Solving 3+1 GRMHD Equations in the eXtended Conformally Flat Condition: the X-ECHO and XNS codes

Niccolo' Bucciantini, Antonio Pili, Luca Del Zanna

INAF Osservatorio Astrofisico di Arcetri, IT

Dipartimento di Fisica ed Astronomia UniFi, IT

INFN Sezione di Firenze, IT

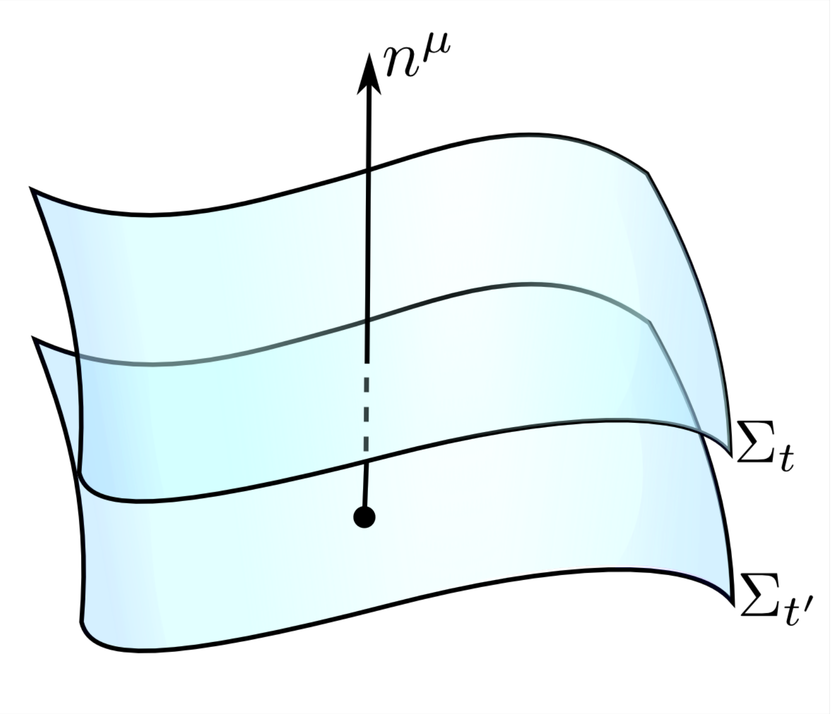
http://www.arcetri.astro.it

Why and How

Scientific Rationale

- *Solving PDE on curved manyfold (not just GR but also expanding systems, non cartesian grids)*
- *Modeling relativistic outflow - High Energy Astrophysical Engines*
- *Strong magnetic field in NS and BH (Magnetars, GRBs, AGN launching)*
- *Mean field effects, resistivity in NS evolution, dynamo action*
- *Relativistically Hot Systems (QGP - BH-MHD)*
- *Model for strongly magnetized NS*

3+1 Formalism - Metric



Space-time is sliced with a set of space-like hyper-surfaces Σ_t with time-like normal vector n^μ

Induced Metric

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

Line element

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

α Lapse Function

β^i Shift-Vector

Extrinsic Curvature

$$K_{\mu\nu} := -\gamma_\mu{}^\alpha \gamma_\nu{}^\beta \nabla_\alpha n_\beta = -(\nabla_\mu n_\nu + n_\mu n^\alpha \nabla_\alpha n_\nu)\,.$$

3+1 Formalism - Fluid

Fluid is described by a stress-energy tensor $T^{\mu\nu}$ $+$ Baryon flux $n_b u^\mu$

3+1 Contractions of the stress-energy tensor

$$E := T_{\mu\nu} n^\mu n^\nu,$$ Energy density

$$S_\alpha = -\gamma_\alpha{}^\mu n^\nu T_{\mu\nu},$$ Momentum density

$$S_{\alpha\beta} = \gamma_\alpha{}^\mu \gamma_\beta{}^\nu T_{\mu\nu},$$ Stress density

Perfect Fluids

$$E = \rho h \Gamma^2 - p, \qquad S^i = \rho h \Gamma^2 v^i, \qquad S^{ij} = \rho h \Gamma^2 v^i v^j + p\gamma^{ij},$$

The 3+1 Equations - Metric

Einstein Equations

$$G^{\mu\nu} = 8\pi T^{\mu\nu}$$

Evolutionary Equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{D}_i \beta_j + \mathcal{D}_j \beta_i,$$

$$\partial_t K_{ij} = \beta^k \partial_k K_{ij} + K_{ki} \partial_j \beta^k + K_{kj} \partial_i \beta^k - \mathcal{D}_i \mathcal{D}_j \alpha +$$
$$+ \alpha [R_{ij} + K K_{ij} - 2K_{il} K^l{}_j] + 4\pi\alpha[\gamma_{ij}(S - E) - 2S_{ij}]$$

Constraints Equations

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$ Hamiltonian

$$\mathcal{D}_j (K^{ij} - K\gamma^{ij}) = 8\pi S^i,$$ Momentum

CFC Approximation

Maximal Slicing

Conformal Flatness

$K^i_{\ i} = 0$

$\gamma_{ij} = \psi^4 f_{ij}$

$f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$

Traceless Curvature

$$2\alpha \tilde{A}^{ij} = (L\beta)^{ij} := \nabla^i \beta^j + \nabla^j \beta^i - \frac{2}{3}\nabla_k \beta^k f^{ij}.$$

Constraint Equations

$$\Delta\psi = -\left[2\pi E + \frac{1}{8} f_{ik} f_{jl} \tilde{A}^{ij} \tilde{A}^{kl}\right] \psi^5 ,$$

$$\Delta(\alpha\psi) = \left[2\pi(E + 2S) + \frac{7}{8} f_{ik} f_{jl} \tilde{A}^{ij} \tilde{A}^{kl}\right] \alpha\psi^5 ,$$

$$\Delta\beta^i + \frac{1}{3}\nabla^i(\nabla_j \beta^j) = \Delta_L \beta^i := 16\pi\alpha\psi^4 S^i + 2\psi^6 \tilde{A}^{ij} \nabla_j \left(\frac{\alpha}{\psi^6}\right) ,$$

Wilson 2003

XCFC Approximation

$$K^{ij} = \frac{1}{\psi^{10}} \hat{A}^{ij} \quad \rightarrow \quad \hat{A}^{ij} = \hat{A}^{ij}_{TT} + (LW)^{ij} \quad \rightarrow \quad \hat{A}^{ij} = (LW)^{ij}$$

XCFC System

$$\Delta_L W^i = 8\pi f^{ij} \hat{S}_j,$$

$$\Delta\psi = -2\pi \hat{E} \psi^{-1} + \frac{1}{8} f_{ik} f_{jl} \hat{A}^{ij} \hat{A}^{kl} \psi^{-7},$$

$$\Delta(\alpha\psi) = [2\pi(\hat{E} + 2\hat{S})\psi^{-2} + \frac{7}{8} f_{ik} f_{jl} \hat{A}^{ij} \hat{A}^{kl} \psi^{-8}]\alpha\psi,$$

$$\Delta_L \beta^i = 16\pi\alpha\psi^{-6} f^{ij} \hat{S}_j + 2\hat{A}^{ij} \nabla_j (\alpha\psi^{-6}),$$

