin\theta}\,. \qquad (22)$$

Spatial structure of the wave-displacement, velocity and magnetic field

Eliminating successively η_θ and η_φ between Eqs. (21) & (22), each of them is expressed in function of $W^{'} = 1/\overline{\rho}\left[P^{'} + \left(\vec{B}_0\cdot\vec{b}^{'}\right)/\mu\right]$ as

$$\eta_\theta^{'} = \frac{1}{r\,\sigma_{\rm M}^2}\frac{1}{1-\nu_{\rm M}^2\cos^2\theta}\left[\partial_\theta W^{'} + m\nu_{\rm M}\frac{\cos\theta}{\sin\theta}W^{'}\right], \qquad (23)$$

$$\eta_\varphi^{'} = i\,\frac{1}{r\,\sigma_{\rm M}^2}\frac{1}{1-\nu_{\rm M}^2\cos^2\theta}\left[\nu_{\rm M}\cos\theta\,\partial_\theta W^{'} + \frac{m}{\sin\theta}W^{'}\right]. \qquad (24)$$

We define $\nu_{\rm M}$ by

$$\nu_{\rm M} = \mathscr{B}\mathscr{A}^{-1} = \nu_s\left(\sigma_s,\Omega_s\right)F_{\rm M}\left(\sigma_s,\Omega_s,A_0\right) \qquad (25)$$

where

$$\nu_s = \frac{2\Omega_s}{\sigma_s} = R_{\rm o}^{-1} \quad \text{and} \quad F_{\rm M} = \frac{1-m\Lambda_{\rm E}}{1-\frac{m^2}{2}\nu_s\Lambda_{\rm E}}\,. \qquad (26)$$

ν_s is the spin parameter which equals to the inverse of the Rossby number, $R_{\rm o} = \sigma_s/2\Omega_s$ (cf. Lee & Saio 1997) and $F_{\rm M} = \nu_{\rm M}/\nu_s$ gives its modification by the magnetic field, $\Lambda_{\rm E} = A_0^2/\left(\Omega_s\sigma_s\right)$ being the Elsasser number that gives the ratio of the Lorentz force by the Coriolis acceleration.

Then, as in the non-magnetic case and because of the equations structure (cf. Lee & Saio 1997), we choose to expand the scalar quantities and the $\vec{\eta}$'s vertical component as follows:

$$X^{'} = \sum_k X_{k,m}^{'}(r)\,\Theta_{k,m}\left(\cos\theta;\nu_{\rm M}\right), \qquad (27)$$

$$\eta_r^{'} = \sum_k \eta_{r;k,m}^{'}(r)\,\Theta_{k,m}\left(\cos\theta;\nu_{\rm M}\right). \qquad (28)$$

The $\Theta_{k,m}$ are the Hough functions (cf. Hough 1898). They are the orthogonal eigenfunctions (with theirs associated eigenvalues $\Lambda_{k,m}$) of the so-called "Laplace Tidal Equation" (hereafter LTE):

$$\mathscr{L}_{\nu_{\rm M}}\left[\Theta_{k,m}\left(x;\nu_{\rm M}\right)\right] = -\Lambda_{k,m}\left(\nu_{\rm M}\right)\Theta_{k,m}\left(x;\nu_{\rm M}\right), \qquad (29)$$

where the Laplace tidal operator is given by:

$$\mathscr{L}_{\nu_{\rm M}} \equiv \frac{\rm d}{{\rm d}x}\left(\frac{1-x^2}{1-\nu_{\rm M}^2x^2}\frac{\rm d}{{\rm d}x}\right) - \frac{1}{1-\nu_{\rm M}^2x^2}\left(\frac{m^2}{1-x^2} + m\nu_{\rm M}\frac{1+\nu_{\rm M}^2x^2}{1-\nu_{\rm M}^2x^2}\right). \qquad (30)$$

For a detailed discussion of the boundary conditions for the LTE, we refer the reader to the Lee & Saio's work (1997).

