Constrained Transport Methods for the 3D Ideal Magnetohydrodynamic Equations

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The MHD equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\mathbf{u} \\ \mathcal{E} \\ \mathbf{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \left(p + \frac{1}{2}\|\mathbf{B}\|^2\right)\mathbb{I} - \mathbf{B}\mathbf{B} \\ \mathbf{u}\left(\mathcal{E} + p + \frac{1}{2}\|\mathbf{B}\|^2\right) - \mathbf{B}\left(\mathbf{u}\cdot\mathbf{B}\right) \\ \mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}\end{bmatrix} = 0$$

$$\nabla\cdot\mathbf{B} = 0,$$

ρ, $\rho\mathbf{u}$, and $\mathcal{E}$ are the total mass, momentum, and energy densities of the plasma system, and $\mathbf{B}$ is the magnetic field

EOS:

$$p = (\gamma - 1)\left(\mathcal{E} - \frac{1}{2}\|\mathbf{B}\|^2 - \frac{1}{2}\rho\|\mathbf{u}\|^2\right),$$

where $\gamma = 5/3$ is the ideal gas constant

The MHD equations

The equation for the magnetic field comes from **Faraday's law**:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$

where the electric field, $\mathbf{E}$, is approximated by **Ohm's law** for a perfect conductor:

$$\mathbf{E} = \mathbf{B} \times \mathbf{u}.$$

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = \partial_t \mathbf{B} + \nabla \cdot (\mathbf{uB} - \mathbf{Bu}) = 0$$

Since the electric field is determined entirely from Ohm's law, we do not require an evolution equation for it; and thus, the only other piece that we need from Maxwell's equations is the divergence-free condition on the magnetic field.

The MHD equations

Remark: *If* $\nabla \cdot \mathbf{B} = 0$ *is true at some time* $t = T$*, then the evolution equation guarantees that* $\nabla \cdot \mathbf{B} = 0$ *is true for all time.*

(take the divergence of Faraday's law)

For this reason $\nabla \cdot \mathbf{B} = 0$ should not be regarded as *constrained* (such as $\nabla \cdot \mathbf{u} = 0$ for the Navier-Stokes equation), but rather an *involution*.

Outline of the talk

Discretization of $\nabla \cdot \mathbf{B} = 0$:

- Projection methods and divergence cleaning methods
- **Constrained transport methods**

  Numerical difficulties:

  - weak hyperbolicity of the magnetic vector potential equation
  - limiting of the magnetic potential

Discretization of $\nabla \cdot \mathbf{B} = 0$

Constrained transport:

Evans and Hawley, 1988

step 1: take a time step using some finite volume method which produces cell average values $(\rho^{n+1}, \rho\mathbf{u}^{n+1}, \mathcal{E}^{n+1}, \mathbf{B}^*)$

step 2: Using the ideal Ohm's law relationship, $\mathbf{E} = \mathbf{B} \times \mathbf{u}$, and some space and time interpolation scheme for $\mathbf{B}$ and $\mathbf{u}$, reconstruct a space and time staggered electric field value $\mathbf{E}^{n+\frac{1}{2}}$

step 3: compute the corrected magnetic field value

$$\mathbf{B}^{n+1} = \mathbf{B}^n - \Delta t \nabla \times \mathbf{E}^{n+\frac{1}{2}}$$

Discretization of $\nabla \cdot \mathbf{B} = 0$

Alternative formulation of step 3:

step 1: take a time step using some finite volume method which produces cell average values $(\rho^{n+1}, \rho\mathbf{u}^{n+1}, \mathcal{E}^{n+1}, \mathbf{B}^{*})$

step 2: Using the ideal Ohm's law relationship, $\mathbf{E} = \mathbf{B} \times \mathbf{u}$, and some space and time interpolation scheme for $\mathbf{B}$ and $\mathbf{u}$, reconstruct a space and time staggered electric field value $\mathbf{E}^{n+\frac{1}{2}}$

step 3a: Produce the magnetic potential value, $\mathbf{A}^{n+1}$, from the induction equation written in potential form

$$\mathbf{A}^{n+1} = \mathbf{A}^{n} - \Delta t \mathbf{E}^{n+\frac{1}{2}}$$

step 3b: Compute

$$\mathbf{B}^{n+1} = \nabla \times \mathbf{A}^{n+1}$$

Unstaggered Constrained transport methods

Consider the induction equation

$$\mathbf{B}_t + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$$

and assume for the moment that $\mathbf{u}$ is a given vector valued function.
Set $\mathbf{B} = \nabla \times \mathbf{A}$ to obtain

$$\nabla \times (\mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u}) = 0$$
$$\Rightarrow \qquad \mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u} = -\nabla\psi$$

where ψ is an arbitrary scalar function.
Different choices of ψ represent different *gauge condition* choices.