Stable - Hierarchical - Use Proper Conserved Variables

$$\hat{S}_j := \psi^6 S_j, \quad \hat{E} := \psi^6 E, \quad \hat{S} := \psi^6 S\,.$$

Cordero-Carrion et al 2009, Bucciantini & Del Zanna 2011

Metric Solvers - Scalar

$$\Delta q = h(1+q) := H$$

- $q = (\psi - 1)$ or $q = (\alpha\psi - 1)$
- h Source Terms

Semi-spectral decomposition of the scalar quantities

$$q(r,\theta) = \sum_{l=0}^{\infty} A_l Y_l(\theta)\,, \quad \Rightarrow \quad \sum_{l=0}^{\infty}\left(\frac{d^2 A_l}{dr^2} + \frac{2}{r}\frac{dA_l}{dr} - \frac{l(l+1)}{r^2}A_l\right)Y_l(\theta) = H,$$

Radial Equation for each harmonic solved using FD - tridiagonal inversion

$$\frac{d^2 A_l}{dr^2} + \frac{2}{r}\frac{dA_l}{dr} - \frac{l(l+1)}{r^2}A_l = H_l,$$

Metric Solvers - Vector

$$(\tilde{\Delta} X)^{\phi} = H^{\phi}(X^{\phi})$$

- $\tilde{\Delta} = \nabla(\nabla \cdot\,) - \nabla \times (\nabla \times\,)$

Semi-spectral decomposition of the vector quantities

$$\mathbf{X} = \sum_{l=0}^{\infty} \left(A_l(r) Y_l(\theta) \mathbf{e}_{\hat{r}} + B_l(r) Y'_l(\theta) \mathbf{e}_{\hat{\theta}} + C_l(r) Y'_l(\theta) \mathbf{e}_{\hat{\phi}} \right).$$

Radial Equation for each harmonic solved using FD - Matrix inversion

$$\frac{4}{3}\frac{d^2}{dr^2}A_l + \frac{8}{3r^2}\left(r\frac{d}{dr}A_l - A_l \right) - \frac{l(l+1)}{r^2}\left(A_l + \frac{r}{3}\frac{d}{dr}B_l - \frac{7}{3}B_l \right) = H_l^{\hat{r}},$$

$$\frac{d^2}{dr^2}B_l + \frac{2}{r}\frac{d}{dr}B_l + \frac{1}{3r}\frac{d}{dr}A_l + \frac{8}{3}\frac{A_l}{r^2} - \frac{4}{3}\frac{l(l+1)}{r^2}B_l = H_l^{\hat{\theta}},$$

$$\frac{d^2}{dr^2}C_l + \frac{2}{r}\frac{d}{dr}C_l - \frac{l(l+1)}{r^2}C_l = H_l^{\hat{\phi}},$$

The GRMHD equations

GR-MHD stress-energy tensor

$$T^{\mu\nu} = \rho h u^{\mu} u^{\nu} + p g^{\mu\nu} + F^{\mu}{}_{\alpha} F^{\nu\alpha} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu}$$

GR-MHD equations + Ohm's law

$$\nabla_{\mu}(\rho u^{\mu}) = 0 \qquad \nabla_{\mu} F^{\mu\nu} = -I^{\nu}$$

$$\nabla_{\mu} T^{\mu\nu} = 0 \qquad \nabla_{\mu} {}^{*}F^{\mu\nu} = 0$$

$$F^{\mu\nu} u_{\nu} = 0 \,.$$

3+1 Splitting

$$E = \rho h \Gamma^2 - p + \frac{1}{2}\left(E_i E^i + B_i B^i\right),$$

$$S^i = \rho h \Gamma^2 v^i + \epsilon^{ijk} E_j B_k,$$

$$S^{ij} = \rho h \Gamma^2 v^i v^j + p\gamma^{ij} - E^i E^j - B^i B^j + \frac{1}{2}(E_k E^k + B_k B^k)\gamma^{ij}$$

Algorithm

Initial Condition are derived by solving the XCFC equation for a given energy distribution

