MHD Turbulence Under the Virtual Microscope

Gregory L. Eyink
Applied Mathematics & Statistics
and
Physics & Astronomy
The Johns Hopkins University

8th International Conference on Numerical Modeling of Space Plasma Flows
July 1st - July 5th 2013, Biarritz, France

Collaborators: JHU Turbulence Database Group (E. Vishniac, C. Meneveau, A. Szalay, R. Burns, H. Aluie, C. Lalescu, K. Kanov) & A. Lazarian

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This talk will complement yesterday's by Alex Lazarian!

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Stochastic Flux-Freezing in MHD Turbulence

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Representative Calculation (global GR MHD simulation of a thin disk)

THE ASTROPHYSICAL JOURNAL, 769:156 (20pp), 2013 June 1

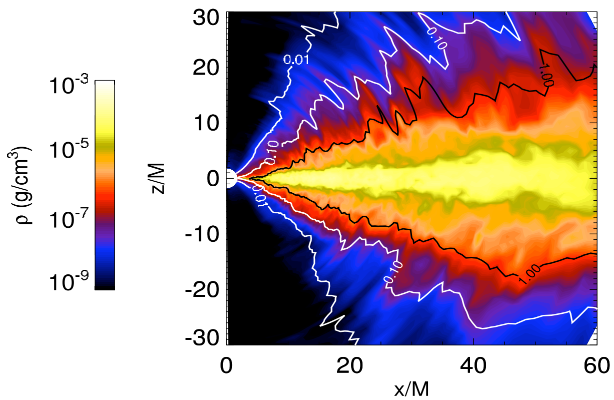

Figure 2. Fluid density profile for a slice of `Harm3d` data in the (r, z) plane at simulation time $t = 12{,}500M$. Contours show surfaces of constant optical depth with $\tau = 0.01, 0.1, 1.0$. Fiducial values for the black hole mass $M = 10\ M_\odot$ and accretion rate $\dot{m} = 0.1$ were used.

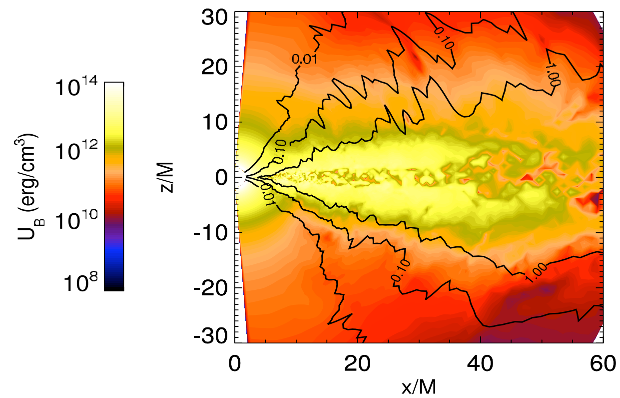

Figure 3. Magnetic energy density profile for a slice of `Harm3d` data in the (r, z) plane corresponding to the same conditions as in Figure 2.

(A color version of this figure is available in the online journal.)

"...because an adequate description of MHD turbulence requires a wide dynamic range in length scales (Hawley et al. 2011; Sorathia et al. 2012), the spatial resolution necessary to simulate disks as thin as some of those likely to occur in nature remains beyond our grasp. Thus, in some respects, our calculations represent an intermediate step toward drawing a complete connection between fundamental physics and output spectra."

-Schnittman et al. (2013)

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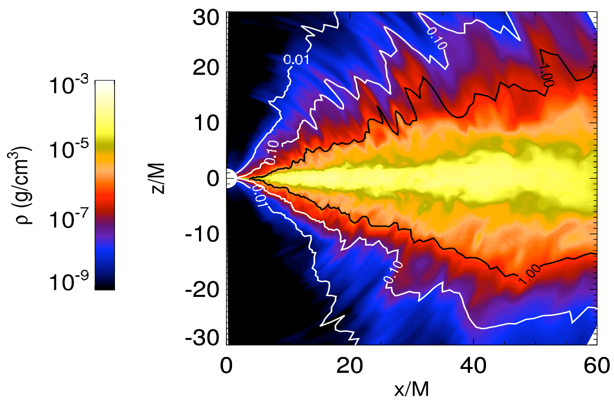
SCHNITTMAN, KROLIK, & NOBLE



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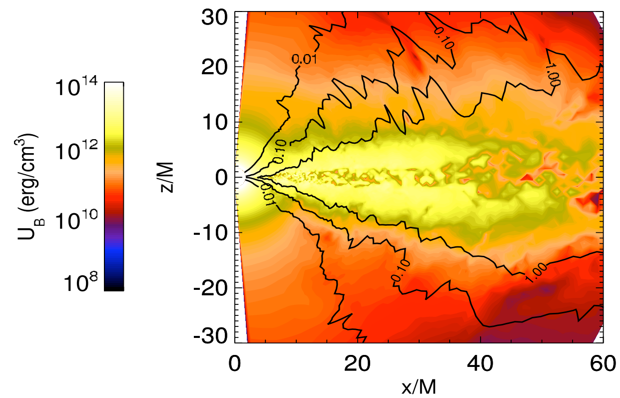

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turbulence.pha.jhu.edu

Welcome to the JHU Turbulence Database Cluster (TDC) site

This website is a portal that enables access to multi-Terabyte turbulence databases. The data reside on several nodes and disks on our database cluster computer and are stored in small 3D subcubes. Positions are indexed using a Z-curve for efficient access.

Access to the data is facilitated by a Web services interface that permits numerical experiments to be run across the Internet. We offer C, Fortran and Matlab interfaces layered above Web services so that scientists can use familiar programming tools on their client platforms. Calls to fetch subsets of the data can be made directly from within a program being executed on the client's platform. Manual queries for data at individual points and times via web-browser are also supported. Evaluation of velocity and pressure at arbitrary points and time is supported using interpolations executed on the database nodes. Spatial differentiation using various order approximations (up to 8th order) are also supported (for details, see documentation page). Other functions such as spatial filtering are being developed.

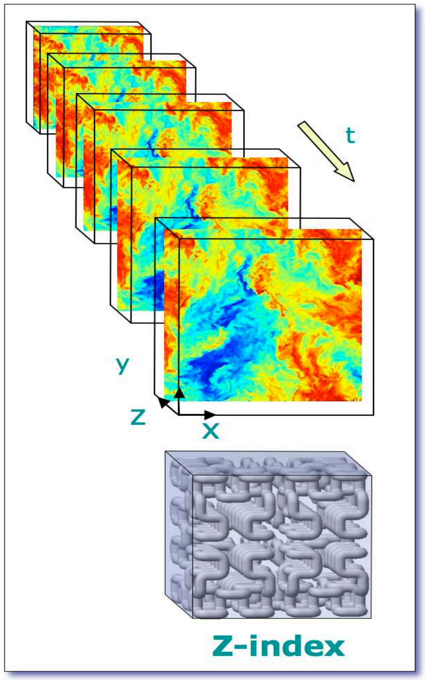
So far the database contains a 1024^4 space-time history of a direct numerical simulation (DNS) of isotropic turbulent flow, in incompressible fluid in 3D, and a DNS of the incompressible magneto-hydrodynamic (MHD) equations. The simulations were performed using 1024 grid points in each direction using a pseudo-spectral method, and forcing at large scales. The database allows access to 1024 time steps covering about one integral turn-over time-scale of the turbulence. The datasets comprise 27 Terabytes for the isotropic turbulence data and 56 Terabytes for the MHD data. Basic characteristics of the data sets can be found in the datasets description page. Technical details about the database techniques used for this project are described in the publications.