Unstaggered Constrained transport methods

$$\mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u} = -\nabla \psi$$

The 2d case (e.g., in the x-y plane) is much simpler:

The only component of $\mathbf{A}$ that influences the evolution is A^3 (i.e., the component of the potential that is perpendicular to the evolution plane).
All gauge choices lead to the same equation:

$$A^3_t + u^1 A^3_x + u^2 A^3_y = 0$$

Ref.: Rossmanith, SISC 2006

The Weyl gauge

The **Weyl gauge:** $\psi = 0$

$$\mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u} = \mathbf{0}$$

which can be written in the form

$$\mathbf{A}_t + N_1(\mathbf{u})\mathbf{A}_x + N_2(\mathbf{u})\mathbf{A}_y + N_3(\mathbf{u})\mathbf{A}_z = \mathbf{0}$$

$$N_1 = \begin{bmatrix} 0 & -u^2 & -u^3 \\ 0 & u^1 & 0 \\ 0 & 0 & u^1 \end{bmatrix}, \; N_2 = \begin{bmatrix} u^2 & 0 & 0 \\ -u^1 & 0 & -u^3 \\ 0 & 0 & u^2 \end{bmatrix}, \; N_3 = \begin{bmatrix} u^3 & 0 & 0 \\ 0 & u^3 & 0 \\ -u^1 & -u^2 & 0 \end{bmatrix}$$

The Weyl gauge

Flux Jacobian in direction $\mathbf{n} \in S^2$

$$n^1 N_1 + n^2 N_2 + n^3 N_3 = \begin{bmatrix} n^2u^2 + n^3u^3 & -n^1u^2 & -n^1u^3 \\ -n^2u^1 & n^1u^1 + n^3u^3 & -n^2u^3 \\ -n^3u^1 & -n^3u^2 & n^1u^1 + n^2u^2 \end{bmatrix}$$

eigenvalues and eigenvectors:

$$\lambda = \{0, \mathbf{n}\cdot\mathbf{u}, \mathbf{n}\cdot\mathbf{u}\}$$

$$R = \left[\mathbf{r}^{(1)} \middle| \mathbf{r}^{(2)} \middle| \mathbf{r}^{(3)}\right] = \begin{bmatrix} n^1 & n^2u^3 - n^3u^2 & u^1(\mathbf{u}\cdot\mathbf{n}) - n^1\|\mathbf{u}\|^2 \\ n^2 & n^3u^1 - n^1u^3 & u^2(\mathbf{u}\cdot\mathbf{n}) - n^2\|\mathbf{u}\|^2 \\ n^3 & n^1u^2 - n^2u^1 & u^3(\mathbf{u}\cdot\mathbf{n}) - n^3\|\mathbf{u}\|^2 \end{bmatrix}$$

for $\|\mathbf{u}\| \neq 0$, $\mathbf{n} \in S^2$ we get

$$\det(R) = -\|\mathbf{u}\|^3 \cos(\alpha) \sin^2(\alpha),$$

where α is the angle between the vectors $\mathbf{n}$ and $\mathbf{u}$

$\Rightarrow$ The system is weakly hyperbolic.

Example of a weakly hyperbolic system

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} -\varepsilon & 1 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0, \quad \varepsilon \in \mathbb{R}$$

$$\begin{bmatrix} -\varepsilon & 1 \\ 0 & \varepsilon \end{bmatrix} = R\Lambda R^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2\varepsilon \end{bmatrix} \cdot \begin{bmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} \cdot \frac{1}{2\varepsilon} \begin{bmatrix} 2\varepsilon & -1 \\ 0 & 1 \end{bmatrix}.$$

Exact solution for the Cauchy problem for all ε:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_0(x+\varepsilon t) - \frac{1}{2\varepsilon}\left\{v_0(x+\varepsilon t) - v_0(x-\varepsilon t)\right\} \\ v_0(x-\varepsilon t) \end{bmatrix}.$$

In the weakly hyperbolic limit we obtain:

$$\lim_{\varepsilon \to 0} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_0(x) - t\, v_0'(x) \\ v_0(x) \end{bmatrix},$$

i.e. the amplitude of the solution grows linearly in time.

