Inflationary gravitational waves

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Quantum harmonic oscillator

- Consider a **quantum harmonic oscillator**:

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \qquad\qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \ , \qquad (m = 1)$$

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- The canonical **position** and **momentum** are **quantised:**

$$x(t) = \sqrt{\frac{\hbar}{2\omega}}\left(a_i e^{-i\omega t} + a_i^\dagger e^{i\omega t}\right)$$

$$p(x) = i\sqrt{\frac{\omega\hbar}{2}}\left(a_i^\dagger e^{i\omega t} - a_i e^{-i\omega t}\right)$$

$$[a, a^\dagger] = 1 \quad \Leftrightarrow \quad [x, p] = i\hbar$$

$$a(t) = a_i e^{-i\omega t}$$

$$a^\dagger(t) = a_i^\dagger e^{i\omega t}$$

- **Vacuum**: $a|0\rangle = 0$

$$\langle x^2(t)\rangle \equiv \langle 0|x^2(t)|0\rangle = \frac{\hbar}{2\omega} \, ,$$

$$\langle p^2(t)\rangle = \frac{\omega\hbar}{2} \, ,$$

$$\langle x^2(t)\rangle\langle p^2(t)\rangle = \frac{\hbar^2}{4}$$

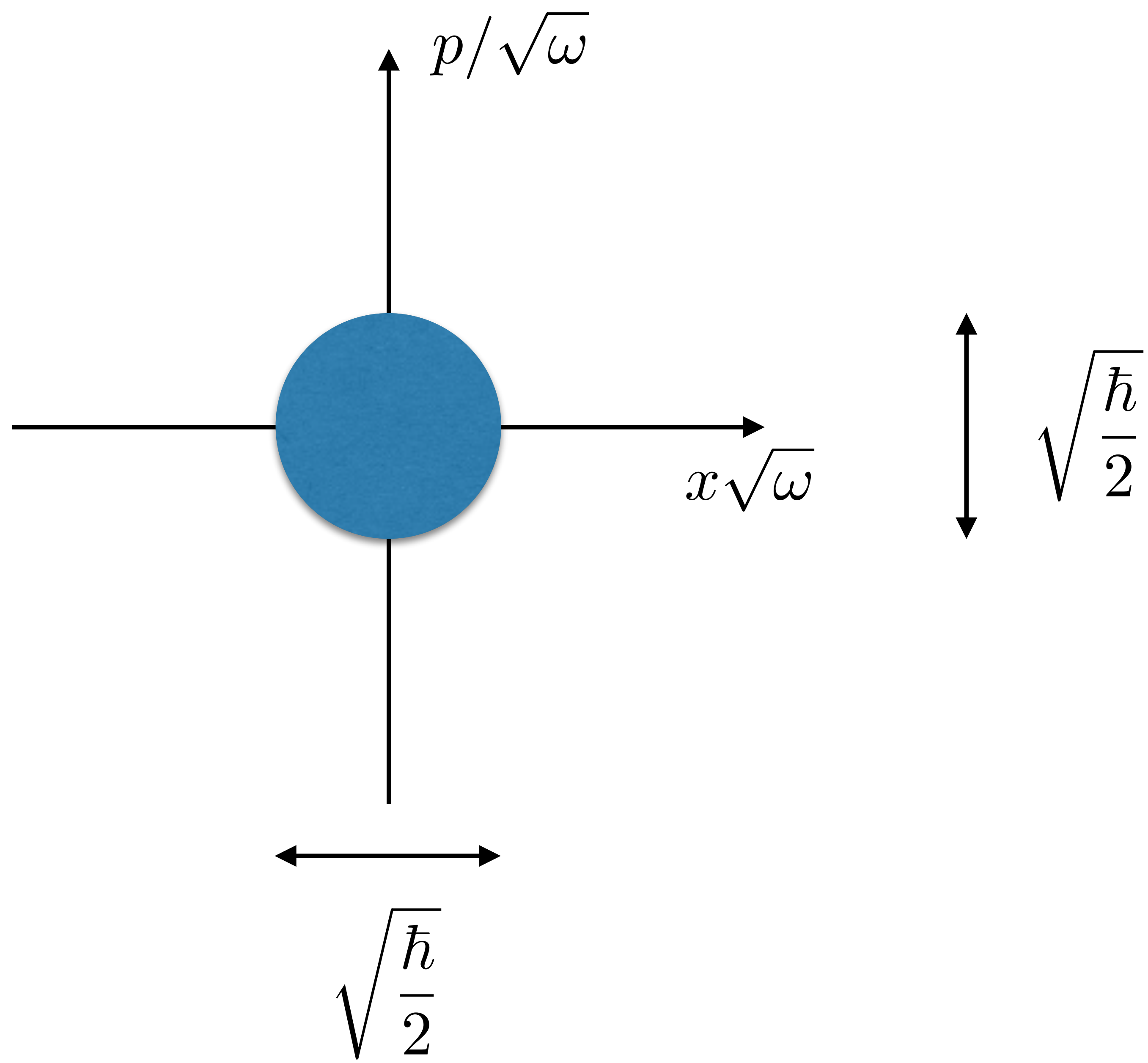
minimal uncertainty relation

Phase space

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \ , \qquad (m = 1)$$

$$\langle x^2(t) \rangle = \frac{\hbar}{2\omega} \ ,$$

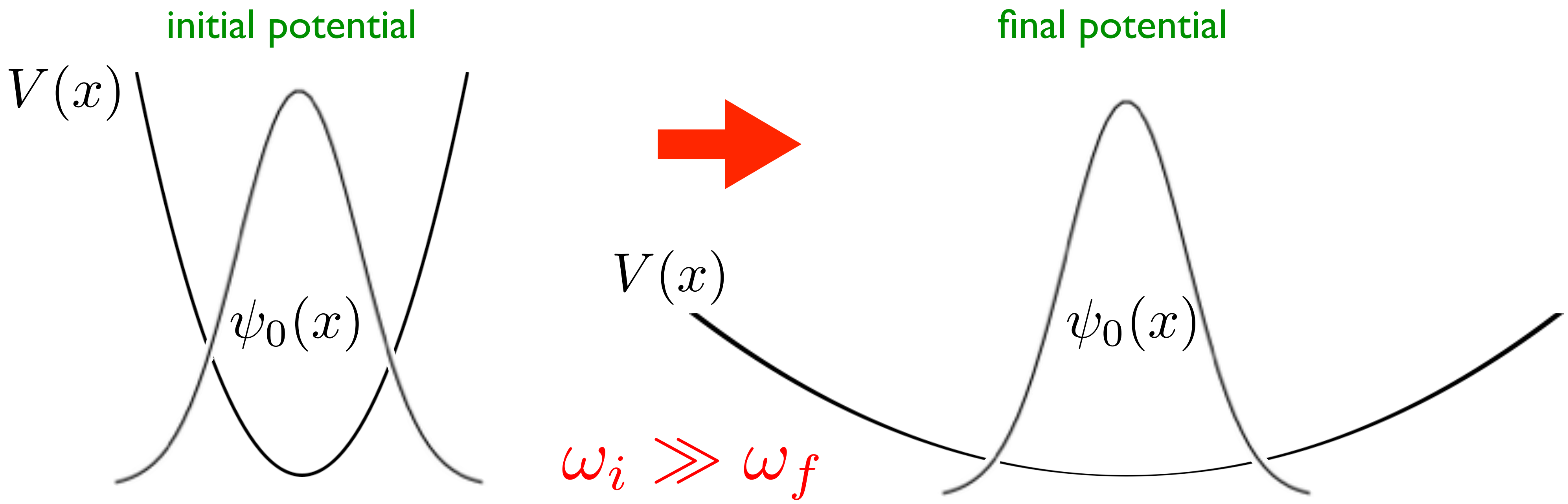
$$\langle p^2(t) \rangle = \frac{\omega\hbar}{2} \ ,$$