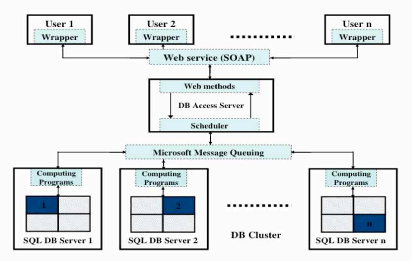
The Turbulence Database Cluster project is funded by the US National Science Foundation .

Questions and comments? turbulence@pha.jhu.edu

269903764678 points queried

Please excuse our dust as we continue to develop this site. The Turbulence Database is on-line but may periodcally be unavailable as we continue to add functionalities.





JHU Turbulence Database Cluster Architecture

```
! Intialize the gSOAP runtime.
CALL soapinit()

points(1,:)=(/(2.*PI/npnt*ii, ii=0,npnt-1)/)
time=0.
do ii=0,nline-1
  points(2,:)=(1024/nliney)*modulo(ii,nliney)*dx
  points(3,:)=(1024/nlinez)*int(ii/nliney)*dx

  ! database query to find the velocities on the points
  rc=getvelocity(authkey, dataset, time, NoSInt, &
            NoTInt, npnt, points, velocity)

  ! FFT of ux and calculate the longitudinal energy spectrum
  call rfftw_f77_one(R2C1d, velocity(1,:), uxo)
  uxo=uxo/npnt
  E11k1(1)=E11k1(1)+uxo(1)*uxo(1)
  do jj=2,npnt/2
    E11k1(jj)=E11k1(jj) + uxo(jj)*uxo(jj) + uxo(npnt+2-jj)*uxo(npnt+2-jj)
  end do
  E11k1(n1)=E11k1(n1)+uxo(n1)*uxo(n1)

  ! FFT of uy and uz and transverse energy spectra
  ......
end do
```

Figure 6. Snippet of the FORTRAN code running on local user machine. Bold font highlights the lines invoking the Web-services method. The authkey has been intentionally marked out.

Energy Spectra for JHU MHD Turbulence Database

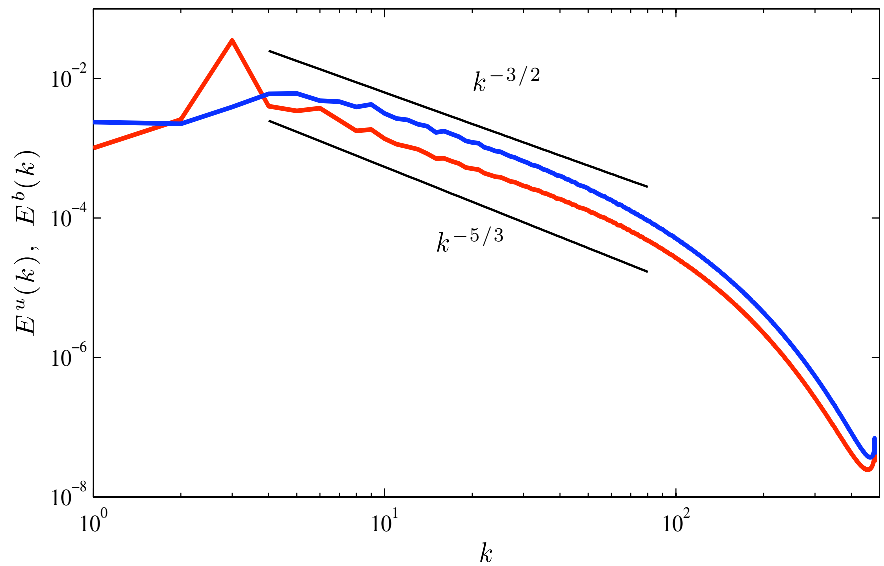


Figure 1: Spectra of velocity (red) and magnetic (blue) fields

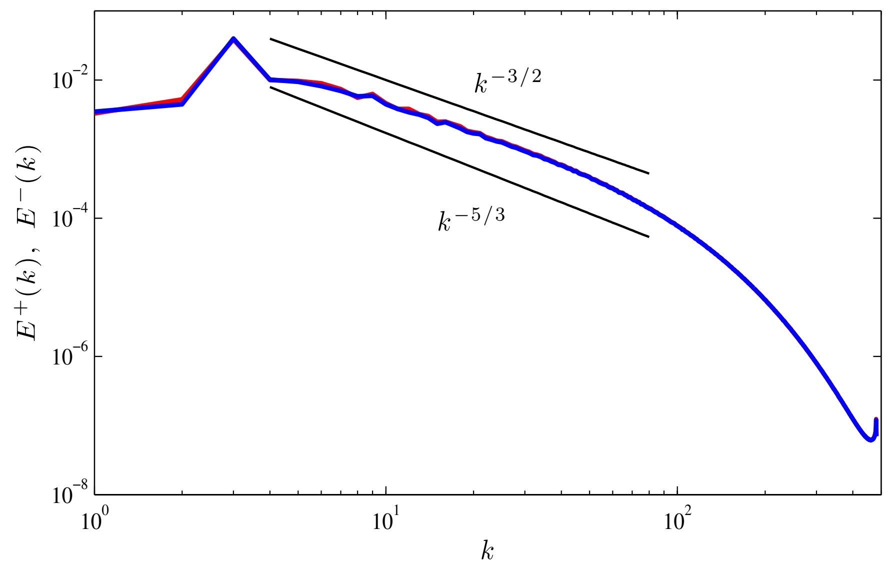


Figure 2: Spectra of Elsasser variables, $\mathbf{z}^+=\mathbf{u}+\mathbf{b}$ (red) and $\mathbf{z}^-=\mathbf{u}-\mathbf{b}$ (blue)

The spectral exponents are closer to -3/2 than to -5/3, as usual for MHD simulations at these Reynolds numbers ($Re \approx 1170$). This fact motivated the Boldyrev theory with $\alpha = 1$, which gives $\delta u(r_\perp) \sim r_\perp^{1/4}$ and $-3/2$ spectrum.

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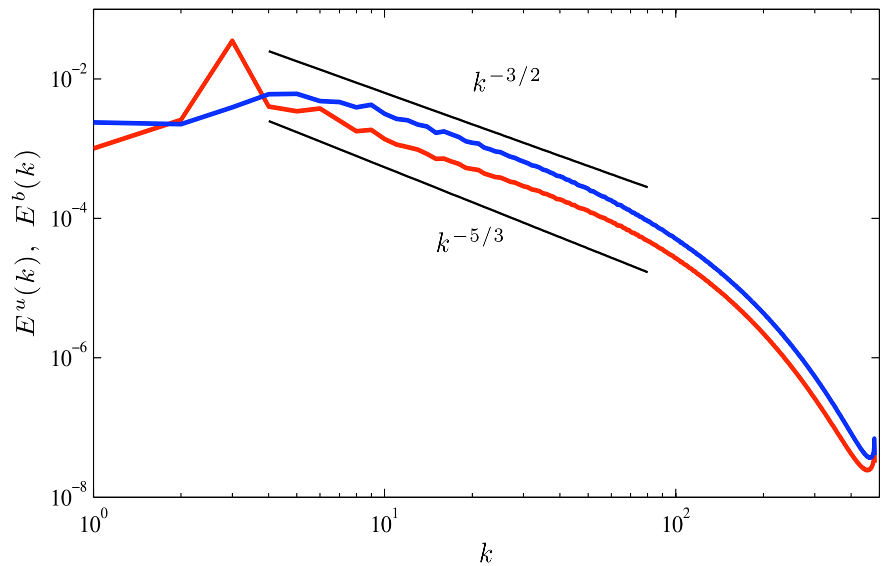