Discretization of the vector potential equation

We have constructed methods for the weakly hyperbolic vector potential equation which are based on

- An operator splitting approach
- The idea of path conservative methods

Operator splitting

First splitting approach:

Sub-problem 1:

$$\begin{aligned} \mathbf{A}_t^1 + u^2\mathbf{A}_y^1 + u^3\mathbf{A}_z^1 &= 0, \\ \mathbf{A}_t^2 - u^1\mathbf{A}_y^1 &= 0, \\ \mathbf{A}_t^3 - u^1\mathbf{A}_z^1 &= 0, \end{aligned}$$

Sub-problem 2:

$$\begin{aligned} \mathbf{A}_t^1 - u^2\mathbf{A}_x^2 &= 0, \\ \mathbf{A}_t^2 + u^1\mathbf{A}_x^2 + u^3\mathbf{A}_z^2 &= 0, \\ \mathbf{A}_t^3 - u^2\mathbf{A}_z^2 &= 0, \end{aligned}$$

Sub-problem 3:

$$\begin{aligned} \mathbf{A}_t^1 - u^3\mathbf{A}_x^3 &= 0, \\ \mathbf{A}_t^2 - u^3\mathbf{A}_y^3 &= 0, \\ \mathbf{A}_t^3 + u^1\mathbf{A}_y^3 + u^2\mathbf{A}_x^3 &= 0. \end{aligned}$$

Operator splitting

Second splitting approach:

Sub-problem 1:

$$\begin{aligned}
&\mathbf{A}_t^1 - u^2\mathbf{A}_x^2 - u^3\mathbf{A}_x^3 = 0,\\
&\mathbf{A}_t^2 + u^1\mathbf{A}_x^2 = 0,\\
&\mathbf{A}_t^3 + u^1\mathbf{A}_x^3 = 0,
\end{aligned}$$

Sub-problem 2:

$$\begin{aligned}
&\mathbf{A}_t^1 + u^2\mathbf{A}_y^1 = 0,\\
&\mathbf{A}_t^2 - u^1\mathbf{A}_y^1 - u^3\mathbf{A}_y^3 = 0,\\
&\mathbf{A}_t^3 + u^2\mathbf{A}_y^3 = 0,
\end{aligned}$$

Sub-problem 3:

$$\begin{aligned}
&\mathbf{A}_t^1 + u^3\mathbf{A}_z^1 = 0,\\
&\mathbf{A}_t^2 + u^3\mathbf{A}_z^2 = 0,\\
&\mathbf{A}_t^3 - u^1\mathbf{A}_z^1 - u^2\mathbf{A}_z^2 = 0.
\end{aligned}$$

The 2.5 dimensional problem - a useful test

$\mathbf{u}, \mathbf{B} \in \mathbb{R}^3$, but all conserved quantities are functions of two spatial variables $\mathbf{x} = (x, y)^t$.

1st approach: update $\mathbf{A}^3$ as in 2d case and set $B^1 = \mathbf{A}^3_y$, $B^2 = -\mathbf{A}^3_x$; use B^3 from the base scheme;

2nd approach: solve

$$
\begin{aligned}
\mathbf{A}^1_t - u^2\mathbf{A}^2_x - u^3\mathbf{A}^3_x + u^2\mathbf{A}^1_y &= 0, \\
\mathbf{A}^2_t + u^1\mathbf{A}^2_x - u^1\mathbf{A}^1_y - u^3\mathbf{A}^3_y &= 0, \\
\mathbf{A}^3_t + u^1\mathbf{A}^3_x + u^2\mathbf{A}^3_y &= 0.
\end{aligned}
$$

and update $B^1 = \mathbf{A}^3_y$, $B^2 = -\mathbf{A}^3_x$, $B^3 = \mathbf{A}^2_x - \mathbf{A}^1_y$

