Statistics of cosmic fields in the large deviation regime

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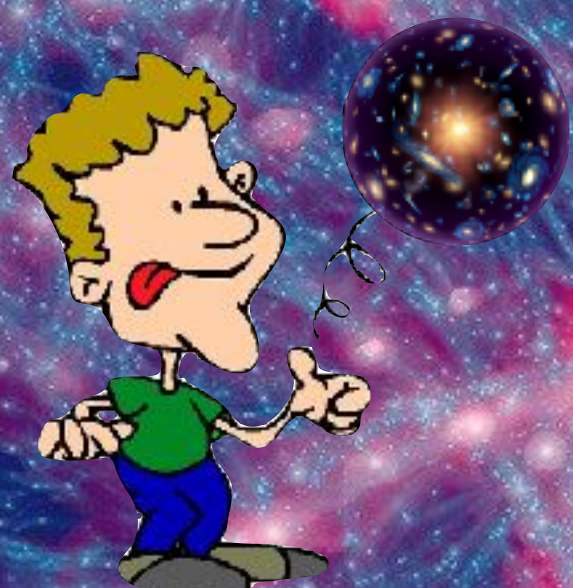
horizon-AGN

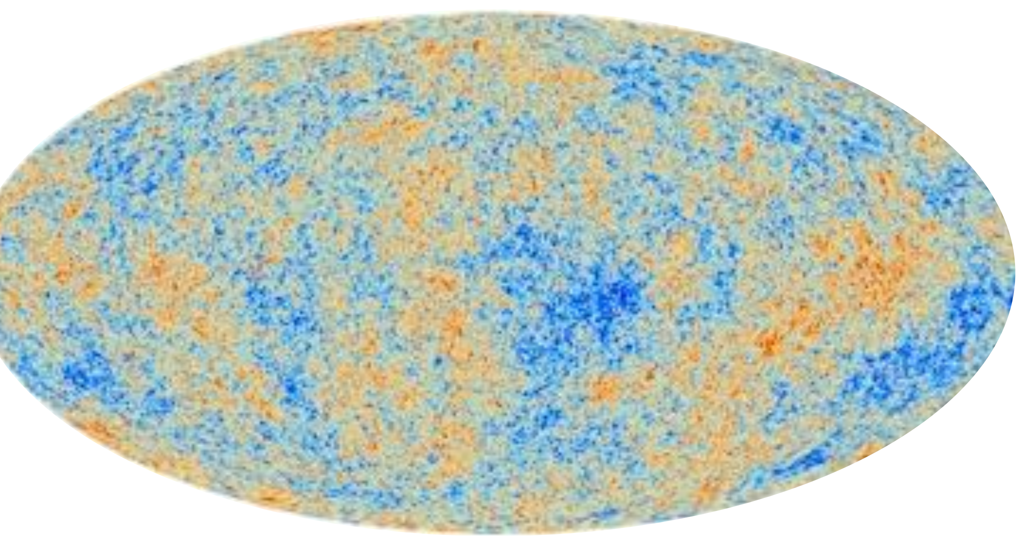


20/11/2018

CosmoClub CEA

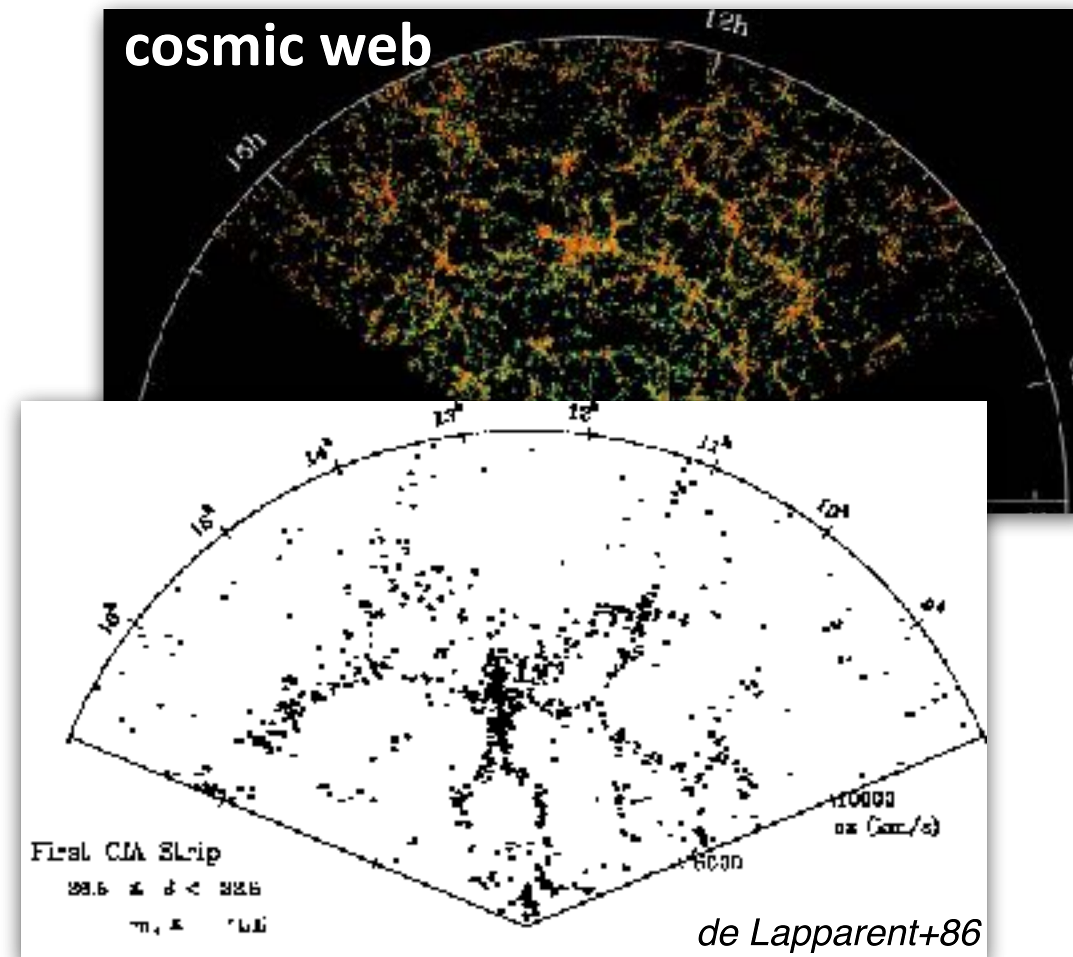
How is the cosmic web woven?

Gaussian primordial fluctuations



gravity

expansion



Vlasov-Poisson equations: dynamics of a self-gravitating collisionless fluid

Liouville theorem:

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{ma^2}\frac{\partial}{\partial \mathbf{x}}-m\nabla\phi\frac{\partial}{\partial \mathbf{p}}\right]f(\mathbf{x},\mathbf{p},\mathbf{t})=\mathbf{0}$$

Poisson equation:

$$\Delta\phi=4\pi a^2 G(\rho-\bar{\rho})$$

These highly non-linear equations can be solved using numerical simulations or analytically in some specific regimes. Exact solutions are crucial to understand the details of structure formation.

Before shell-crossing, moments>2 can be neglected (velocity dispersion,...) and we get evolution equations for the cosmic density and velocity fields:

continuity equation:

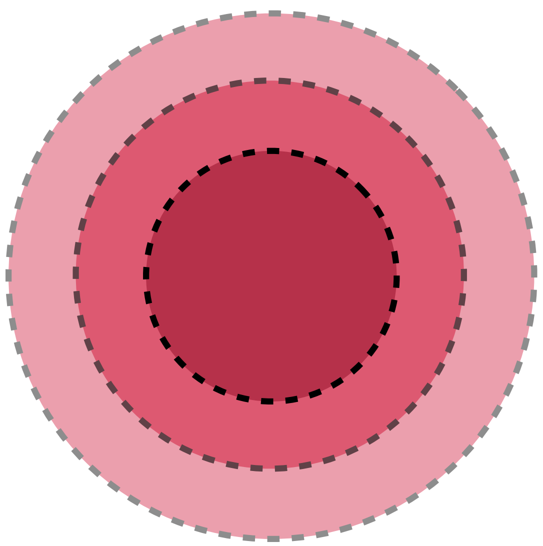
$$\frac{\partial\delta}{\partial t}+\frac{1}{a}\nabla\cdot[(1+\delta)\mathbf{u}]=0$$

Euler equation:

$$\frac{\partial u_i}{\partial t}+\frac{\dot{a}}{a}u_i+\frac{u_j\partial_j u_i}{a}=-\frac{\partial_i\phi}{a}-\cancel{\frac{\partial_j[\rho\sigma_{ij}]}{\rho a}}$$

The spherical collapse dynamics

A solution is known for an initial spherically symmetric fluctuation thanks to Gauss theorem.



The evolution of the radius of the shell of mass M is given by

$$\ddot{R} = -\frac{GM}{R^2}$$

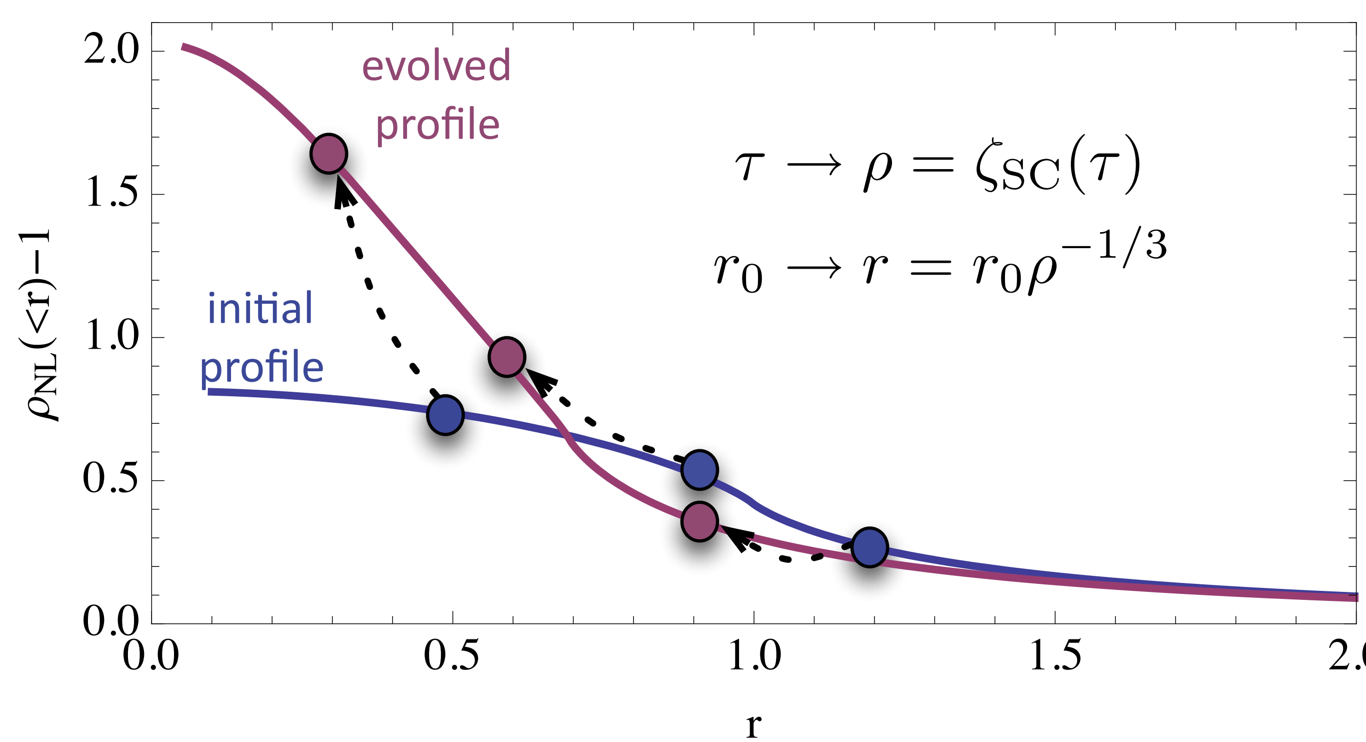
where M is $$M = \frac{4}{3}\pi R^3 \left(\rho - \frac{\Lambda}{8\pi G}\right).$$

Parametric solutions are known in an EdS Universe (numerical integration has to be done in the general case).

$$\frac{R}{R_m} = \frac{1}{2}(1 - \cos\eta)$$

$$\frac{t}{t_m} = \frac{1}{\pi}(\eta - \sin\eta)$$

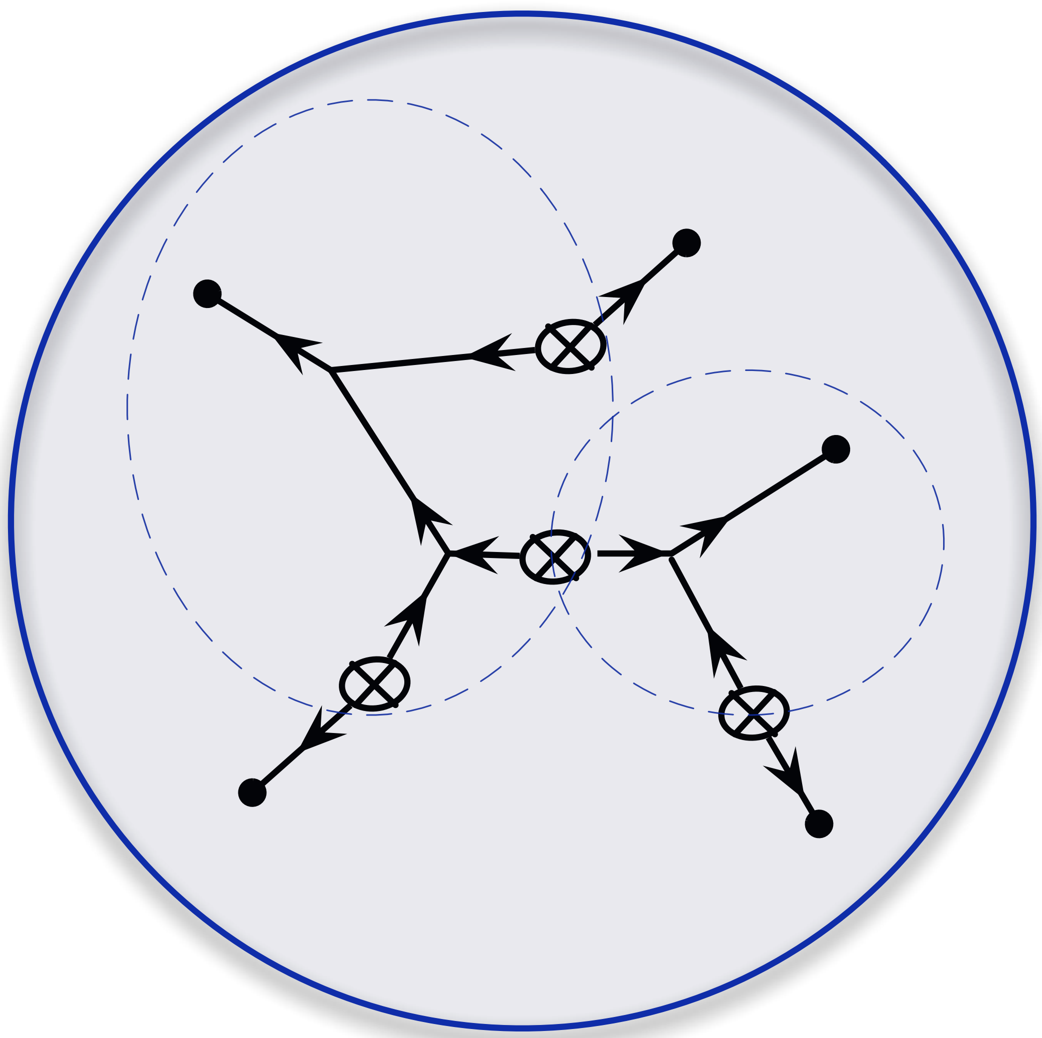
An initially overdense sphere expands until turnaround and collapses for a linearly interpolated density $\delta_c \approx 1.686$.

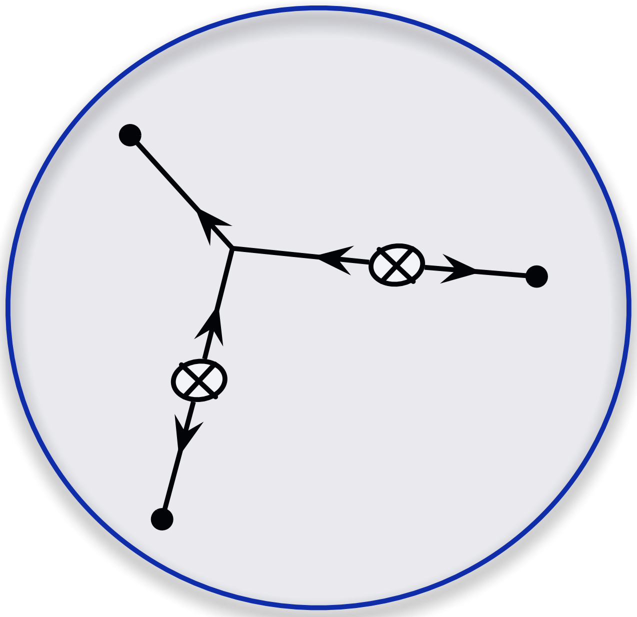


Perturbation theory

Assumption: cosmic fields can be expanded wrt initial fields $\delta(\mathbf{x},t) = \delta_1(\mathbf{x},t) + \delta_2(\mathbf{x},t) + \cdots$
All orders can then be computed hierarchically

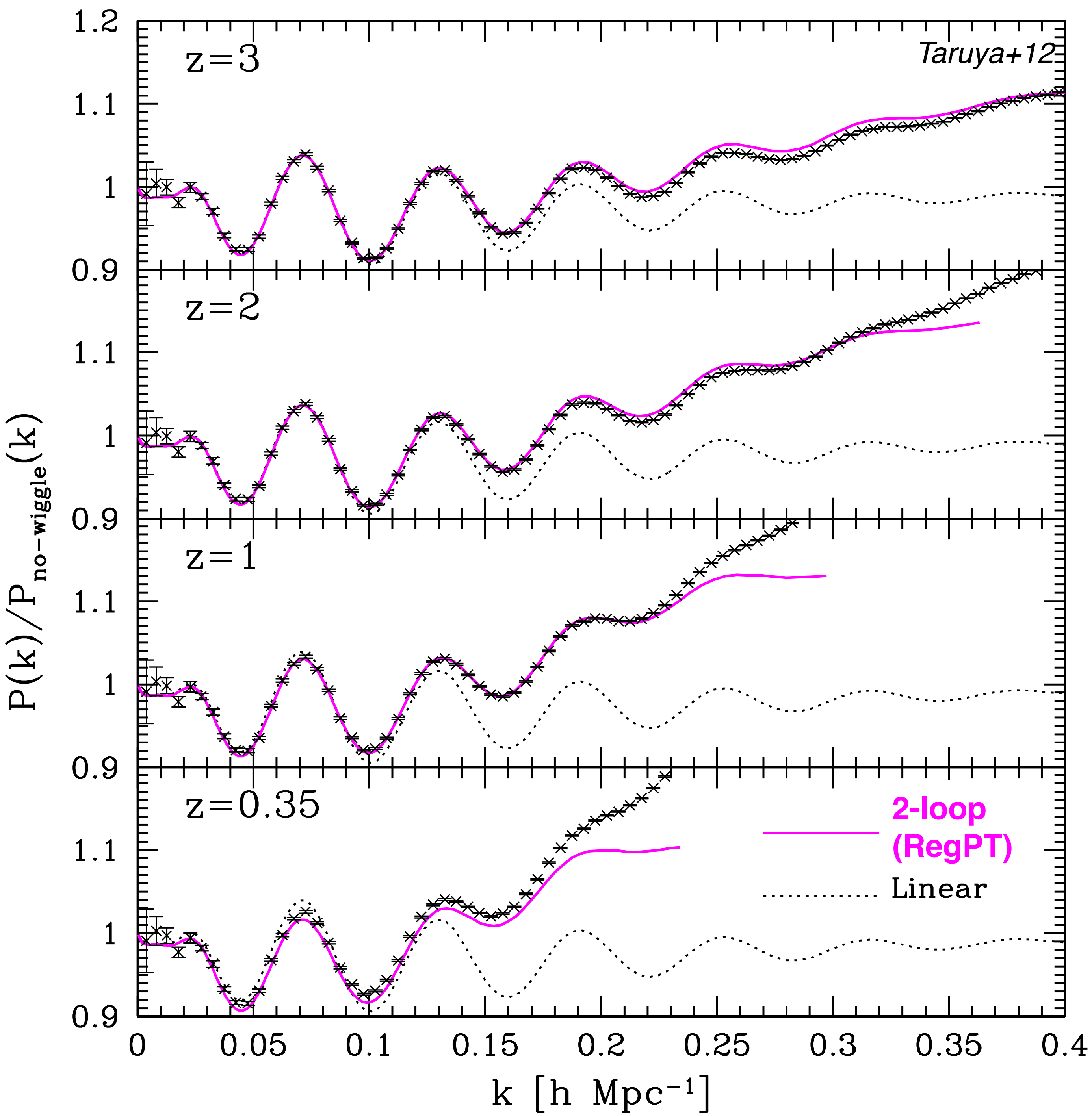
$$\delta_n(\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}_1 \ldots \int \mathrm{d}^3\mathbf{q}_n \, \delta_D(\mathbf{k} - \mathbf{q}_{1\ldots n}) \underbrace{F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n)}_{\text{PT kernels}} \delta_1(\mathbf{q}_1) \ldots \delta_1(\mathbf{q}_n)$$





Perturbation theory

matter power spectrum:



This approach is valid in the weakly non-linear regime where $|\delta| << 1$ i.e at high redshift / large scale.

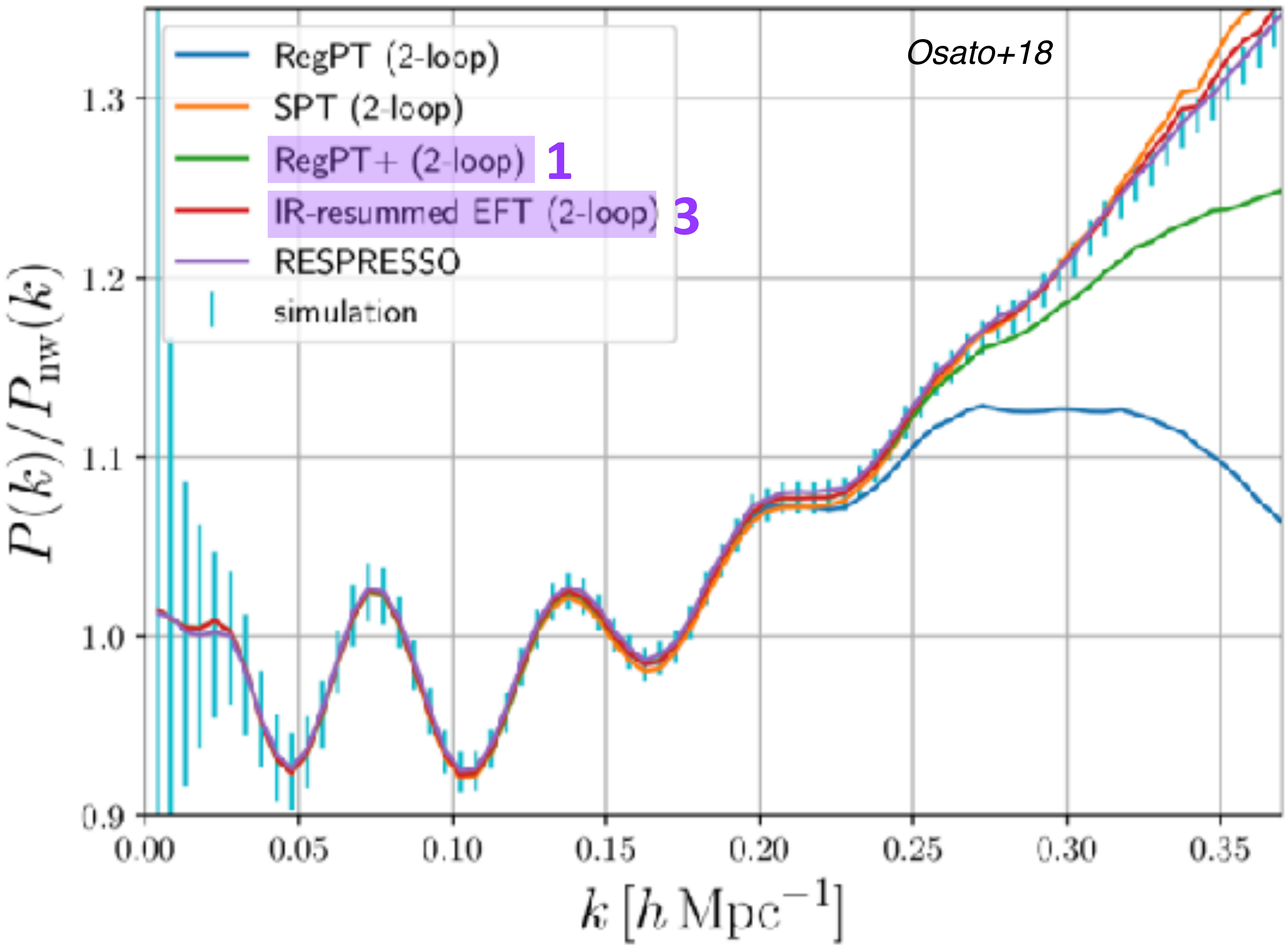
How to go beyond?