GR-MHD equations are solved on a fixed metric for conserved quantities

The solution for W and the conformal factor are solved

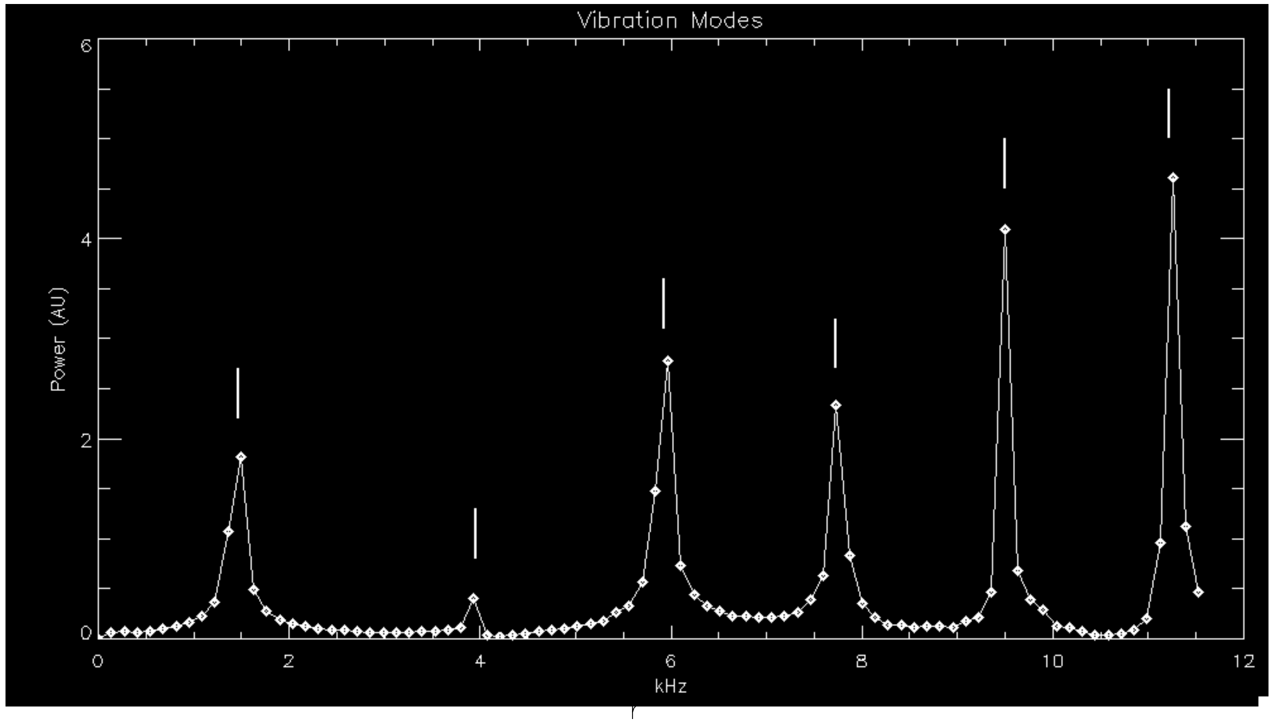
Primitive variables are recomputed

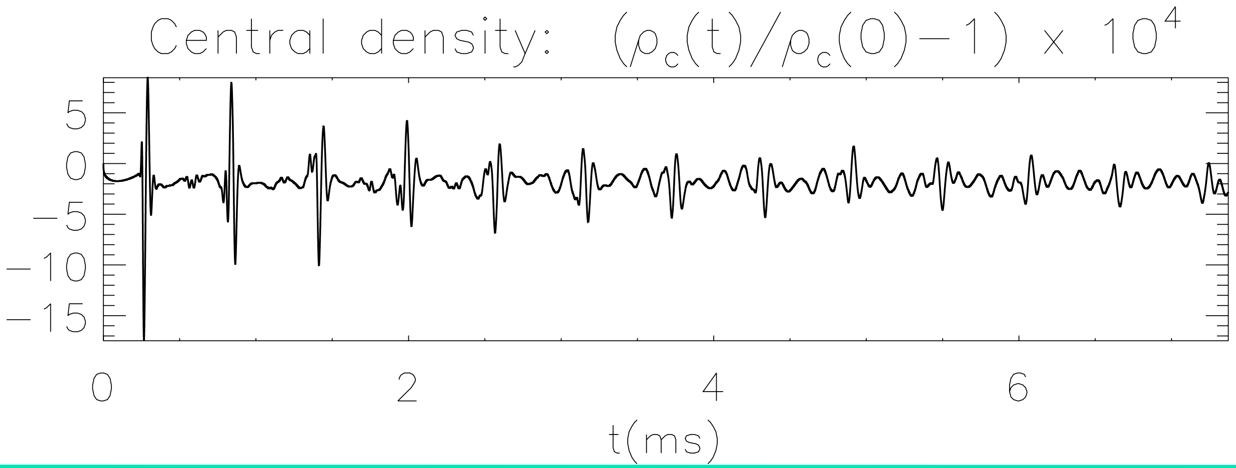
The equation for the lapse and shift vector are solved - metric is updated

Metric update non sync with fluid

NS dynamics - vibrations

Simple Test of the metric solver in the perturbative regime
NS -Oscillations





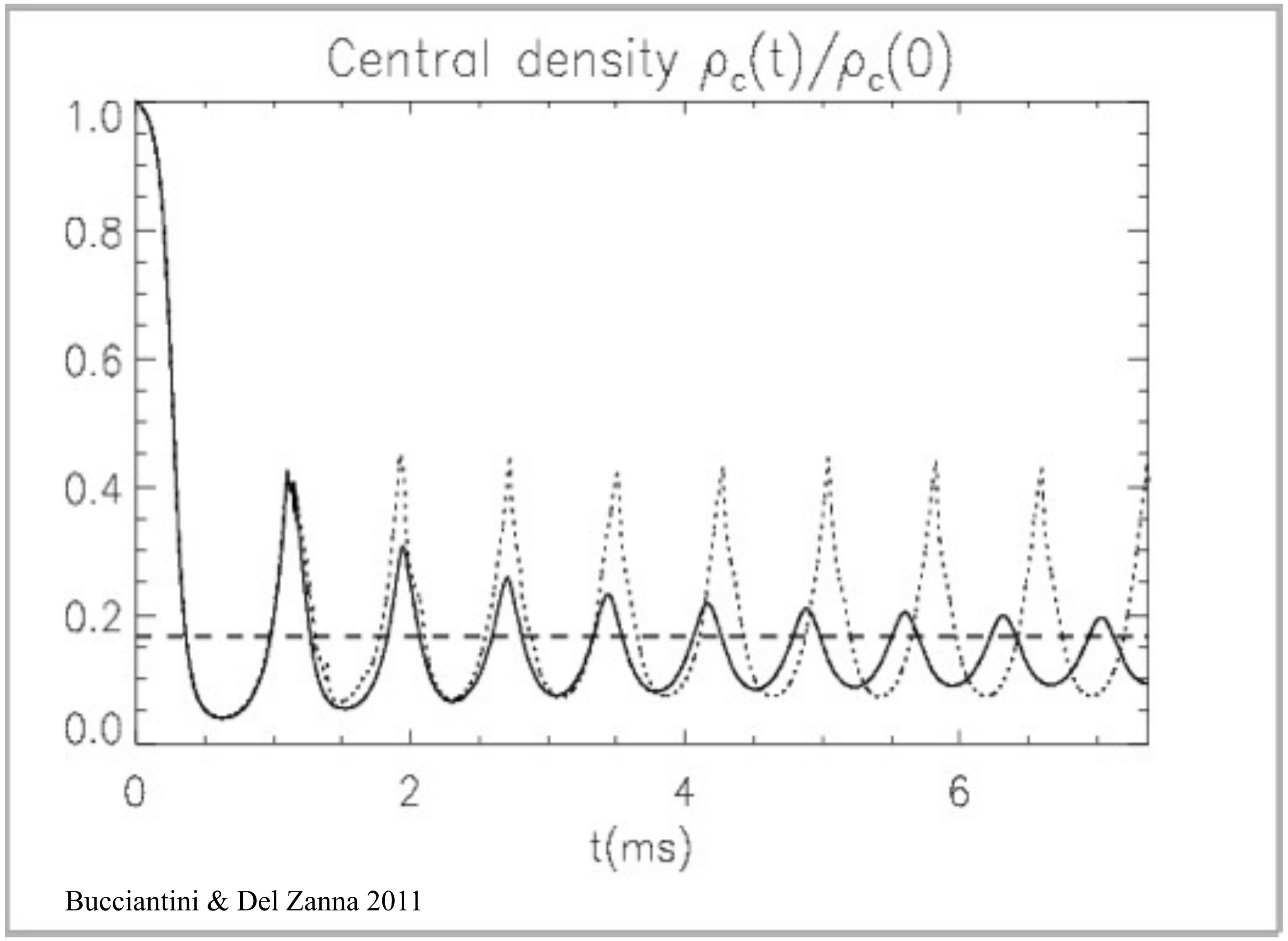
All the harmonic frequencies for l = 0,2,4 are recovered with high Q even for very small amplitude of the oscillations ~ 1.e-4