Test computations: cloud-shock interaction

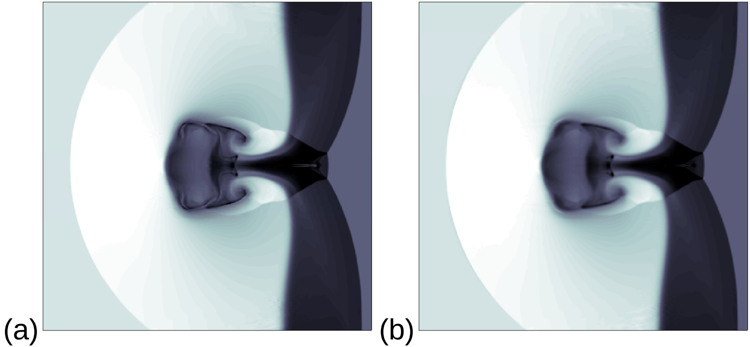


Figure: The 2.5-dimensional cloud-shock interaction problem. Shown here are the out-of-plane magnetic field at time $t = 0.06$ as calculated on a 512$\times$512 mesh using (a) a 2d approach that only uses $\mathbf{A}^3$, and (b) the proposed scheme using the full vector potential $\mathbf{A}$.

Unsplit discretization of weakly hyperbolic system

Consider 1d weakly hyperbolic system

$$q_t + A(x) q_x = 0$$

$\tilde{q}$, $\tilde{A}$: piecewise polynomial reconstructions

$$q^+_{i-\frac{1}{2}} := \lim_{\varepsilon \to 0} \tilde{q}_i(x_{i-\frac{1}{2}} + \varepsilon), \quad q^-_{i+\frac{1}{2}} := \lim_{\varepsilon \to 0} \tilde{q}_i(x_{i+\frac{1}{2}} - \varepsilon)$$

$$A^+_{i-\frac{1}{2}} := \lim_{\varepsilon \to 0} \tilde{A}_i(x_{i-\frac{1}{2}} + \varepsilon), \quad A^-_{i+\frac{1}{2}} := \lim_{\varepsilon \to 0} \tilde{A}_i(x_{i+\frac{1}{2}} - \varepsilon)$$

Unsplit discretization of weakly hyperbolic system

Semi-discrete form of the method:

$$Q_i'(t) = -\frac{1}{\Delta x}\left(\mathcal{A}^-\Delta q_{i+\frac{1}{2}} + \mathcal{A}^+\Delta q_{i-\frac{1}{2}} + \mathcal{A}\Delta q_i\right)$$

with

$$\mathcal{A}\Delta q_i := \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{A}_i(x)(\tilde{q}_i)_x dx \approx \lim_{\varepsilon\to 0}\int_{x_{i-\frac{1}{2}}+\varepsilon}^{x_{i+\frac{1}{2}}-\varepsilon} A(x) q_x dx$$

$$\mathcal{A}^+\Delta q_{i-\frac{1}{2}} \approx \lim_{\varepsilon\to 0}\int_{x_{i-\frac{1}{2}}}^{x_{i-\frac{1}{2}}+\varepsilon} A(x) q_x dx$$

$$\mathcal{A}^-\Delta q_{i+\frac{1}{2}} \approx \lim_{\varepsilon\to 0}\int_{x_{i+\frac{1}{2}}-\varepsilon}^{x_{i+\frac{1}{2}}} A(x) q_x dx$$

Unsplit discretization of weakly hyperbolic system

Introduce a regularization of q at each grid cell interface

$$q^{\varepsilon}_{i-\frac{1}{2}}(t,x) = \begin{cases} q^{-}_{i-\frac{1}{2}}(t) & : \quad x \le x_{i-\frac{1}{2}} - \varepsilon, \\ \Psi_{i-\frac{1}{2}}\left(\frac{x - x_{i-\frac{1}{2}} + \varepsilon}{2\varepsilon}, t\right) & : \quad x \in \left(x_{i-\frac{1}{2}} - \varepsilon, x_{i-\frac{1}{2}} + \epsilon\right), \\ q^{+}_{i-\frac{1}{2}}(t) & : \quad x \ge x_{i-\frac{1}{2}} + \varepsilon, \end{cases}$$

straight-line path: $\Psi_{i-\frac{1}{2}} = q^{-}_{i-\frac{1}{2}} + s\left(q^{+}_{i-\frac{1}{2}} - q^{-}_{i-\frac{1}{2}}\right)$, $s \in (0,1)$

leads to

$$\mathcal{A}^{-}\Delta q_{i-\frac{1}{2}} + \mathcal{A}^{+}\Delta q_{i-\frac{1}{2}} = A\big|_{\Psi_{i-\frac{1}{2}}}(q^{+}_{i-\frac{1}{2}} - q^{-}_{i-\frac{1}{2}})$$

with $A\big|_{\Psi_{i-\frac{1}{2}}} := \frac{1}{2}A^{-}_{i-\frac{1}{2}} + \frac{1}{2}A^{+}_{i-\frac{1}{2}}$