Changing the frequency

- Let us **rapidly** (non-adiabatically) change the **frequency**:

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega_i^2 x^2 \;, \qquad \Rightarrow \qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega_f^2 x^2$$



- Non adiabatic change, $\dot{\omega} \gg \omega^2$: the final state is not in the vacuum, **many particles created**!

$$a(t_f)|0\rangle \neq 0$$

Quantum harmonic oscillator

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- Evolution of the **annihilation** and **creation** operators **at** transition:

$$a_f = a_i \cosh r - a_i^\dagger \sinh r$$
$$a_f^\dagger = a_i^\dagger \cosh r - a_i \sinh r$$

$$e^r \equiv \sqrt{\frac{\omega_i}{\omega_f}}$$

- Particle creation: $\quad n_f = \langle 0 | a_f^\dagger a_f | 0 \rangle = \sinh^2 r \simeq \frac{1}{4}\frac{\omega_i}{\omega_f}$

$$E = \left(n + \frac{1}{2} \right) \hbar\omega$$

initial state

$$n_i = 0 \ \Rightarrow \ E_i = \frac{1}{2}\hbar\omega_i$$

final potential

$$E_f = \frac{1}{4}\hbar\omega_i \quad \Rightarrow \quad n_f \simeq \frac{1}{4}\frac{\omega_i}{\omega_f} \gg 1$$

Quantum harmonic oscillator

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- The canonical position and momentum uncertainties are **queezed**:

$$\langle x_f^2 \rangle = \frac{\hbar}{2\omega_f} \times \frac{\omega_f}{\omega_i}$$

$$\langle p_f^2 \rangle = \frac{\omega_f \hbar}{2} \times \frac{\omega_i}{\omega_f}$$

$$\langle x_f^2 \rangle \langle p_f^2 \rangle = \frac{\hbar^2}{4}$$

minimal uncertainty relation

Phase space

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega_i^2 x^2 \, , \qquad \Rightarrow \qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega_f^2 x^2$$

