INTEGRAL METHOD FOR TRANSIENT HE II HEAT TRANSFER IN A SEMI-INFINITE DOMAIN

B. Baudouy

CEA/Saclay, DSM/DAPNIA/STCM
91191 Gif-sur-Yvette Cedex, France

ABSTRACT

Integral methods are suited to solve a non-linear system of differential equations where the non-linearity can be found either in the differential equations or in the boundary conditions. Though they are approximate methods, they have proven to give simple solutions with acceptable accuracy for transient heat transfer in He II. Taking in account the temperature dependence of thermal properties, direct solutions are found without the need of adjusting a parameter. Previously, we have presented a solution for the clamped heat flux and in the present study this method is used to accommodate the clamped-temperature problem. In the case of constant thermal properties, this method yields results that are within a few percent of the exact solution for the heat flux at the axis origin. We applied this solution to analyze recovery from burnout and find an agreement within 10 % at low heat flux whereas at high heat flux the model deviates from the experimental data suggesting the need for a more refined thermal model.

INTRODUCTION

Exact solutions for transient heat transfer in He II have been already studied in a semi-infinite domain with constant properties by Dresner [1-2]. In his approach, it is necessary to define an average temperature in order to evaluate thermal properties, thus adjusting the solution to the experimental data. Such operation makes this tool unsuitable for designers and engineers. An alternative method, the integral method, has been proposed for the clamped heat flux problem in He II with temperature dependent properties [3]. This method is analogous to that employed to solve boundary layer problems in fluid mechanics and non-linear heat transfer problems [4-6]. The previous treatment, using the Kirchhoff transformation, leads to an approximation which is acceptable in heat transfer provided that the temperature variation of the properties are not too large. This is not necessarily the case in He II. Therefore we reformulate a general solution for linear boundary conditions with temperature dependent thermal properties with a variable change proposed by Goodman [5].

GENERAL CASE FOR LINEAR BOUNDARY CONDITIONS

We examine the case where the thermodynamic properties are temperature dependent and the domain is considered semi-infinite. For a fully developed turbulent state, the heat flux is given by the Gorter-Mellink law. Neglecting the dissipation effects in He II, the partial differential equation modeling the system in one space dimension is,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(f \frac{\partial T}{\partial x} \right)^{1/3} \text{ in } 0 \le x \le \infty \text{ for } t>0, \quad (1)$$

where ρ is the density, C_p the specific heat at constant pressure and f the He II turbulent thermal conductivity function. The boundary condition at the axis origin is defined as

$$T=T_0 \text{ at } x=0 \text{ for } t>0, \quad (2\text{-a})$$

for the case of a clamped-temperature problem, and for the clamped heat flux it is

$$-\left(f \frac{\partial T}{\partial x} \right)^{1/3} = q_0 \text{ at } x=0 \text{ for } t>0. \quad (2\text{-b})$$

Initially, we assume that the domain is at constant temperature; that is,

$$T=T_b \text{ in } 0 \le x \le \infty \text{ at } t=0. \quad (3)$$

For a semi-infinite domain, the necessary second boundary condition for both clamped temperature and heat flux problems is a constant temperature when $x \to \infty$. Namely,

$$T=T_b \text{ for } x \to \infty \text{ for } t>0. \quad (4)$$

We are interested here by the solution for the disturbed temperature field, which is limited by the thermal layer $\delta(t)$ inferior to the length of the domain. Now the condition (4) can be rewritten as,

$$T=T_b \text{ at } x=\delta(t) \text{ for } t>0. \quad (5)$$

By applying the transformation used by Goodman [5],

$$\Theta = \int_{T_b}^{T} \rho C_p(T) dT \, , \quad (6)$$

the system (1), (2), (3) and (5) is transformed into

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial}{\partial \chi}\left(\alpha \frac{\partial \Theta}{\partial \chi} \right)^{1/3} \text{ in } 0 \le x \le \infty \text{ for } t>0, \quad (7\text{-a})$$

with $\alpha = f/\rho C_p$. The boundary and initial conditions are now,

$$\Theta=1 \text{ or } -\left(\alpha \frac{\partial \Theta}{\partial x} \right)^{1/3} = q_0 \text{ at } x=0 \text{ for } t>0, \quad (7\text{-b-1}) \text{ and } (7\text{-b-2})$$