Unsplit discretization of weakly hyperbolic system

Definition of the fluctuations using generalized Rusanov flux:

$$\mathcal{A}^{-}\Delta q_{i+1/2} = \frac{1}{2}\left[A\big|_{\Psi_{i+1/2}} - \alpha_{i+1/2}\,\mathbb{I}\right]\left(q^{+}_{i+\frac{1}{2}} - q^{-}_{i+\frac{1}{2}}\right)$$

and

$$\mathcal{A}^{+}\Delta q_{i-1/2} = \frac{1}{2}\left[A\big|_{\Psi_{i-1/2}} + \alpha_{i-1/2}\,\mathbb{I}\right]\left(q^{+}_{i-\frac{1}{2}} - q^{-}_{i-\frac{1}{2}}\right),$$

with $|\lambda^k| \leq \alpha,$ for $k = 1, \ldots, m$, λ^k is eigenvalue of $A\big|_{\Psi}$

Extension to 2d and 3d is straight forward.

Outline of the method using a MOL approach

Stage 1. Start with Q^n_{MHD} and Q^n_A, then update via:

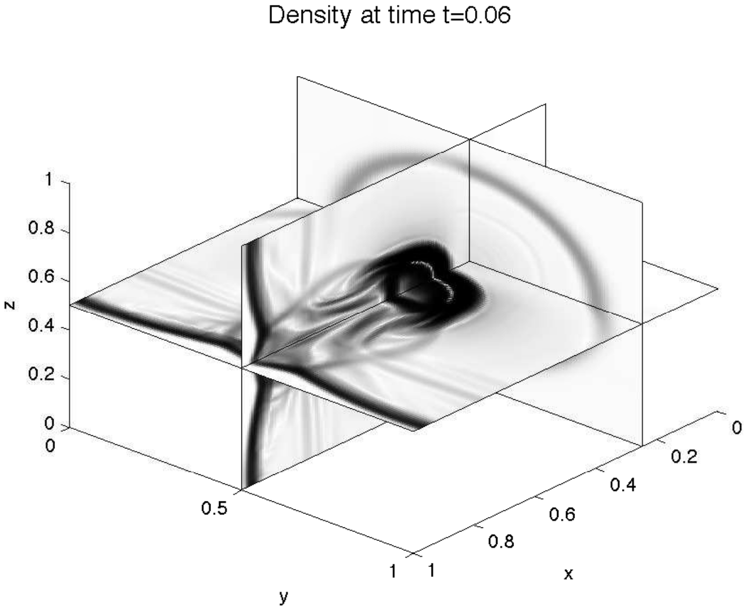
$$Q^{(1\star)}_{MHD} = Q^n_{MHD} + \Delta t\, \mathcal{L}_1\left(Q^n_{MHD}\right),$$
$$Q^{(1)}_{A} = Q^n_{A} + \Delta t\, \mathcal{L}_2\left(Q^n_{A}, Q^n_{MHD}\right),$$

where $Q^{(1\star)}_{MHD} = \left(\rho^{(1)}, \rho\mathbf{u}^{(1)}, \mathcal{E}^{(1)}, \mathbf{B}^{(1\star)}\right)$ and $\mathbf{B}^{(1\star)}$ denotes the *predicted* value of the magnetic field in the first Runge-Kutta stage. The magnetic field components of $Q^{(1\star)}_{MHD}$ are then *corrected* by $\nabla \times Q^{(1)}_A$; we denote this result by $Q^{(1)}_{MHD} = \left(\rho^{(1)}, \rho\mathbf{u}^{(1)}, \mathcal{E}^{(1)}, \mathbf{B}^{(1)}\right)$.

Continue with Stage 2 and Stage 3.

3rd order accurate for smooth Alfvén wave problem.

Cloud-shock interaction problem



References

- J.A. Rossmanith, *An unstaggered, high-resolution constrained transport method for magnetohydrodynamic flows*, SISC, 28:1766-1797, 2006.

- C. Helzel, J.A. Rossmanith, B. Taetz, *An unstaggered constrained transport method for the 3d ideal magnetohydrodynamic equations*, JCP, 230:3803-3829, 2011.

- C. Helzel, J.A. Rossmanith, B. Taetz, *A high-order unstaggered constrained transport method for the 3d ideal magnetohydrodynamic equations based on the method of lines*, SISC in press.