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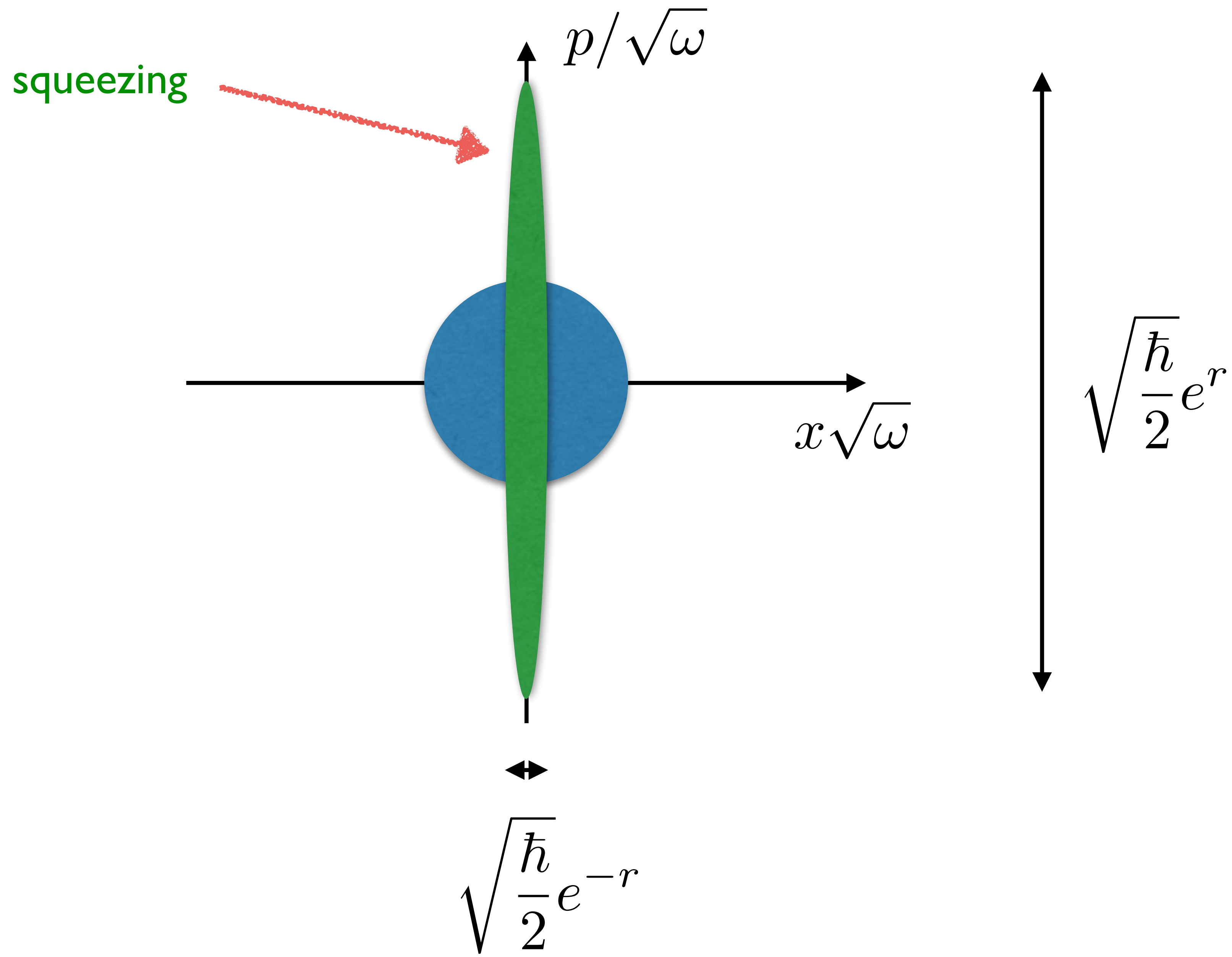


Quantum harmonic oscillator

- Time-evolution of **annihilation** and **creation** operators **after** transition:

$$x(t) = \sqrt{\frac{\hbar}{2\omega_f}}\left(a_f e^{-i\omega_f t} + a_f^\dagger e^{i\omega_f t}\right)$$

$$p(x) = i\sqrt{\frac{\omega_f \hbar}{2}}\left(a_f^\dagger e^{i\omega_f t} - a_f e^{-i\omega_f t}\right)$$

↓

$$x(t) = i\sqrt{\frac{\hbar}{2\omega_f}} e^r \left[(a_i^\dagger - a_i)\sin\omega_f t - i\frac{\omega_f}{\omega_i}(a_i^\dagger + a_i)\cos\omega_f t\right]$$

$$p(x) = i\sqrt{\frac{\omega_f \hbar}{2}} e^r \left[(a_i^\dagger - a_i)\cos\omega_f t + i\frac{\omega_f}{\omega_i}(a_i^\dagger + a_i)\sin\omega_f t\right]$$

- **Quantum**: mathematically, variables do not commute.

- From a “**corse-grained**” experimental point of view they commute and the **evolution follows** the **classical (deterministic) solution**. **Decoherence**: no correlation between “growing” and “decaying” modes.