Perturbation theory

This approach is valid in the weakly non-linear regime where $|\delta| \ll 1$ i.e at high redshift / large scale.

How to go beyond?
Adding **new degrees of freedom** in the modeling?

Perturbation theory



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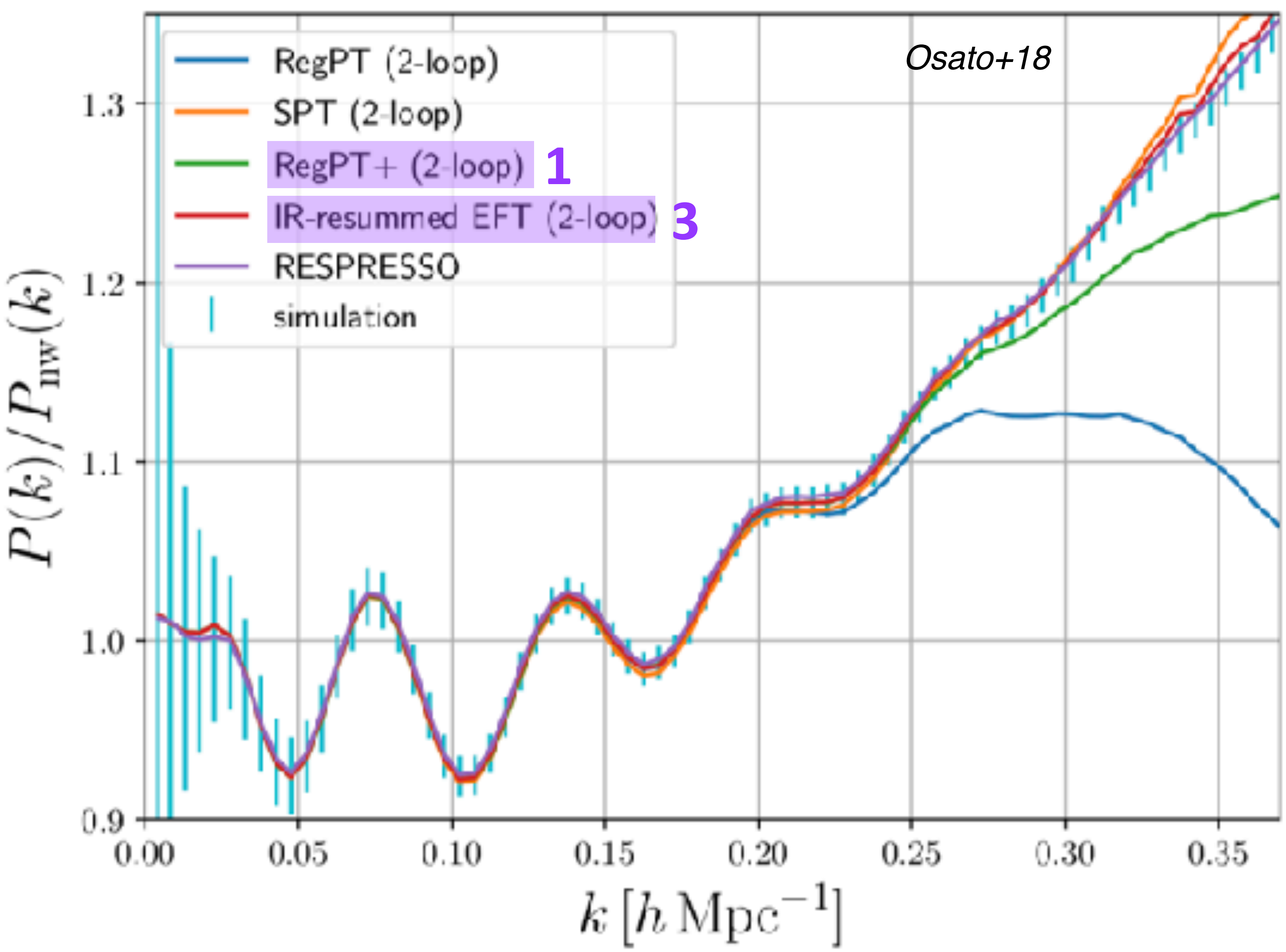

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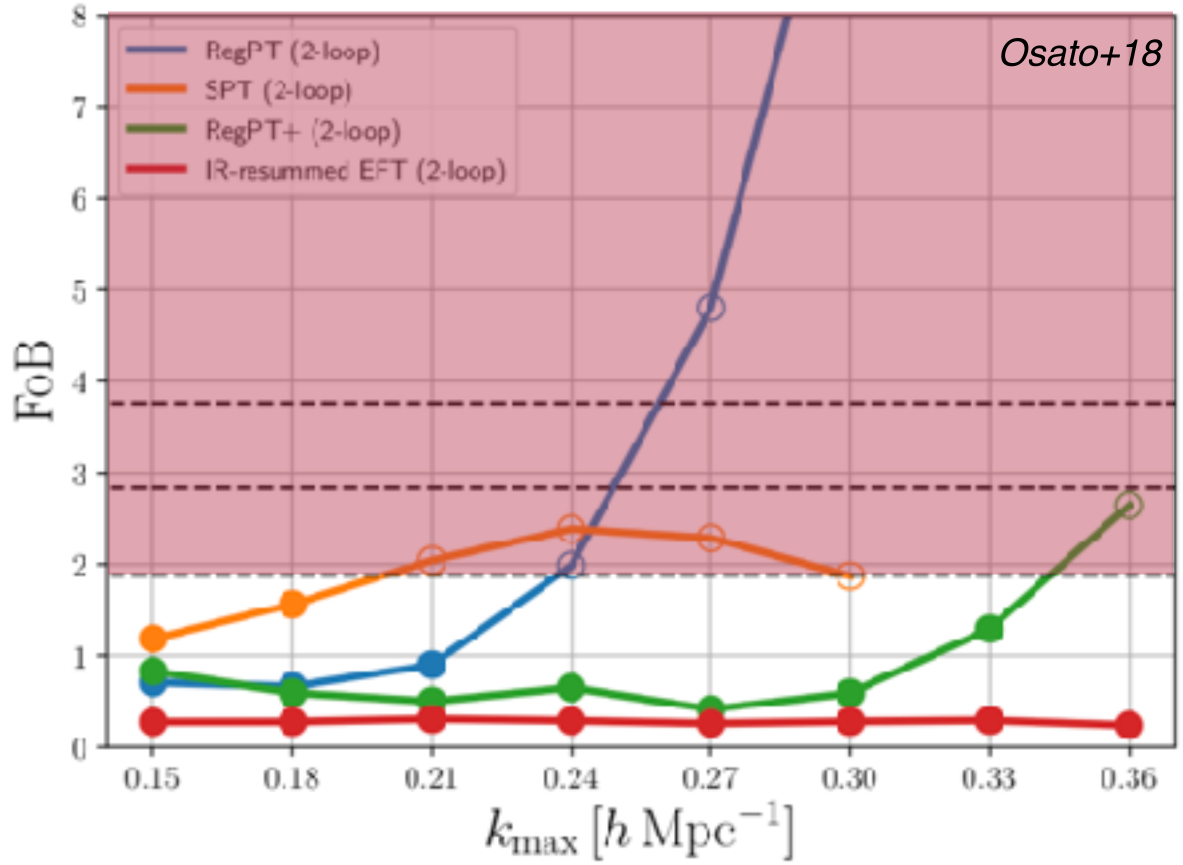
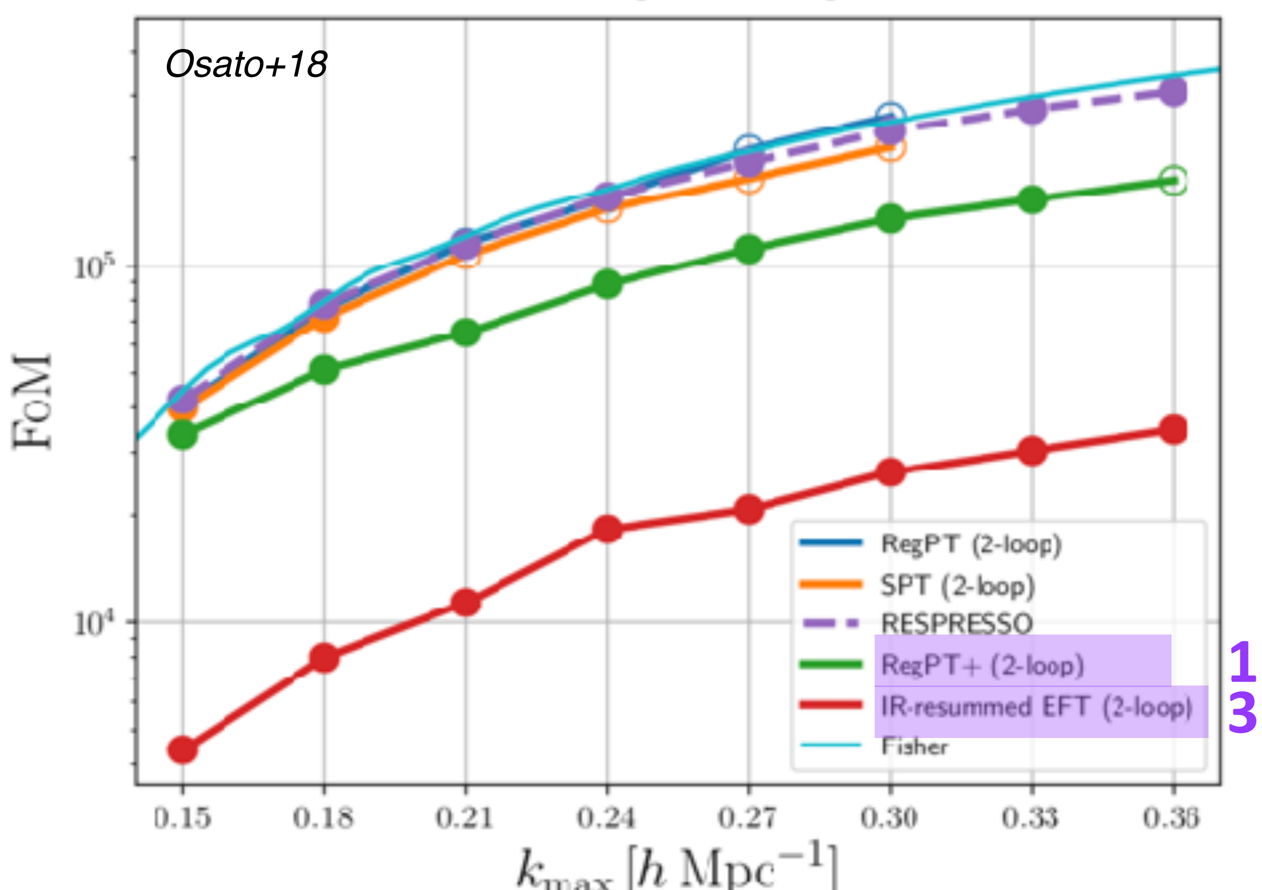
How to go beyond?
Adding **new degrees of freedom** in the modeling?

But… does it really allow us to get tighter cosmological constraints?

Perturbation theory





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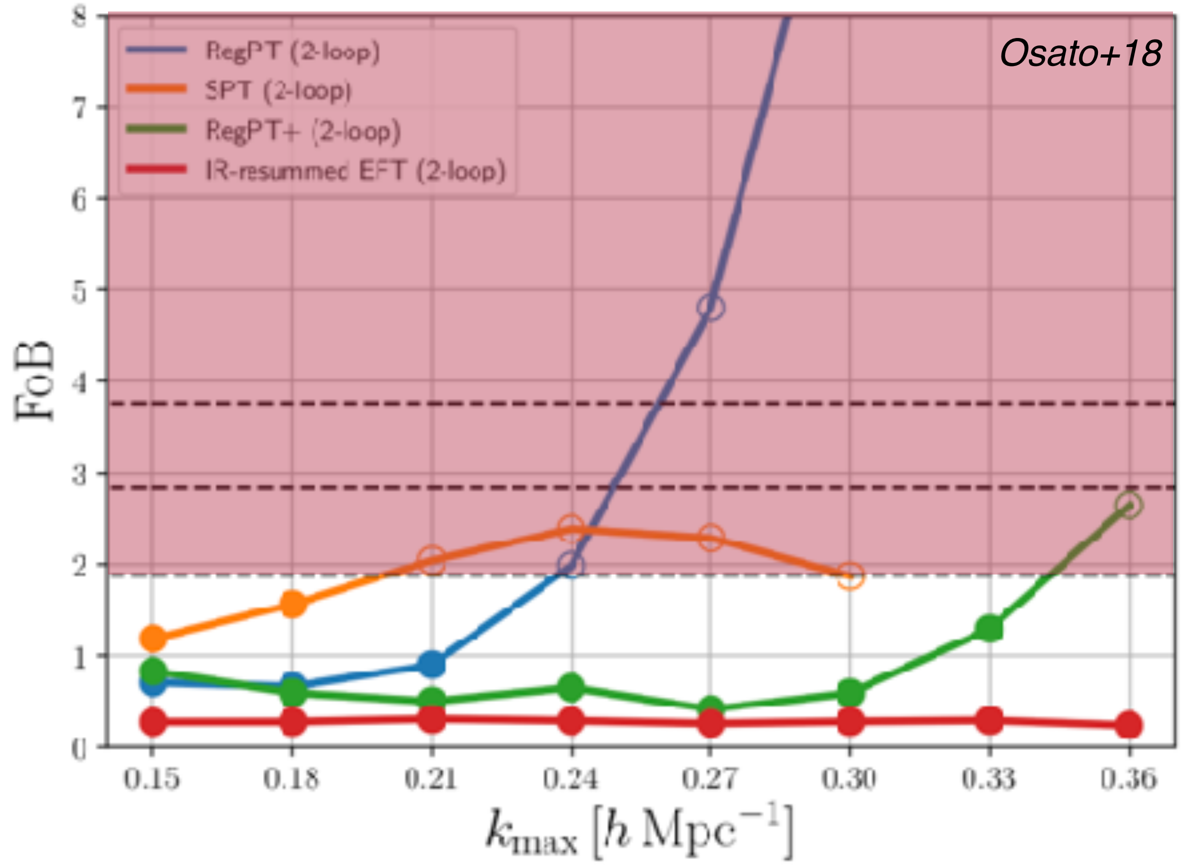

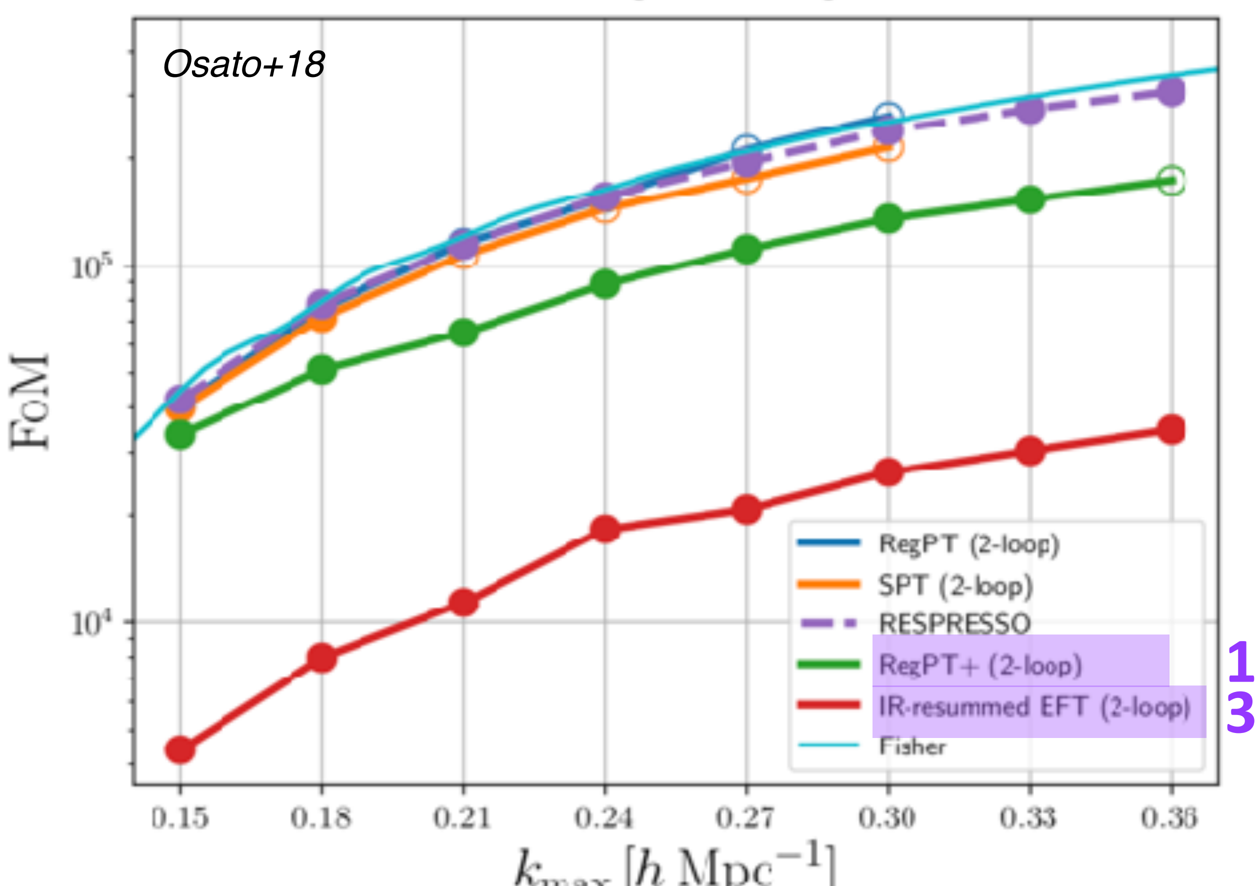

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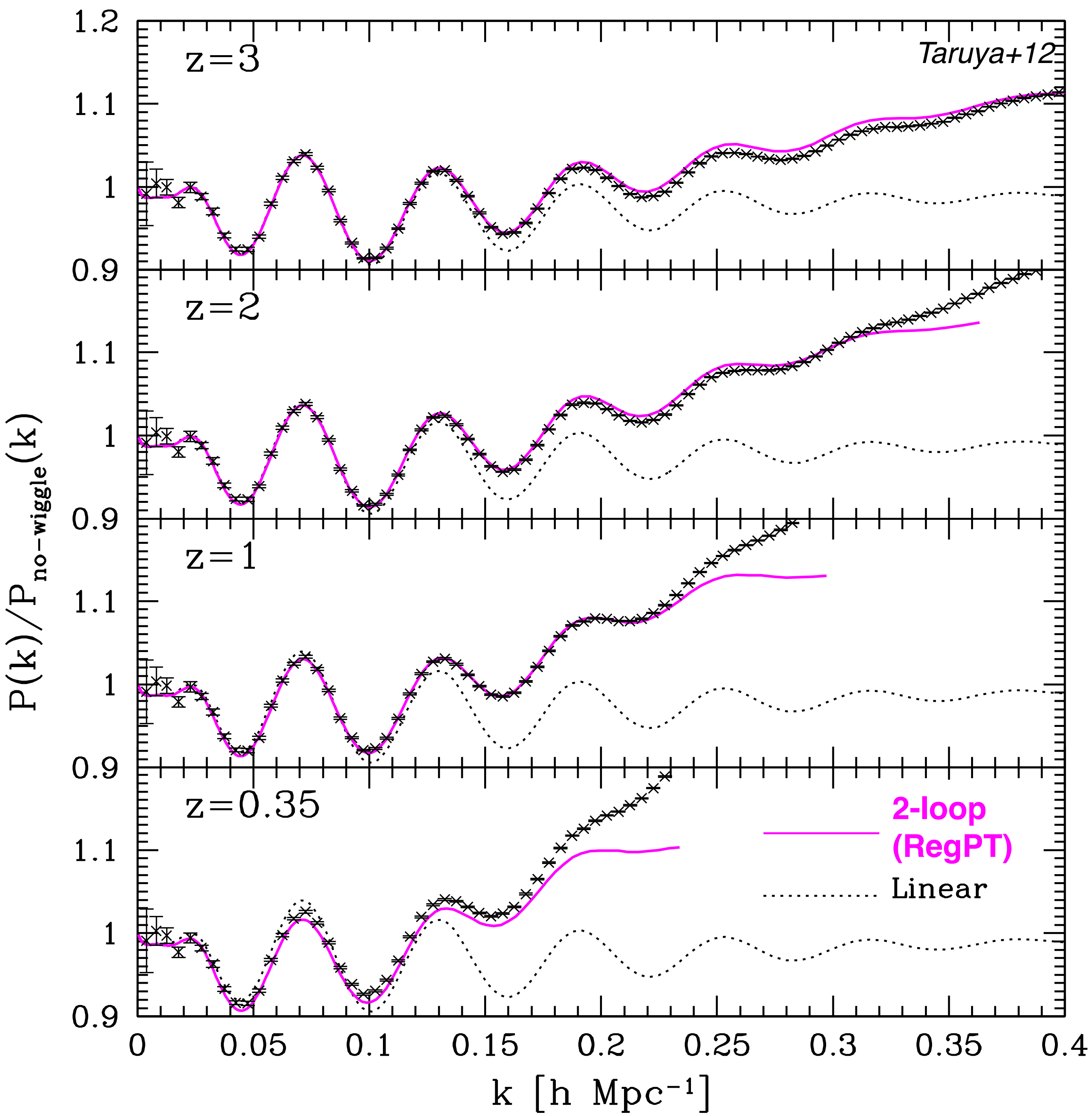
How to go beyond?
Adding **new degrees of freedom** in the modeling?

But… does it really allow us to get tighter cosmological constraints?

No!

Perturbation theory

matter power spectrum:



This approach is valid in the weakly non-linear regime where $|\delta|<<1$ i.e at high redshift / large scale.

How to go beyond **without introducing a myriad of free parameters**?

How to go beyond the weakly non-linear regime?