No evidence for drift in the central density

Bucciantini & Del Zanna 2011

NS Dynamics - migration

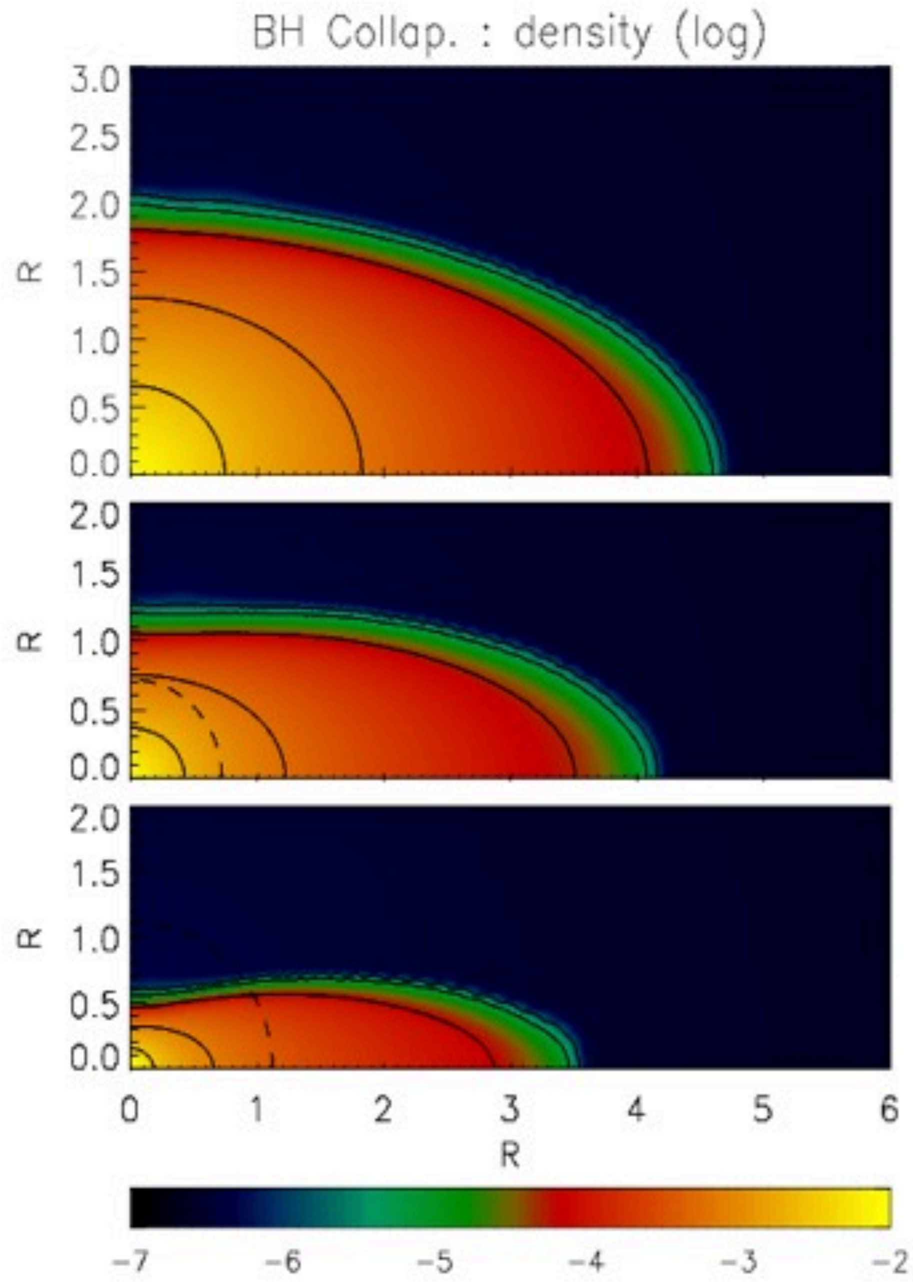
Non perturbative regime - Migration of an unstable NS toward the stable configuration



Both dissipative and non-dissipative regime are correctly reproduced in terms of:
Oscillation Periods.
Amplitude of the Oscillations
Shape of the Peaks
Shocks

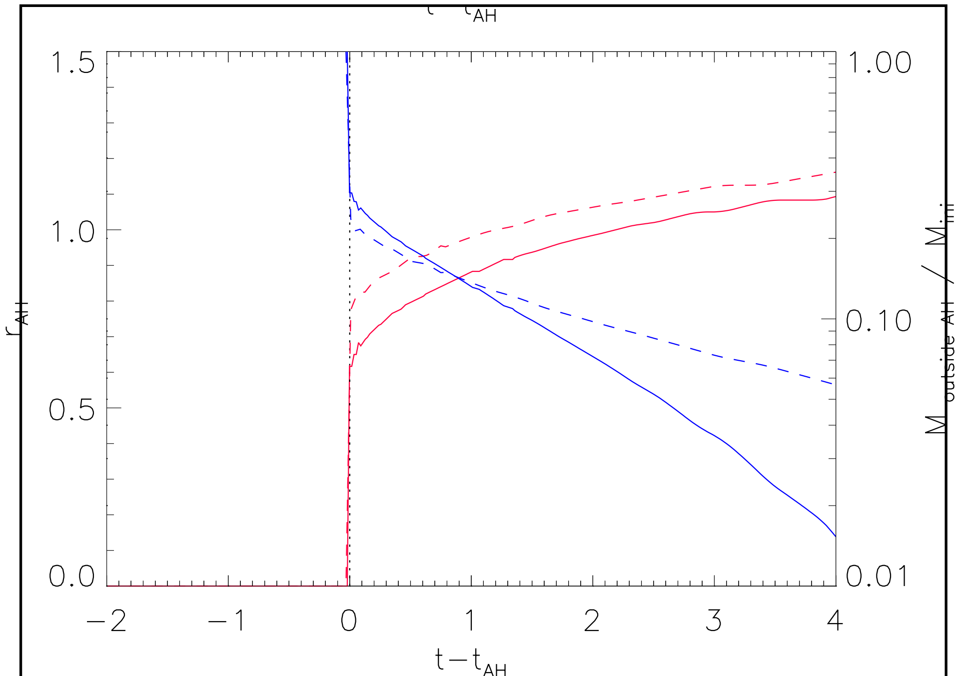
This evolution is not stable in the original CFC