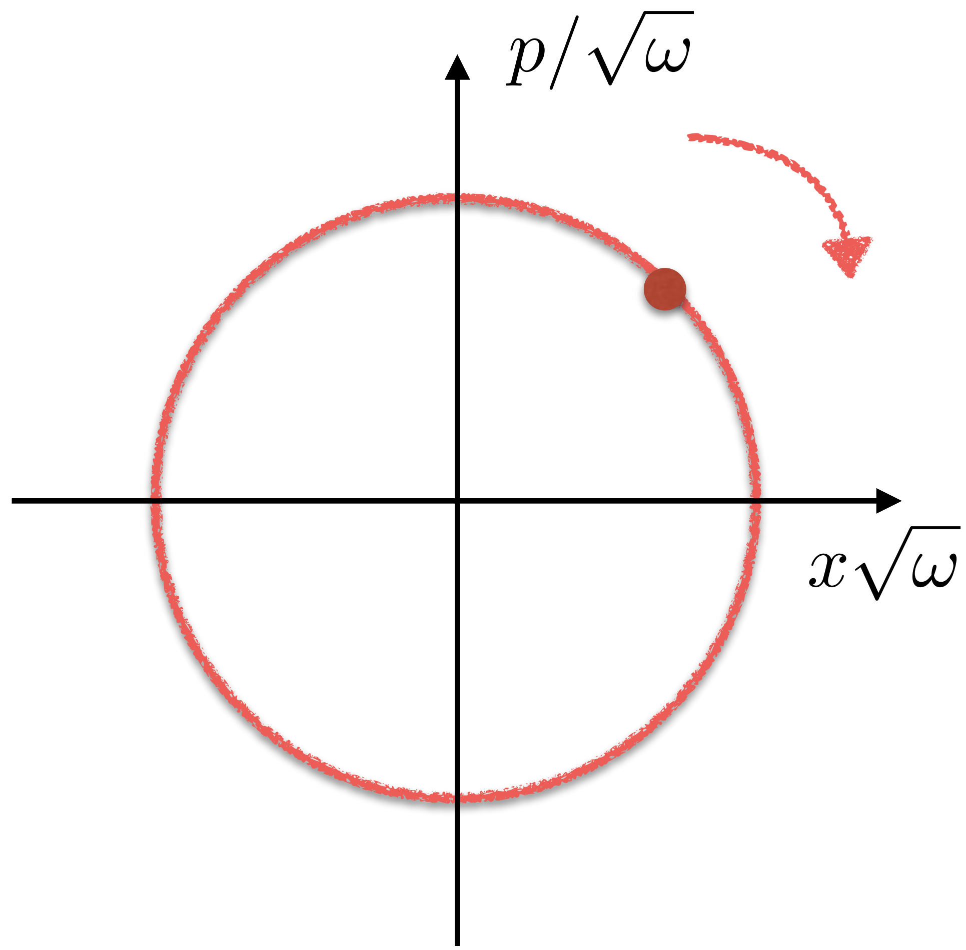
Classical stochastic H.O.

$$x(t) = \hat{e}\,\mathcal{A}\,\sin\omega_f t$$
$$p(x) = \hat{e}\,\mathcal{A}\,\omega_f\,\cos\omega_f t$$