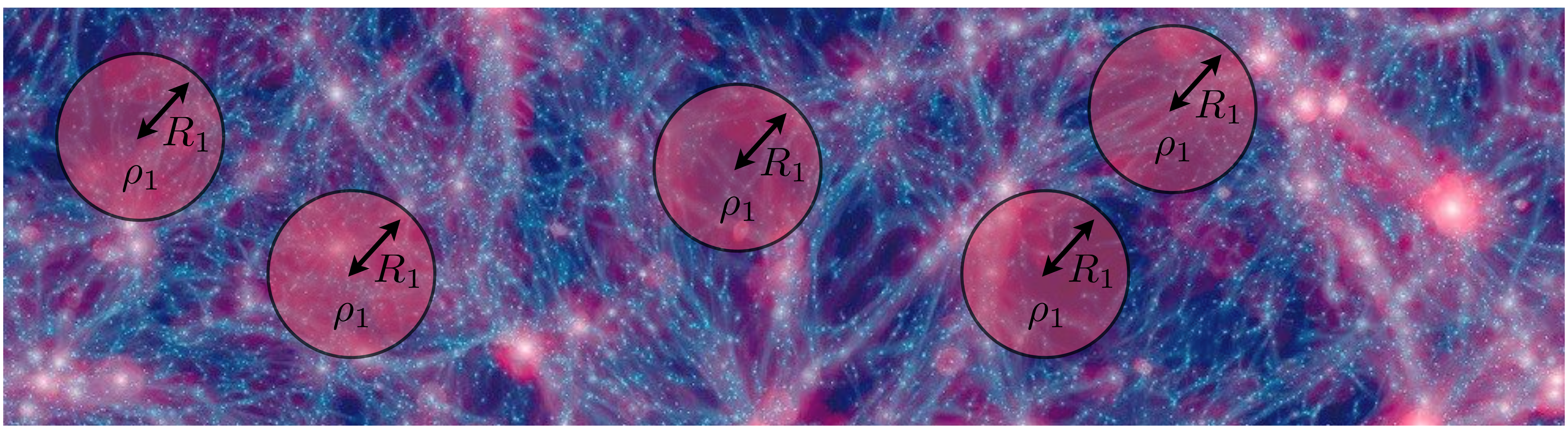
Need: configurations in which
-solutions from first principles can be found
-solutions are accurate as deep as possible in the non-linear regime

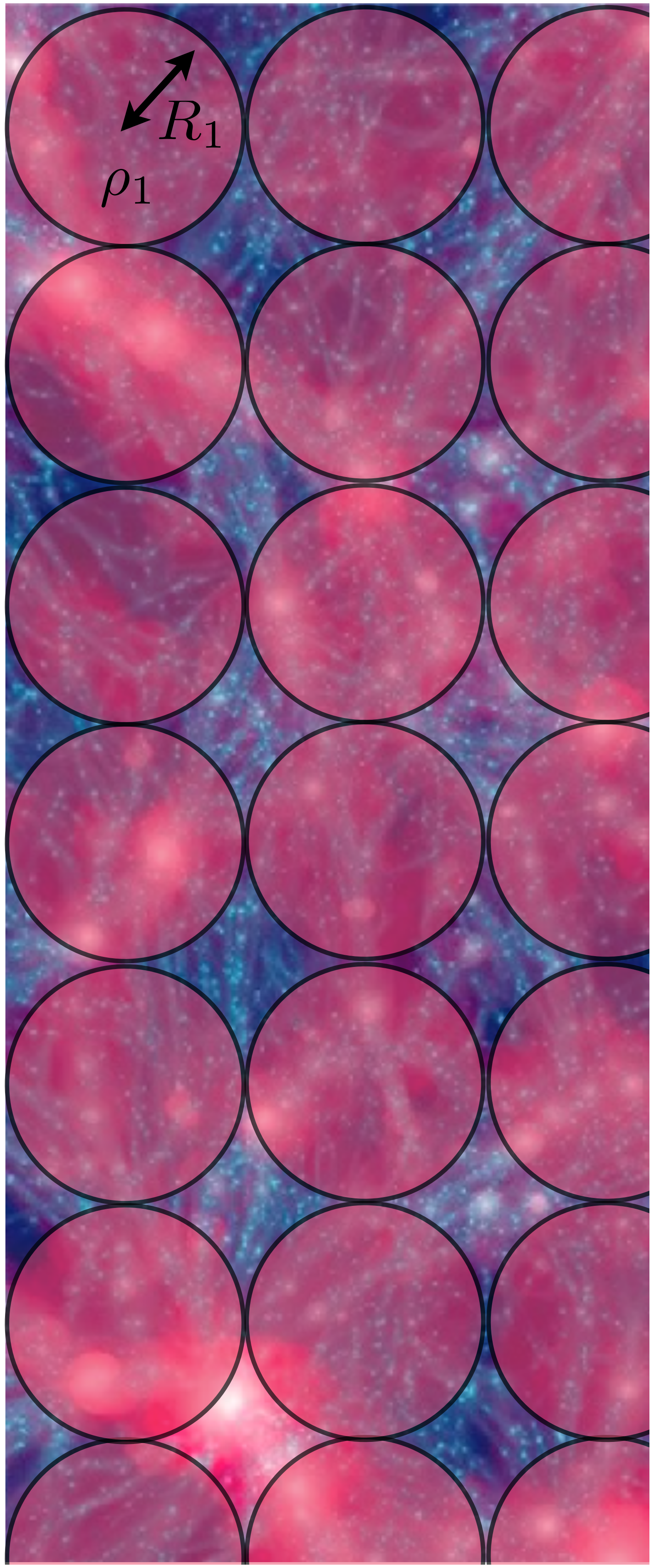
Motivation:
-theorists: we want to understand the physical processes driving structure formation!
-galaxy surveys: huge datasets that will need to be modelled very precisely to optimally extract the underlying cosmological information

Idea: use the symmetry!

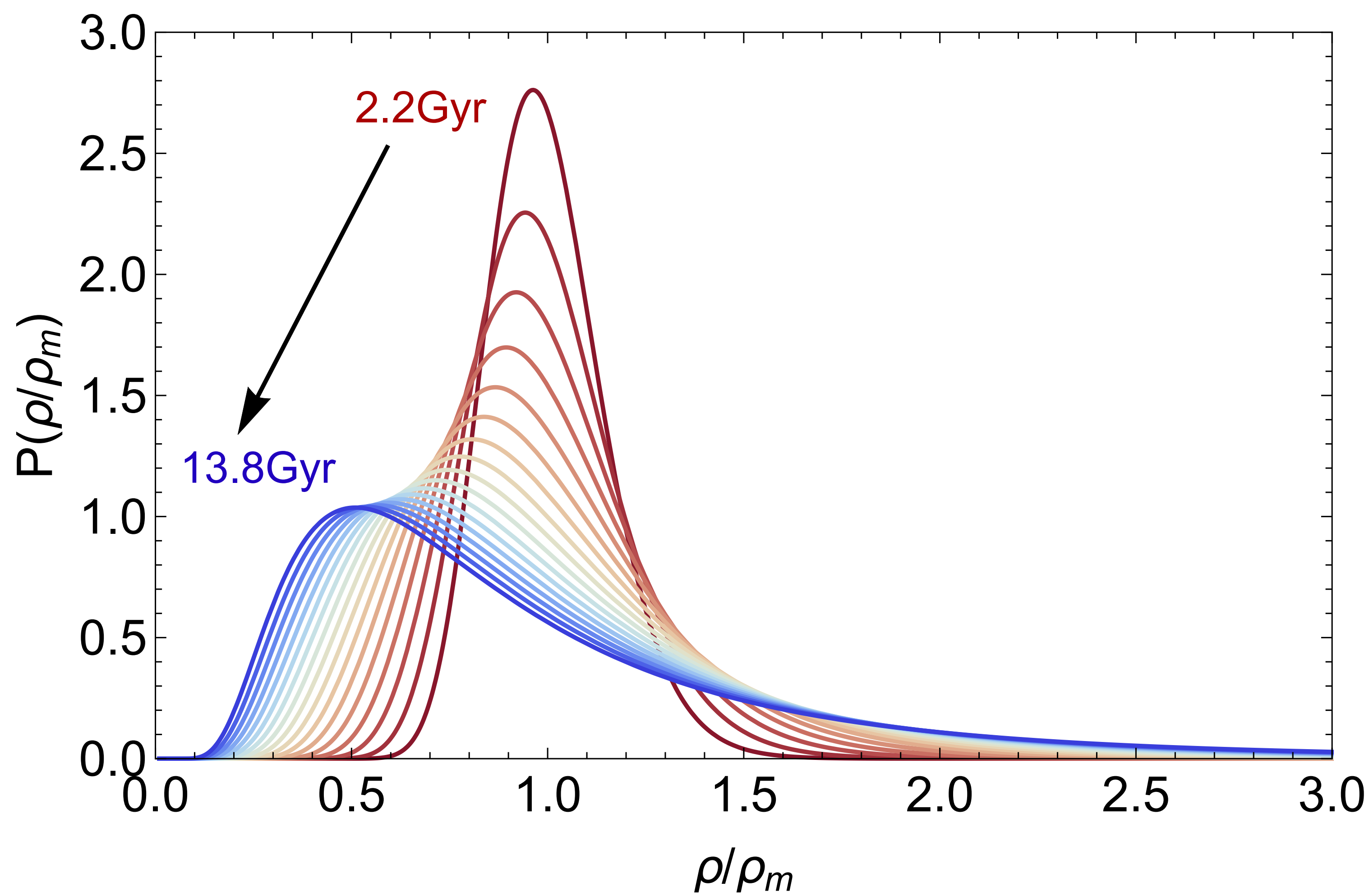
Proposed configurations: count-in-(spherical)cells

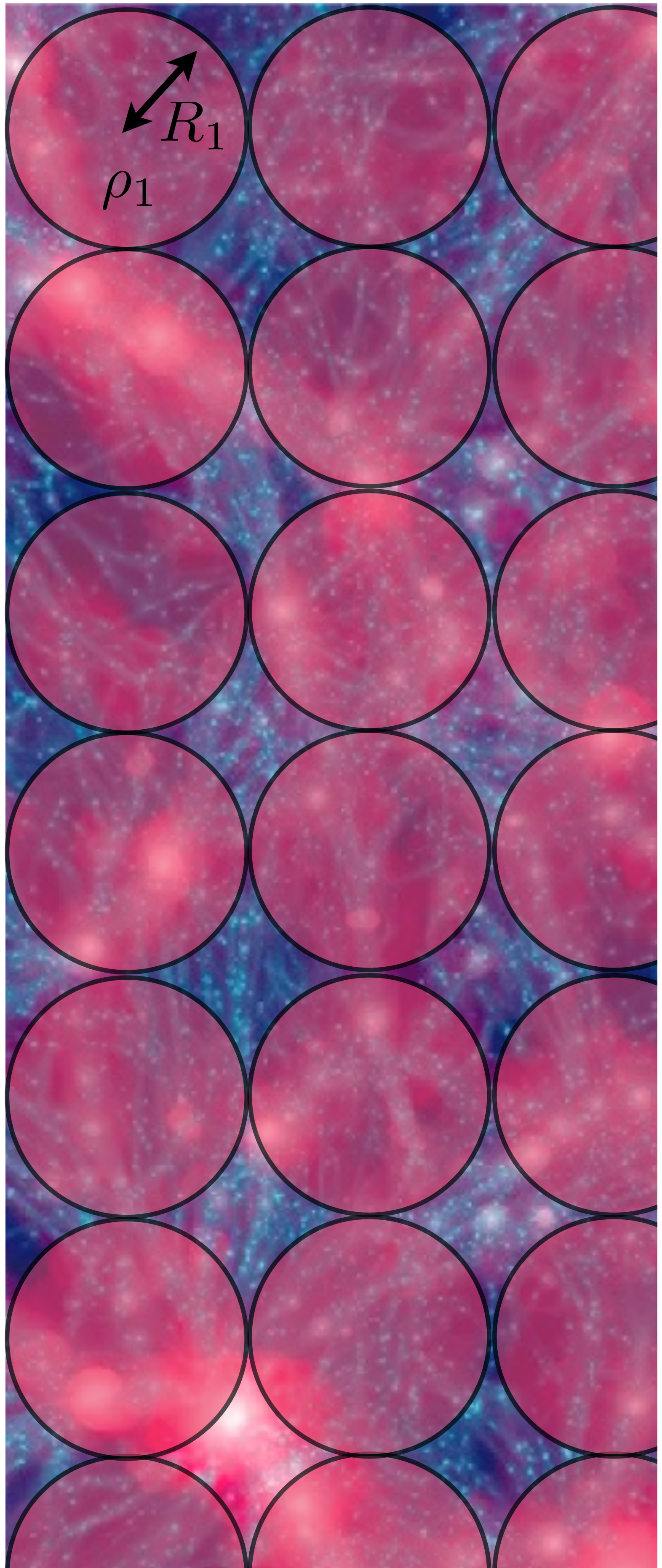
$$\mathcal{P}(\rho_1) = ?$$





Cosmic density PDF





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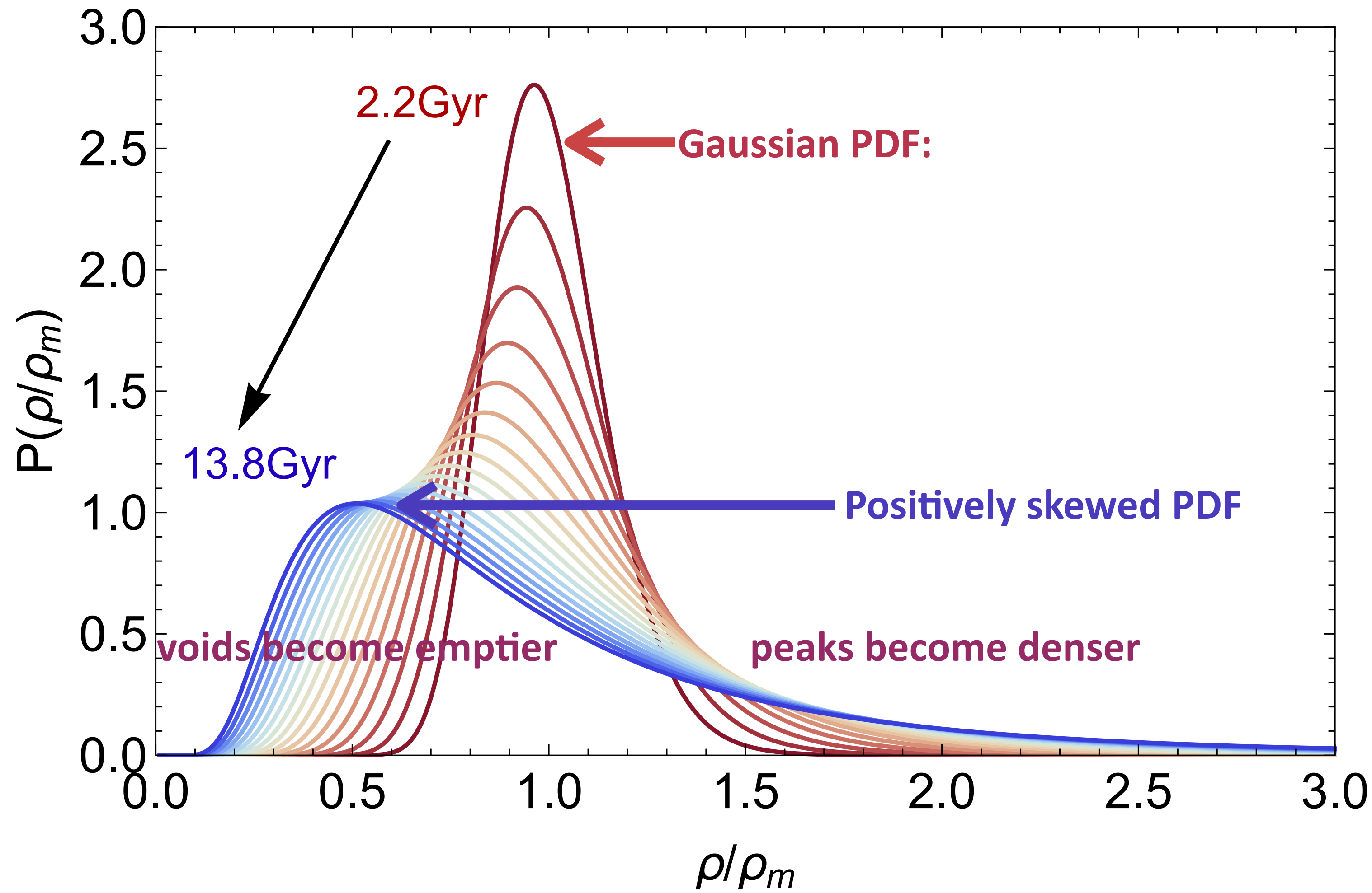


Driving parameter: variance σ (=amplitude of fluctuations)

From cumulants to PDF

PT can predict the n-th order cumulants whose ratios

$$S_n = \frac{\langle \delta^n \rangle_c}{\sigma^{2n-2}}$$

are almost z-independent. In particular, if the density field is smoothed with a top-hat filter

Bernardeau 94

$$S_3 = \frac{34}{7} + \gamma_1,$$

$$S_4 = \frac{60712}{1323} + \frac{62\,\gamma_1}{3} + \frac{7\,\gamma_1{}^2}{3} + \frac{2\,\gamma_2}{3},$$

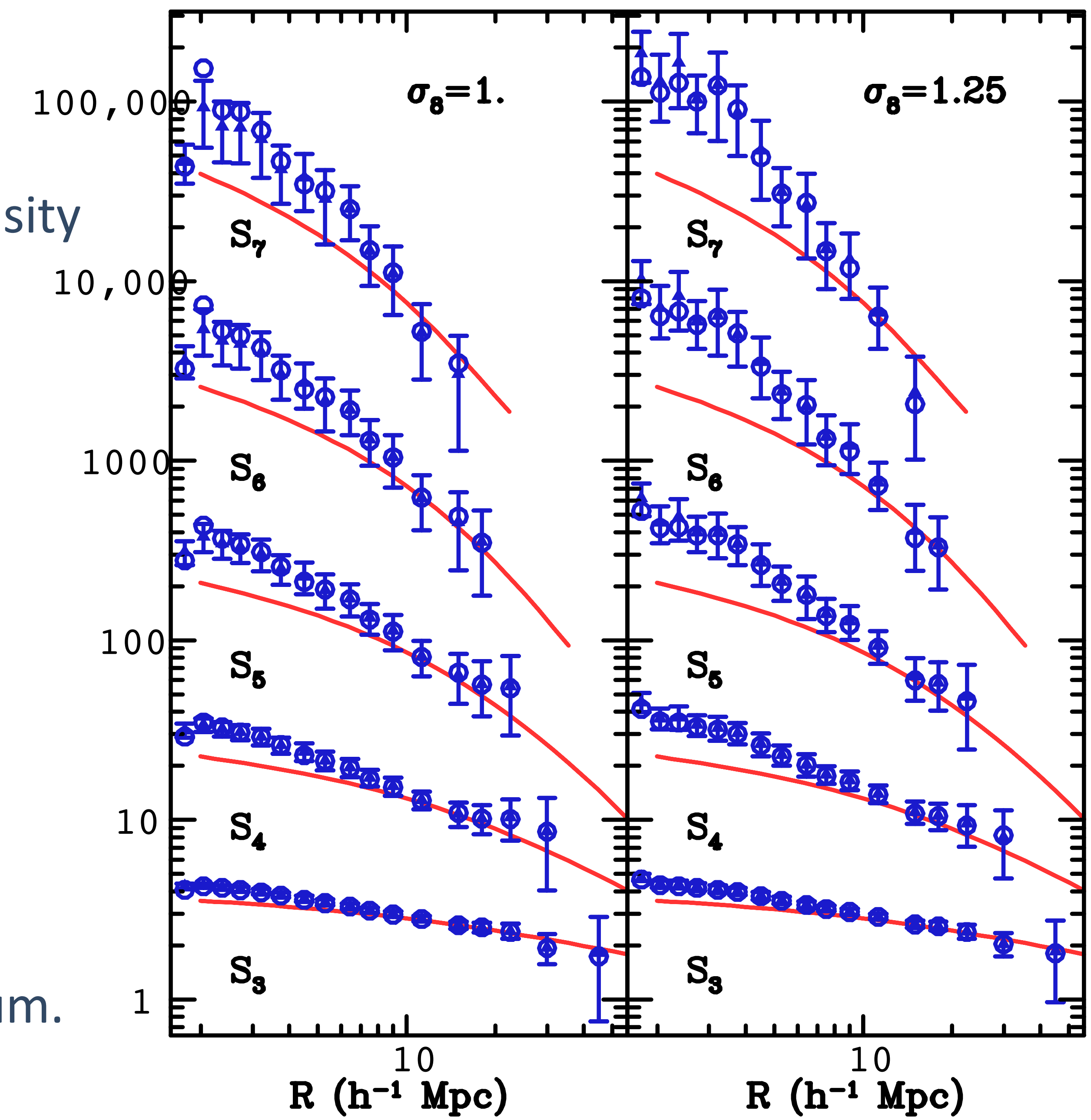
$$S_5 = \frac{200575880}{305613} + \frac{1847200\,\gamma_1}{3969} + \frac{6940\,\gamma_1{}^2}{63} + \frac{235\,\gamma_1{}^3}{27} + \frac{1490\,\gamma_2}{63} + \frac{50\,\gamma_1\,\gamma_2}{9} + \frac{10\,\gamma_3}{27},$$

where

$$\gamma_p = \frac{\mathrm{d}^p \log \sigma^2(R_0)}{\mathrm{d} \log^p R_0}$$

depends on the shape of the linear power spectrum.

Baugh & Gaztañaga 95



Hierarchy of cumulants:

σ^2, $\langle\delta^3\rangle_c \propto \sigma^4$, $\langle\delta^4\rangle_c \propto \sigma^6$, ...

From cumulants to PDF

$$S_n = \frac{\langle \delta^n \rangle_c}{\sigma^{2n-2}}$$

The PDF of x=δ/σ can then be written as an Edgeworth expansion (in powers of σ):

$$P(x) = G(x)\left[1 + \sigma \frac{S_3}{3!} H_3(x) + \sigma^2 \left(\frac{S_4}{4!} H_4(x) + \frac{1}{2}\left(\frac{S_3}{3!}\right)^2 H_6(x) \right) + \cdots \right]$$

which can be derived from the cumulant generating function of ρ=1+δ

$$\exp \varphi(\lambda) = \int P(\rho) \exp(\lambda\rho) \leftrightarrow P(\rho) = \int_{-\imath\infty}^{\imath\infty} \frac{\mathrm{d}\lambda}{2\imath\pi} \exp(\lambda\rho - \varphi(\lambda))$$

Laplace transform **inverse Laplace transform**

where $\varphi(\lambda) = \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} \langle \rho^i \rangle_c$.