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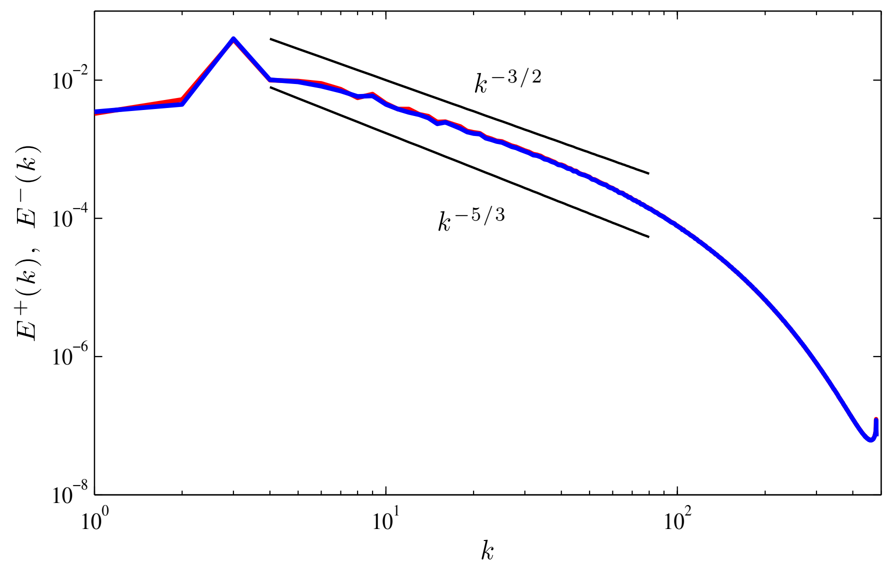

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However, see A. Beresnyak, PRL **106** 075001 (2011)! The spectral scaling of MHD turbulence at astrophysically relevant Reynolds numbers is still being debated....

Magnetic Flux-Freezing

"In view of the infinite conductivity, every motion (perpendicular to the field) of the liquid in relation to the lines of force is forbidden because it would give infinite eddy currents. Thus the matter of the liquid is 'fastened' to the lines of force." (H. Alfvén, 1942)

Field-lines do not really move! It is permissible to ascribe a velocity $\mathbf{u}$ to the lines of force of magnetic field $\mathbf{B}$ if and only if $\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = -\boldsymbol{\nabla}\Phi$, or

$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}). \tag{$*$}$$

(W. A. Newcomb, 1958). A flux-preserving velocity $\mathbf{u}$ is not usually unique, cf. Newcomb (1958), Vasyliunas (1972), Alfvén (1976).

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In fact, even if (*) holds to an extremely good approximation, standard flux-freezing is generally false, under realistic astrophysical conditions!

In turbulent plasmas with power-law spectra of velocity and magnetic fields, flux-freezing does not hold in the standard sense but neither is it completely broken. Instead, flux-freezing becomes intrinsically stochastic.

Stochastic Flux-Freezing for Resistive MHD

The exact solution of the resistive induction equation

$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \lambda\triangle\mathbf{B}$$

is given by a *stochastic Lundquist formula* (Eyink 2009, 2011)

$$\mathbf{B}(\mathbf{x},t) = \left\langle \left.\frac{\mathbf{B}_0(\mathbf{a})\boldsymbol{\cdot}\boldsymbol{\nabla}_a\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a})}{\det(\boldsymbol{\nabla}_a\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a}))}\right|_{\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a})=\mathbf{x}} \right\rangle.$$

Here the average $\langle\cdot\rangle$ is over an ensemble of stochastic flows generated by

$$d_t\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a}) = \mathbf{u}(\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a}),t)dt + \sqrt{2\lambda}\,d\tilde{\mathbf{W}}(t), \quad \tilde{\mathbf{x}}_{t_0,t_0}(\mathbf{a}) = \mathbf{a},$$

where $\tilde{\mathbf{W}}(t)$ is a random Brownian motion.

This is equivalent to a path-integral formula or "sum-over-histories",

$$\mathbf{B}(\mathbf{x},t) = \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\ \mathbf{B}_0(\mathbf{a}(t_0))\boldsymbol{\cdot}\mathbf{J}(\mathbf{a},t,t_0)\exp\left(-\frac{1}{4\lambda}\int_{t_0}^{t} d\tau\,|\dot{\mathbf{a}}(\tau)-\mathbf{u}^{\nu}(\mathbf{a}(\tau),\tau)|^2\right)$$

where the matrix $\mathbf{J}$ satisfies the ODE along the stochastic trajectory $\mathbf{a}(\tau)$

$$\frac{d}{d\tau}\mathbf{J}(\mathbf{a},\tau,t_0) = \mathbf{J}(\mathbf{a},\tau,t_0)\boldsymbol{\nabla}_x\mathbf{u}(\mathbf{a}(\tau),\tau) - \mathbf{J}(\mathbf{a},\tau,t_0)(\boldsymbol{\nabla}_x\boldsymbol{\cdot}\mathbf{u})(\mathbf{a}(\tau),\tau),, \quad \mathbf{J}(\mathbf{a},t_0,t_0) = \mathbf{I}.$$

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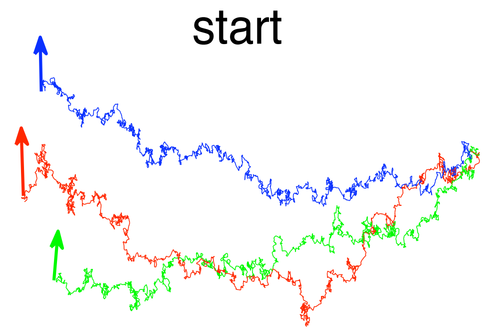
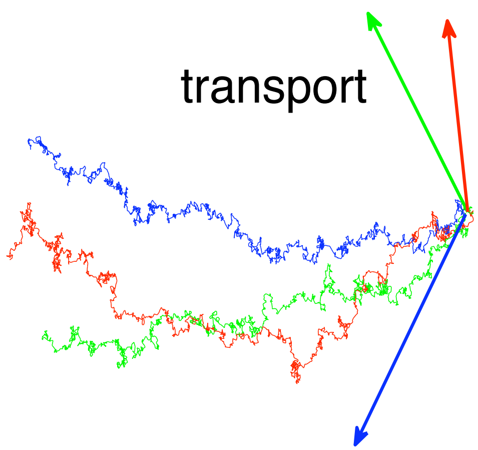
We shall see that magnetic field evolution in the presence of fluid turbulence becomes as indeterministic as quantum mechanics and requires similar methods for its description!

Stochastic Lundquist Formula

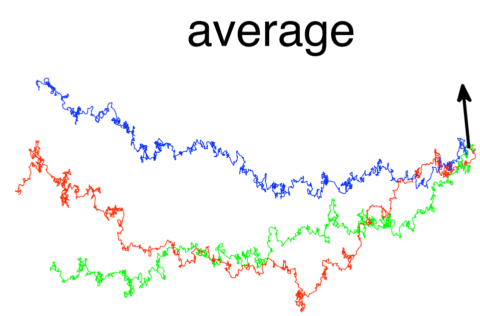
$$\mathbf{B}(\mathbf{x},t) = \left\langle \left. \frac{\mathbf{B}(\mathbf{a},t_0)\boldsymbol{\cdot}\boldsymbol{\nabla}_a \tilde{\mathbf{x}}_{t,t_0}(\mathbf{a})}{\det(\boldsymbol{\nabla}_a \tilde{\mathbf{x}}_{t,t_0}(\mathbf{a}))} \right|_{\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a})=\mathbf{x}} \right\rangle$$



$$d\tilde{\mathbf{x}}(\tau) = \mathbf{u}(\tilde{\mathbf{x}},\tau)d\tau + \sqrt{2\lambda}\, d\tilde{\mathbf{W}}(\tau)$$



$$(d/d\tau)\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\boldsymbol{\cdot}\boldsymbol{\nabla}\mathbf{u} - (\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u})\tilde{\mathbf{B}}$$



$$\mathbf{B}(\mathbf{x},t) = \left\langle \tilde{\mathbf{B}}(\mathbf{x},t) \right\rangle$$