NS Dynamics - BH collapse



BH Collapse of a supra-massive rotating NS

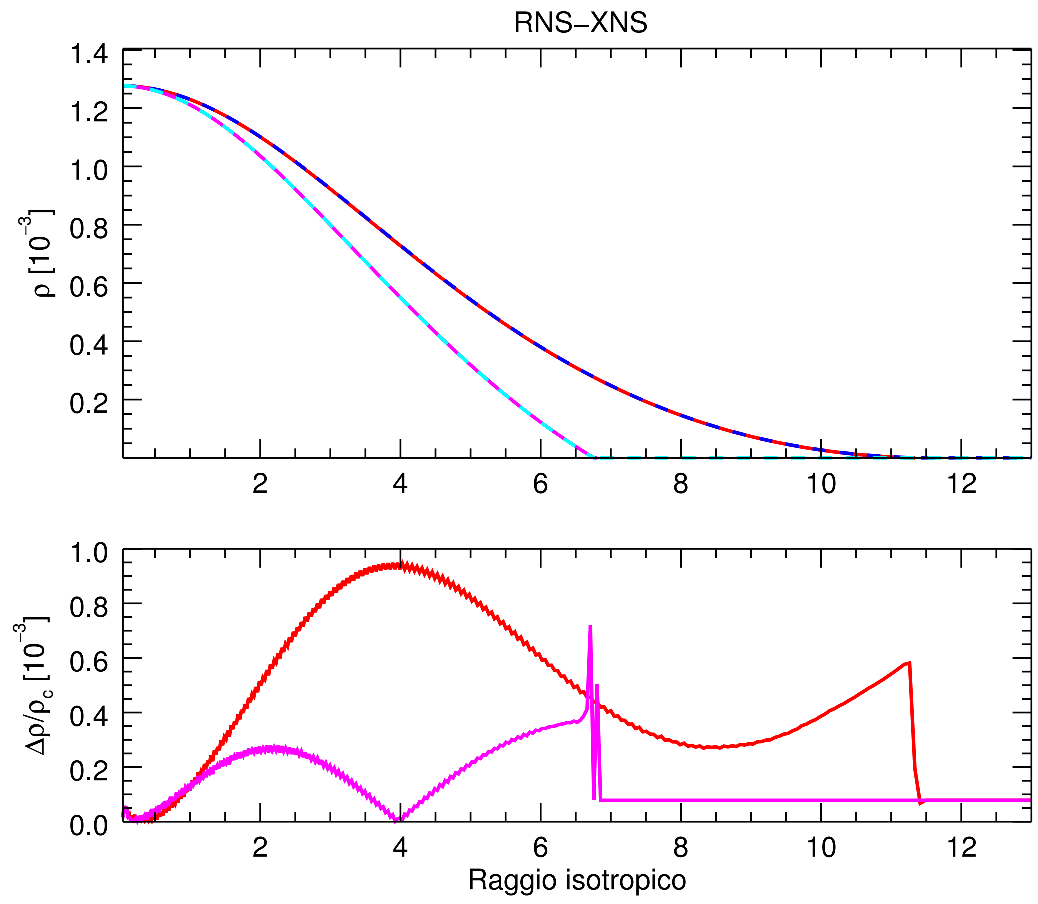
AH and Disk

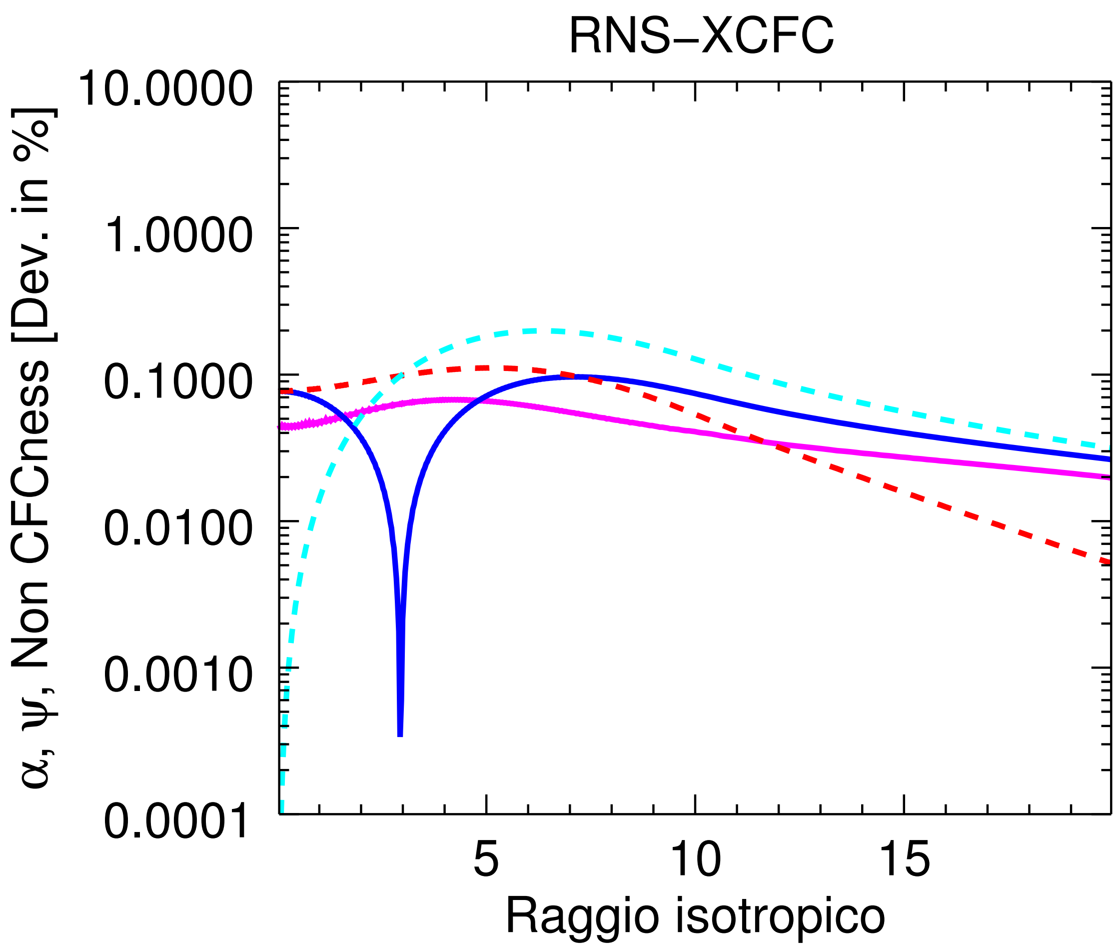


Bucciantini & Del Zanna 2011

Goodness of XCFC

Comparison with Exact Solutions for Rotating NS





Level of accuracy at worst 0.1% = Level of Non-Flatness

The XNS code

- Ohm's law $\Rightarrow$ $E^i = \epsilon^{ijk} v_j B_k = 0$ and $F^{ij} = \epsilon^{ijk} B_k$
- Maxwell Eqs. $\Rightarrow$ $\partial_i(\sqrt{\gamma} B^i) = 0$ e $J^i = \frac{1}{\alpha}\epsilon^{ijk}\partial_j(\alpha B_k)$
  - B^r, B^θ Can be expressed as a function of just the ϕ component of the vector potential A_ϕ
- $\nabla_\mu T^{\mu\nu} = 0$ $\Rightarrow$ Euler's Equation $\frac{\partial_i \alpha}{\alpha} + \frac{\partial_i h}{h} - \frac{L_i}{\rho h} = 0$
  - Integrability of Lorenz Force
    $L_i = \epsilon_{ijk} J^j B^k = \partial_i \mathscr{M}$
  - Axisymmetry $L_\phi = 0 \Rightarrow [B^r = B^\theta = 0 \text{ o } B_\phi = \mathscr{F}(A_\phi)/\alpha]$