From cumulants to PDF

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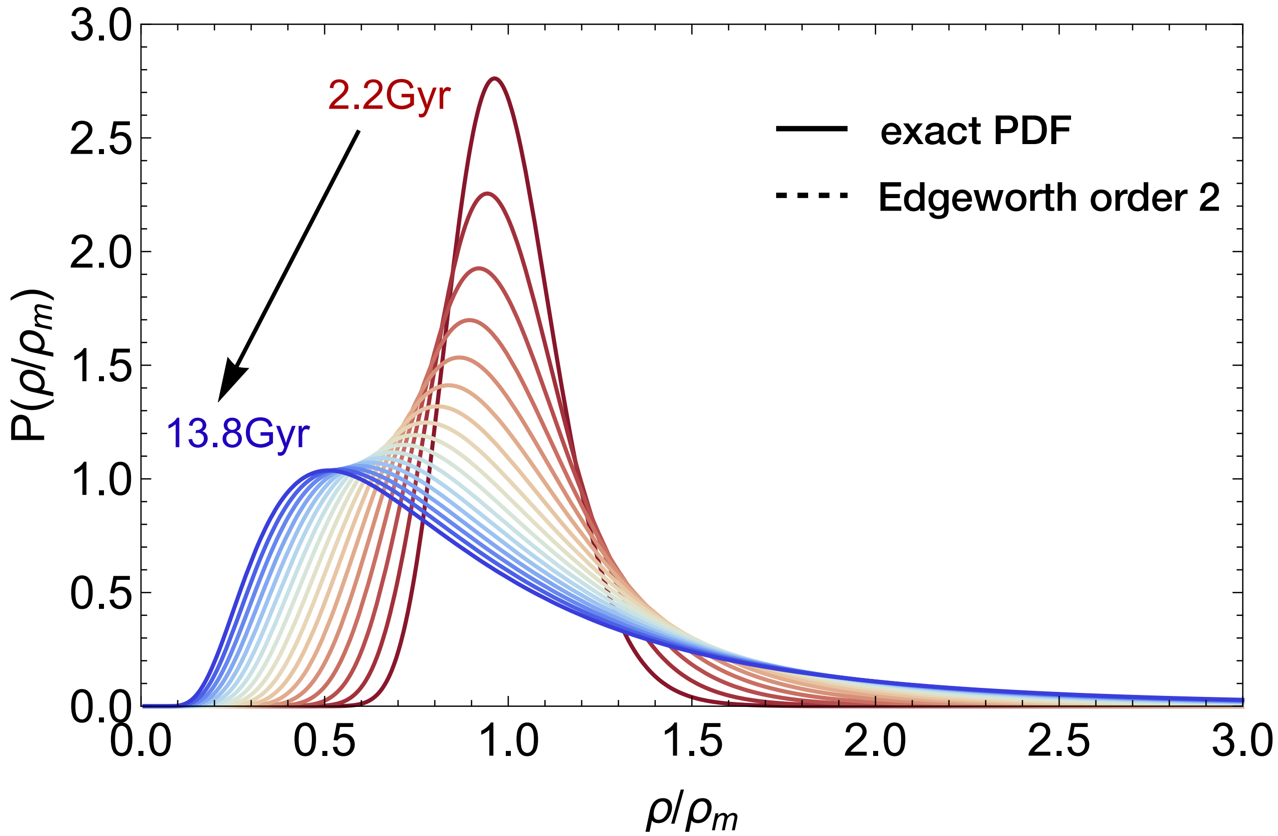

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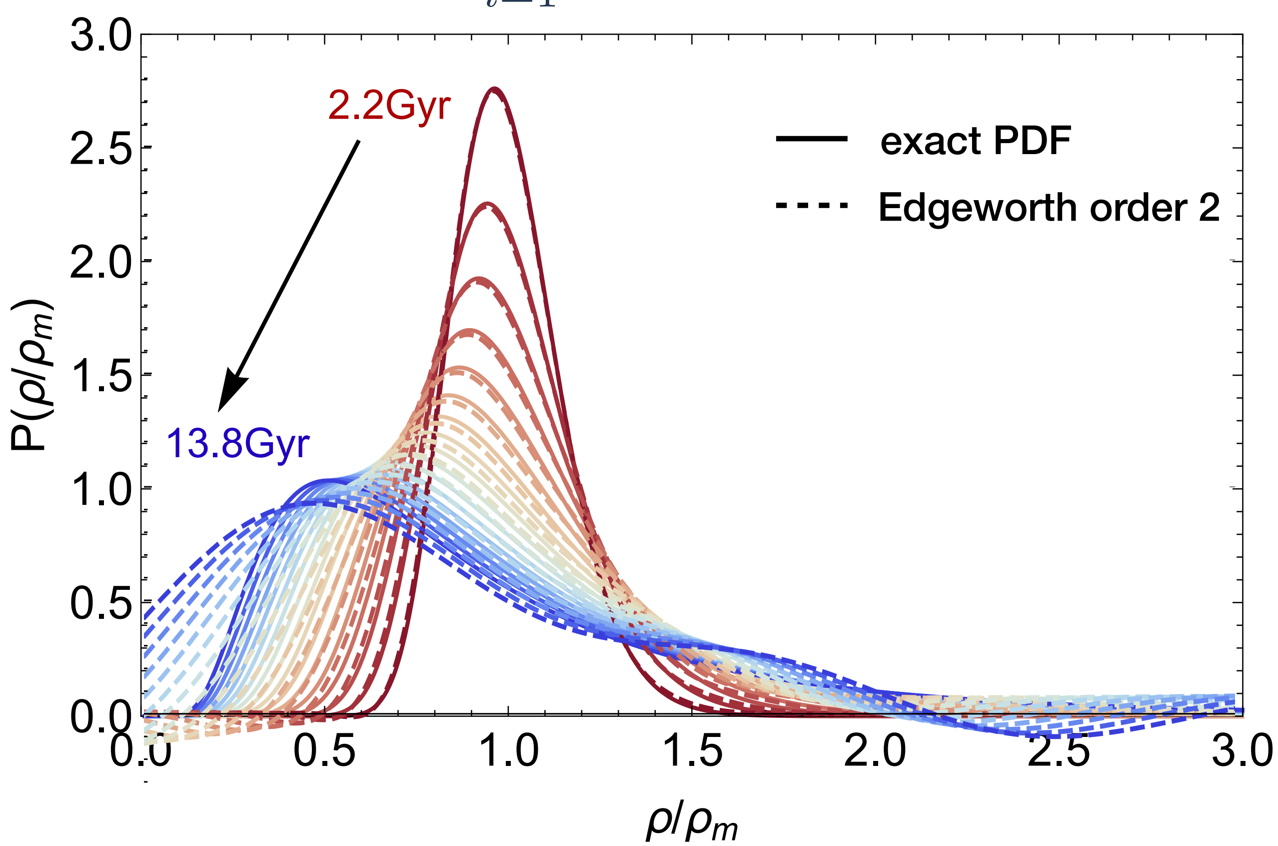

Problem : When this series is truncated at some orders, the PDF is unphysical : it is not normalised and can take negative values.

Solution : **large-deviation theory** provides us with a model for the PDF which does not suffer from those issues. All cumulants are exact at tree-order.

«An unlikely fluctuation is brought about by the least unlikely among all unlikely paths »

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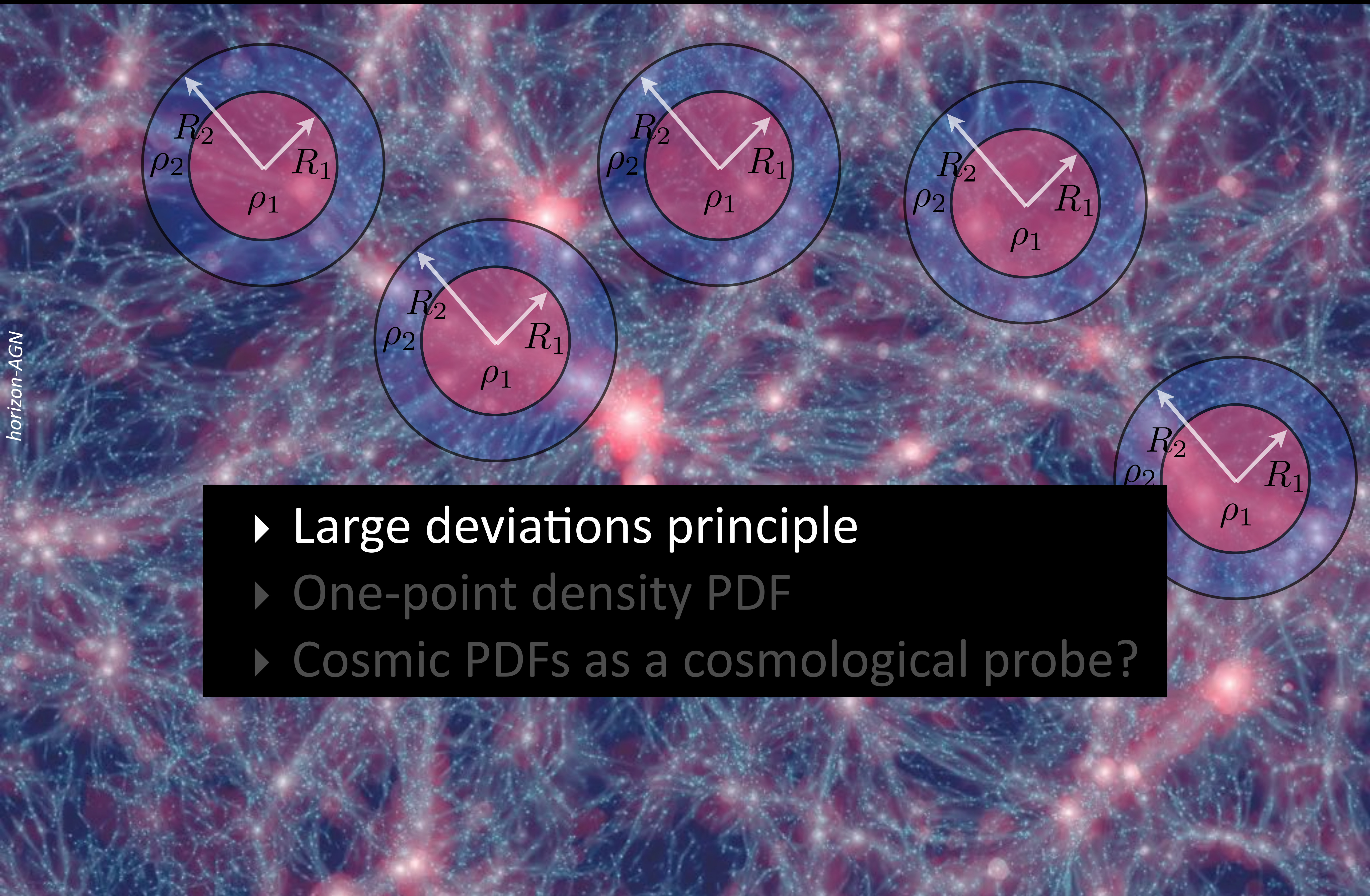
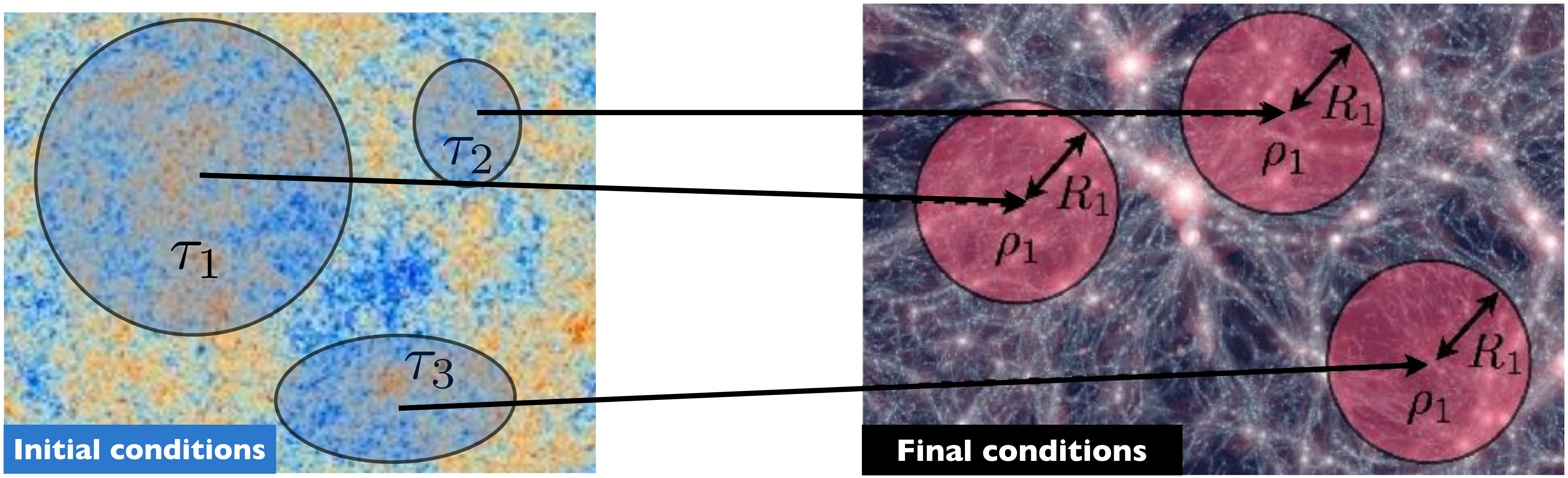


- Large deviations principle
- One-point density PDF
- Cosmic PDFs as a cosmological probe?

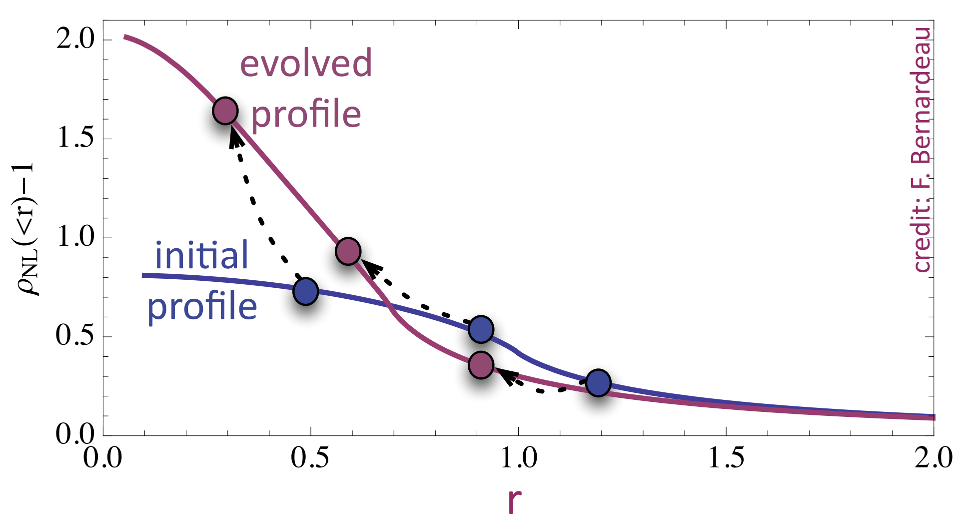
Bernardeau 94
Valageas 02
Bernardeau&Reimberg 16

Large-deviation Theory: what is the most likely initial configuration a final density originates from?

In principle, one has to sum over all possible paths:



Different initial configurations can lead to the same final state! What is the most likely one?
Conjecture: Spherical symmetry enforces this most likely path to be the ***Spherical Collapse dynamics***.



$$\tau \rightarrow \rho = \zeta_{\rm SC}(\tau)$$

$$r_0 \rightarrow r = r_0 \rho^{-1/3}$$

Large-deviation Theory: in a nutshell

LDP tells us how to compute the **cumulant generating function** from the initial conditions using the spherical collapse as the « mean dynamics »:

$$\varphi(\{\lambda_k\}) = \sup_{\rho_i}(\lambda_i \rho_i - I(\rho_i))$$

Varadhan's theorem

The density **PDF** is then obtained via an inverse Laplace transform of the CGF

$$\exp \varphi(\lambda) = \int P(\rho)\exp(\lambda\rho) \leftrightarrow P(\rho) = \int_{-\imath\infty}^{\imath\infty} \frac{\mathrm{d}\lambda}{2\imath\pi}\exp(\lambda\rho - \varphi(\lambda))$$

- This is exact in the zero variance limit. We then extrapolate to non zero values.
- **Parameter-free** theory which depends on cosmology through : the spherical collapse dynamics, the linear power spectrum and growth of structure.
- Predictions are fully **analytical** if one applies the LDP to the log *Uhlemann+16*

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Why?

$$\varphi(\lambda_k) = \Big\langle \underbrace{\exp(\Sigma_i \lambda_i \rho_i)}_{\simeq \lambda_i \langle \rho_i \rangle + \lambda_i \lambda_j \langle \rho_i \rho_j \rangle + \ldots} \Big\rangle = \int_0^\infty \Pi_i \mathrm{d}\rho_i P(\{\rho_k\}) \exp\left(\Sigma_i \lambda_i \rho_i\right)$$

initial density contrast

$$= \int \mathcal{D}\left[\tau(\vec{x})\right] \mathcal{P}\left[\tau(\vec{x})\right] \exp(\lambda_i \rho_i \left[\tau(\vec{x})\right])$$

known Gaussian PDF $\mathcal{P}(\tau) \propto e^{-I(\tau)}$

contraction principle

$$= \int \mathrm{d}\tau_i \exp\left(\lambda_i \zeta_{\rm SC}(\tau_i) - I(\tau_i)\right)$$

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Uhlemann+16

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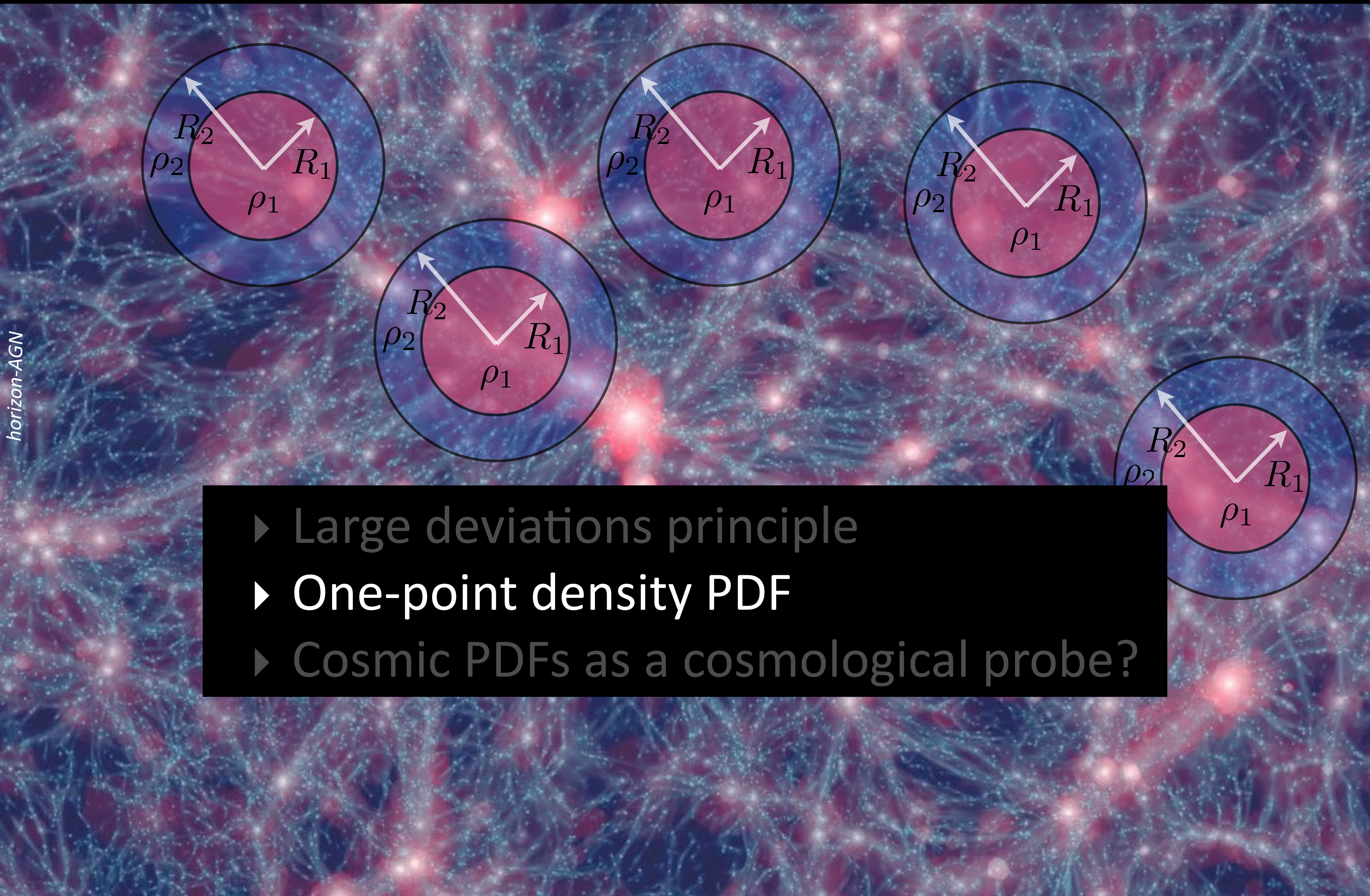

- Large deviations principle
- One-point density PDF
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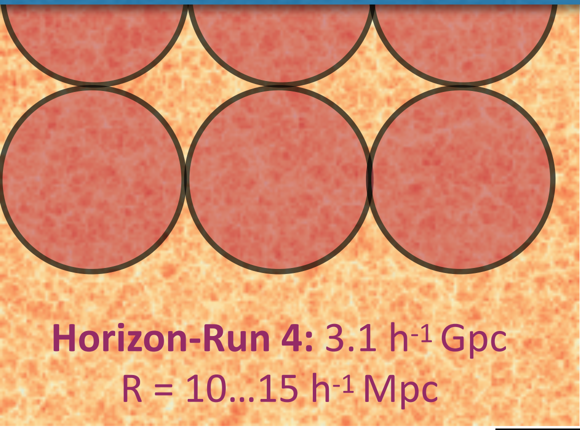
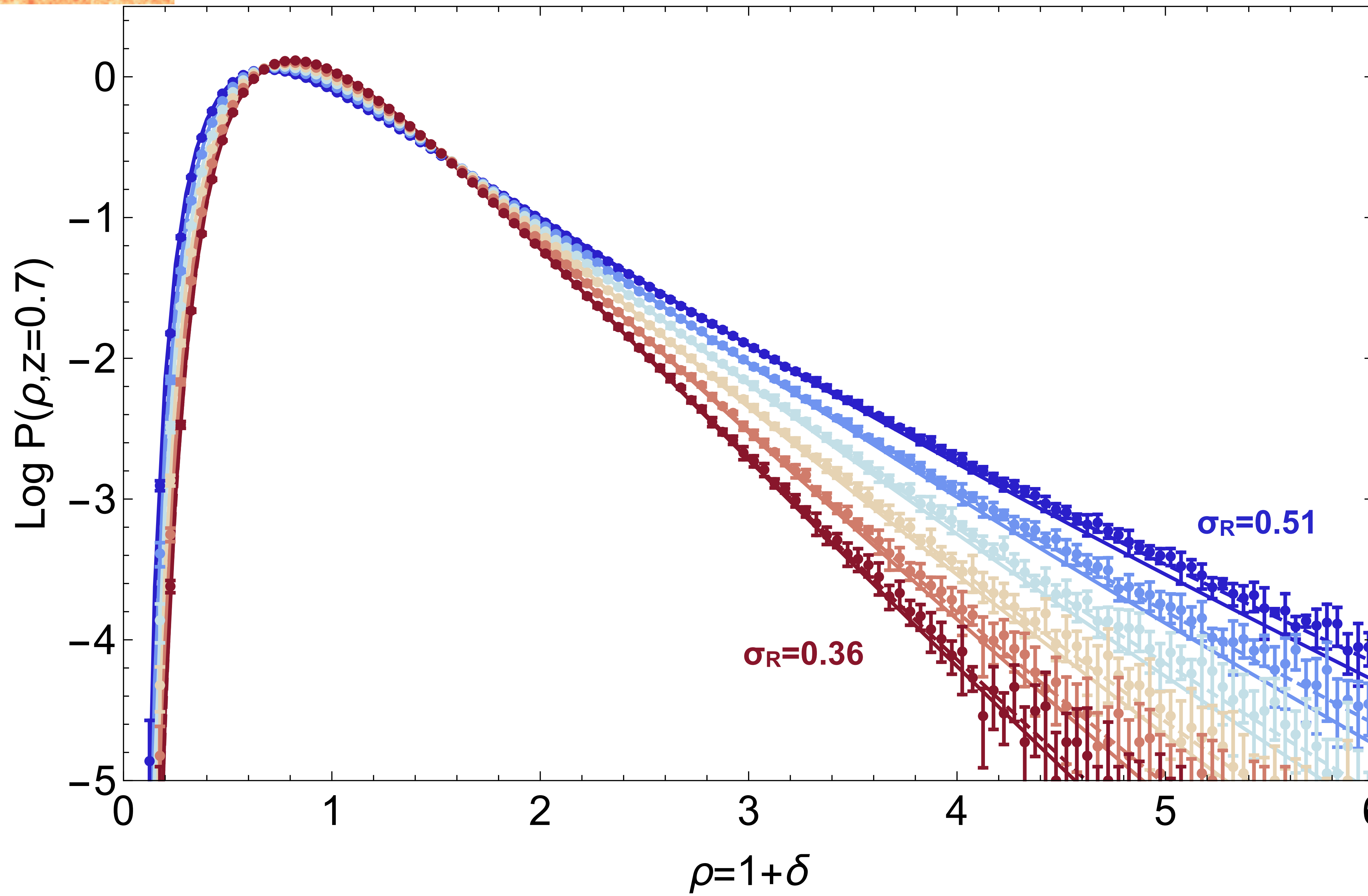
One-cell density PDF

Bernardeau+14
Uhlemann+16
SC+16b





We have developed a fast and easy-to-use public code…

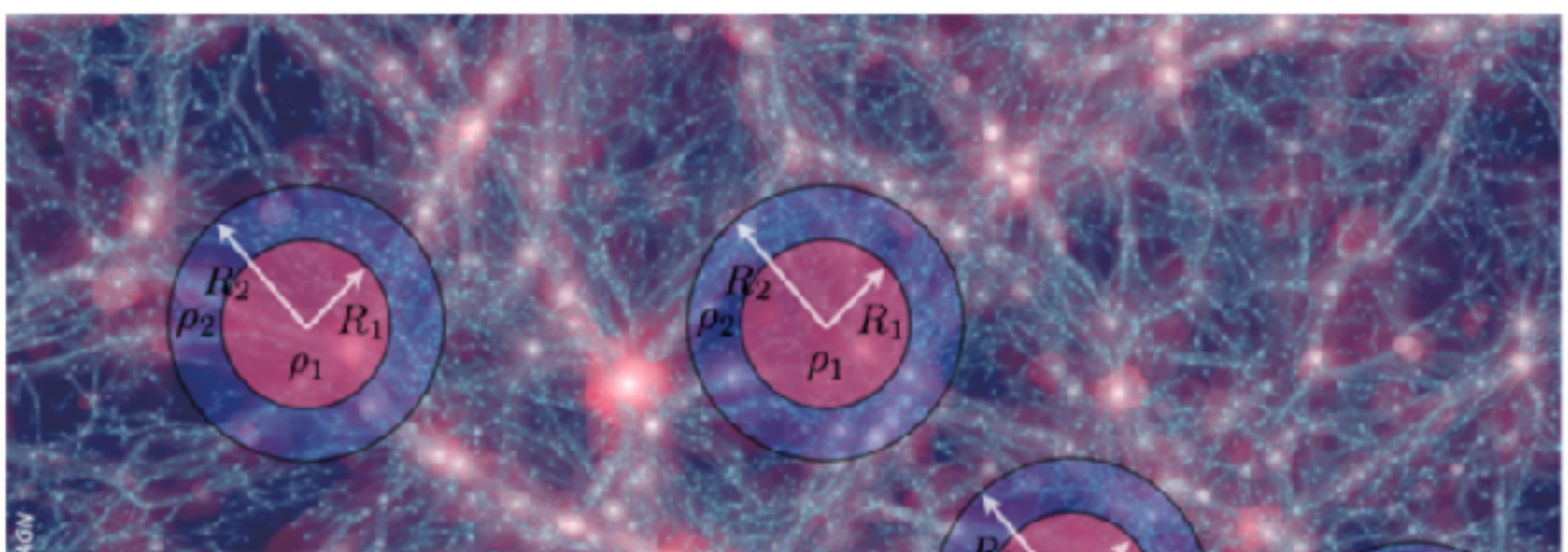
http://cita.utoronto.ca/~codis/LSSFast.html