The Standard View of Flux-Freezing at High Conductivity

It is not hard to show for a *smooth* velocity field $\mathbf{u}$ satisfying

$$|\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{x}',t)| \leq K|\mathbf{x}-\mathbf{x}'|$$

that

$$\left\langle |\tilde{\mathbf{x}}_{t,t_0}(\mathbf{a})-\mathbf{x}_{t,t_0}(\mathbf{a})|^2 \right\rangle \leq 3\lambda \frac{e^{2Kt}-1}{K} \doteq 6\lambda t \ \text{ for } \ Kt \ll 1.$$

Here $\mathbf{x}_{t,t_0}(\mathbf{a})$ is the deterministic flow that solves

$$\frac{d}{dt}\mathbf{x}_{t,t_0}(\mathbf{a}) = \mathbf{u}(\mathbf{x}_{t,t_0}(\mathbf{a}),t), \quad \mathbf{x}_{t_0,t_0}(\mathbf{a}) = \mathbf{a}.$$

Cf. Freidlin & Wentzell (1984), Chapter 2. In particular, $\lim_{\lambda\to 0} \tilde{\mathbf{x}}_{t,t_0}(\mathbf{a}) = \mathbf{x}_{t,t_0}(\mathbf{a})$.

"Flux freezing is a very strong constraint on the behavior of magnetic fields in astrophysics. As we show in chapter 3, this implies that lines do not break and their topology is preserved. The condition for flux freezing can be formulated as follows: In a time t, a line of force can slip through the plasma a distance

$$\ell = \sqrt{\frac{\eta c^2 t}{4\pi}} \tag{1}$$

If this distance ℓ is small compared to δ, the scale of interest, then flux freezing holds to a good degree of approximation."—R. M. Kulsrud (2005), Ch.13, Magnetic Reconnection

Richardson Two-Particle Dispersion

Volcanic ash plume over Kīlauea volcano

Meteorologist, physicist and applied mathematician Lewis Fry Richardson proposed in 1926 that particle-pairs advected by turbulence (e.g. a pair of soot particles in a volcanic plume) would have mean-square separation increasing with time as the cube power

$$\langle |\mathbf{x}_1(t) - \mathbf{x}_2(t)|^2 \rangle \sim t^3.$$

This is Richardson's t^3-law.

Scale-Dependent Eddy-Diffusivity

Reference.	K cm.^{2} sec^{-1}	l cm.
K from molecular diffusion of oxygen into nitrogen (Kaye and Laby's 'Physical and Chemical Constants'). For l see preceding discussion.	$1 \cdot 7 \times 10^{-1}$	5×10^{-2}
K at 9 metres above ground from anemometers at heights of 2, 16 and 32 metres (W. Schmidt, 'Wien. Akad. Sitzb.,' IIa, vol. 126, p. 773 (1917)).	$3 \cdot 2 \times 10^3$	$1 \cdot 5 \times 10^3$
K from anemometers at heights of 21 to 305 metres (Åkerblom, F., 'Nova Acta Reg. Soc. Upsaliensis' (1908)).	$1 \cdot 2 \times 10^5$	$1 \cdot 4 \times 10^4$
K from pilot balloons at heights between 100 and 800 metres (Taylor, 'Phil. Trans.,' A, vol. 215, p. 21 (1914), also Hesselberg and Sverdrup, 'Leipzig Geophys. Inst.,' Ser. 2, Heft 10 (1915)).	6×10^4	5×10^4
K from tracks of balloons either manned (L. F. Richardson, 'Weather Prediction by Numerical Process,' p. 221) or not manned (Richardson & Proctor, 'Royal Meteorological Society Memoirs,' No. 1).	10^8	2×10^6
Volcano ash, same reference as last	5×10^8	5×10^6
Diffusion due to cyclones regarded as deviations from the mean circulation of the latitude (Defant, 'Geog. Ann.,' H. 3, also (1921), 'Wien. Akad. Wiss. Sitzb.,' IIa, vol. 130, p. 401 (1921)).	10^{11}	10^8

Richardson's table of raw data

Richardson's approach was semi-empirical. By estimating "effective diffusivity" $K = \langle|\Delta \mathbf{x}|^2\rangle/t$ as a function of $\ell = \sqrt{\langle|\Delta \mathbf{x}|^2\rangle}$, he found from data that

$$K(\ell) \sim K_0 \ell^{4/3}.$$

He proposed that the probability density function of the separation vector $\boldsymbol{\ell} = \mathbf{x}_1 - \mathbf{x}_2$ would satisfy a diffusion equation

$$\partial_t P(\boldsymbol{\ell}, t) = \frac{\partial}{\partial \ell_i}\left(K(\ell)\frac{\partial P}{\partial \ell_i}(\boldsymbol{\ell}, t)\right)$$

with scale-dependent 2-particle eddy-diffusivity. This equation predicts at long times that

$$\langle|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^2\rangle \sim t^3,$$

averaging over velocity realizations.

Similarity Solution

Richardson (1926) observed that there is an exact similarity solution of his equation, given by the stretched-exponential PDF

$$P_*(\boldsymbol{\ell},t) = \frac{A}{(K_0 t)^{9/2}} \exp\left(-\frac{9\ell^{2/3}}{4K_0 t}\right)$$

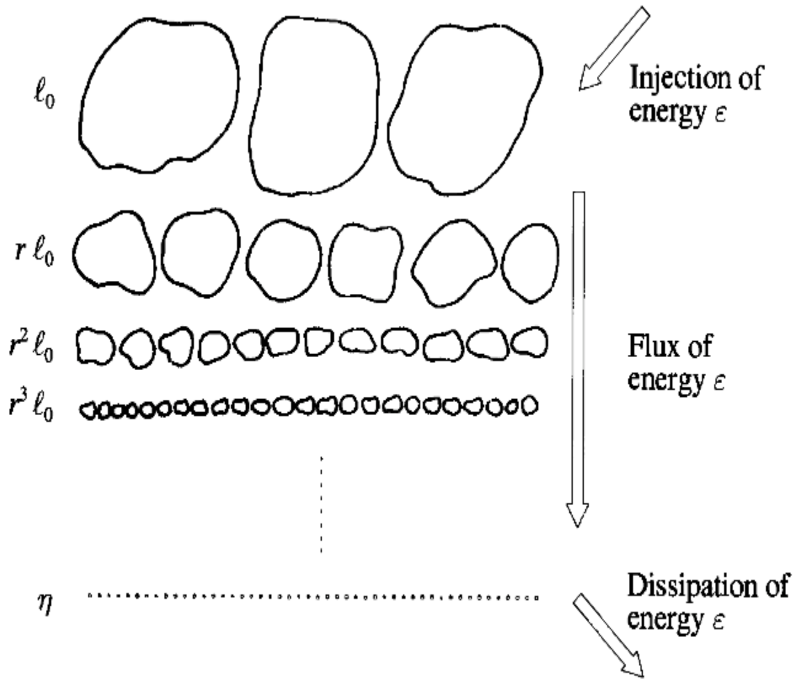
in three space dimensions. All solutions approach this self-similar form asymptotically at long times.

Averaging ℓ^2 with respect to this density yields

$$\langle \ell^2(t) \rangle = \gamma_0 t^3$$

with $\gamma_0 = 1144 K_0^3/81$.

Kolmogorov Cascade Picture



A cartoon of the Kolmogorov cascade

In the Kolmogorov (1941) picture, velocity differences across eddies of size ℓ have magnitude

$$\delta u(\ell) \sim (\varepsilon \ell)^{1/3}.$$

This increases with ℓ, so that larger turbulent eddies have larger velocities.