$$\Theta=0 \text{ at } x=\delta(t) \text{ for } t>0, \quad (7\text{-c})$$

$$\Theta=0 \text{ in } 0 \le x \le \infty \text{ at } t=0, \quad (7\text{-d})$$

By integrating (7-a) with respect to x over the thermal layer $\delta(t)$, one obtains the Heat-Integral equation of the system,

$$\int_0^{\delta} \frac{\partial \Theta}{\partial t} dx = \int_0^{\delta} \frac{\partial}{\partial x} \left(\alpha \frac{\partial \Theta}{\partial x} \right)^{1/3} dx = \left(\alpha_{\delta} \left. \frac{\partial \Theta}{\partial x} \right|_{\delta} \right)^{1/3} - \left(\alpha_0 \left. \frac{\partial \Theta}{\partial x} \right|_0 \right)^{1/3} . \tag{8}$$

Note that in our system $\left. \frac{\partial \Theta}{\partial x} \right|_{\delta}$ is null by the definition of the thermal boundary $\delta(t)$ thus using the condition (7-b) the equation (8) is reduced to

$$\frac{d}{dt} \int_0^{\delta} \Theta dx = - \left(\alpha_0 \left. \frac{\partial \Theta}{\partial x} \right|_0 \right)^{1/3} . \tag{9}$$

As before, we have used a temperature profile $\Theta = a + bx + cx^2 + dx^3$ where the coefficients a, b, c and d are functions of $\delta(t)$. To find the coefficients we need two additional boundary conditions. The first, already used to construct the Heat-Integral equation, is straightforward and comes directly from the definition of the thermal layer,

$$\frac{\partial \Theta}{\partial x} = 0 \text{ at } x = \delta(t) \text{ for } t > 0. \tag{10}$$

To construct the second condition, we differentiate the condition (10) with respect to x,

$$\frac{\partial^2 \Theta}{\partial x^2} = 0 \text{ at } x = \delta(t) \text{ for } t > 0. \tag{11}$$

Using the boundary conditions (7-b), (7-c), (10) and (11), we can formulate a solution of Θ as a function of $\delta(t)$,

$$\Theta = \Theta_0 \left(1 - \frac{x}{\delta} \right)^3 . \tag{12}$$

For the clamped-temperature problem Θ_0 is already known and given by (7-b-1) whereas for the clamped heat flux problem it is given by (7-b-2),

$$\Theta_0 = 1 \text{ and } \Theta_0 = \frac{q_0{}^3}{3\alpha_0} \delta \qquad \text{(13-a) and (13-b)}$$

By substituting equation (12) into the Heat-Integral equation (9), we obtain the thermal layer as a function of time, subjected to the initial condition (7-d),

$$\delta = \frac{8}{\sqrt{3}} \frac{\alpha_0{}^{1/4}}{\sqrt{\Theta_0}} t^{3/4} \text{ and } \delta = 2\sqrt{3} \frac{\sqrt{\alpha_0}}{q_0} \sqrt{t} , \qquad \text{(14-a) and (14-b)}$$

for the clamped-temperature problem (14-a) and for the clamped heat flux (14-b). The reader can notice that the solution for the clamped heat flux problem is more rigorous than presented in [3] where the Kirchhoff transformation was used, leading to the approximation of α independent of Θ. The solution for the clamped-temperatures problem is straightforward whereas a transcendental equation for Θ_0 has to be solved because the temperature at the axis origin is unknown in the clamped heat flux problem,