LSSFAST

A Mathematica package to compute cosmic density PDF in the large-deviation regime

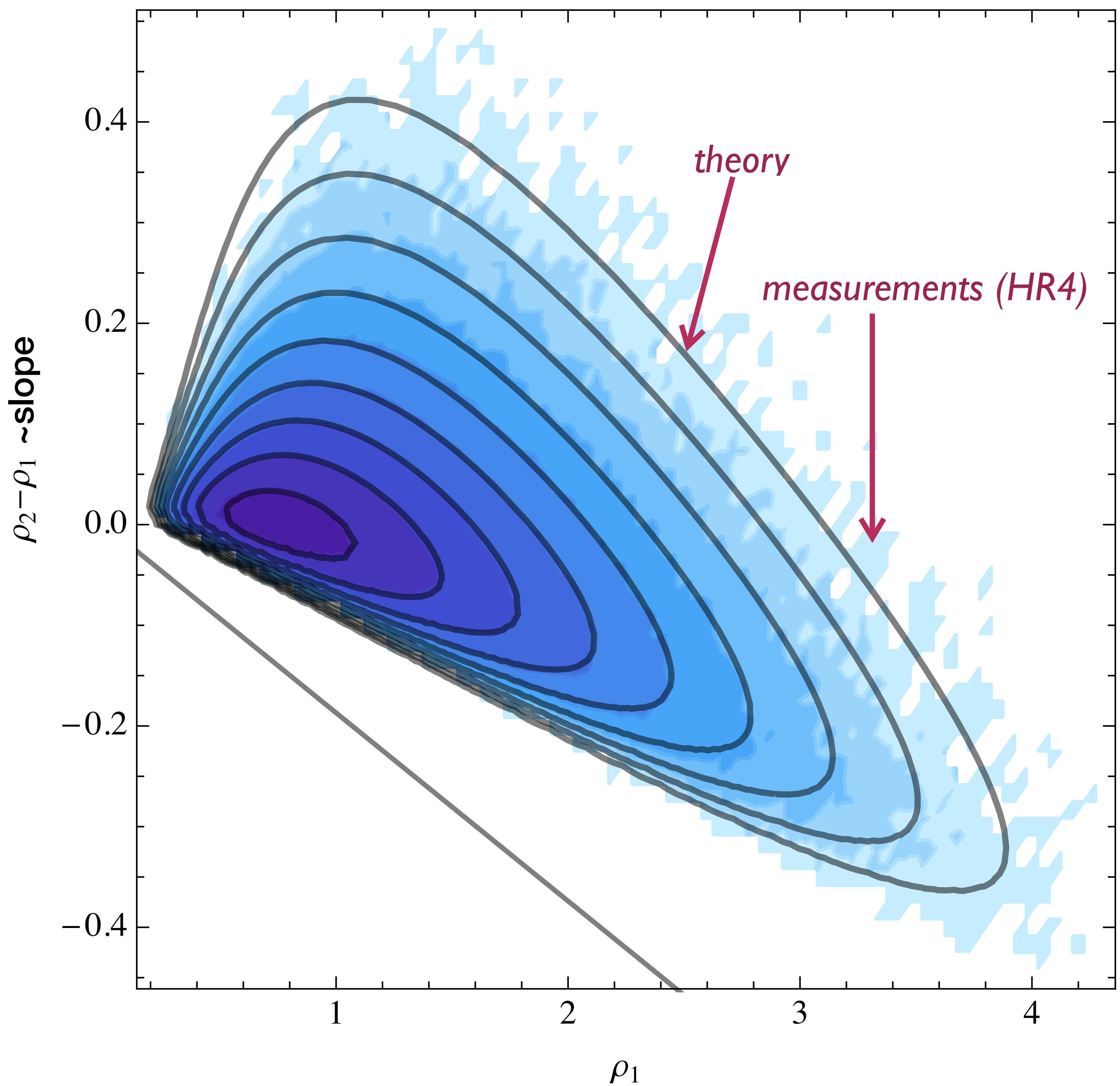
Author: Sandrine Codis (CITA)
Last modified: 04/03/2016
Code: LSSFast.tar.gz
This code is based on theoretical works in collaboration with Francis Bernardeau (IAP, CEA-Saclay) , Christophe Pichon (IAP, KIAS) and Cora Uhlemann (Utrecht University)

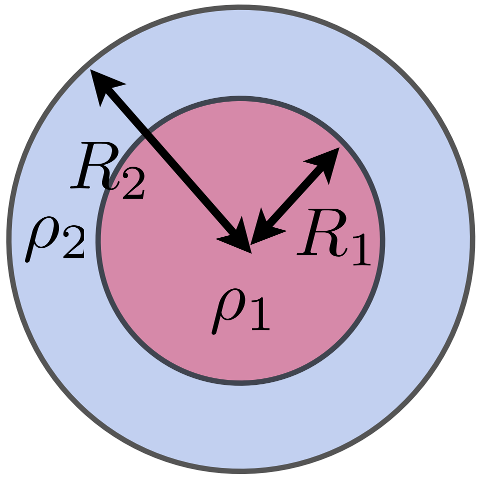
The LSSFast code is a free software distributed under the terms of the GNU-General Public License 3. It can be redistributed and modified at your own risk.
This program is made publicly available in the hope that it will be useful in scientific research but without any warranty.
The companion paper "Constraining the nature of dark energy via density PDF" by S. Codis, F. Bernardeau, C. Pichon, C. Uhlemann and S. Prunet illustrates the possible use of LSSFast for cosmological data analysis.
Any questions or remarks can be emailed to codis@cita.utoronto.ca



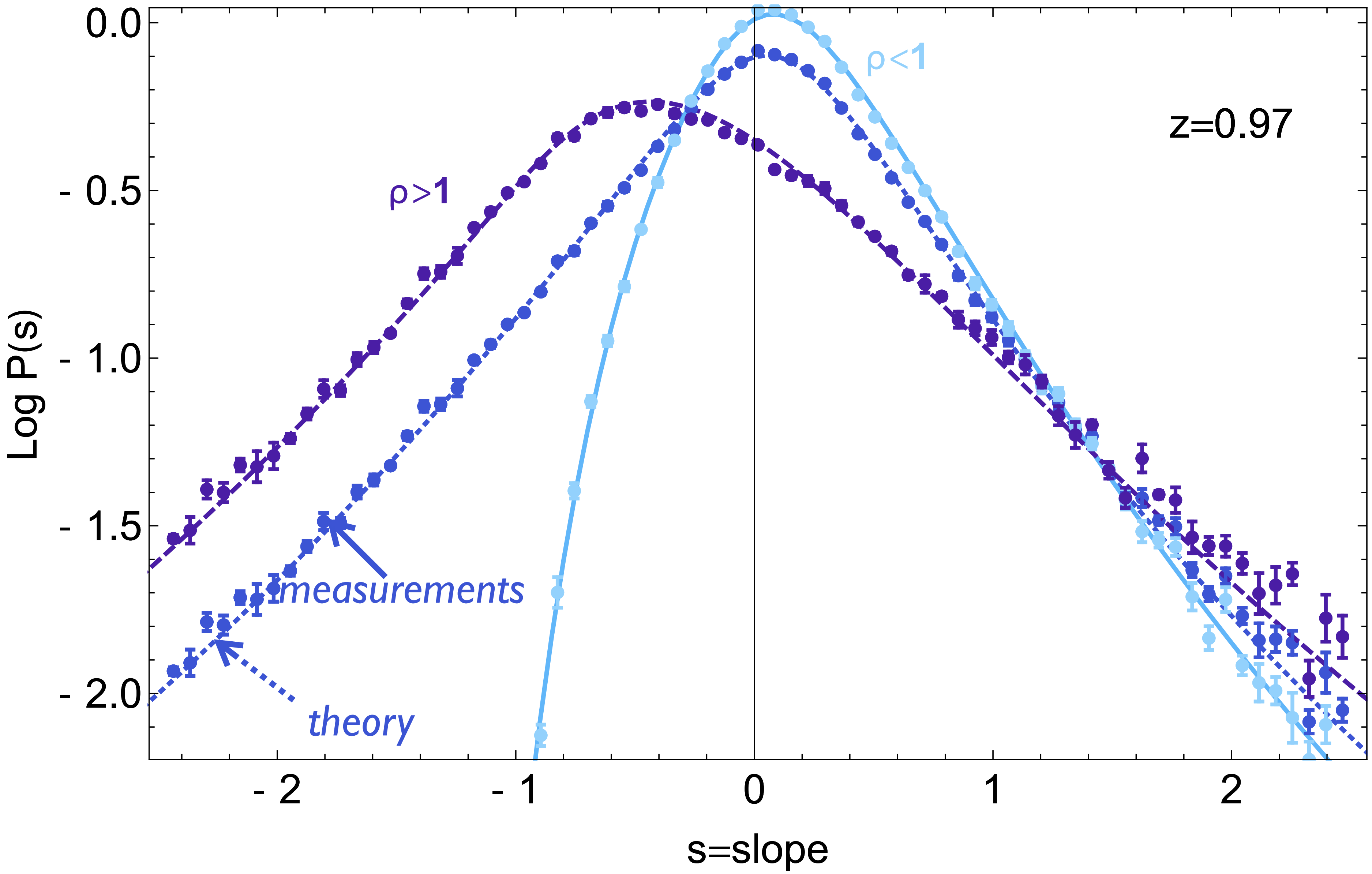
Bernardeau+15
Uhlemann+16

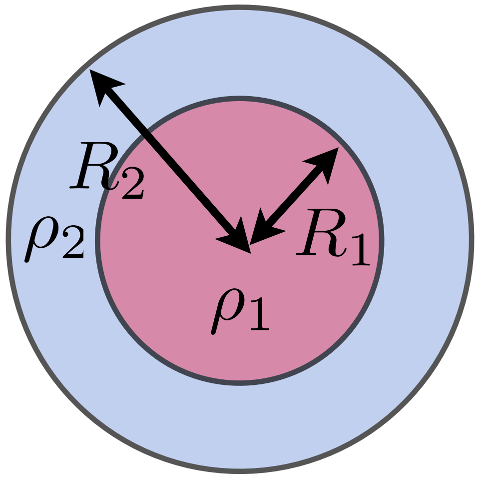
Two-cell PDF





Two-cell PDF
statistics of the slope





Higher density environments have more negative slopes (peaks!).

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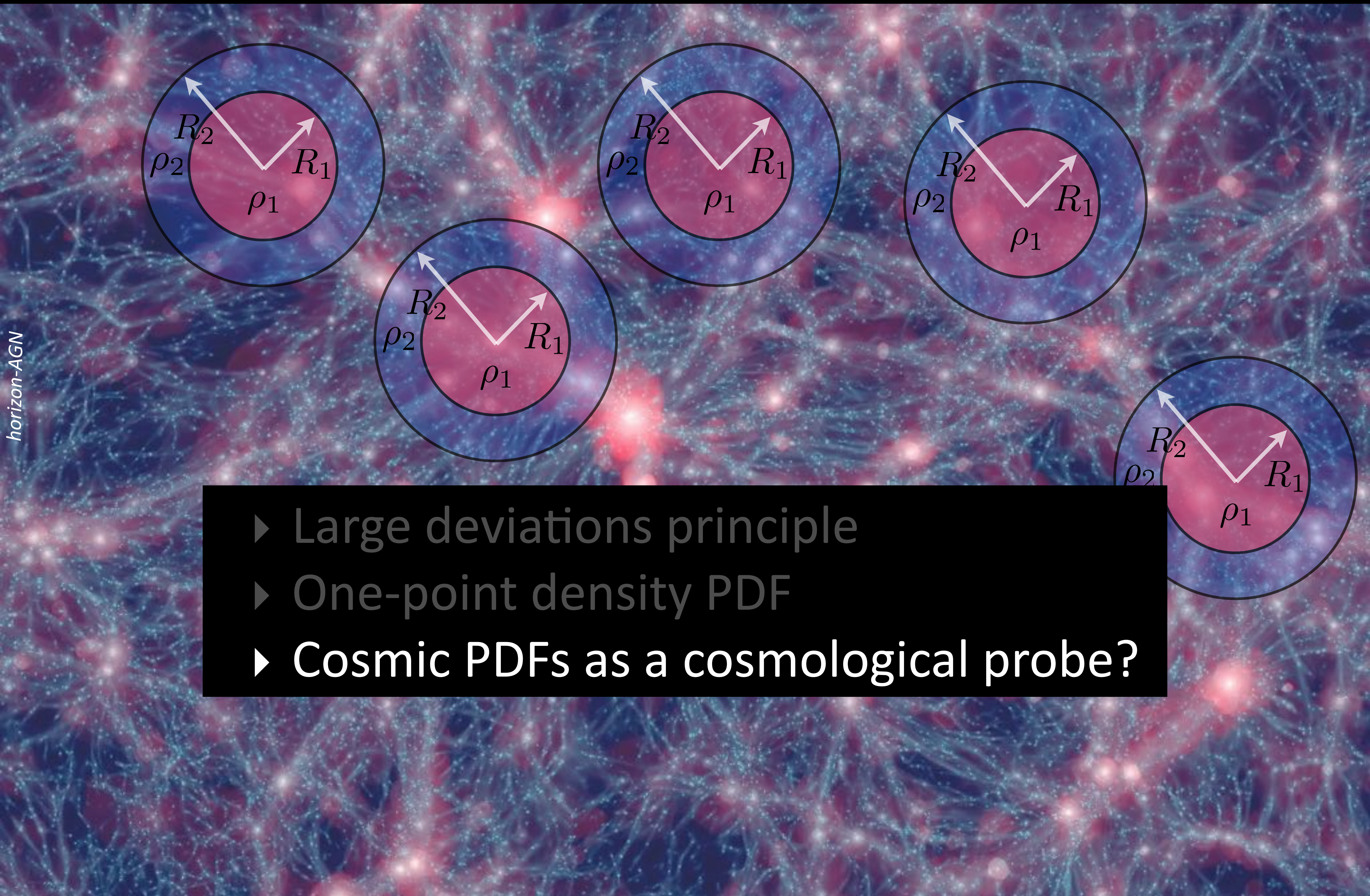

- Large deviations principle
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Where is the cosmology dependence?

To get one-cell PDF, one has to:

1) know the rate function of the initial conditions e.g (Gaussian):

$$I(\tau(R_0)) = \sigma^2(R_p) \times 1/2\tau(R_0)^2/\sigma^2(R_0)$$

where the initial variance is a function of the **linear power spectrum**

$$\sigma^2(R) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, P_{\rm lin}(k) W^2_{\rm TH}(kR)$$

2) deduce the rate function of the final densities from the Contraction Principle

$$I(\rho) = I(\tau = \zeta^{-1}(\rho))$$

spherical collapse dynamics

3) compute CGF and then PDF

$$\boxed{P(\rho|\nu, P_{\rm lin}, \sigma_{\rm NL}(R,z))}$$

growth of structure dark energy

modifications of gravity

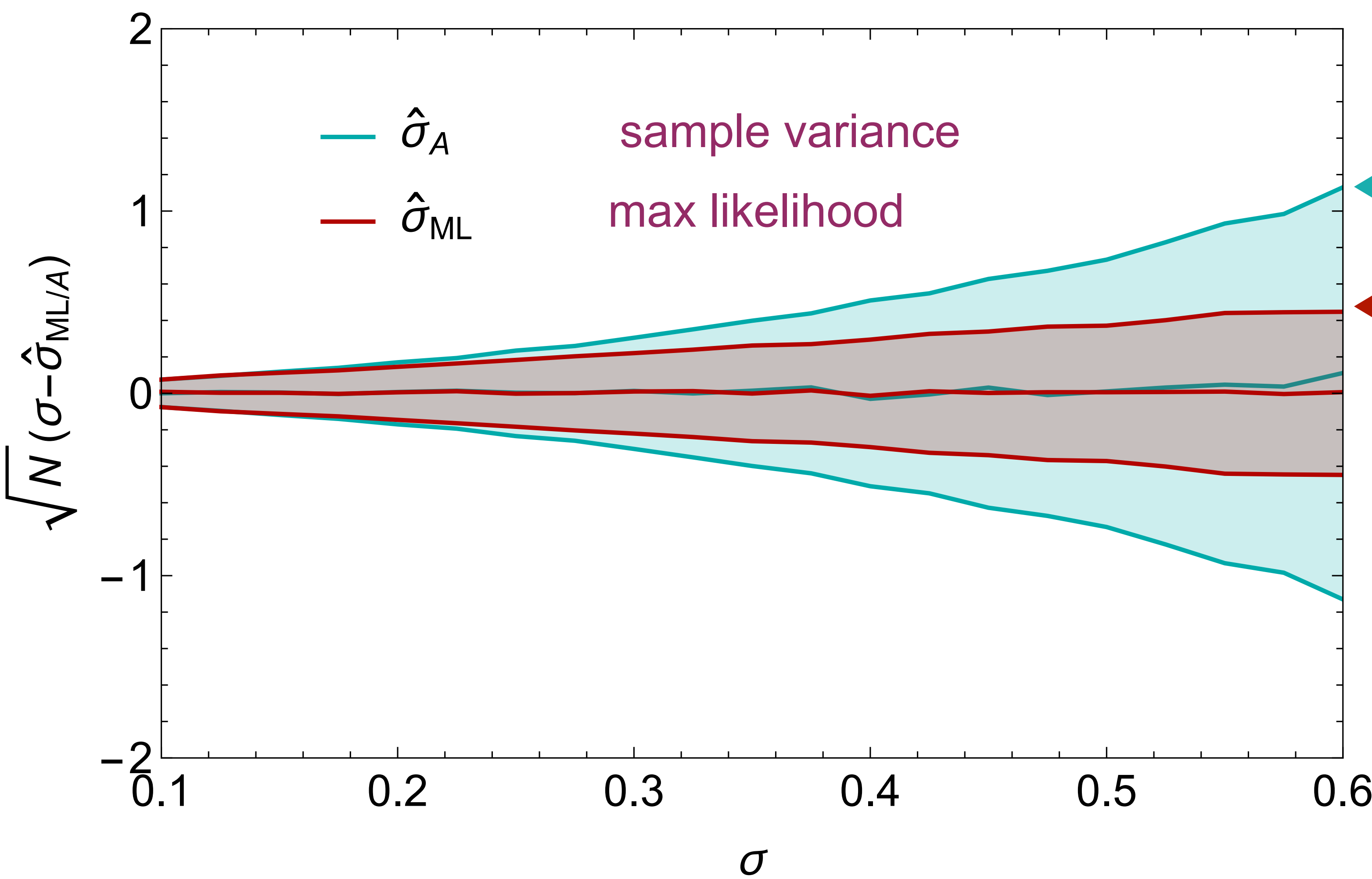
initial statistics primordial non-Gaussianities

ML estimator for the variance

The full knowledge of the PDF can be used to estimate the redshift evolution of the density variance σ and therefore the DE e.o.s through D(z).

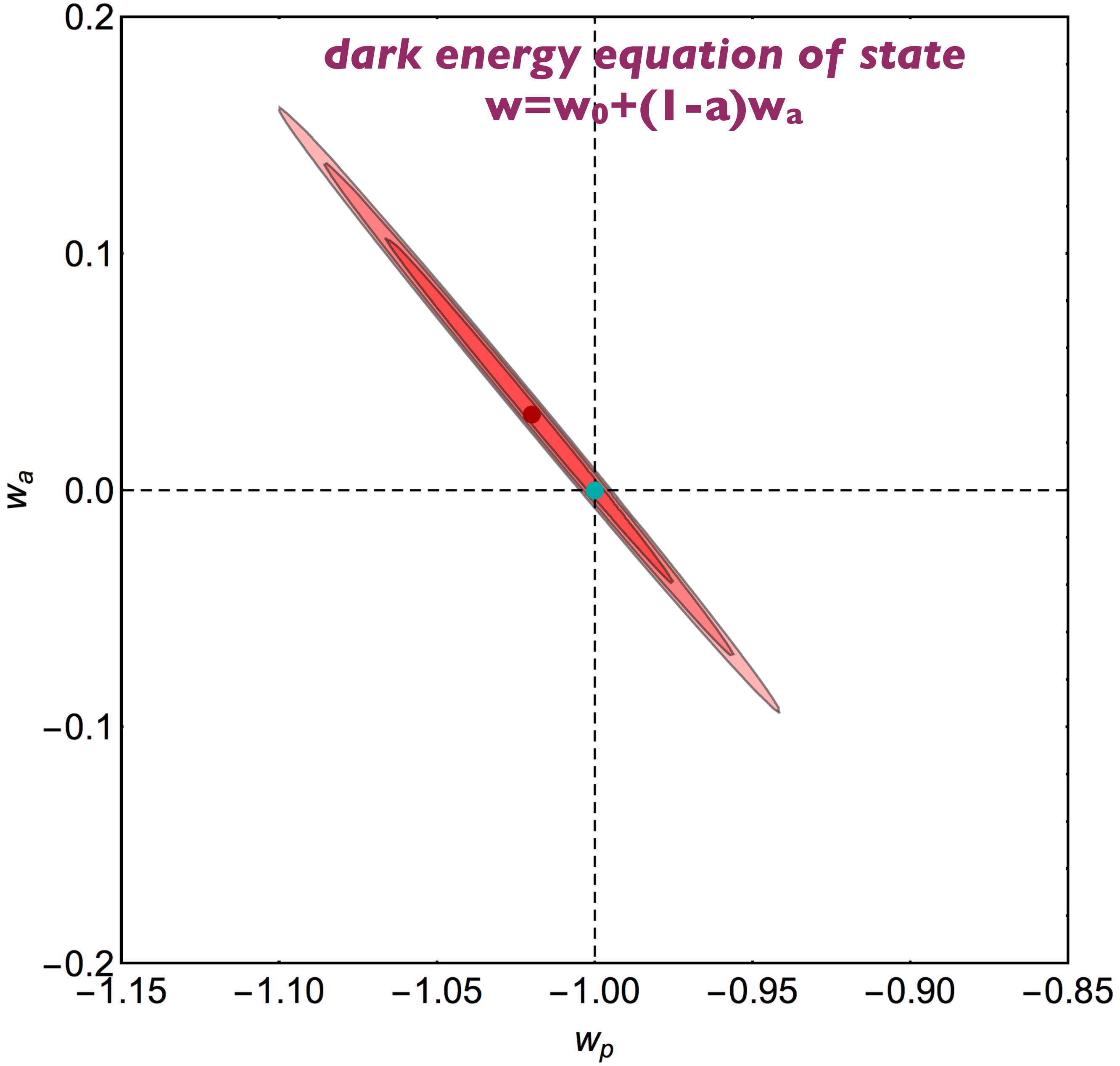
Maximum Likelihood estimator : $\hat{\sigma}^2_{\rm ML} = {\rm argmax}_{\tilde{\sigma}^2} \prod_{i=1}^{\rm N} \mathcal{P}(\rho_i|\tilde{\sigma}^2)$

Sample variance : $\hat{\sigma}^2_A = \frac{1}{N}\sum_{i=1}^{\rm N}(\rho_i - 1)^2$