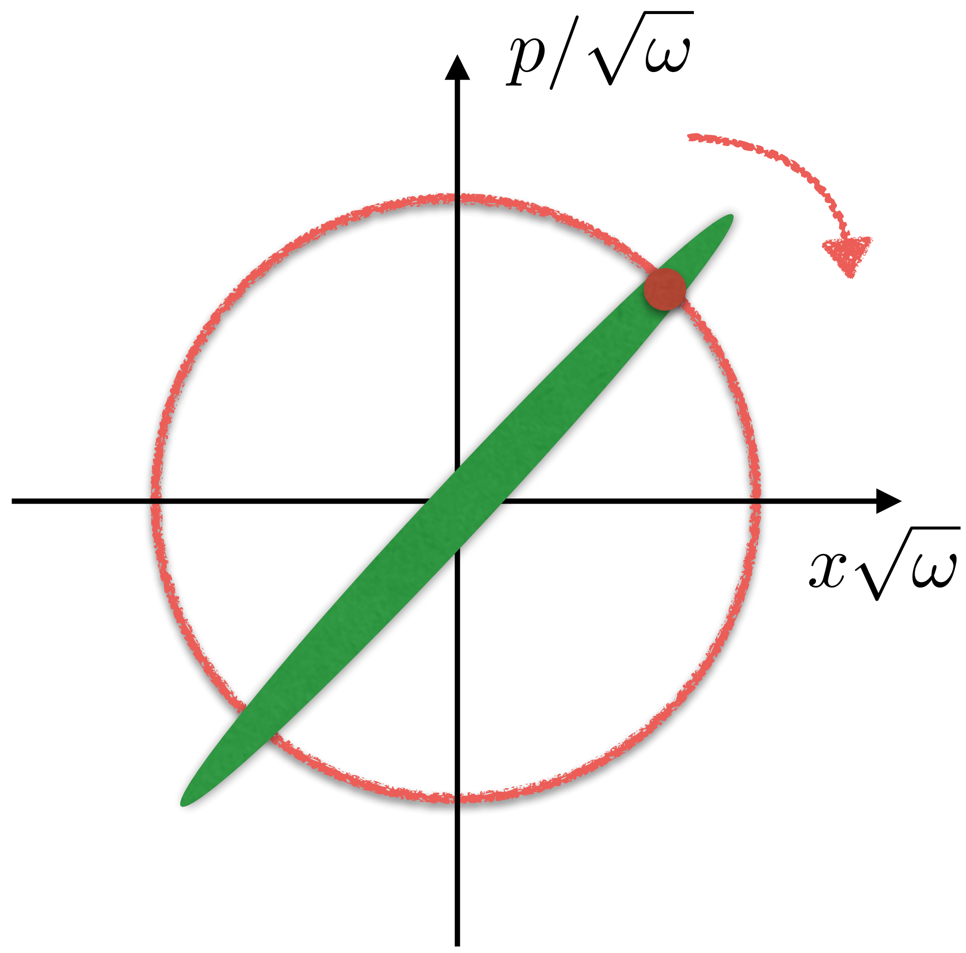
where $\hat{e}$ is a Gaussian stochastic variable with zero average, $\langle\hat{e}\rangle$, and unit variance: $\langle\hat{e}\hat{e}^*\rangle = 1$

- If we neglect the “decaying” mode, the **quantum** harmonic oscillator is indistinguishable from a **classical stochastic** one.

Phase space



Phase space



Why gravity waves

- We can count the number of degrees of freedom:

Metric (4x4 symmetric) = 10

Gauge freedom = - 4

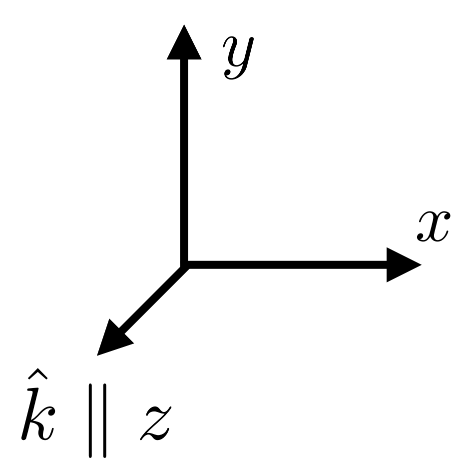
Constraints = - 4

Total = 2 tensors (gravity waves polarisations)

- Gravity waves are traceless and transverse (helicity-2). Expand around FRW:

$$ds^2 = -dt^2 + a^2(t)\left(\delta_{ij} + \gamma_{ij}\right)dx^i dx^j \;, \qquad \gamma_{ii} = 0 = \partial_i \gamma_{ij}$$

$$\gamma_{ij} = \begin{pmatrix} \gamma_+ & \gamma_\times & 0 \\ \gamma_\times & -\gamma_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Quantising gravity