A pair of particles as they separate thus experience greater relative velocities as they move further apart. The outcome is an explosive separation

$$\langle \ell^2(t) \rangle \sim g_0 \varepsilon t^3,$$

even much faster than ballistic ($\propto t^2$).

The (presumed universal) constant g_0 is now usually called "Richardson's constant".

Advection by Kolmogorov Velocity

A toy calculation: Assume that $\ell(t)$ satisfies

$$\frac{d}{dt}\ell(t) = \delta u(\ell) = \frac{3}{2}(g_0\varepsilon\ell)^{1/3}.$$

Separation of variables gives the exact solution

$$\ell(t) = \left[\ell_0^{2/3} + (g_0\varepsilon)^{1/3}(t - t_0)\right]^{3/2}.$$

For $t - t_0 \gg \ell_0^{2/3}/(g_0\varepsilon)^{1/3} \equiv T_0$

$$\ell^2(t) \sim g_0\varepsilon t^3.$$

The condition for this behavior, $\delta u(\ell) \propto \ell^{1/3}$, is equivalent to the Kolmogorov energy spectrum

$$E(k) \propto k^{-5/3}$$

which is very common in astrophysical plasmas.

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Fate of Particles Initially at the Same Point?

The odd feature of the previous result is that, if $\ell_0 = 0$, then

$$\ell^2(t) = g_0 \varepsilon (t - t_0)^3 > 0.$$

Two particles started at the *same* point at time t_0 separate to a finite distance at any time $t > t_0$!

The same oddity may be seen in Richardson's similarity solution, which satisfies at initial time $t_0 = 0$

$$P_*(\boldsymbol{\ell}, 0) = \delta^3(\boldsymbol{\ell}).$$

All particles start with separation $\ell(0) = 0$. However, $P_*(\boldsymbol{\ell}, t)$ is a smooth density for $t > 0$, so that $\ell(t) > 0$ with probability one.

Breakdown of Laplacian Determinism

According to Richardson's results, Lagrangian fluid particles that are advected by the fluid velocity $\mathbf{u}(\mathbf{x},t)$ starting at $\mathbf{x}_0$

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t),t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

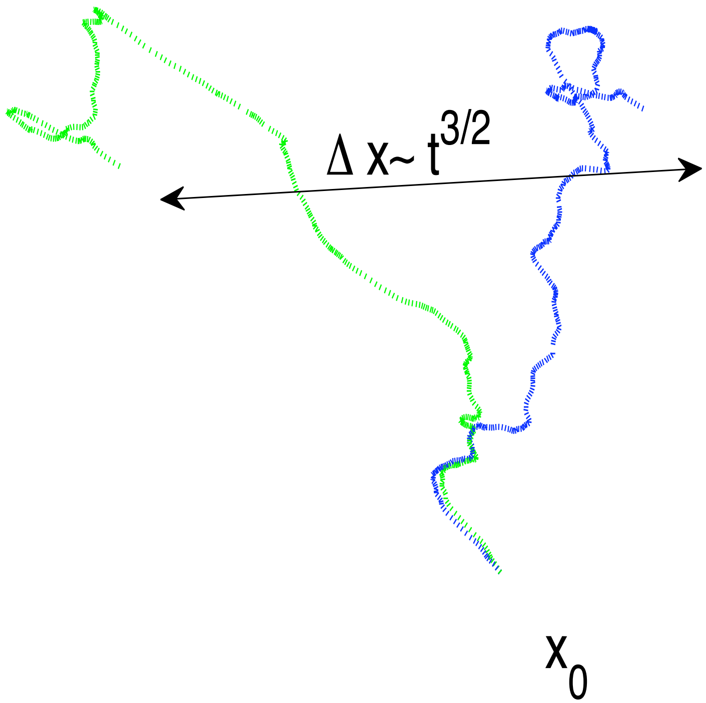
have the property that there is more than one solution. Doesn't this violate the theorem on uniqueness of solutions of initial-value problems for ODE's? No!

Loophole: The theorem requires that $\mathbf{u}(\mathbf{x},t)$ be $\mathbf{x}-$differentiable. A turbulent velocity field in a Kolmogorov inertial range is only Hölder continuous

$$|\mathbf{u}(\mathbf{x}_1,t) - \mathbf{u}(\mathbf{x}_2,t)| \leq C|\mathbf{x}_1 - \mathbf{x}_2|^h$$

with exponent $h \doteq 1/3$.

Spontaneous Stochasticity



Consider

$$d\tilde{\mathbf{x}} = \mathbf{u}^\nu(\tilde{\mathbf{x}},t)dt + \sqrt{2\lambda}d\tilde{\mathbf{W}}(t), \quad \tilde{\mathbf{x}}(t_0) = \mathbf{x}_0$$

where ν is a viscosity which smooths the velocity. What happens as $\lambda \to 0$ with Prandtl number $Pr = \nu/\lambda$ fixed?

At least in the Kazantsev-Kraichnan kinematic dynamo model there is a nontrivial limiting distribution $P_{\mathbf{u}}(\mathbf{x},t|\mathbf{x}_0,t_0)$ over an infinite family of solutions to the (deterministic) initial-value problem $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x},t), \ \mathbf{x}(t_0) = \mathbf{x}_0$.

There is an obvious analogy with *spontaneous symmetry-breaking*, e.g. a non-vanishing mean-magnetization in a ferromagnet even in the limit of zero external magnetic field.

See Falkovich et al. Rev. Mod. Phys. (2001), Section II.C

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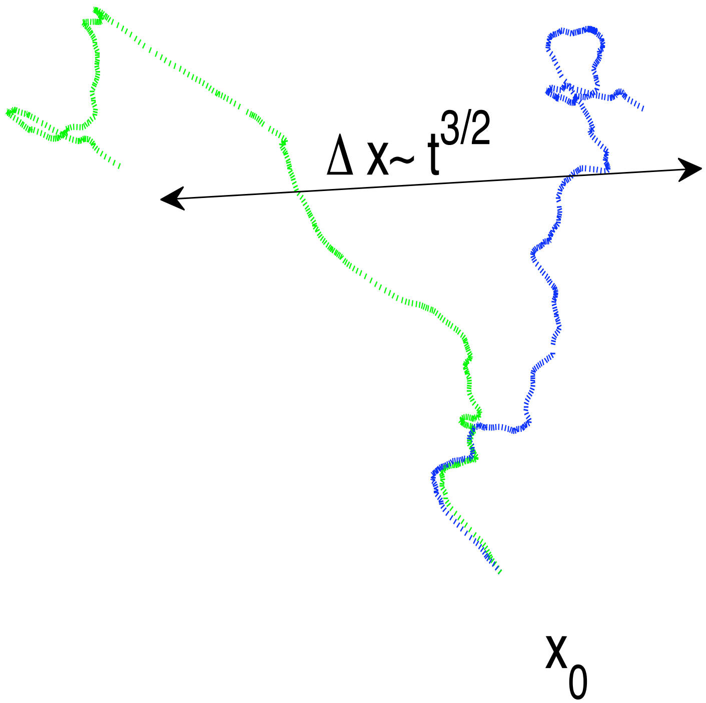

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The distribution does not collapse!