Equilibrium is given by Bernoulli Integral

$$\ln\frac{\alpha}{\alpha_c} + \ln\frac{h}{h_c} - \mathscr{M} = 0$$

The Grad-Shafranov

Purely Toroidal

Integrability of Euler's Equation requires $B = \sqrt{B_\phi B^\phi}$:

$$\alpha\psi^2 r \sin\theta B = K_m(\alpha^2\psi^4 r^2 \sin^2\theta\rho h)^m$$

$$\mathscr{M} = -\frac{mK_m^2}{2m-1}(\alpha^2\psi^4 r^2 \sin^2\theta\rho h)^{2m-1}$$

Poloidal Fields

$\mathscr{F}, \mathscr{M}, A_\phi$ Related by Grad-Shafranov:

$$\Delta_* A_\phi = -\rho h \psi^8 r^2 \sin^2\theta \frac{d\mathscr{M}}{dA_\phi} - \partial A_\phi \partial \ln\left(\frac{\alpha^2}{\psi^2}\right) - \frac{\psi^4}{\alpha^2}\frac{d\mathscr{F}}{dA_\phi}$$

$$\Delta_* = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 - \frac{1}{r^2\tan\theta}\partial_\theta, \qquad \partial\partial = \partial_r \cdot \partial_r \cdot + \frac{1}{r^2}\partial_\theta \cdot \partial_\theta \cdot$$

Purely Poloidal

$$\mathscr{M} = k_{\rm POL}(A_\phi + \tfrac{1}{2}\xi A_\phi^2)$$

$$\mathscr{F} = 0$$

Twisted-Torus

$$\mathscr{M} = k_{\rm POL} A_\phi$$

$$\mathscr{F} = a(A_\phi - A_\phi^{\rm max})\Theta(A_\phi - A_\phi^{\rm max})$$

Solving the GS Equation

$$(\tilde{\Delta} X)^{\phi} = H^{\phi}(X^{\phi})$$

- $\tilde{\Delta} = \nabla(\nabla\cdot\,) - \nabla\times(\nabla\times\,)$

Semi-spectral decomposition of the vector quantities

$$\mathbf{X} = \sum_{l=0}^{\infty}\Big(A_l(r)Y_l(\theta)\mathbf{e}_{\hat{r}} + B_l(r)Y'_l(\theta)\mathbf{e}_{\hat{\theta}} + C_l(r)Y'_l(\theta)\mathbf{e}_{\hat{\phi}}\Big)\,.$$

Only the phi-component need to be solver - Matrix inversion -

$$\frac{d^2}{dr^2}C_l + \frac{2}{r}\frac{d}{dr}C_l - \frac{l(l+1)}{r^2}C_l = H_l^{\hat{\phi}}\,,$$

Non Linear in the source term - Iterative

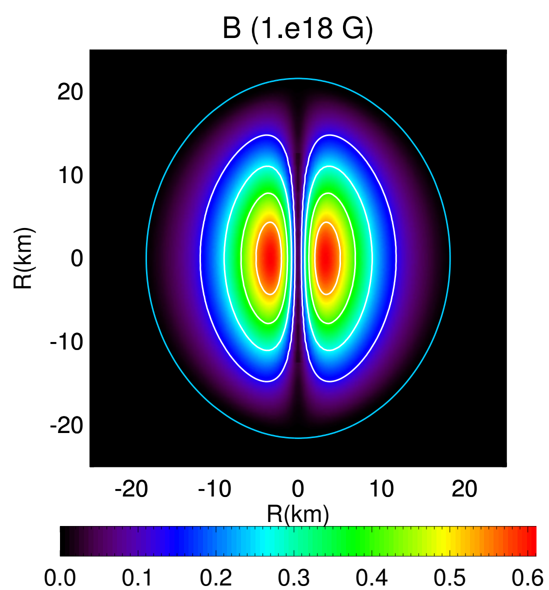
Parametrizing NS

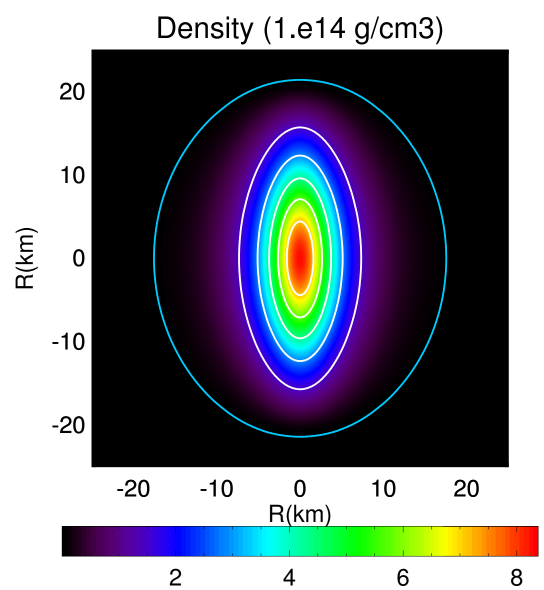
NS are parametrized using several quantities

- Baryon Mass
- Proper Mass
- Gravitational Mass
- Kinetic Energy
- Magnetic Energy
- Magnetic Azimuthal Flux
- Magnetic Dipole Moment
- Helicity
- Binding Energy
- Circumferential Radius
- Eccentricity