- Expand Einstein-Hilbert action around FLRW:

$$ds^2 = -dt^2 + a^2(t)\big(\delta_{ij} + \gamma_{ij}\big)dx^i dx^j \ , \qquad \gamma_{ii} = 0 = \partial_i \gamma_{ij}$$

$$S = \int d^4x \frac{R}{16\pi G} \approx \int d^3x dt \frac{a^3}{64\pi G}\left[\dot{\gamma}_{ij}\dot{\gamma}^{ij} - \frac{1}{a^2}\partial_k \gamma_{ij}\partial^k \gamma^{ij}\right]$$

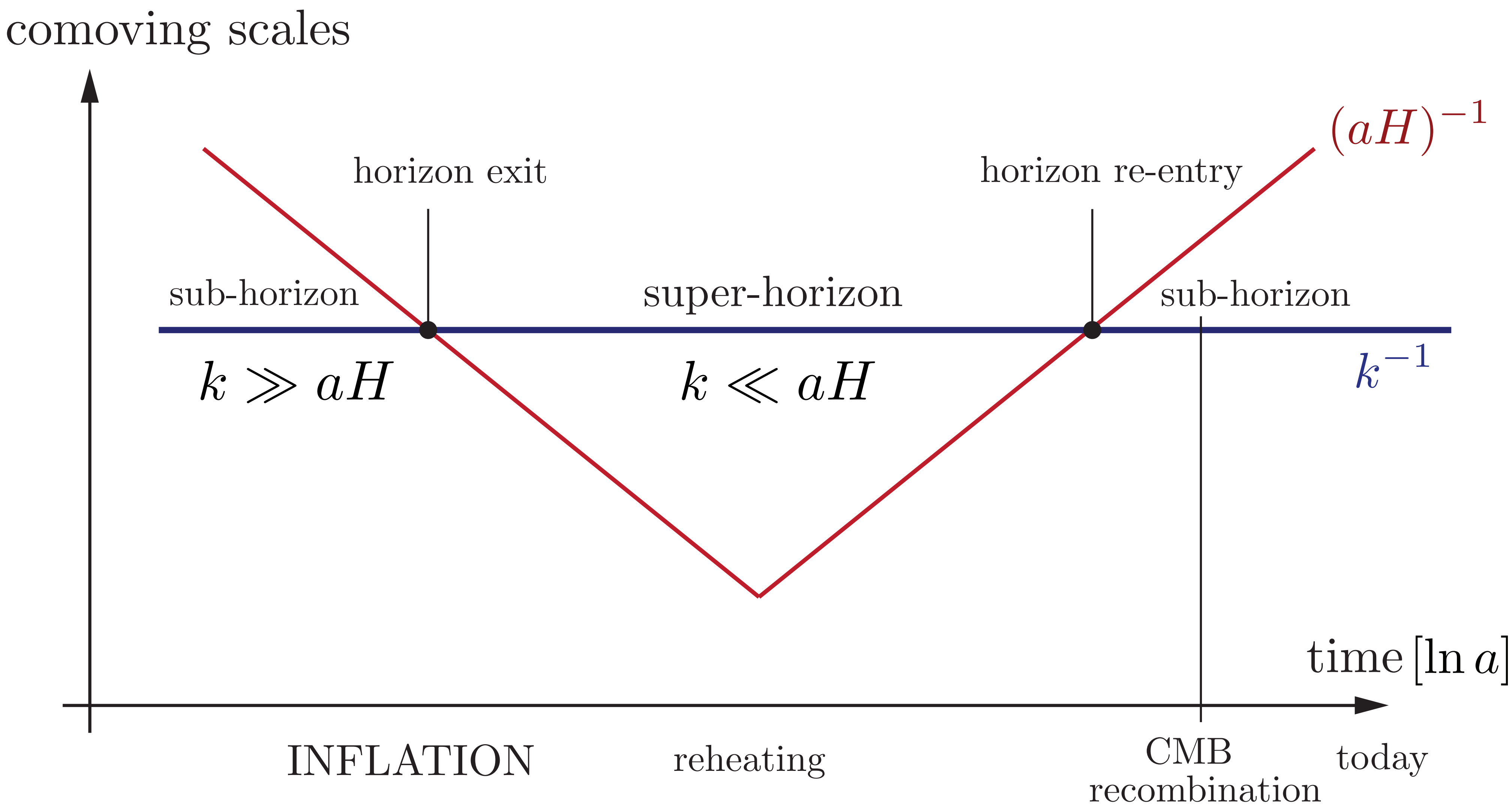
- Two helicity-2 states. In Fourier space, for each polarisation:

