Structure preserving schemes

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Joint work with S. Mishra

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Outline

- **Introduction**
  - Astrophysical motivation: core-collapse SN
- **Structure Preserving schemes**
  - Structures?
  - Well-balanced schemes
  - Angular momentum conserving schemes
- **Conclusions**

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Core-collapse supernova

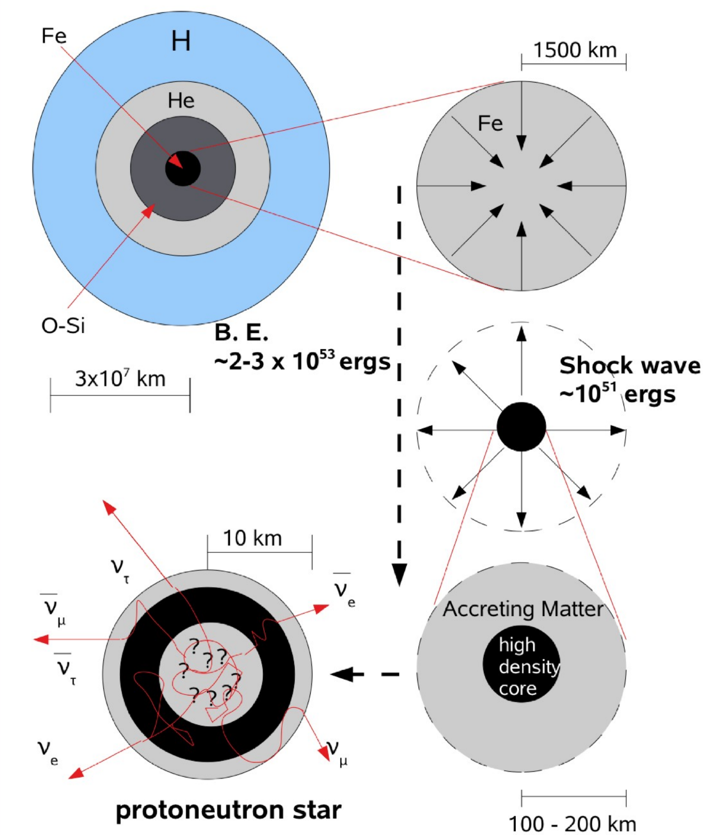
- General idea:
  - Implosion of iron core of massive $M \gtrsim 8 M_\odot$ at the end of thermonuclear evolution
  - Explosion powered by gravitational binding energy of forming compact remnant:

$$E_b \approx 3 \times 10^{53} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{10\mathrm{km}}\right)^{-1} \mathrm{erg}$$