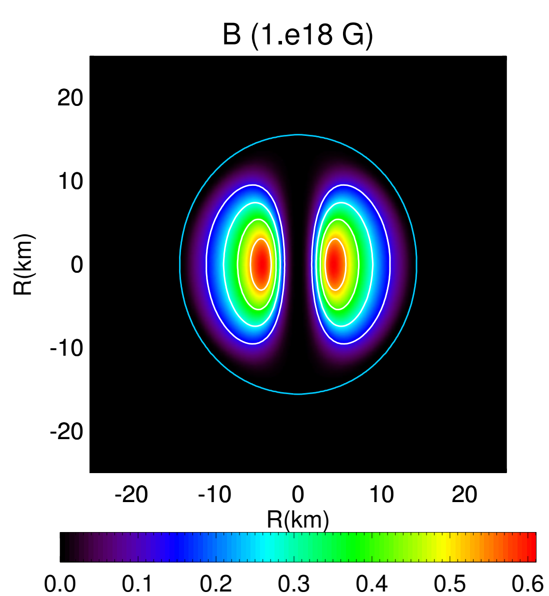
Toroidal Case

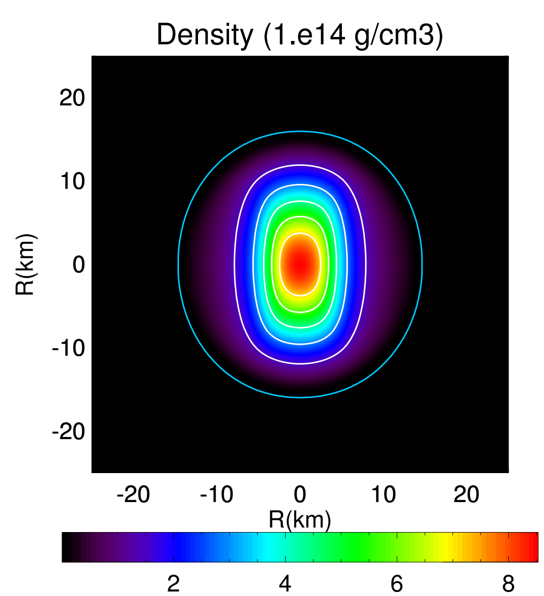
m=1





m=2

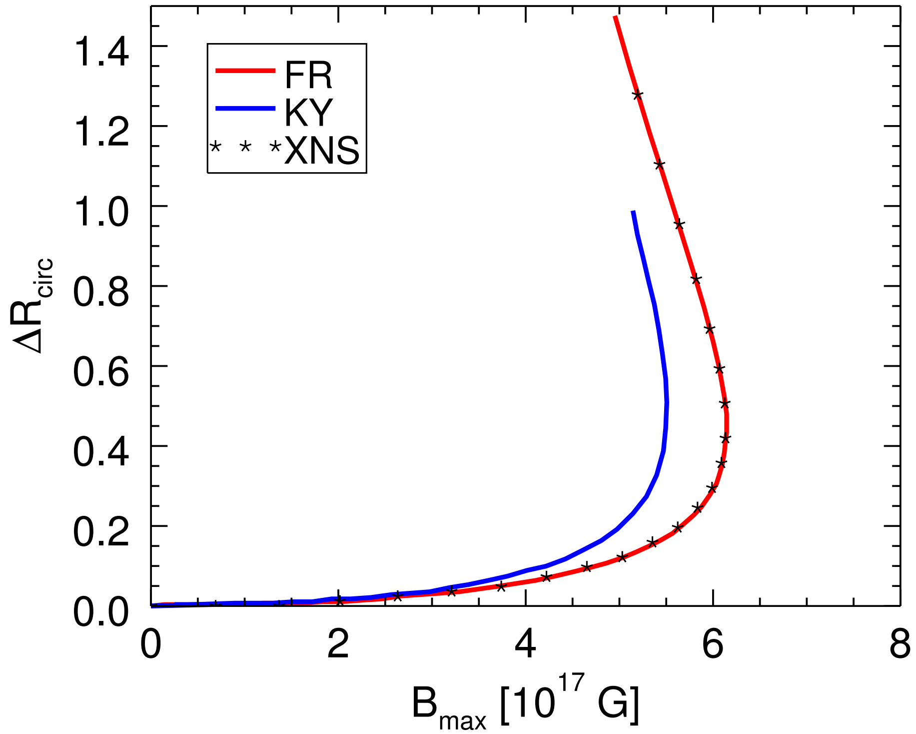




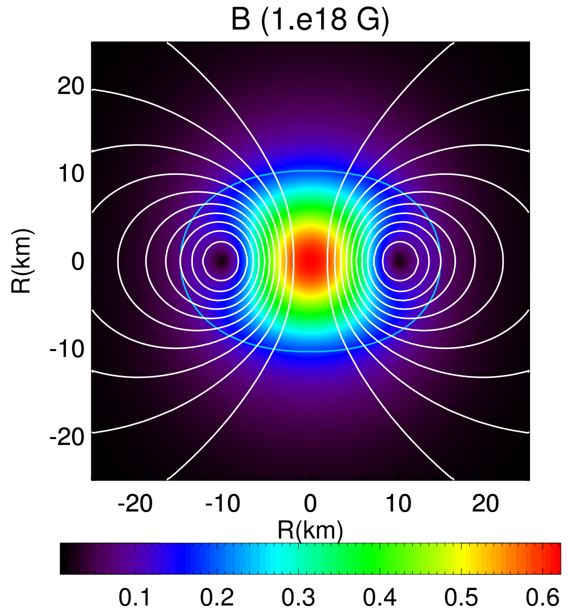
$$\mathscr{M} = -\frac{mK_m^2}{2m-1}(\alpha^2\psi^4 r^2 \sin^2\theta\rho h)^{2m-1}$$

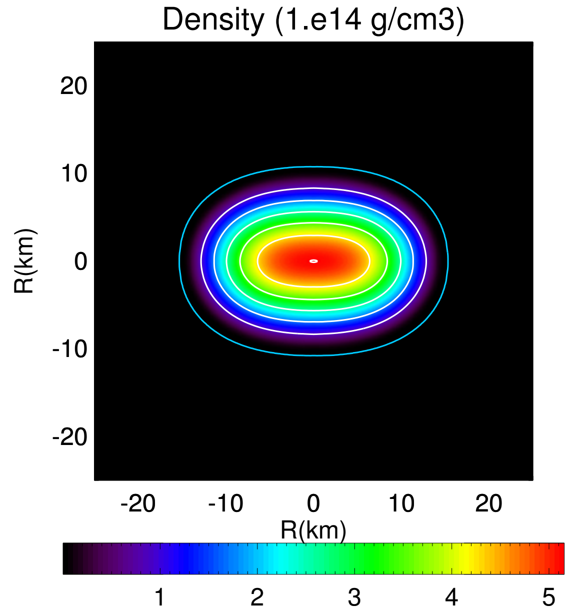
Concentrated fields (low m) have stronger effects.