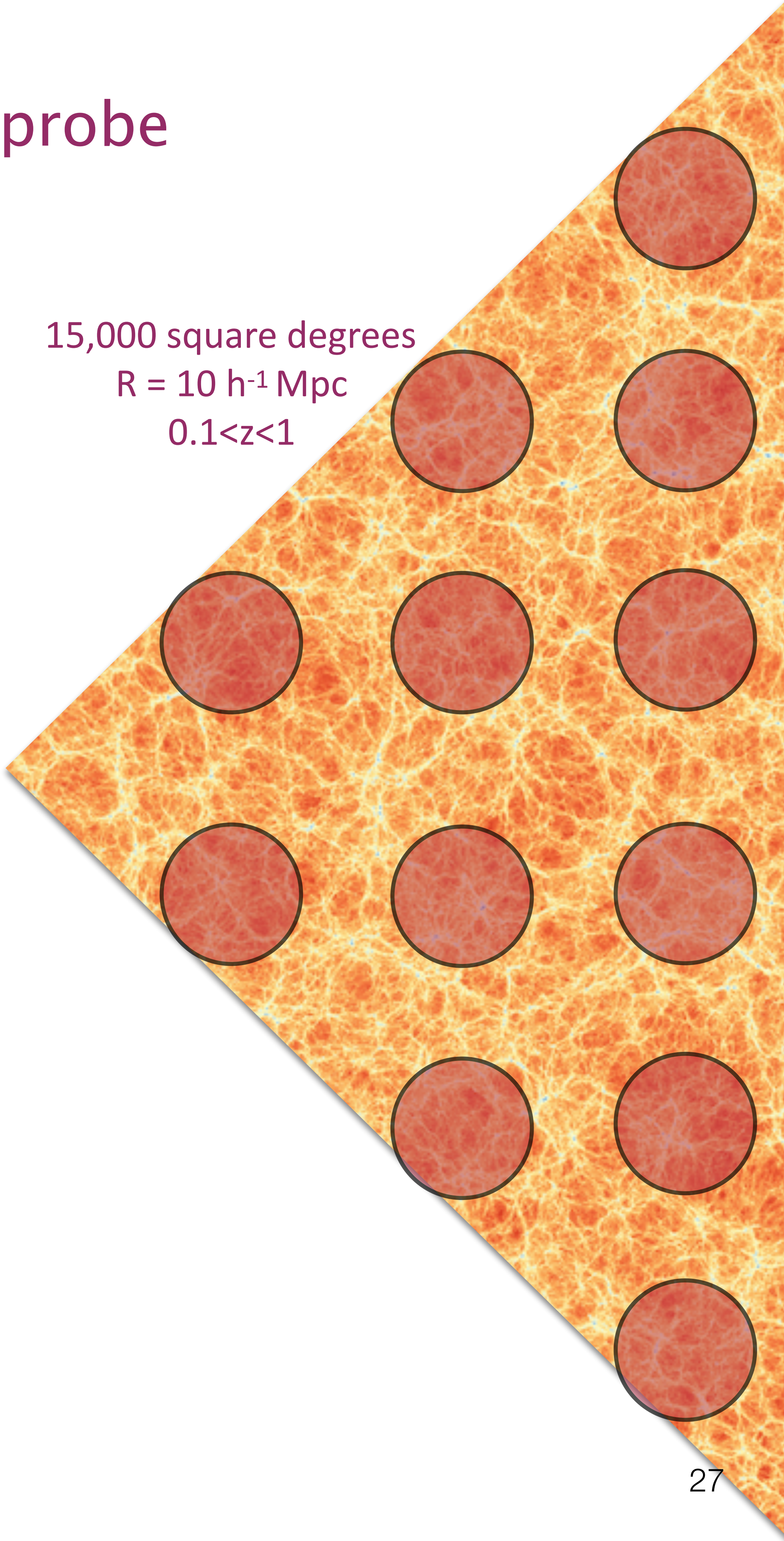
When the PDF becomes non-Gaussian (high σ), the sample variance is sub-optimal compared to the ML estimator

PDF as a cosmological probe



FoM for a Euclid-like survey

15,000 square degrees
R = 10 h^{-1} Mpc
0.1<z<1



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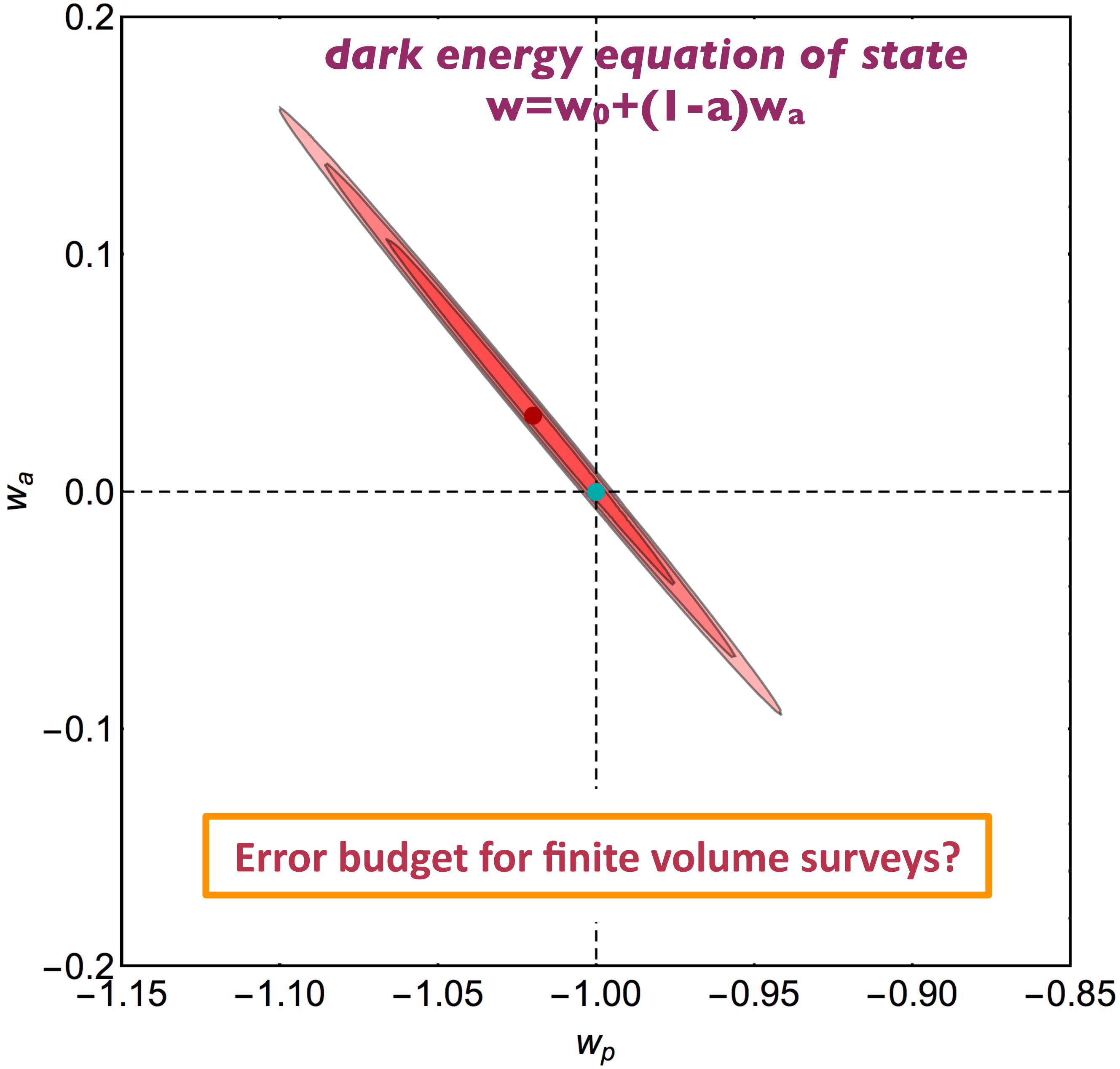

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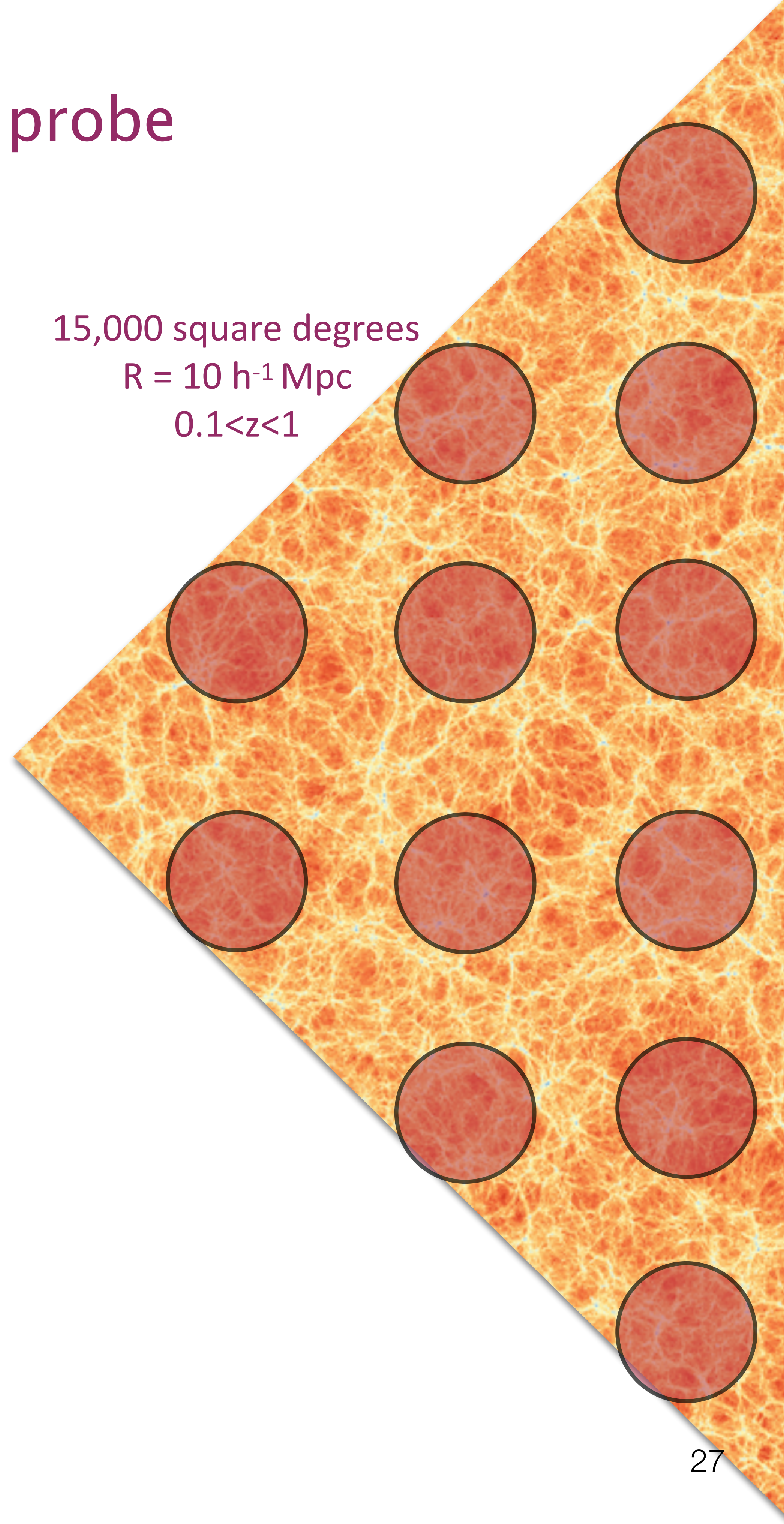
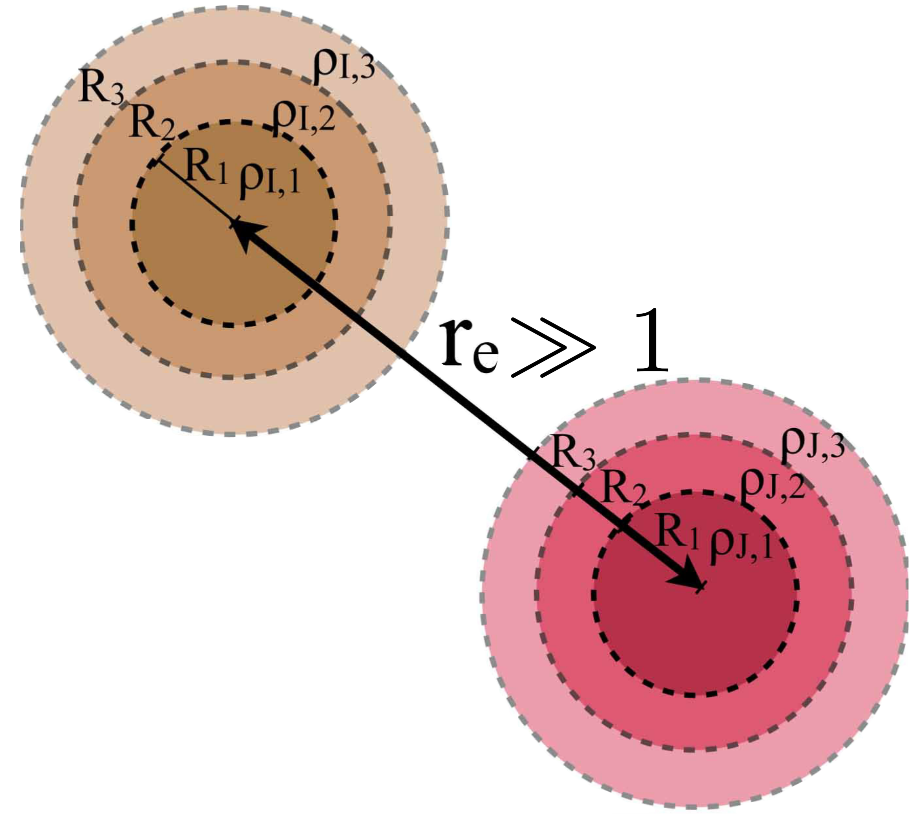


SC+16a

Error budget?



Maximum likelihood requires proper handling of **correlations** between spheres at **finite** separations.
The large-deviation principle provides a framework to compute the expected two-point correlations in the (not so) large separation limit

dark matter correlation **density bias**

$$P(\rho(x), \rho'(x + r_e)) = P(\rho)P(\rho')[1 + \xi(r_e)b(\rho)b(\rho')]$$

where the large-deviations bias is

spherical collapse

$$b(\rho) = \frac{\zeta_{\mathrm{SC}}^{-1}(\rho)}{\sigma^2(R\rho^{1/3})}$$

encodes $P_{lin}(k)$

SC+16a

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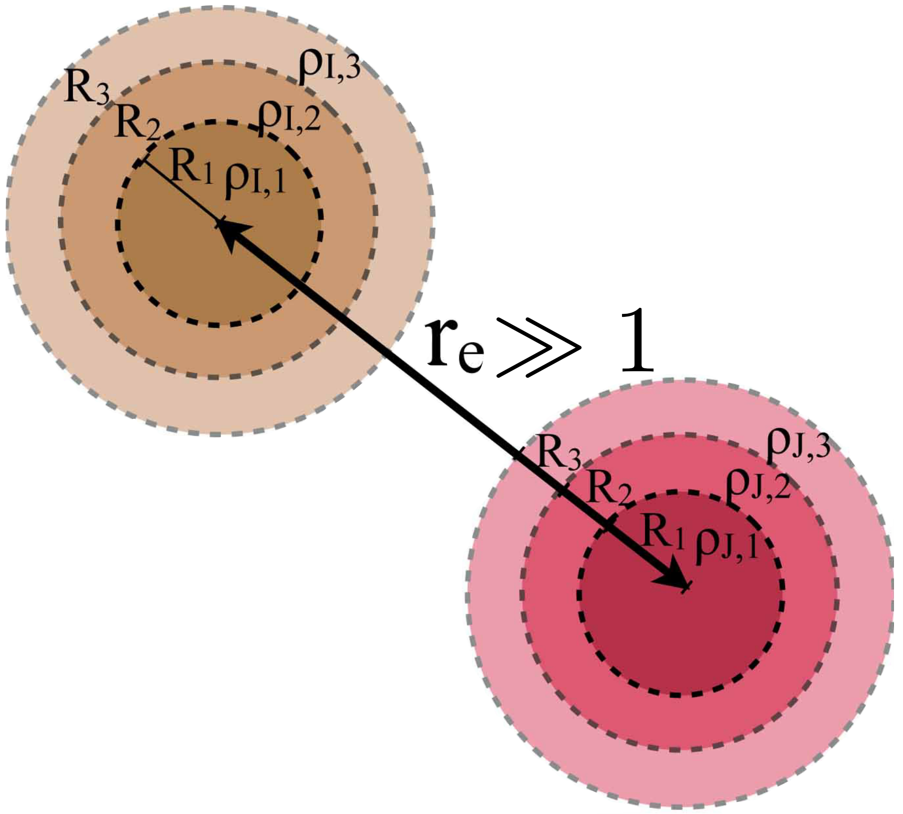

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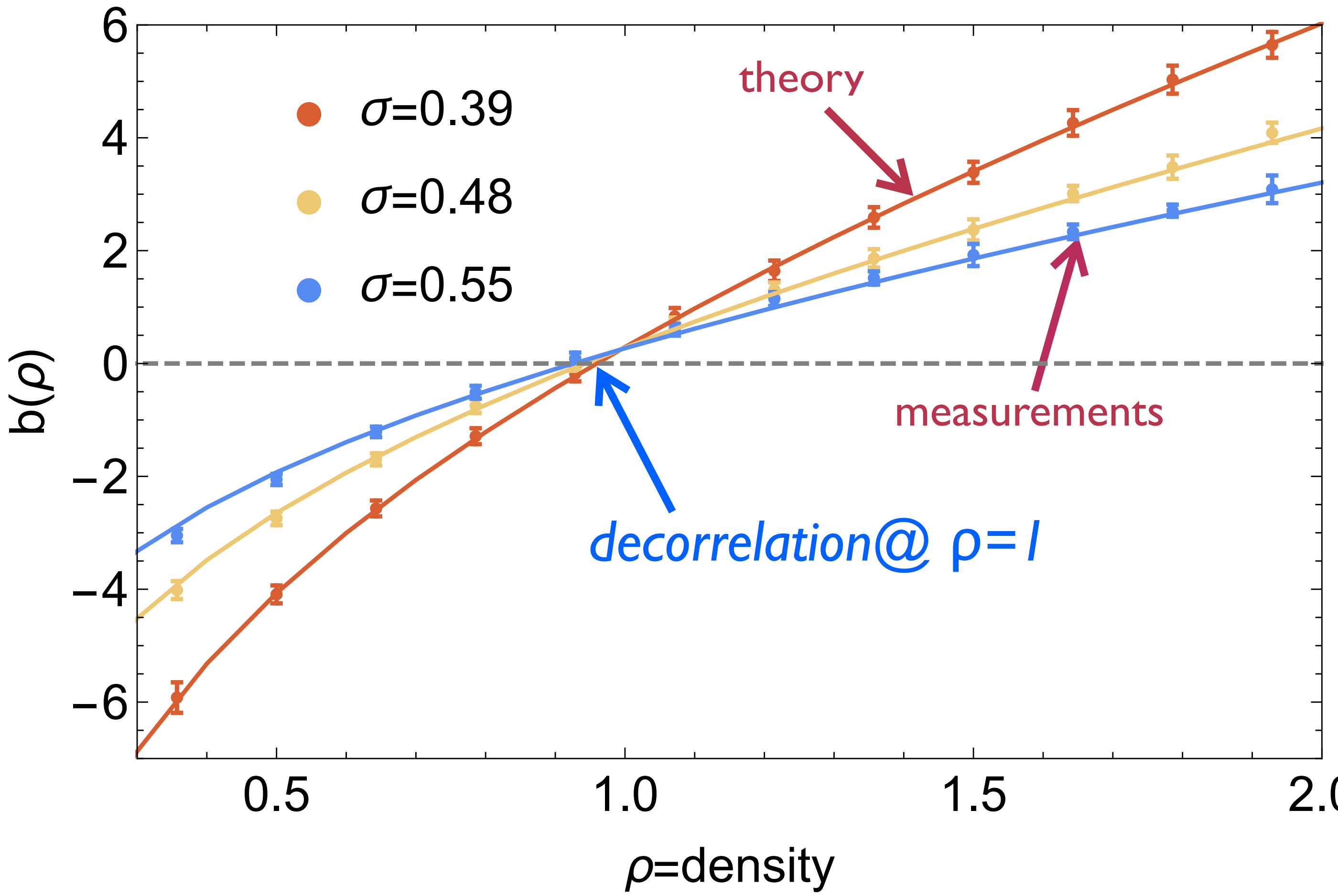

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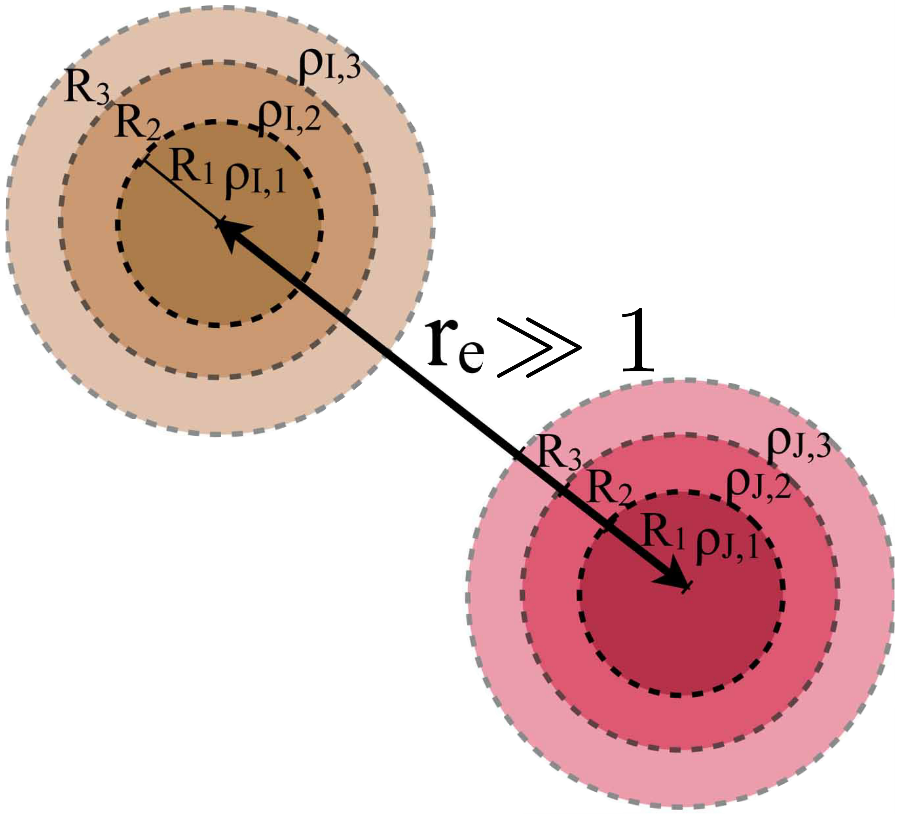