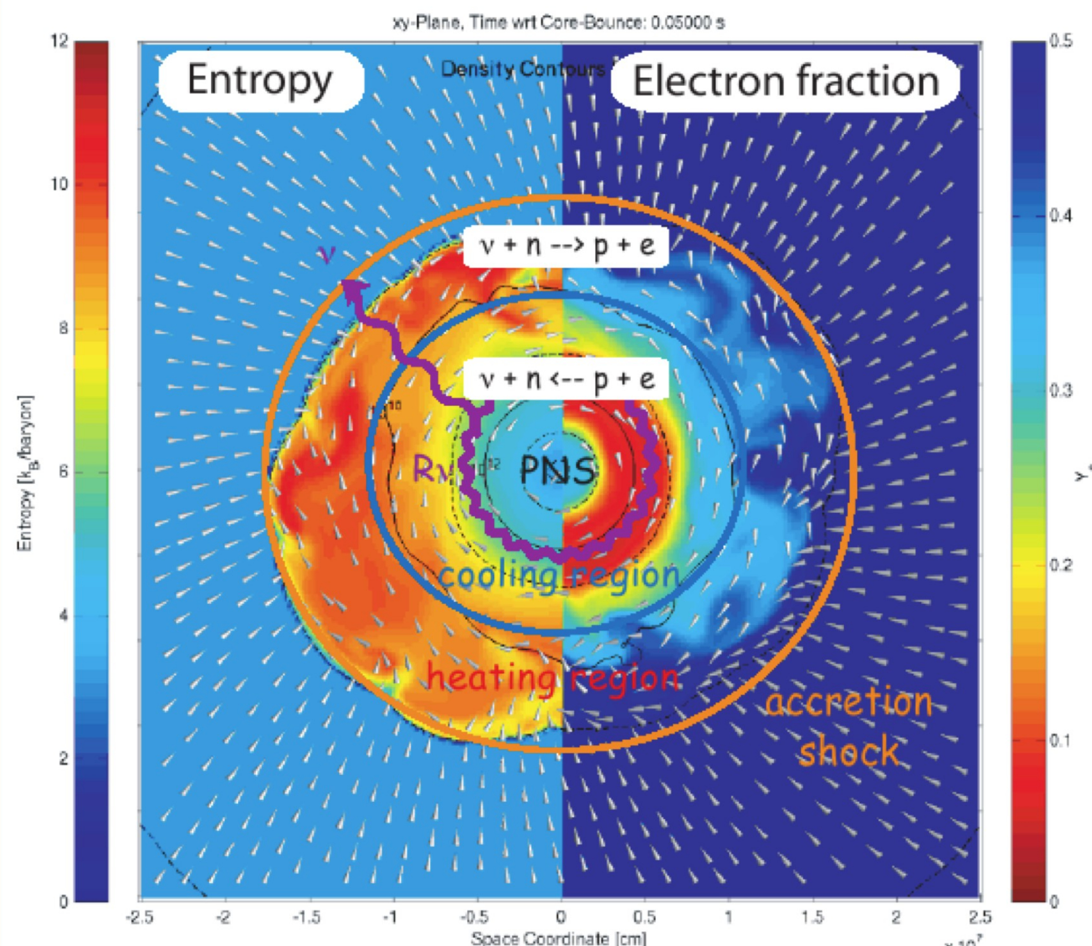
GRAVITY BOMB!

M Mass of remnant
R Radius of remnant

Core-collapse supernova



From S. Scheidegger



From M. Liebendoerfer

CCSN Explosion Mechanism?

- Discussed explosion mechanisms:
  - "Enhanced" neutrino-driven explosion mechanism

    Hydro. instabilities: convection, Standing Accretion Shock Instabilities (SASI) e.g. Blondin et al. 2003, Blondin & Shaw 2007, Foglizzo et al. 2008, Iwakami et al. 2008, Marek & Janka 2009, ...
  - MHD mechanism

    Rapid rotation + Magnetic field amplification (Flux compression, winding, MRI, dynamos) e.g. Akiyama et al. 2003, Wilson et al. 2005, Kotake et al. 2006, Burrows et al. 2007, Winteler et al. 2012, ...
  - Acoustic mechanism

    Excitation of ProtoNeutron Star (PNS) oscillations by accretion/SASI generating acoustic power to reheat the stalled shock Burrows et al. 2006,2007
  - Phase transition induced explosion mechanism

    Additional compactification of PNS due to phase transition from hadronic matter to quark matter Migdal et al. 1971, … Sagert, Fischer et al. 2009, Fischer et al. 2011, ...

(MHD) CCSN model

Actual model's ingredients list:

1) Multi-D hydro. →
2) Plasma physics →

Parallel 3D ideal MHD code (Assume infinite conductivity)
Käppeli et al. 2011

3) Weak interactions →
4) Neutrino transport →

Spectral leakage scheme
developed by A. Perego
Rosswog & Liebendörfer 2003
"Not so bad"... 2D simulations shown that ν contribute only 10-25% to explosion energy

5) Nuclear physics → EoS e.g. Lattimer & Swesty 1991, Shen et al. 1998, Hempel et al. 2011

6) General relativity → Spherical effective GR potential Marek et al. 2006
+
2D axisymmetric Newton potential

7) "Accurate" initial conditions

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  - Well-balanced schemes
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- **Conclusions**

Structures

- Examples
  - Divergence of the magnetic field constraint

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times \boldsymbol{v} \times \boldsymbol{B} = 0 \longrightarrow \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) = 0$$

  - (Pseudo-) vorticity conservation (inviscid)

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \rho \boldsymbol{v}) + \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v})] = 0$$

  - Angular momentum conservation

$$\frac{\partial}{\partial t}(\boldsymbol{x} \times \rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot [\boldsymbol{x} \times (\rho \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I})] = 0$$

  - Steady/stationary solutions

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{S} \longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{S}$$

Analytically preserved structures

Structures

- Examples
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Discretely preserved structures?