This allows to separate the radial and latitudinal variables for $\eta_\theta^{'}$ and $\eta_\varphi^{'}$. Introducing the amplitude of the horizontal displacement

$$\eta_{H;k,m}^{'}(r) = \frac{1}{r\,\sigma_{\rm M}^2} W_{k,m}^{'}\,, \tag{31}$$

the latidudinal component of $\vec{\eta}^{\,'}$ is written

$$\eta_{\theta;k,m}^{'} = \eta_{H;k,m}^{'} \mathscr{H}_{k,m}^{\theta}(x;\nu_{\rm M})\,, \tag{32}$$

where

$$\mathscr{H}_{k,m}^{\theta}(x;\nu_{\rm M}) = \frac{1}{\left(1-\nu_{\rm M}^2 x^2\right)\sqrt{1-x^2}}\left[-\left(1-x^2\right)\frac{\rm d}{{\rm d}x}+m\nu_{\rm M}x\right]\Theta_{k,m}(x;\nu_{\rm M}). \tag{33}$$

In the same way, its azimuthal component is given by

$$\eta_{\varphi;k,m}^{'} = i\,\eta_{H;k,m}^{'} \mathscr{H}_{k,m}^{\varphi}(x;\nu_{\rm M})\,, \tag{34}$$

where

$$\mathscr{H}_{k,m}^{\varphi}(x;\nu_{\rm M}) = \frac{1}{\left(1-\nu_{\rm M}^2 x^2\right)\sqrt{1-x^2}}\left[-\nu_{\rm M}x\left(1-x^2\right)\frac{\rm d}{{\rm d}x}+m\right]\Theta_{k,m}(x;\nu_{\rm M}). \tag{35}$$

Then, the r-component of the momentum equation in the Traditional Approximation (Eq. 20), the continuity one (Eq. 11) and the energy transport equation (Eq. 12) give, eliminating $\rho_{k,m}^{'}$ and using the Cowling approximation, the following system for $r^2\eta_{r;k,m}$ and $W_{k,m}^{'}$:

$$\frac{{\rm d}W_{k,m}^{'}}{{\rm d}r} = \frac{N^2}{g}W_{k,m}^{'} + \frac{1}{r^2}\left(\sigma_{\rm M}^2 - N^2\right)\left(r^2\eta_{r;k,m}^{'}\right), \tag{36}$$

$$\frac{\rm d}{{\rm d}r}\left(r^2\eta_{r;k,m}^{'}\right) = \left[\frac{\Lambda_{k,m}(\nu_{\rm M})}{\sigma_{\rm M}^2} - \frac{\overline{\rho}r^2}{\Gamma_1\overline{P}}\right]W_{k,m}^{'} - \frac{1}{\Gamma_1\overline{P}}\frac{{\rm d}\overline{P}}{{\rm d}r}\left(r^2\eta_{r;k,m}^{'}\right). \tag{37}$$

This is a strict generalization of the system obtained by Press (1981) in the non-rotating and non-magnetic case where $\sigma_{\rm M}$ and $\Lambda_{k,m}(\nu_{\rm M})$ respectively replace σ and $l\,(l+1)$, l being the orbital number of the spherical harmonics.

Adopting the anelastic approximation where sonic waves are filtered out ($\vec{\nabla}\cdot(\overline{\rho}\,\vec{\eta})=0$), and following the procedure given by Press (1981), we get finally for the vertical displacement

$$\frac{{\rm d}^2\Psi_{k,m}(r)}{{\rm d}r^2} + \left[\left(\frac{N^2}{\sigma_{\rm M}^2}\right)\frac{\Lambda_{k,m}(\nu_{\rm M})}{r^2}\right]\Psi_{k,m}(r) = 0\,, \tag{38}$$

where $\Psi_{k,m} = \overline{\rho}^{1/2} r^2 \eta_{r;k,m}^{'}$, and for the pressure fluctuation

$$\frac{{\rm d}^2\mathscr{W}_{k,m}(r)}{{\rm d}r^2} + \left[\left(\frac{N^2}{\sigma_{\rm M}^2}\right)\frac{\Lambda_{k,m}(\nu_{\rm M})}{r^2}\right]\mathscr{W}_{k,m}(r) = 0\,, \tag{39}$$

where $\mathscr{W}_{k,m} = \left(\frac{\overline{\rho}r^2}{N^2}\right)^{1/2} W_{k,m}^{'}$. This last equation is the Poincaré equation in the Tradiational Approximation's framework.

Then, the wave velocity field and magnetic field are straightforwardly derived using

$$\vec{u} = i\,\sigma_s\vec{\eta} \quad \text{and} \quad \vec{b} = im\sqrt{\mu\overline{\rho}}A_0\vec{\eta}. \tag{40}$$

Therefore, one of the main important conclusion of this work is that the wave-induced perturbations can be completely expressed in function of the Hough functions used in the non-magnetic case in the one of strongly stably stratified rotating magnetic spherical shells, the spin parameter (ν_s) being now replaced by $\nu_{\rm M}$.

We have now to discuss the framework in which the Traditional approximation can be applied with its associated variables separation.