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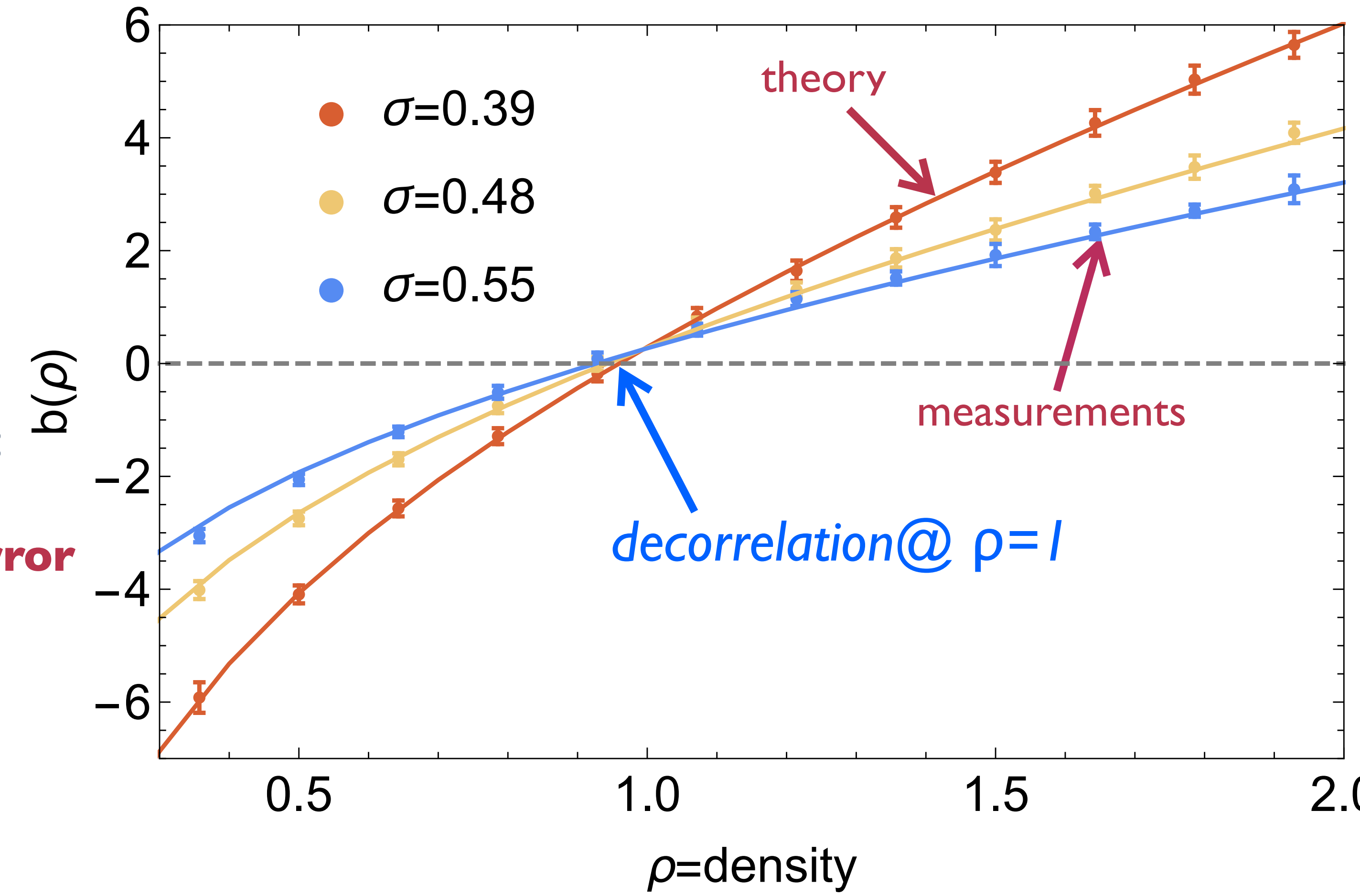
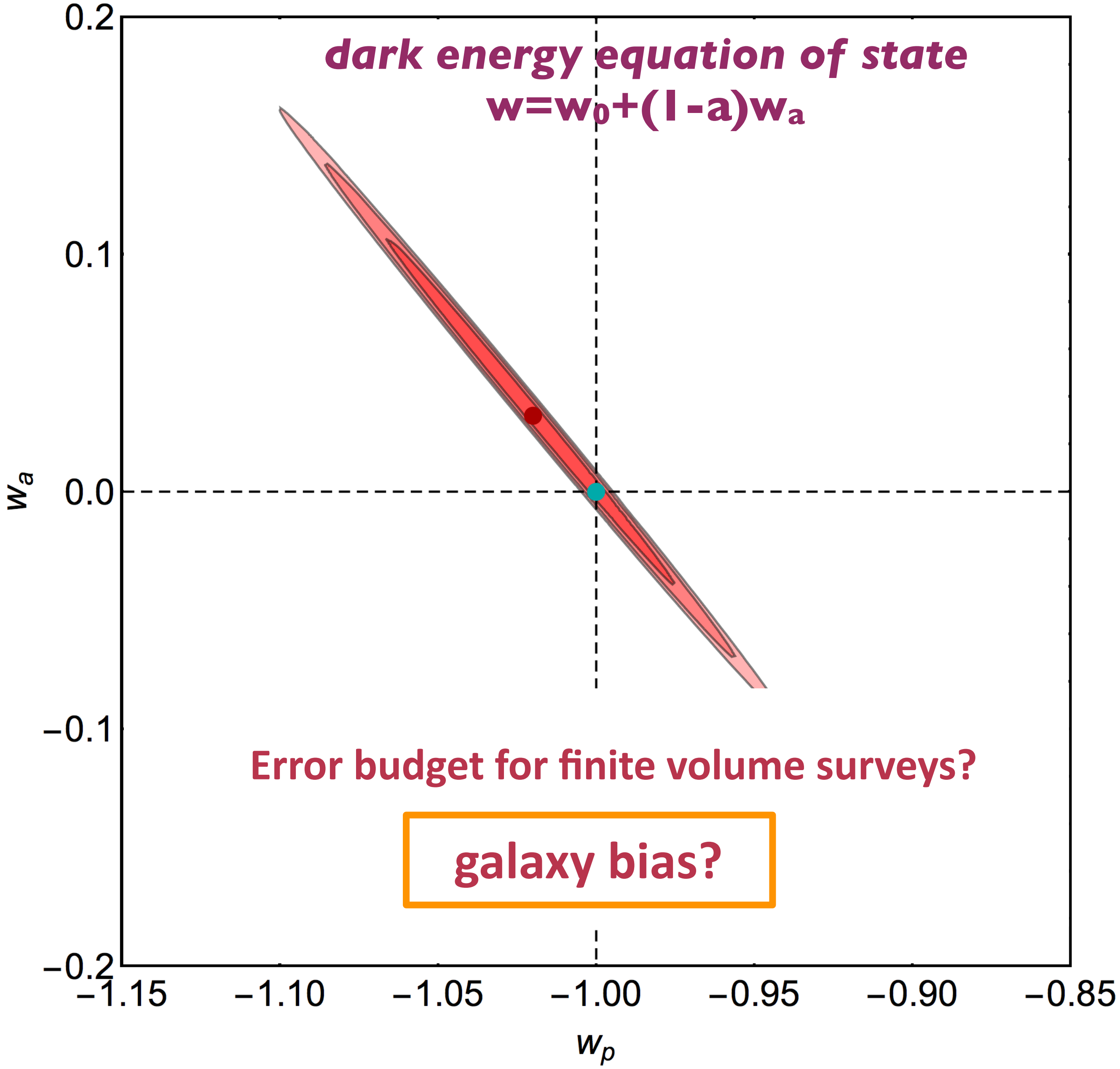
The typical cosmic variance on the density PDF is then:

shot noise **finite volume error**

$$\langle \hat{\mathcal{P}}^2(\rho)\rangle - \langle \hat{\mathcal{P}}(\rho)\rangle^2 = \frac{\mathcal{P}(\rho)}{N\Delta\rho} + \xi b^2(\rho)\mathcal{P}^2(\rho)$$



PDF as a cosmological probe



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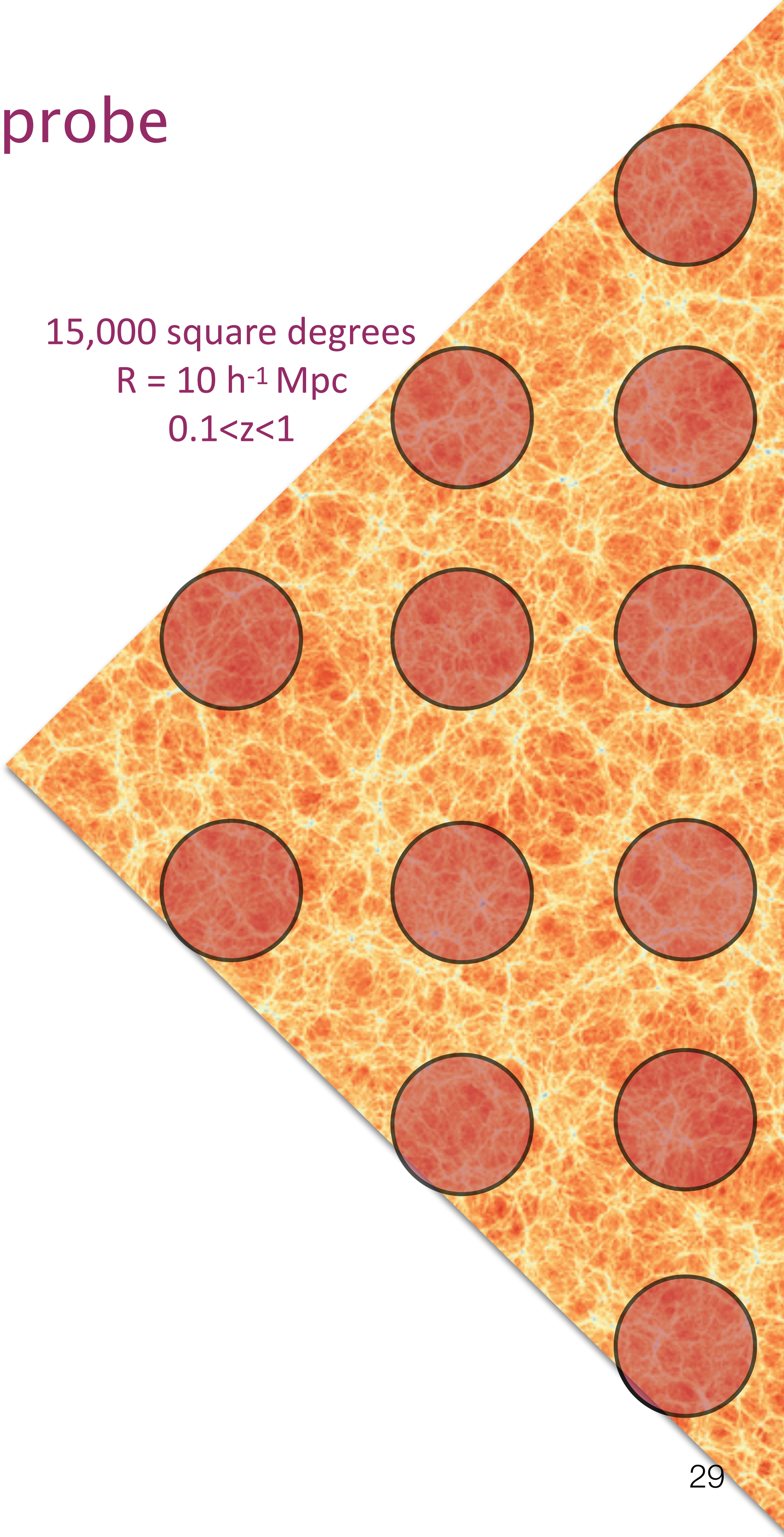
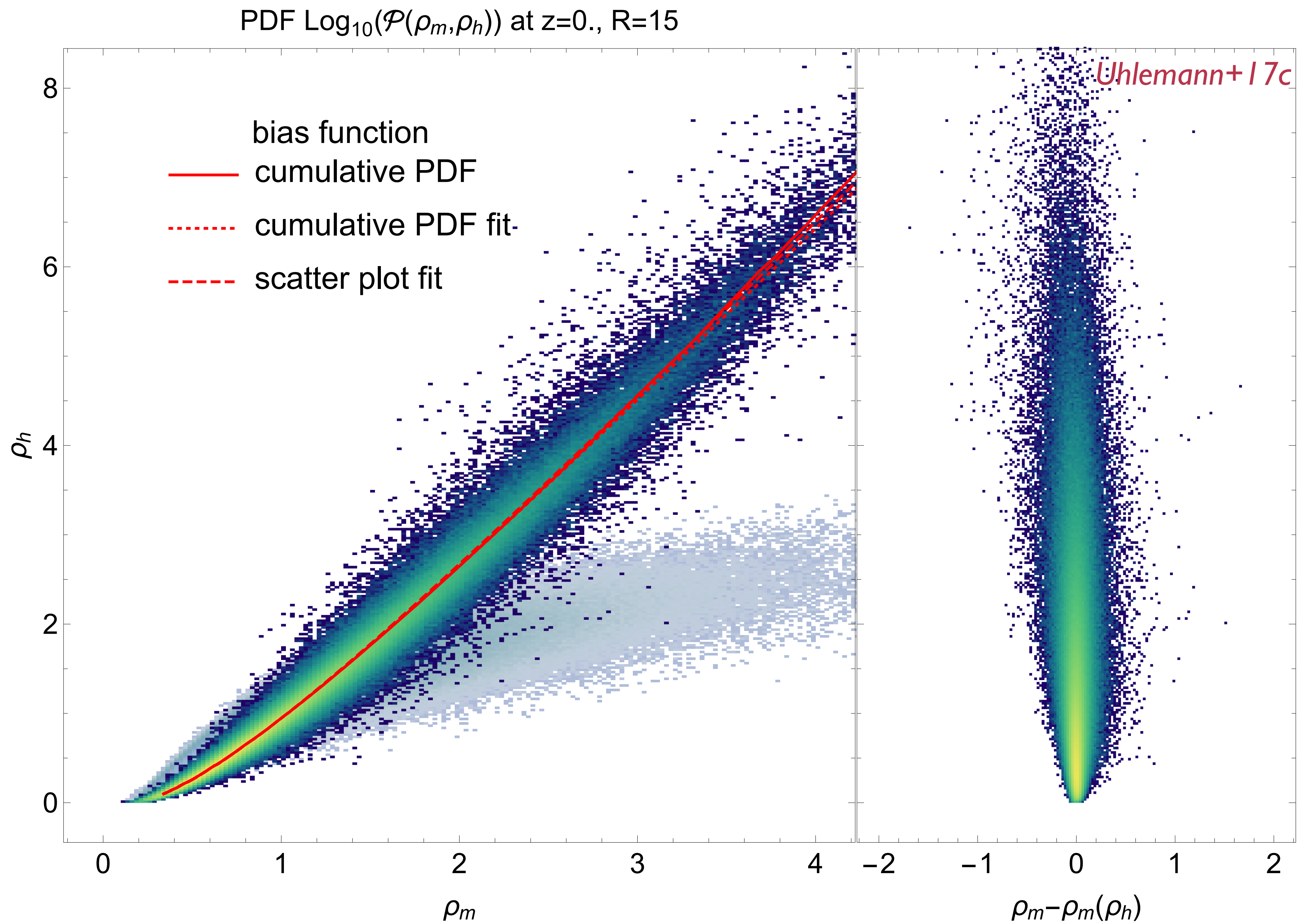


How to deal with biased tracers?

Halo bias can be accounted for and marginalised over for cosmological experiments…

We use a quadratic log bias model: $\log \rho_m = b_0 + \beta_1 \sigma \log \rho_h + \beta_2 \sigma \log^2 \rho_h$



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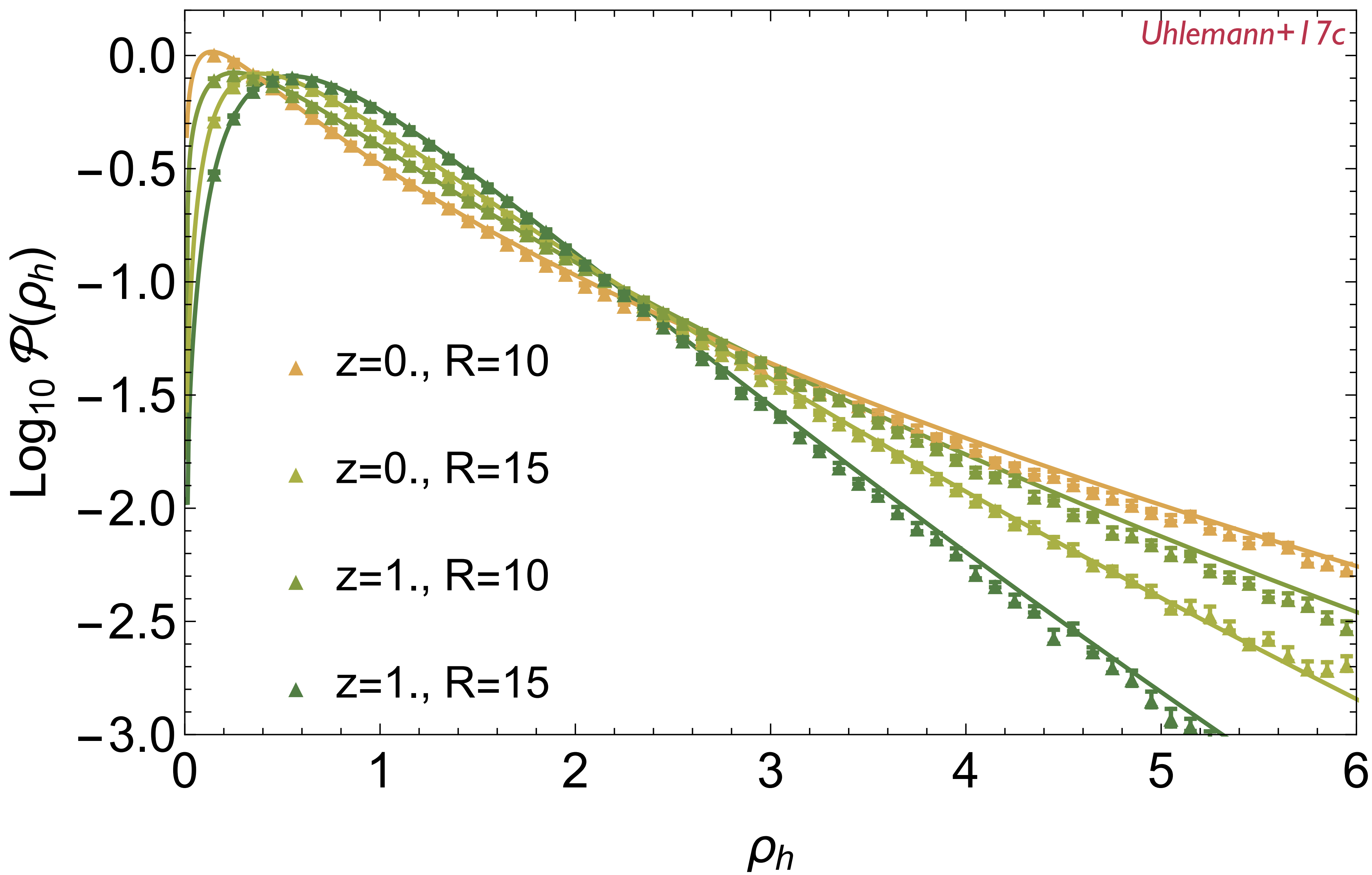

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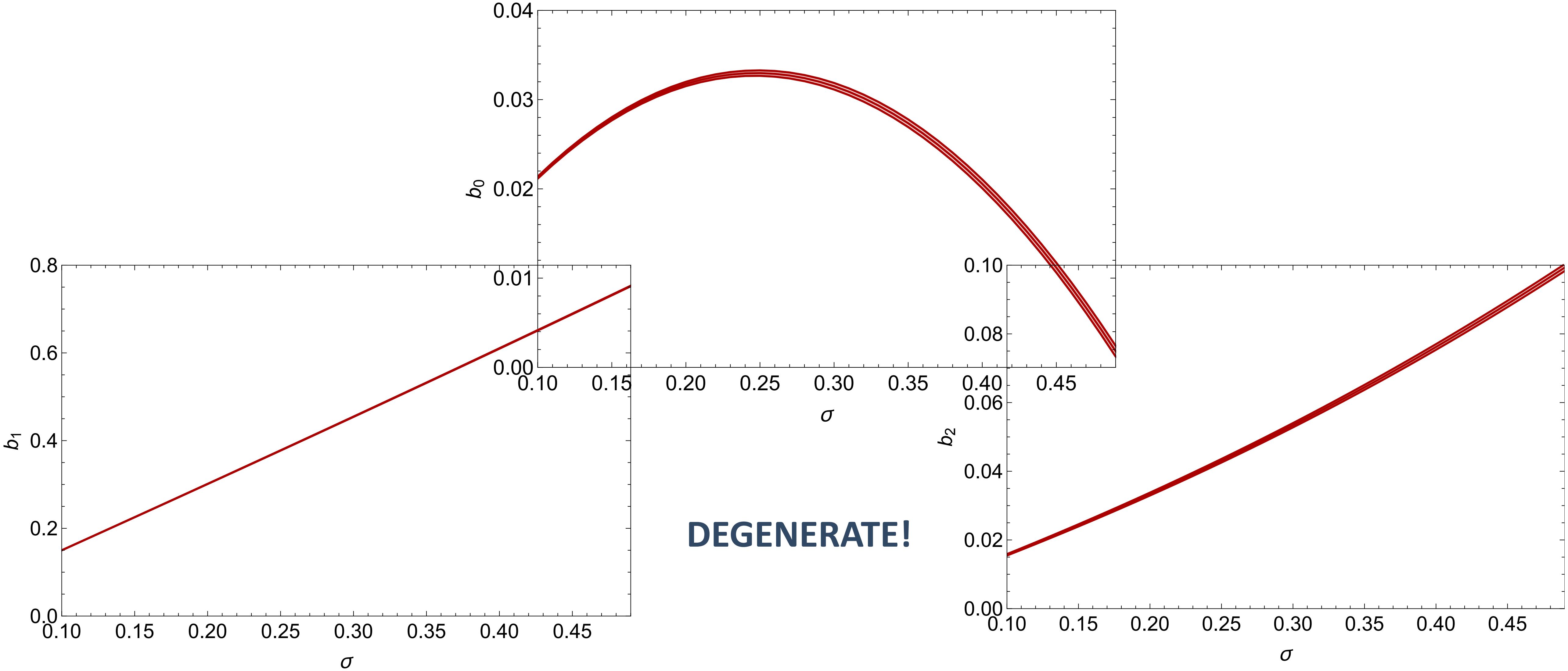
Measuring the PDF then allows us to constrain σ and the bias parameters:



DEGENERATE!