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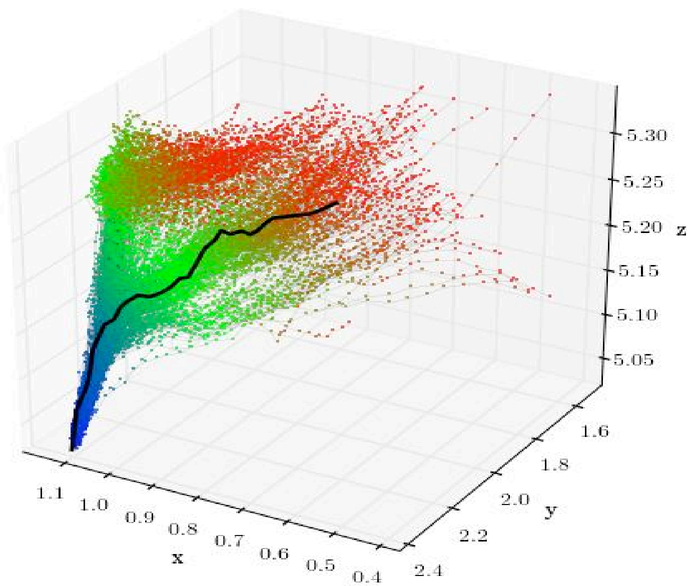
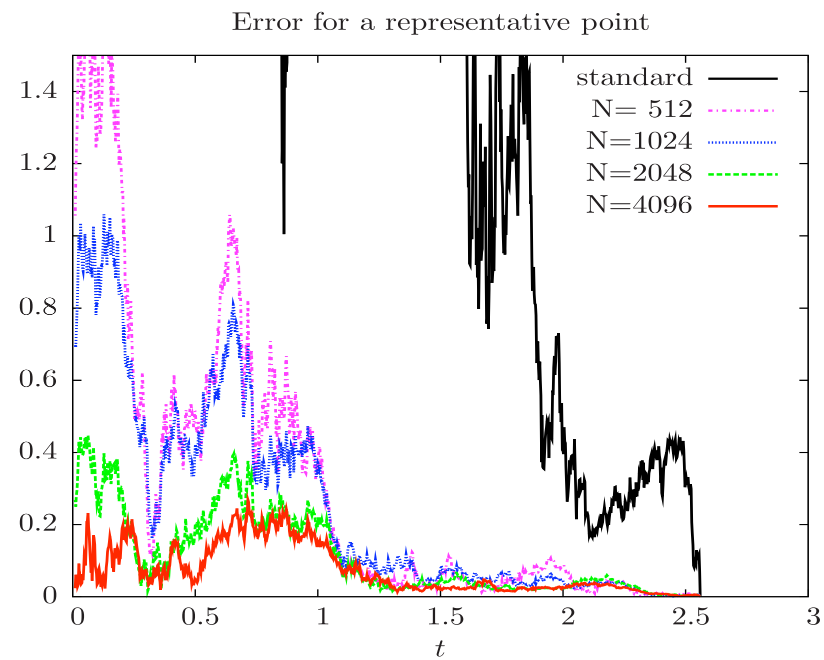
Stochastic Flux-Freezing in MHD Turbulence



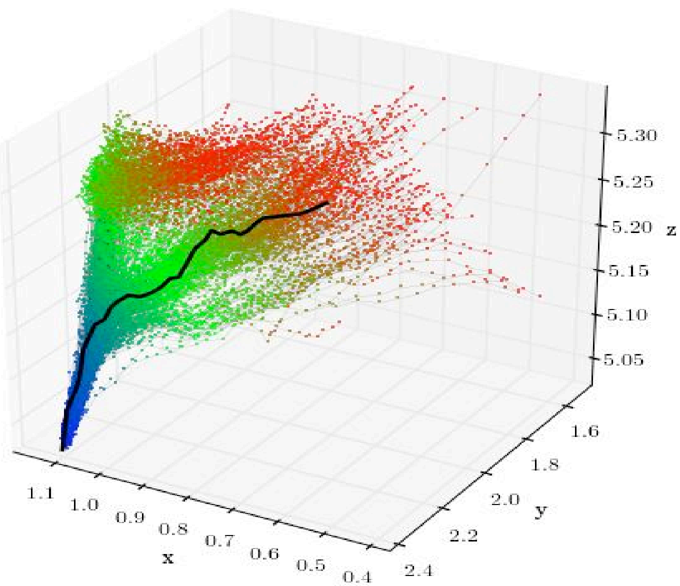


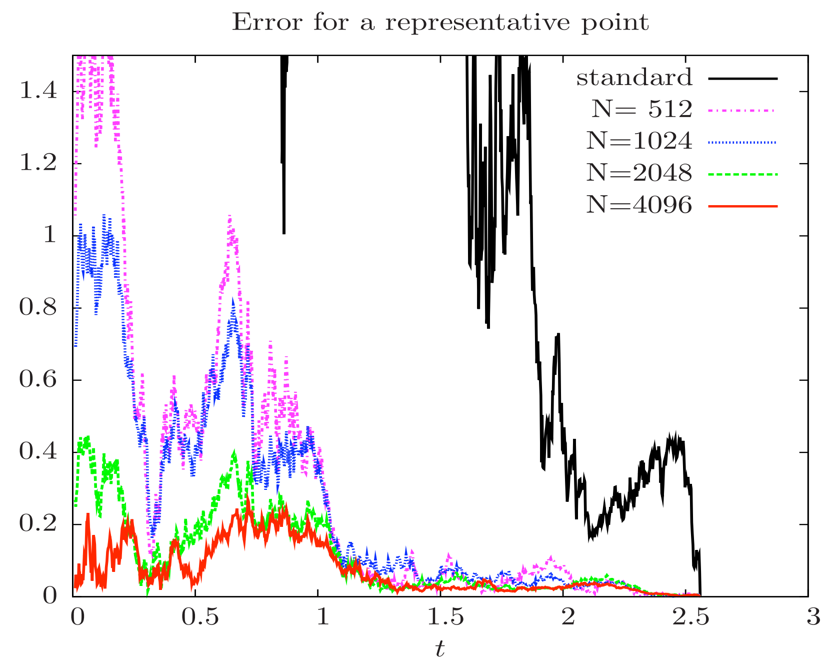
The stochastic flux-freezing theorem is well-satisfied in the MHD database. Despite the fact that the conductivity (or magnetic Reynolds number) is high, standard flux-freezing is not even approximately valid. (G. E. et al., *Nature*, 2013)

START

Shown is the point $\mathbf{x}_f = (1.09, 2.42, 5.03)$ in the archived MHD flow at time $t_f = 2.56$. To calculate the magnetic field $\mathbf{B}(\mathbf{x}_f, t_f)$, we generate the stochastic trajectories that arrive at $\mathbf{x}_f$ at time t_f by integrating $d\tilde{\mathbf{x}} = \mathbf{u}(\tilde{\mathbf{x}}, t)dt + \sqrt{2\lambda}d\tilde{\mathbf{W}}(t)$ from $\mathbf{x}_f$ at time t_f back to time t_0. The particles change color from **blue**, to **green**, to **red** as they go from time t_f to time t_0. The magnetic field vectors at time t_0 are then sampled from the database at the random ensemble of points $\tilde{\mathbf{x}}(t_0)$.

Stochastic Flux-Freezing in MHD Turbulence





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Does Richardson Dispersion Exist in Homogeneous MHD Turbulence?

Unlike in hydrodynamic turbulence, there are expected to be a significant effects of the Lorentz force in nonlinear MHD turbulence. Particle separations will be different parallel and perpendicular to the magnetic field. The laws of 2-particle dispersion will depend upon the theory of MHD turbulence. Assuming the generalized Boldyrev (2005) scaling

$$\delta u(r_\perp) \sim v_A M_A^{\frac{2(2+\alpha)}{3+\alpha}} \left(\frac{r_\perp}{L_f}\right)^{\frac{1}{3+\alpha}},$$

with $M_A = b_{rms}/\bar{B}$, one obtains from $dr_\perp/dt \sim \delta u(r_\perp)$ the **Richardson-type law**

$$\langle r_\perp^2(t)\rangle \sim L_f^2 M_A^4 \left(\frac{v_A t}{L_f}\right)^{\frac{2(3+\alpha)}{2+\alpha}},$$

for the transverse slippage of magnetic field lines in MHD turbulence.