Existence of max. fields



Poloidal Case

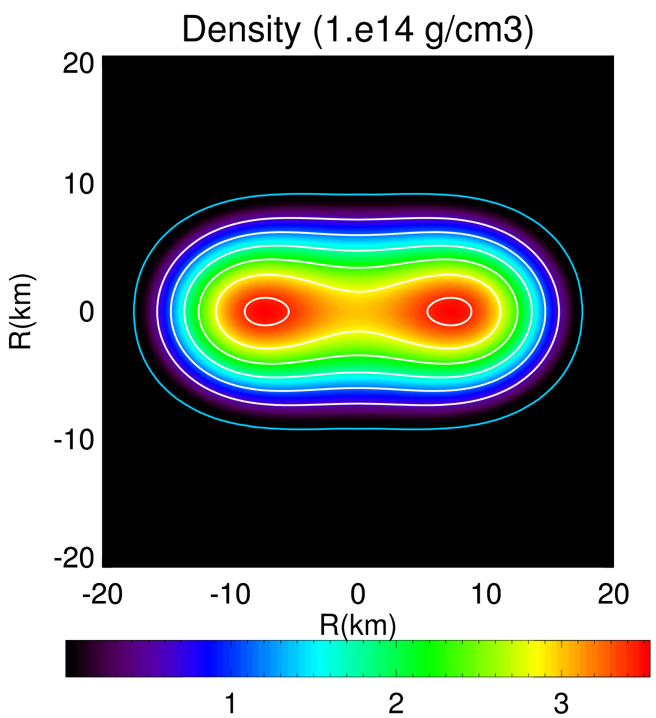


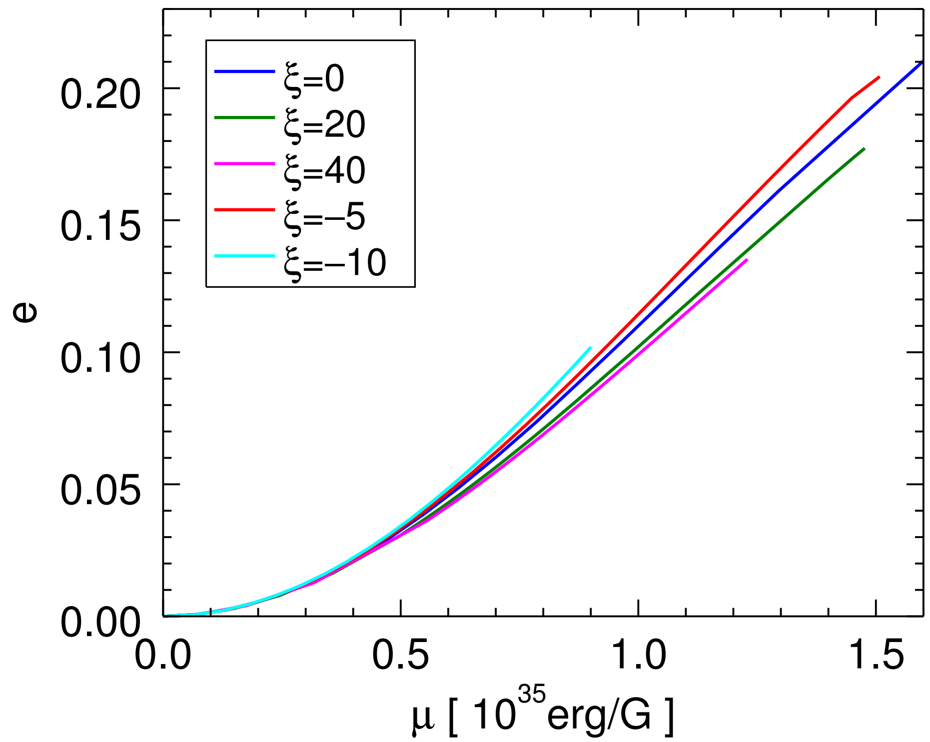


$$\mathscr{M} = k_{\rm POL}(A_\phi + \tfrac{1}{2}\xi A_\phi^2)$$

$$\mathscr{F} = 0$$

- ξ Non-Linear Currents



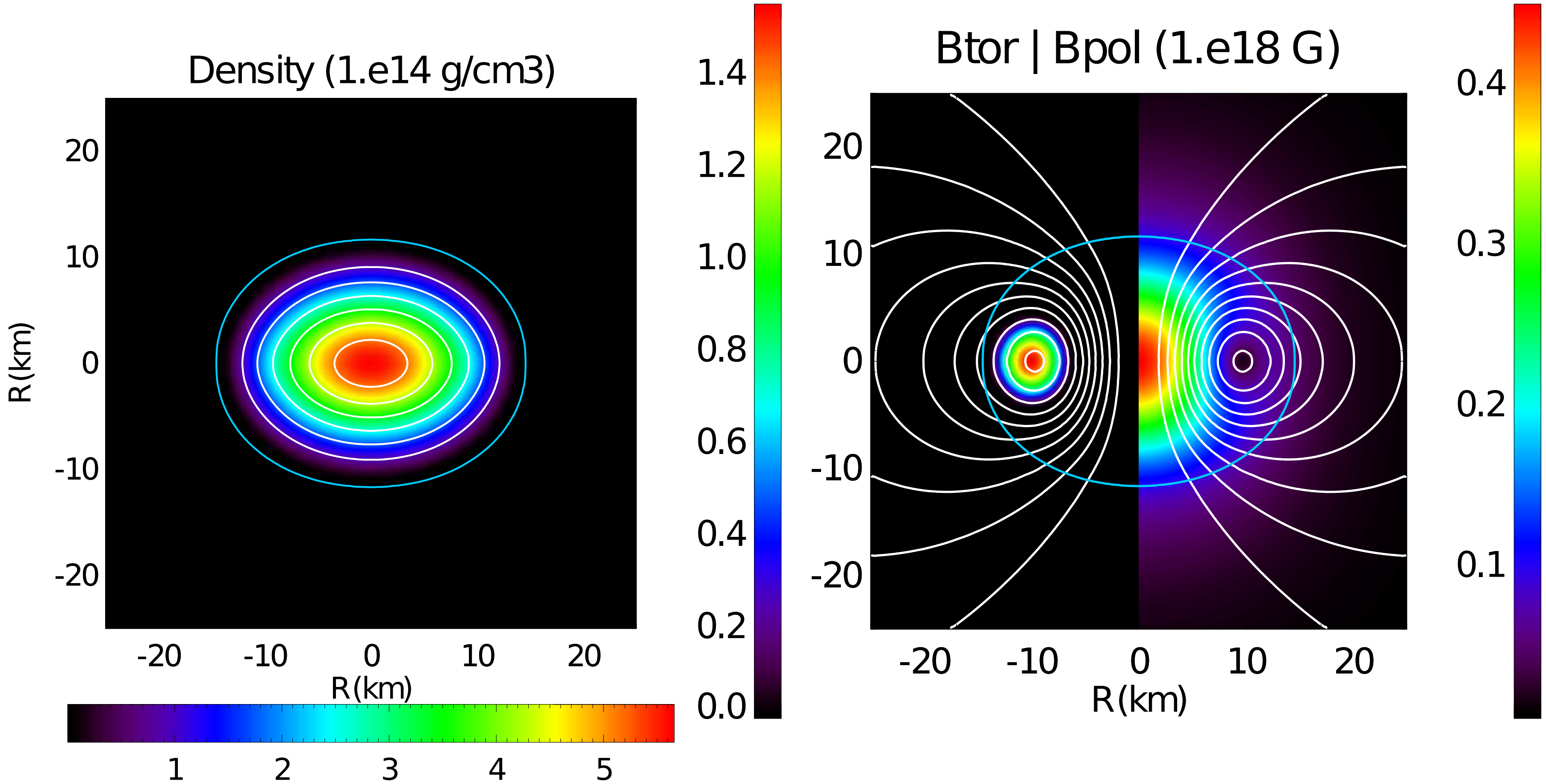


Twisted Torus

$$\mathscr{M} = k_{\mathrm{POL}} A_\phi$$

$$\mathscr{F} = a(A_\phi - A_\phi^{\mathrm{max}})\Theta(A_\phi - A_\phi^{\mathrm{max}})$$

Currents are fully confined
Toroidal field is fully confined
Deformation is oblate



Conclusion & Developments

Algorithm is fast and accurate
Simulations show accuracy with more sophisticated codes
XNS soon to be made public (first public code for magnetized NS)

Among future developments:

More physically motivated EoS

Analysis of stability of magnetized configuration (Tayler, Kink, MRI)

Evaluation of GW emission in Core-Collapse events and for rotating NS

Inclusion of Non-Ideal Effect (thermal conduction)

Neutrino & Radiation Transport