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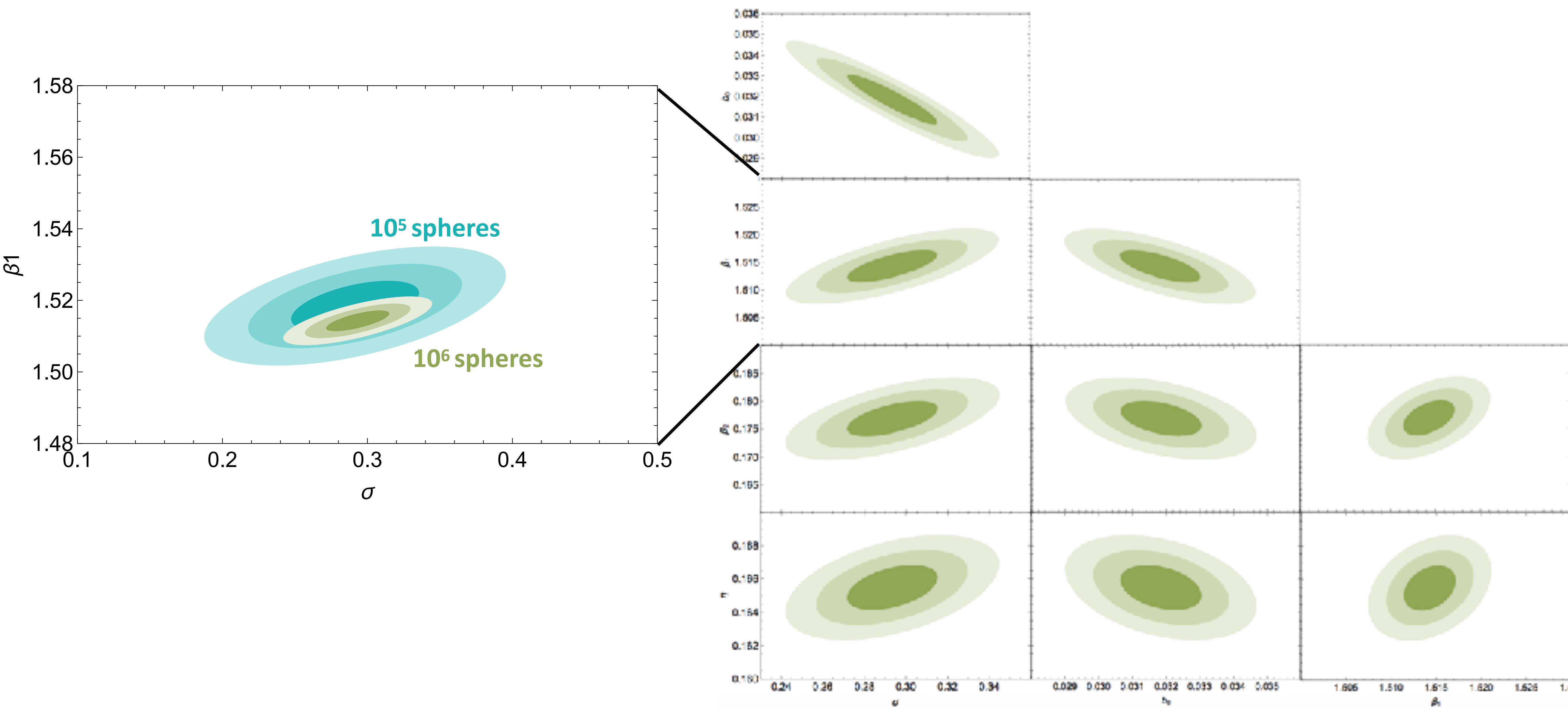
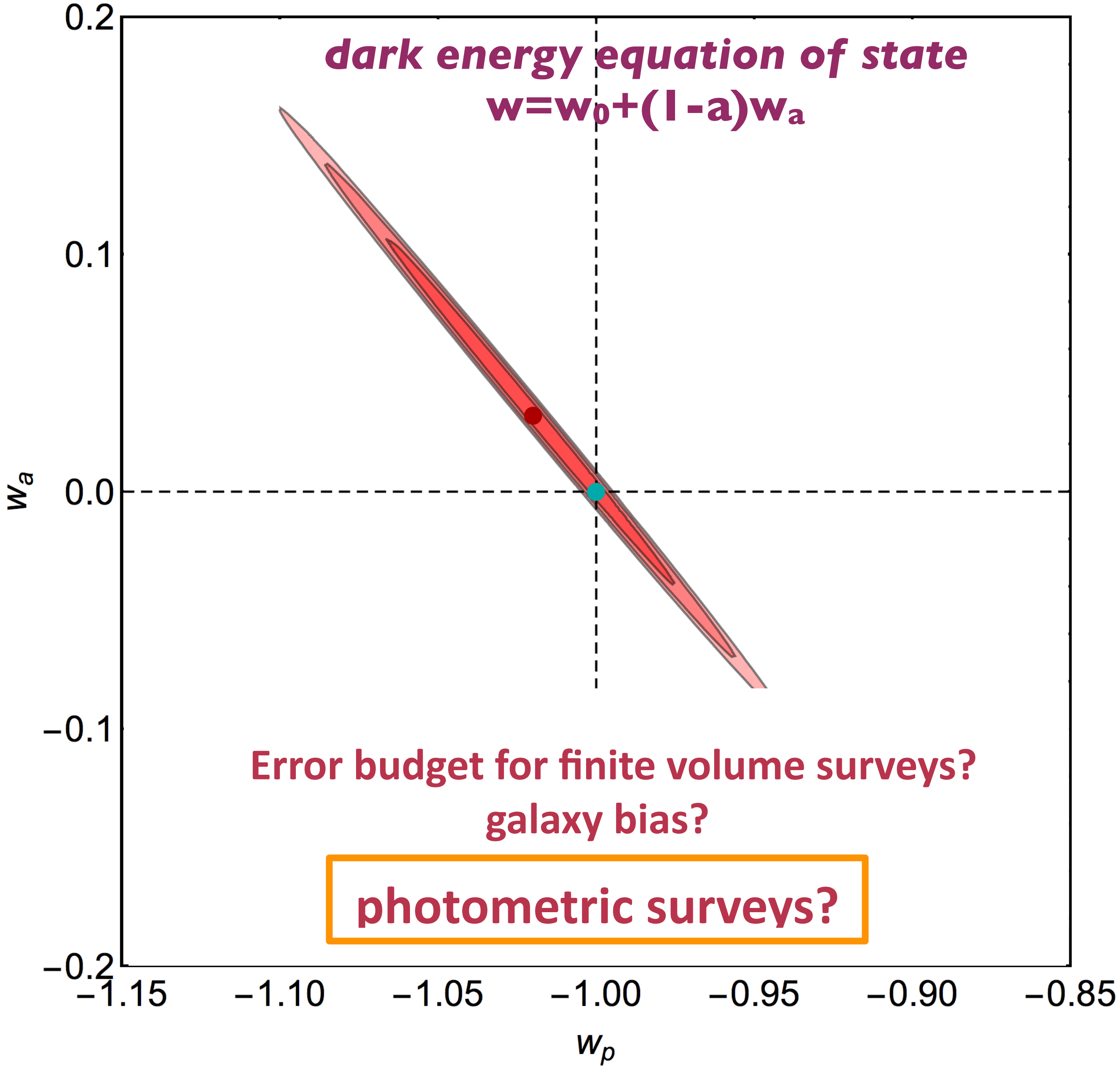
+ 2pt PDF



PDF as a cosmological probe



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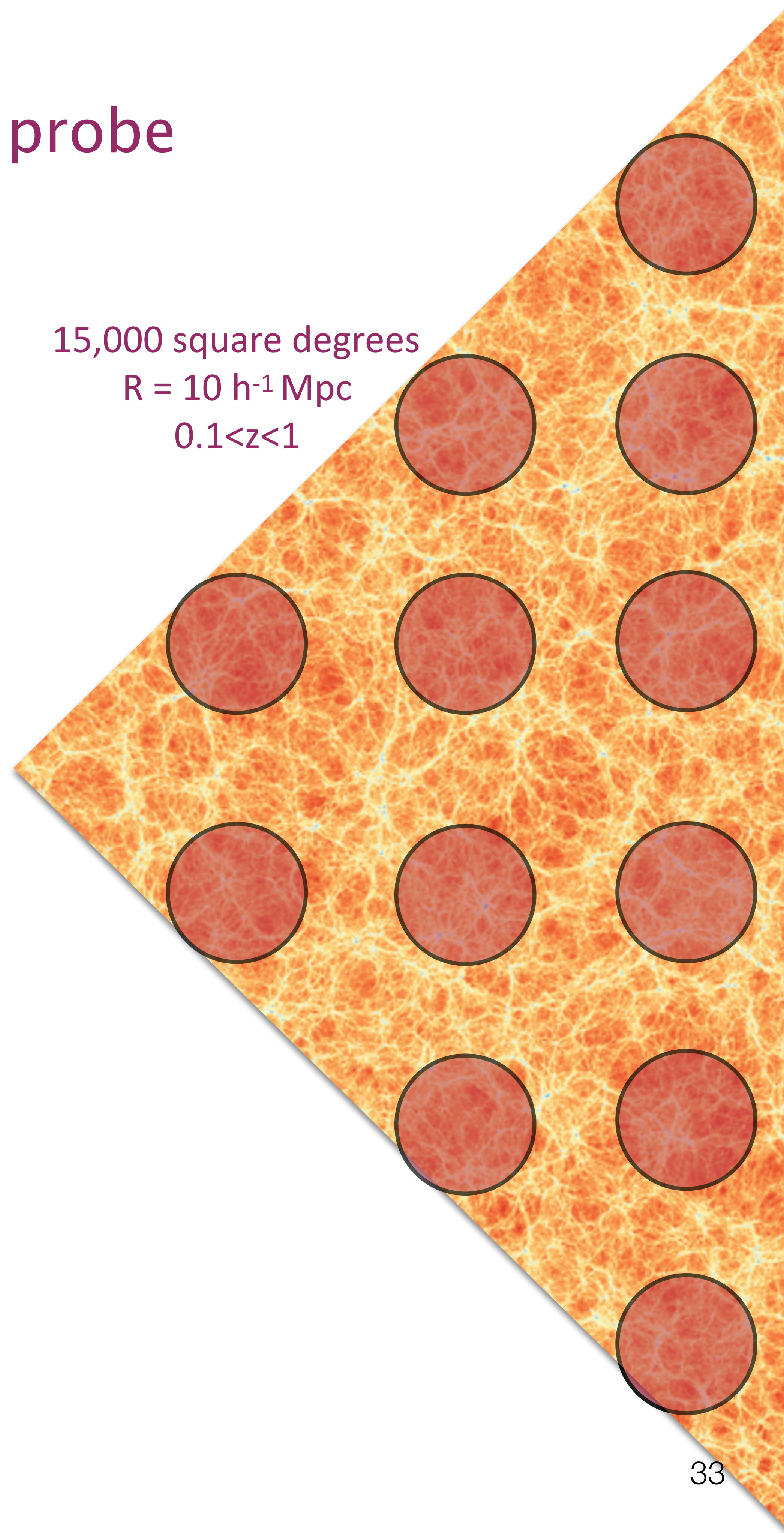
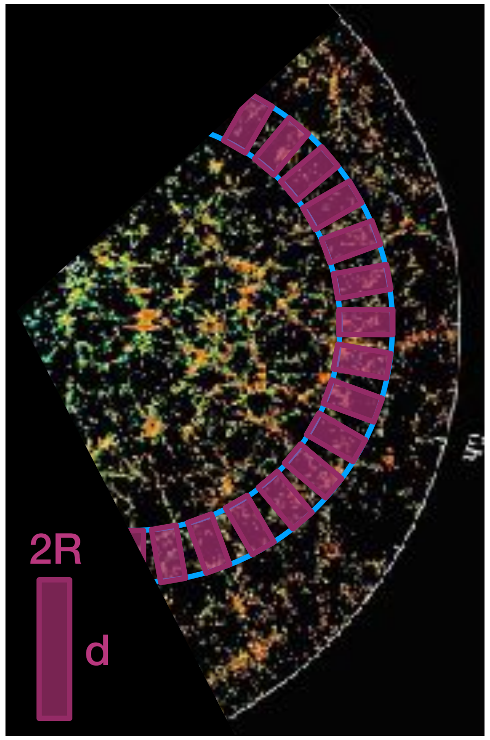


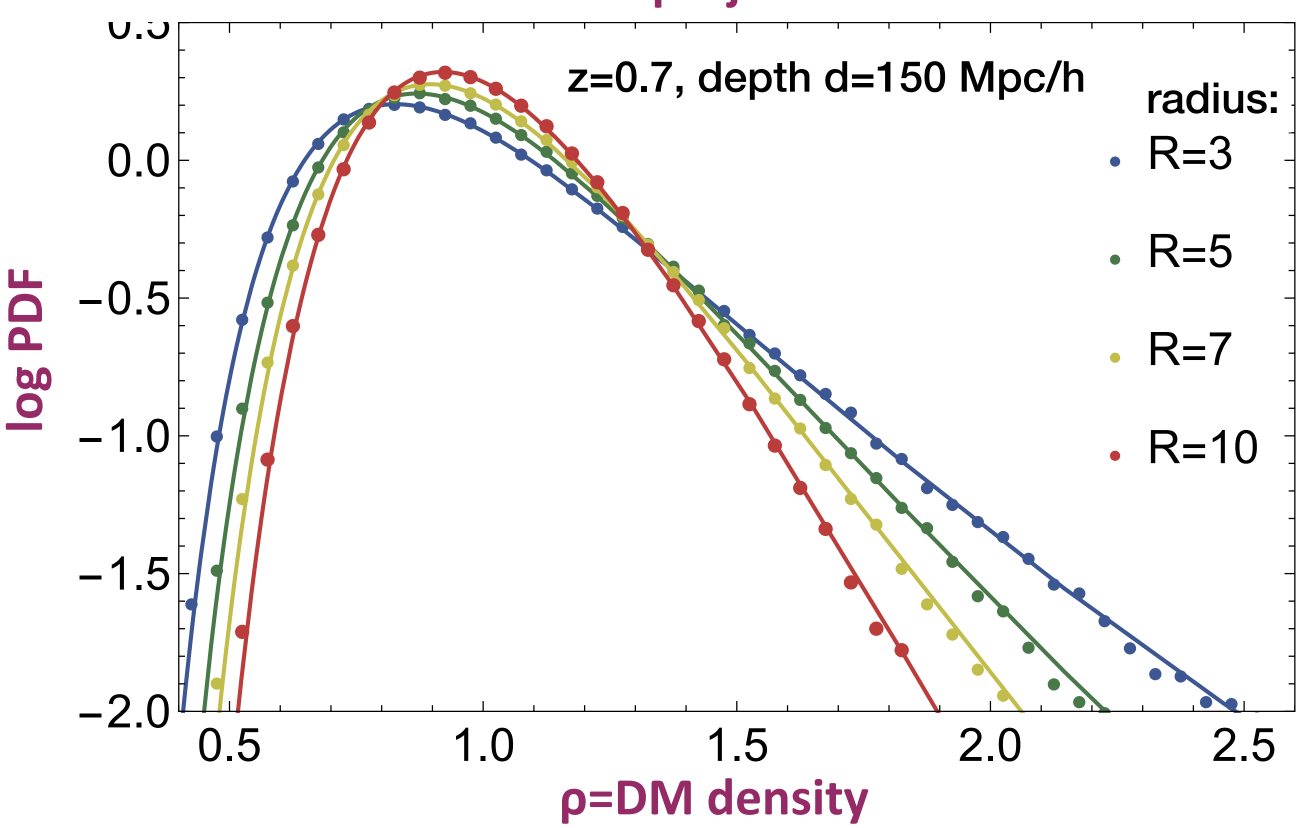
Uhlemann+17e

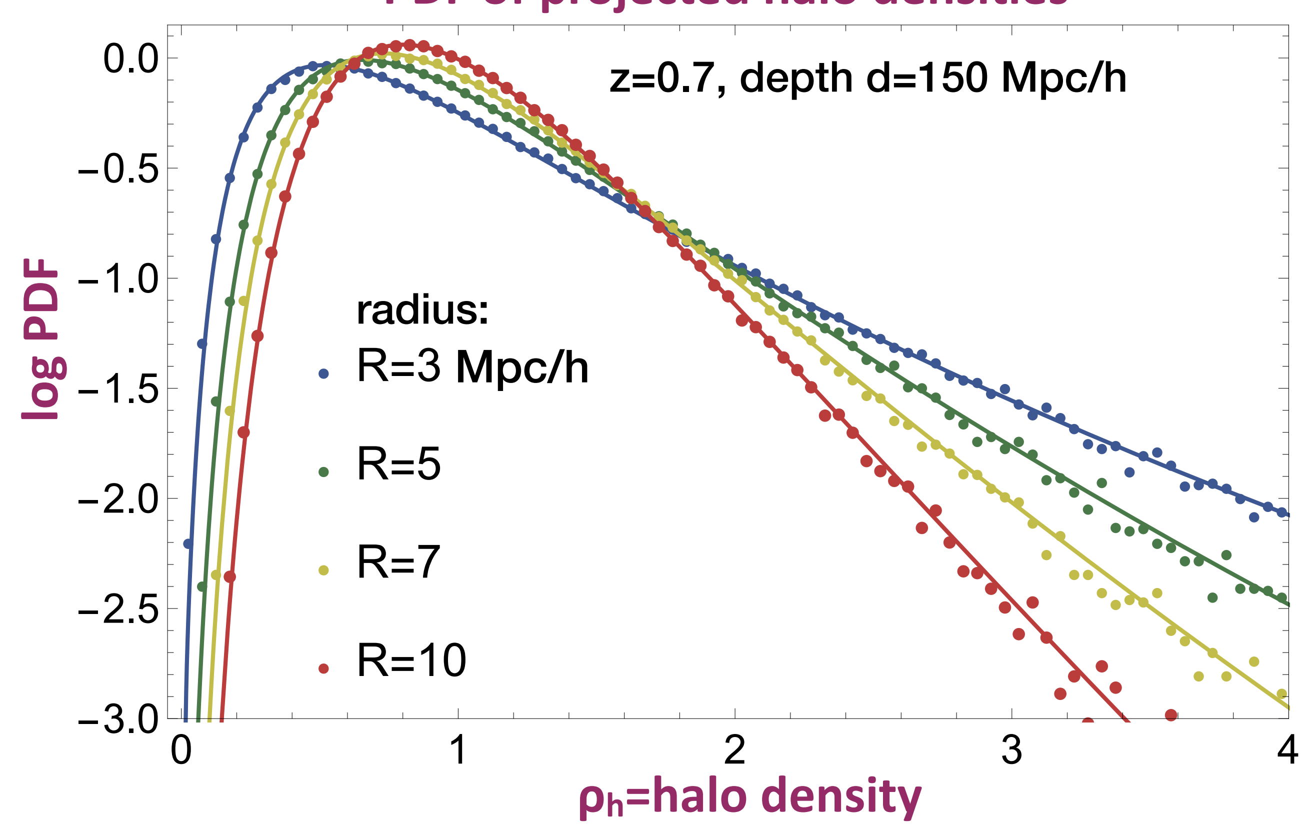
densities in redshift bins



Densities in long cylinders: same formalism applies with cylindrical collapse

$$\zeta_{CC}(\tau_{2D}) = \left(1 - \frac{\tau_{2D}}{\nu}\right)^{-\nu}$$
$$\nu \approx 1.3$$





Conclusion

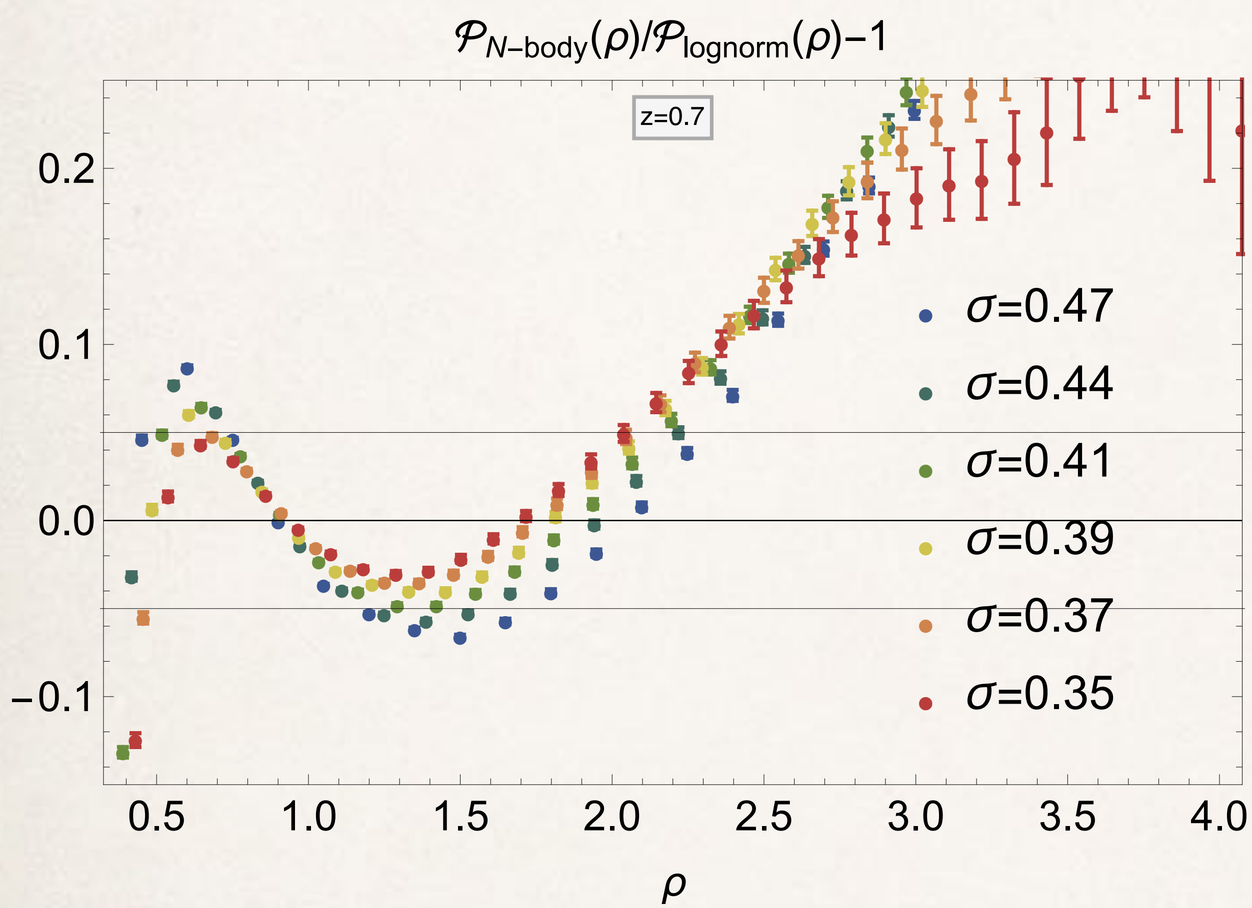
- Multi-scale density PDF can be predicted in the mildly non-linear regime with surprising accuracy (<1% for σ=O(1)) even in the rare event tails
- Predictions are fully analytical, parameter-free and explicitly cosmology-dependent
- Cosmic variance can be predicted from first principle
- We have an accurate model for biased density tracers, velocities, projected densities and (in progress) cosmic shear maps, including primordial non-Gaussianities

horizon-AGN

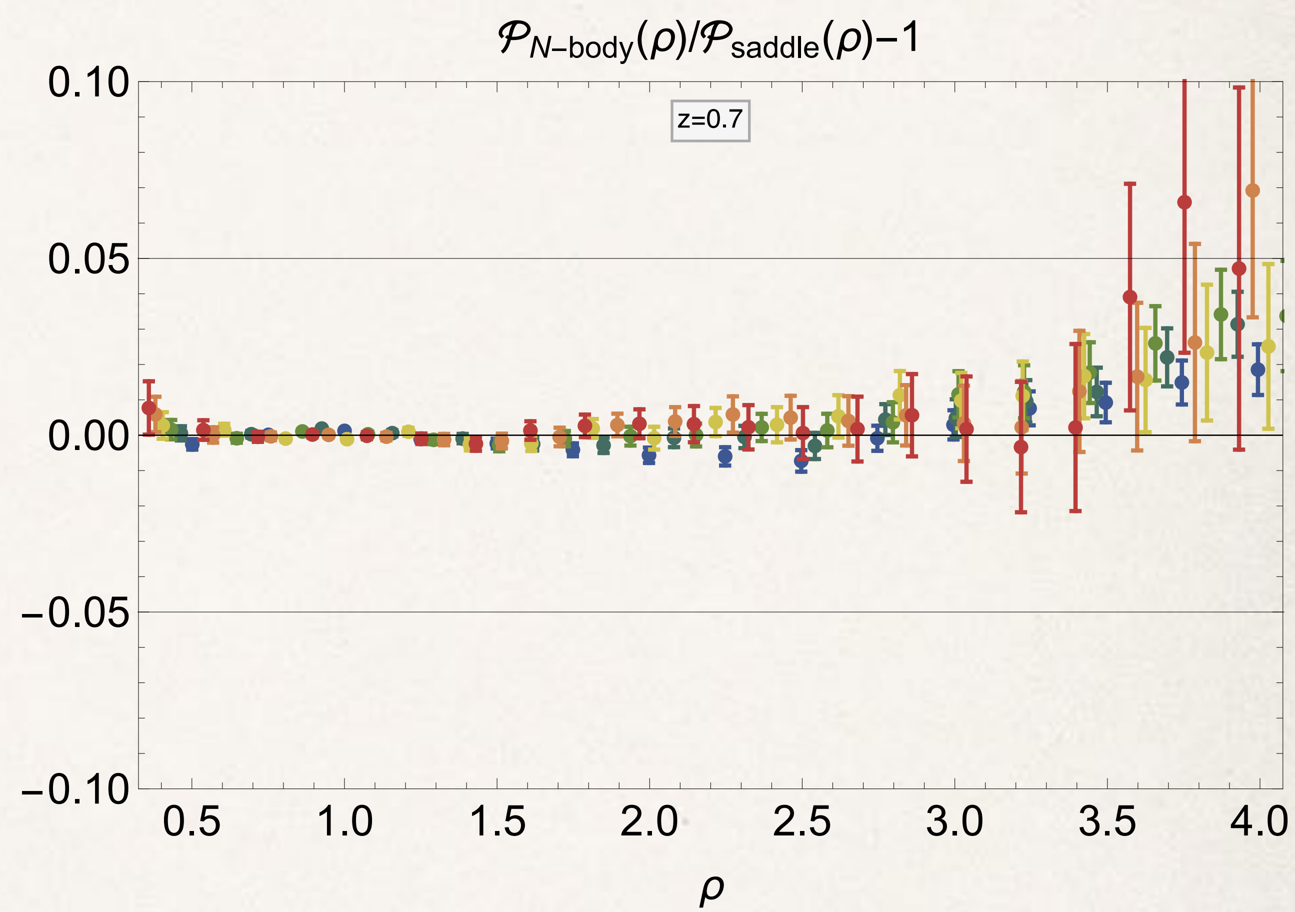
Large deviation principle:

an unlikely fluctuation is brought about by the least unlikely of all unlikely paths.

comparison with log normal



log-normal accuracy



large-deviation theory accuracy

comparison with log normal : biased tracers