Richardson dispersion has **not** *yet been observed in MHD simulations, despite prior attempts:*

A. Busse et al., "Statistics of passive tracers in three-dimensional magnetohydrodynamic turbulence," Phys. Plasmas **14** 122303 (2007)

A. Busse and W.-C. Müller, "Diffusion and dispersion in magnetohydrodynamic turbulence: The influence of mean magnetic fields," Astron. Nachr. **329** 714 (2008)

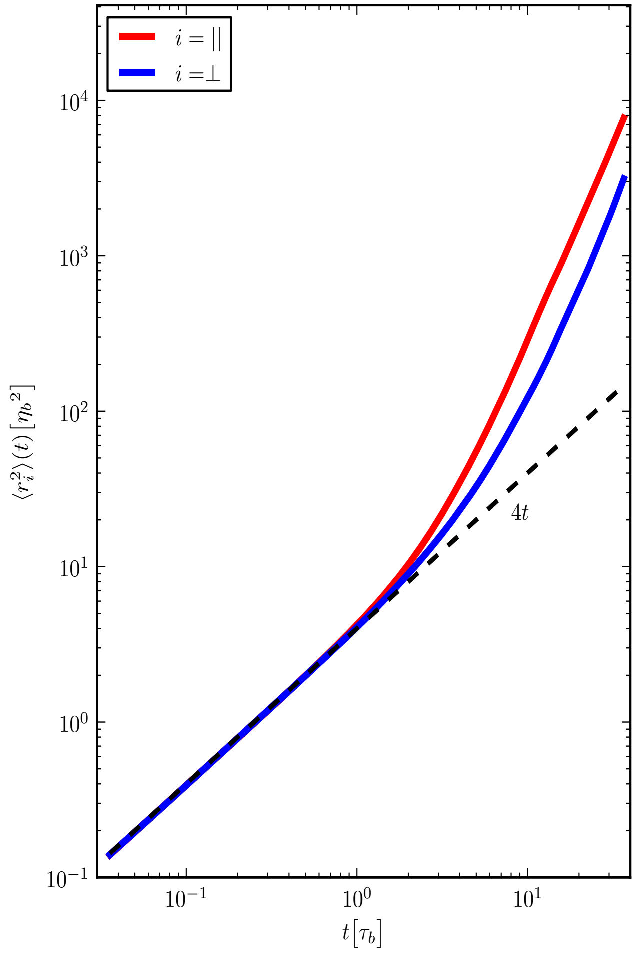
Richardson Dispersion of Field-Lines

Field-lines disperse through the plasma faster along the direction of the local mean magnetic field than they disperse perpendicular to the field, but the growth is the same power

$$\langle r_i^2(t)\rangle \sim L_f^2 M_A^4 \left(\frac{v_A t}{L_f}\right)^{\frac{8}{3}},$$

in both directions, $i = \|, \perp$ .This growth-law is consistent with the $-3/2$ energy spectra of the database flow, or $h = 1/4$ scaling exponent of velocity and magnetic fields.

The standard diffusive estimate $\sim 4\lambda t$, proportional to microscopic plasma resistivity, is valid for only about one resistive time!



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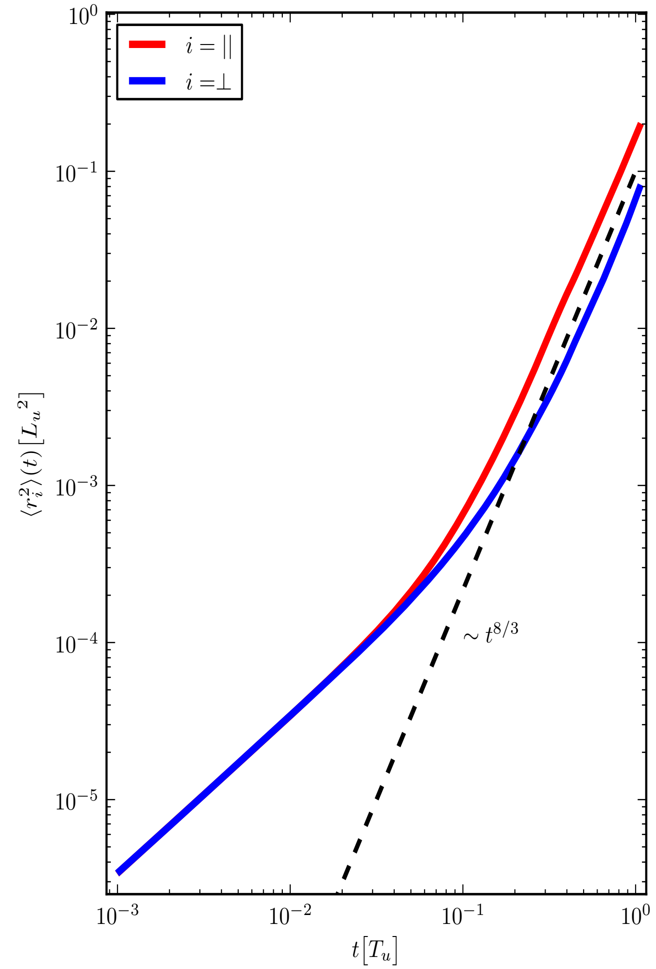