The Traditional approximation in the MHD case and the waves classification

The MHD-Traditional approximation can be applied as long as:

$$\mathscr{D}(x;\nu_{\rm M}) = 1 - \nu_{\rm M}^2 x^2 > 0 \text{ for every x}\,, \tag{41}$$

thus as long $|\nu_{\rm M}| < 1$. In this regime, waves are elliptic and regular. In the other one where $|\nu_{\rm M}| \geq 1$, waves become hyperbolic and trapped in an equatorial belt where $\theta \in [\theta_c, \pi-\theta_c]$, θ_c being the critical colatitude

$$\theta_c = \cos^{-1}\left(|\nu_{\rm M}|^{-1}\right), \tag{42}$$

where $\mathscr{D} = 0$. There, the adiabatic velocity field is singular and the MHD-Traditional approximation can not be applied (see the previous discussion), the wave behaviour being ruled by the dissipation in wave attractors (see Dintrans & Rieutord 2000). This is a strict generalization of the criteria derived in the hydrodynamical case for gravito-inertial waves (Mathis et al. 2008, and Mathis 2009).

Under the MHD Traditional Approximation (as long as $|\nu_{\rm M}| < 1$), four types of Magneto-Gravito-Inertial waves can be identified (see Townsend 2003 and Mathis et al. 2008 and references therein in the hydrodynamical case):

- Class *I* waves: they are internal gravity waves, which exist in the non-rotating and in the non-magnetic cases, that are modified both by the Coriolis acceleration and the Lorentz force; their eigenvalues ($\Lambda_{k,m}$), and hence their radial wave number, $k_{V;k,m}(r) \equiv \left(N^2/\sigma_{\rm M}^2\right) \cdot \left(\Lambda_{k,m}(\nu_{\rm M})/r^2\right)$, are increased.
- Class *II* waves: they are purely retrograde waves ($m > 0$) which exist only in the case of high-values $\nu_{\rm M}$. Their dynamics is driven by the conservation of the specific vorticity combined with the effects of curvature and by the Lorentz force. However, due to the fact that $|\nu_{\rm M}| \geq 1$, they can not be treated using the MHD Traditional Approximation. In the hydrodynamical case, they are called "quasi-inertial" waves that corresponds to the geophysical Rossby waves (see Provost, Berthomieu & Rocca 1981).
- Class *III* waves: they are mixed class *I* and class *II* waves. $m \leq 0$ waves exist in the absence of rotation and of magnetic field. $m > 0$ appear when $\nu_{\rm M} = m+1$ with small eigenvalues while their horizontal eigenfunctions are $\Theta_{k,m}(\nu_{\rm M} = m+1; x) = P_{m+1}^{m}(x)$. When they appear and have small eigenvalues, they behave mostly like class *II* waves; $m \leq 0$ and $m > 0$ waves with large eigenvalues behave rather like class *I* waves. Their eigenvalues are much smaller than those of class *I* waves. Thus, they have smaller vertical wave number. They may be identified with the geophysical Yanai waves.
- *Class IV waves*: they are purely prograde waves ($m < 0$) whose characteristics change little with $\nu_{\rm M}$, their displacement in the θ direction being very small. Their dynamics is driven by the conservation of the specific vorticity combined with the stratification effects and by the Lorentz force; their eigenvalues are smaller than those of both class *I* and class *III* waves. Hence, their vertical wave number is smaller. In the hydrodynamical case, they may be identified with the geophysical Kelvin waves.

CONCLUSION AND PERSPECTIVES

In this work, we have examined low-frequency internal waves behaviour in stellar radiation zones, which are stably stongly stratified rotating magnetic regions. Then, the waves dynamics is driven by the buoyancy force (the Archimedean restoring force), the Coriolis acceleration and the Lorentz force. Internal waves thus become Magneto-Gravito-Inertial waves. In this first work, we have studied the simplified case of radiation zones where both the angular velocity (Ω_s) and the Alfvén pulsation (A_0) are taken constant. This allows to extract the main characteristics of the waves.

First, we have derived the dynamical equations, following Braginsky's work and using stellar notations. Then, the MHD Traditional Approximation, which can be used only in the case where $|\nu_{\rm M}| < 1$ due to the strong stratification, has been introduced and discussed. This allows to simplify the former equations and to obtain the wave's velocity field, pressure fluctuation and magnetic field using the formalism which is used in the non-magnetic case for gravito-inertial waves in the super-inertial regime ($\nu_s < 1$). In fact, the MHD Traditional Approximation allows to used the usual Hough functions and associated special horizontal functions. Finally, four classes of Magneto-Gravito-Inertial waves are isolated, while their asymptotic properties and induced-transport can be obtained.

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