$$\Theta_0 \sqrt{\alpha_0} = \frac{2}{\sqrt{3}} q_0{}^2 \sqrt{t} . \tag{15}$$

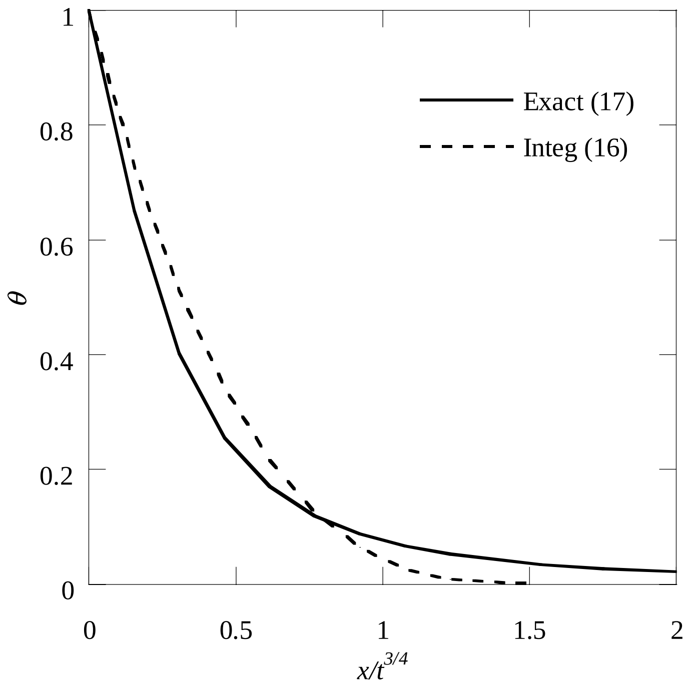

FIGURE 1. Comparison between the integral method and exact solutions for the clamped-temperature problem. Solid line depicts the exact solution given by equation (17) and dotted line is the approximate solution using integral method as described by equation (16)

COMPARISON WITH EXACT SOLUTIONS

Integral method solutions can be compared with exact solutions assuming constant thermal properties. For the clamped-temperature problem such a solution can be deduced from equations (12) and (14-a)

$$\theta = \frac{T - T_b}{T_0 - T_b} = \left(1 - \frac{\sqrt{3}}{8}\frac{(\rho C_p)^{3/4}}{f^{1/4}}(T_0 - T_b)^2 \frac{x}{t^{3/4}}\right)^3 \tag{16}$$

and are compared in Figure 1 with the exact solution given by Dresner in [1],

$$\theta = 1 - \frac{z}{\sqrt{8/3\sqrt{3} + z^2}} \tag{17}$$

with $z = \sqrt{T_0 - T_b}\,\frac{(\rho C_p)^{3/4}}{f^{1/4}}\frac{x}{t^{3/4}}$. The discrepancy between the integral method solution and the exact solution can be as high as 25 % for $x/t^{3/4}<1$ and diverges at higher $x/t^{3/4}$. This discrepancy is due to the fact that the integral equation only satisfies the original partial differential equation averaged over a finite distance; δ. Further inaccuracy is introduced by approximating the temperature profile. Since the integral method solution is obtained by solving a time dependent differential equation for $\delta(t)$ and the temperature variation in the x-direction is only taken into account in the temperature profile, it is expected that inaccuracy lie primarily in the x-direction. The dominant inaccuracy is found in the x-derivatives of the temperature profiles for large x. On the contrary, for small x, the accuracy is found to be within a few percent and it is demonstrated in comparing the heat flux at the axis origin for the clamped-temperature problem. The heat flux at the x-origin is

$$q_0 = \frac{\sqrt{3}}{2}\sqrt{\Theta_0}\alpha_0^{1/4} t^{-1/4}\,. \tag{18}$$

Considering thermal properties independent of temperature, equation (18) is simplified to,

$$q_0 = \frac{\sqrt{3}}{2}(\rho C_p f)^{1/4} t^{-1/4}\,. \tag{19}$$

Equation (19) is identical to Dresner's solution except in the coefficient $\sqrt{3}/2$ (0.87), compared to $(3\sqrt{3}/8)^{1/6}$ (0.93) found in his solution [1]. As mentioned earlier, parameter adjustment is required to fit the data to equation (19) whereas equation (18) can give direct solution. The same comparison can be done for the clamped-heat flux problem and the agreement between the exact solution and this model is similar.

APPLICATION TO BURNOUT RECOVERY WITH CONSTANT HEAT FLUX

Previously, we presented comparison between the integral method solution and experimental results for the case of clamped heat flux. While agreement with the experimental data is reasonable, we have noticed that the inaccuracy increases away from the axis origin [3]. Here, we are interested to apply the integral method solutions to the problem of recovery from burnout studied experimentally by Seyfert [1] and theoretically by Dresner [7]. In Seyfert's experiment, after applying a heat pulse E ($E=qt_p$) in the heater, a constant heat flux q_p (post-heating) is maintained in the heater to simulate joule heating in a superconductor after a quench (see figure 2). In this thermal configuration they were interested to evaluate the maximum heat pulse, E, allowing recovery from burnout as a function of this constant heat flux q_p. Seyfert approached the problem in considering the heat transfer problem only after the heater zone reached burnout meaning that the temperature reaches the lambda temperature, T_λ. When T_λ is reached, a helium phase change appears at the heater surface and a layer of normal helium in sub-cooled conditions or vapor in saturated conditions is created. This phase change creates a thermal barrier between the He II channel and the heater. In his experiment (sub-cooled He II) Seyfert assumed that the thermal barrier has its temperature locked at T_λ. The heat transfer problem is simplified with such assumption to a decrease of heat from a location at constant temperature T_λ into an infinite domain at a temperature lower than T_λ. This can be modeled by the clamped-temperature problem provided that the layer of He I is negligible compared to the size of the domain and also that the phase change regime is not the dominant heat transfer regime in this configuration.