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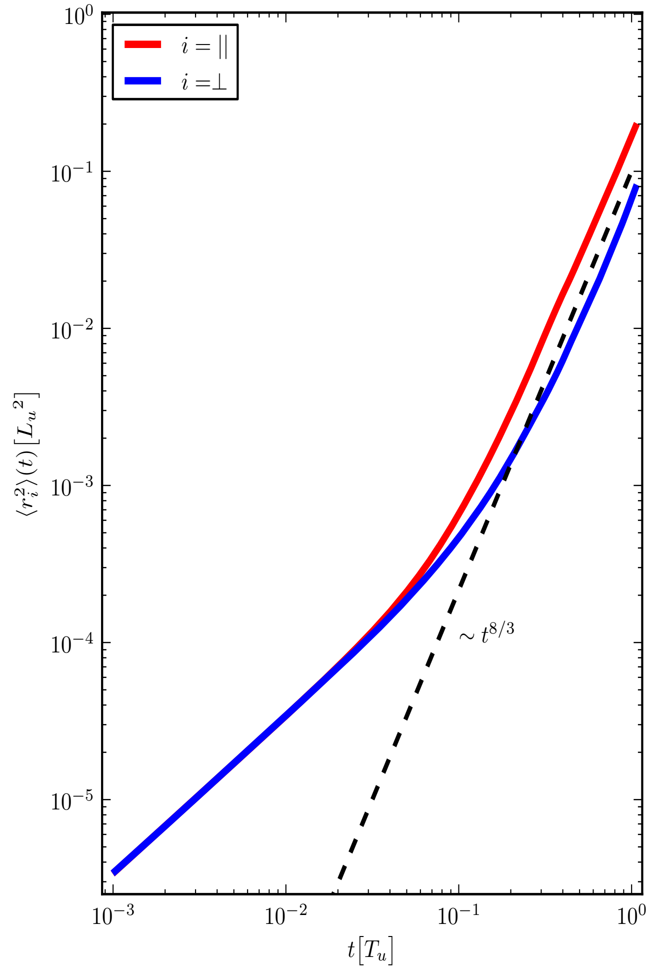
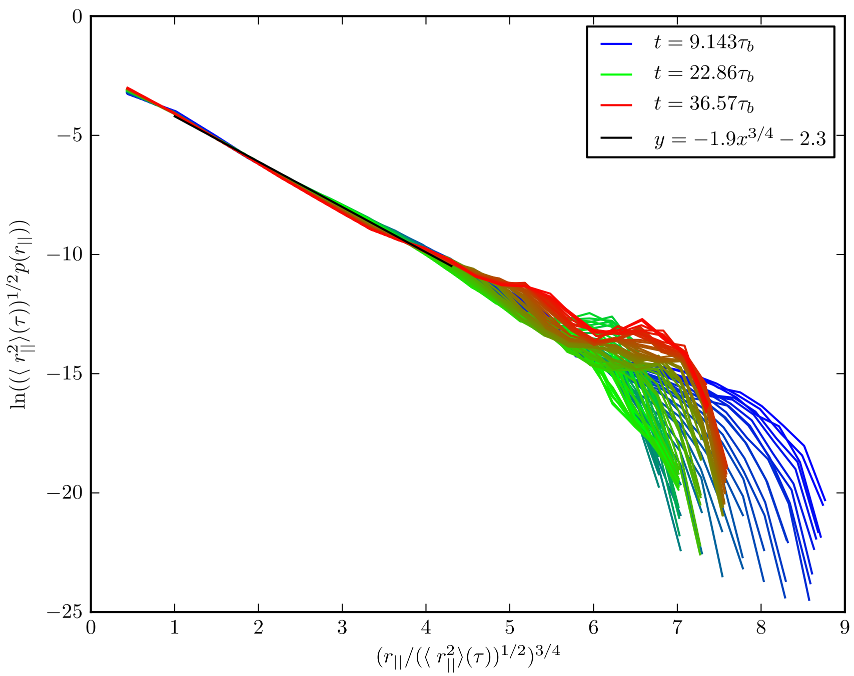
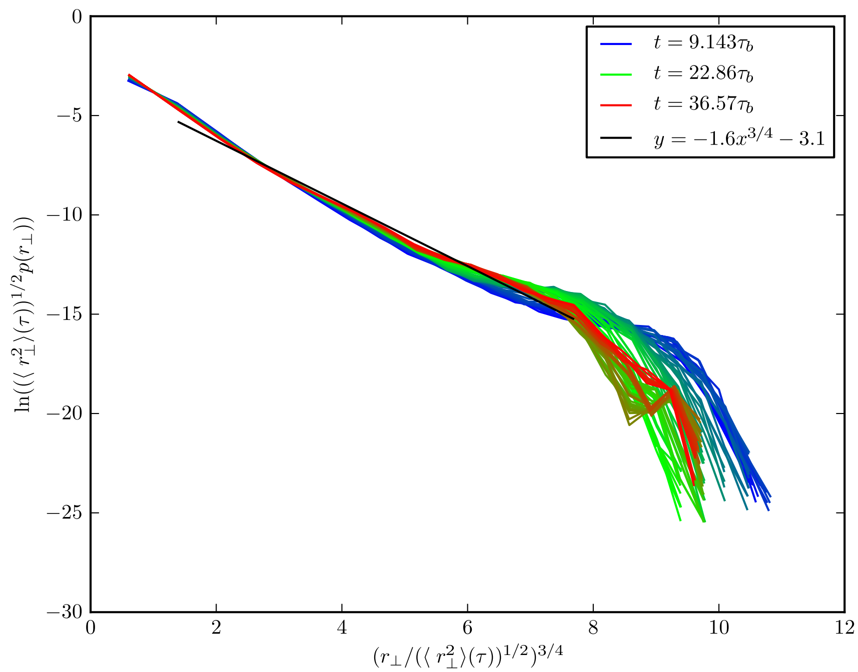
What you learned in the textbooks about magnetic flux-freezing for high-conductivity MHD plasmas is wrong.



Further Evidence of Richardson Dispersion

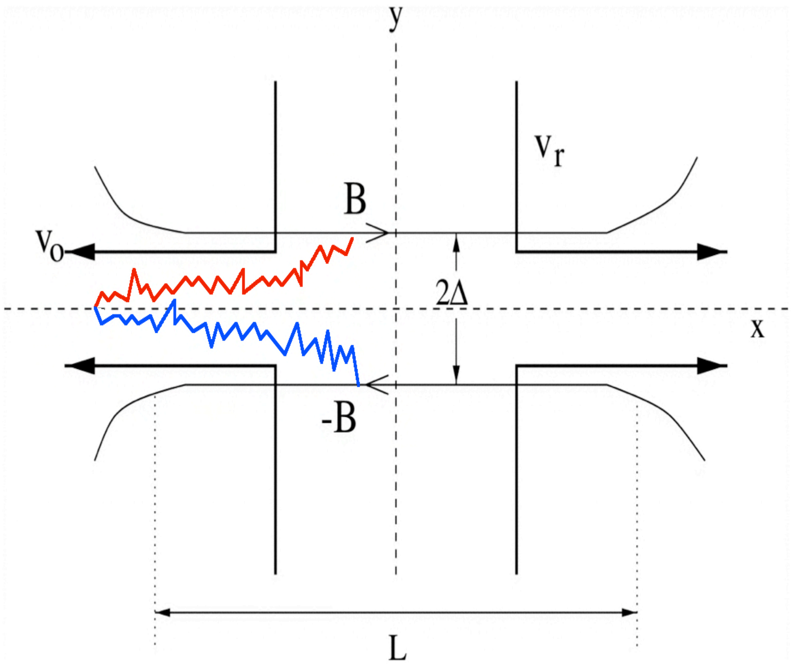
Shown are PDFs of separation distances of field-lines, $r_{\|}$ parallel and $r_{\perp}$ perpendicular to the local mean magnetic field. As expected from Richardson's theory, the PDF's are self-similar in time and have stretched-exponential form, roughly $P(r) \propto \exp(-Cr^{3/4})$ for $h = 1/4$.





Turbulent Magnetic Reconnection

Assume that the reconnection occurs in a background MHD plasma turbulence with rms velocity $u_L < v_A$ and integral length or injection scale $L_f > L$.



Richardson diffusion of field-lines gives

$$\Delta \simeq \sqrt{\langle r_\perp^2(t_A)\rangle} \sim L_f M_A^2 \left(\frac{v_A t_A}{L_f}\right)^{\frac{3+\alpha}{2+\alpha}},$$

with $t_A = L/v_A$ the Alfvén crossing time and $M_A = b_{rms}/\bar{B}$. Mass conservation $v_R L = v_A \Delta$ with $v_0 \simeq v_A$ yields

$$v_R = v_A M_A^2 (L/L_f)^{\frac{1}{2+\alpha}}$$

Now **Lazarian-Vishniac (1999)** theory is obtained, for the case $\alpha = 0$. The reconnection rate is *independent of resistivity*!

Estimating for solar flares that $L_f \simeq L$ and $M_A \simeq 0.1$ (Bemporad, 2008) one obtains a release time of about one hour.

For more details, see Eyink, Lazarian & Vishniac, ApJ. **743** 1-28 (2011)

Selected References

Stochastic Flux-Freezing

G. L. Eyink, Stochastic flux-freezing and magnetic dynamo, Phys. Rev. E. **83** 056405 (2011)

G. L. Eyink, A. Lazarian & E. T. Vishniac, Fast magnetic reconnection and spontaneous stochasticity, Astrophys. J., **743** 1-28 (2011)

G. L. Eyink et al. Flux-freezing breakdown in high-conductivity magnetohydrodynamic turbulence, Nature, **497** 466-469 (2013)

JHU Turbulence Database

Y. Li et al., A public turbulence database cluster and applications to study Lagrangian evolution of velocity increments in turbulence, J. Turbulence, **9**, 31: 1-29 (2008)

H. Yu et al., Studying Lagrangian dynamics of turbulence using on-demand fluid particle tracking in a public turbulence database, J. Turbulence, **13**, 12: 1-29 (2012)

http://turbulence.pha.jhu.edu