$$S = \int \frac{dk^3}{(2\pi)^3} \int dt \frac{a^3}{64\pi G}\left[\dot{\gamma}_{\vec{k}}\dot{\gamma}_{-\vec{k}} - \frac{k^2}{a^2}\gamma_{\vec{k}}\gamma_{\vec{k}}\right] \qquad \text{cf.} \qquad L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$$

- Canonical quantisation: $\left[\gamma^{(c)}_{\vec{k}}(t), \pi^{(c)}_{\vec{k}'}(t)\right] = i\hbar\delta(\vec{k} - \vec{k}')$ $\qquad \gamma^{(c)}_{\vec{k}} \equiv \frac{a^{3/2}}{\sqrt{32\pi G}}\gamma_{\vec{k}}$

$x(t)$ $\qquad$ $p(t)$

Comoving scales vs comoving Hubble



Quantising gravity

- Equation of motion

$$\ddot{\gamma}_{\vec{k}} + 3H\dot{\gamma}_{\vec{k}} + \frac{k^2}{a^2}\gamma_{\vec{k}} = 0$$

$$\omega = \frac{k}{a} \simeq k e^{-Ht}$$

$$a \simeq e^{Ht}$$

quasi de Sitter

- On sub-Hubble scales, vacuum normalisation:

$$\langle \gamma_{\vec{k}} \gamma_{-\vec{k}} \rangle = \frac{32\pi G}{a^3} \times \frac{\hbar}{2k/a} \qquad k \gg aH$$

cf. $\langle x^2 \rangle = \frac{\hbar}{2\omega}$

- On super-Hubble scales, freeze in at $a = k/H$:

$$\langle \gamma_{\vec{k}} \gamma_{-\vec{k}} \rangle = \frac{32\pi G}{c^5} \times \frac{\hbar H^2}{2k^3} = \frac{4}{2k^3} \times \frac{\hbar^2 H^2}{m_P^2 c^4} \qquad k \ll aH$$

Quantum of classical?

Quantum

• No other field than gravity is involved. Gravity waves treated as a quantum field.

• Mathematically, gravity wave polarisations do not commute with their momenta: quantum.

$$\left[\gamma^{(c)}_{\vec{k}}(t), \pi^{(c)}_{\vec{k}'}(t)\right] = i\hbar\delta(\vec{k}-\vec{k}')$$

Classical

• **Squeezing**: a corse-grained experiment which is only sensitive to the **growing mode** will see a classical evolution:

$$\gamma^{(c)}_{\vec{k}}(t) = H\sqrt{\frac{\hbar}{2k^3}}\left[(a^\dagger_{\vec{k}} - a_{-\vec{k}}) + A_{\vec{k}}(a^\dagger_{\vec{k}} + a_{-\vec{k}})\left(\frac{k}{aH}\right)^3\right]$$

$$\pi^{(c)}_{\vec{k}}(t) = \sqrt{\frac{k^3\hbar}{2}}\left[(a^\dagger_{\vec{k}} - a_{-\vec{k}})\frac{aH}{k} + B_{\vec{k}}(a^\dagger_{\vec{k}} + a_{-\vec{k}})\right] \qquad k \ll aH$$