One must be cautious in using a clamped-temperature solution in such case because two boundary conditions are imposed at the x-origin. There is an imposed heat flux q_p after the energy pulse and we consider to solve the problem where a temperature is fixed at the same location. Remark that this is inconsistent with resolution of boundary-value problems, however this approach is motivated by its simplicity. Stating that the temperature is constant at a fixed location implies that the associated heat flux is function of time. It even decreases with increasing time in our case. With increasing time the heat flux, given by a clamped temperature solution, will be eventually underestimated compared to a constant heat flux boundary condition. With these assumptions in mind, in figure 2, the area A is the energy that cannot be removed by diffusion in He II and serves to increase the temperature of the heater and the size of the thermal layer. After the pulse is terminated, the He II can evacuate a quantity of energy indicated by the area B. Seyfert

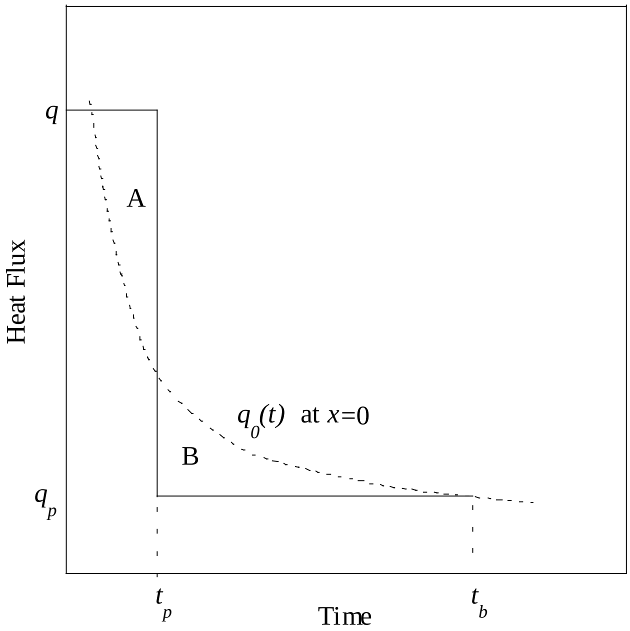

FIGURE 2. Schematic of the thermal configuration for recovery from burnout as proposed by Seyfert. The solid lines schematized the heat input and the dotted line is the heat flux given by the integral method at x=0. The energy is E=qt_p and t_b is the time associated to the recovery from burnout.

that the maximum energy E allowing recovery is attained when the areas A and B are equal and the heat flux at the layer is given by the clamped-temperature solutions. This can be mathematically expressed by [1]

$$E-\int_0^{t_p} q dt=\int_{t_p}^{t_b} q dt-q_p(t_b-t_p)=\frac{1}{3}\frac{q}{q-q_p}q_p t_b . \tag{20}$$

If the initial heat flux is much larger than the post heating heat flux, we have

$$E\approx\frac{1}{3}q_p t_b . \tag{21}$$

From figure 2, it is easy to see that the post heating heat flux is equal to the heat flux given by the integral method solution equation (18) at t_b. When equation (18) is introduced into equation (21), we obtain the expression for the pulse energy E as a function of post heating heat flux q_p,

$$E\approx\frac{3}{16}\Theta_0^{\;2}\alpha_0 q_p^{\;-3} . \tag{22}$$

We may now compare the results of Seyfert's experiment to equation (22) in Figure 3. Experimental results have been taken from Seyfert's work and the best fit to the data, for bath temperatures of 1.8 K and 1.9 K, has been found for a temperature at the x-origin of 2.133 K which is smaller than T_λ≡2.163 K. Our solution agrees within 10 % with the experimental results only for low heat flux below 4×10^{-4} W/m^2. The dependency of the energy E with q_p is well described by equation (22) in this heat flux range. The fact that the best fit is found for $T(x=0)<T_\lambda$ is primarily due because when the temperature approaches T_λ, α_0 tends to zero because the He II thermal conductivity function tends to zero. This inaccuracy comes from the use of the clamped-temperature model. For higher heat flux, the model overestimates the energy because in this region, experimental heat fluxes q_p are

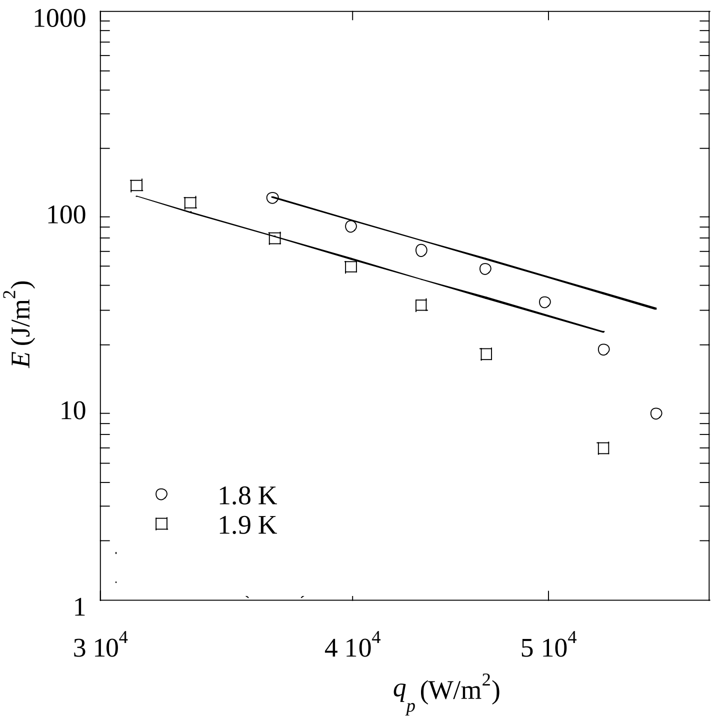

FIGURE 3. Comparison between Seyfert's experiment [7] and equation (22) (solid lines). Best fit is found for $T(x=0)=2.133$ K for both bath temperature.

higher than the heat flux given by the model at the *x*-origin with the temperature difference considered in the low heat flux region. The energy dependency with the heat flux q_p does not follow what is given by equation (22) indicating also the limit of this model. Clearly, the helium change of phase in helium has to be considered to cover the entire heat flux range.

CONCLUSIONS

The solution of the clamped temperature given by the integral method is in good agreement with the exact solution. This solution has been used to reproduce experimental data of recovery from burnout in sub-cooled He II with reasonable accuracy at low heat flux. The agreement is within 10 % in that range and the cubic dependence of the energy with the post-heating heat flux is found. Despite the apparent success in the model for low heat flux, it is not suitable for larger heat flux. This calls for further development of a model for the recovery from burnout that can include the phase change near the heater.

REFERENCES

1. Dresner, L., "Transient Heat Transfer in Superfluid Helium. Part II" in *Advances in Cryogenics Engineering* 29, edited by Plenum Press, Colorado Springs, CO (USA), 1983, pp. 323-333.
2. Dresner, L., "Transient Heat Transfer in Superfluid Helium" in *Advances in Cryogenics Engineering* 27, edited by Plenum Press, San Diego, CA, (USA), 1981, pp. 411-419.
3. Baudouy, B., "Approximate solution for transient heat transfer in static turbulent He II" in *Advances in Cryogenic Engineering* 45, edited by Plenum Press, Montréal, Canada, 1999, pp. 969-976.
4. Schlichting, H., *Boundary-layer theory*, McGraw-Hill, Inc, New-York, 1976.
5. Goodman, T. R., "Application of Integral Methods to Transient Nonlinear Heat Transfer" in *Advances in Heat Transfer* 1, edited by Academic press, 1964, pp. 51-122.
6. Özisik, M. N., *Boundary Value Problems of Heat Conduction*, International Textbook Company, Scranton, 1968, pp. Pages.
7. Seyfert, P., Lafferranderie, J., and Claudet, G., *Cryogenics*, **vol. August**, pp. 401-408, (1982).