Structures

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  - Divergence of the magnetic field constraint

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Discretely preserved structures?

Well-balanced scheme for HSE

- Consider 1D hydrodynamics eqs with gravity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S}$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \quad \boldsymbol{S} = -\begin{bmatrix} 0 \\ \rho \\ \rho v \end{bmatrix} \frac{\partial \phi}{\partial x}$$

- Classical solution algorithm:
  - Solve homogeneous eqs with Godunov type method (i.e. solve Riemann problem)
  - Account for source term in second step (split/unsplit)

Well-balanced scheme for HSE (2)

- Classical solution algorithm:

$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^n - \boldsymbol{F}_{i-1/2}^n\right) + \Delta t \boldsymbol{S}_i^n$$

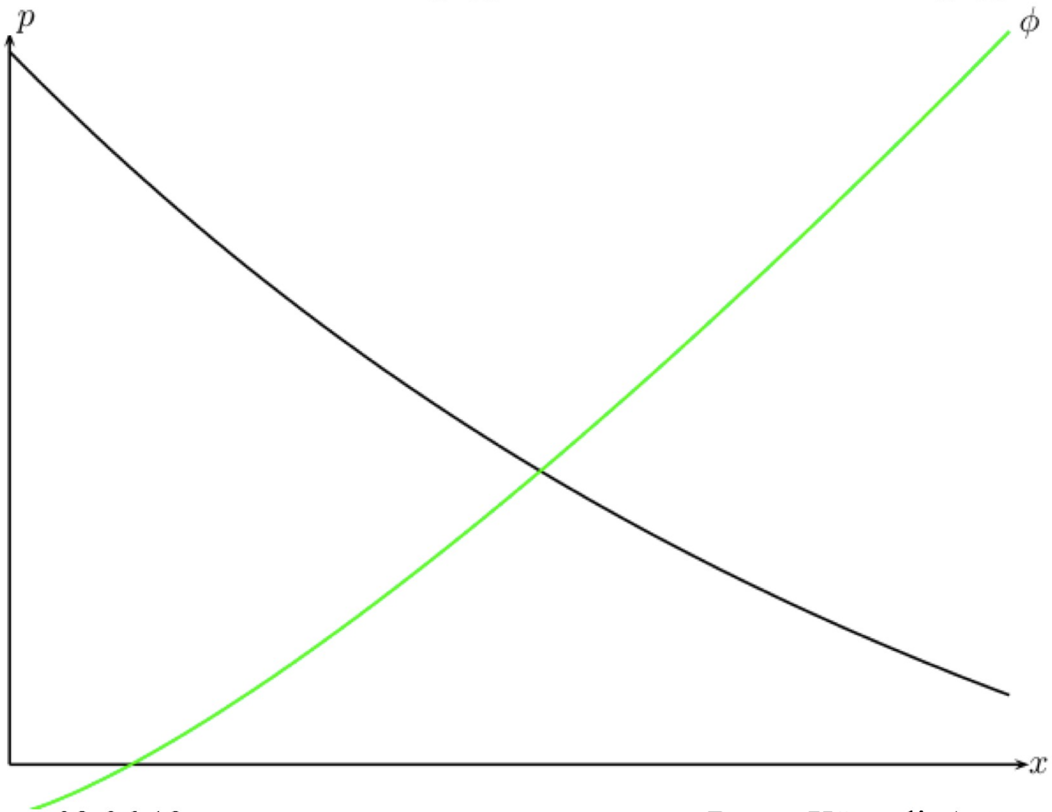
- Numerical flux $\boldsymbol{F}_{i\pm1/2}^n = \mathcal{F}(\boldsymbol{u}_{i\pm1/2}^{n,L}, \boldsymbol{u}_{i\pm1/2}^{n,R})$ from (approximate) Riemann solver, e.g.
  - (Local) Lax-Friedrichs Lax (1954), Rusanov (1961)
  - HLL (C) Harten, Lax and van Leer (1983), Toro et al. (1994)
  - Roe Roe (1981)

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \quad \Longrightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$



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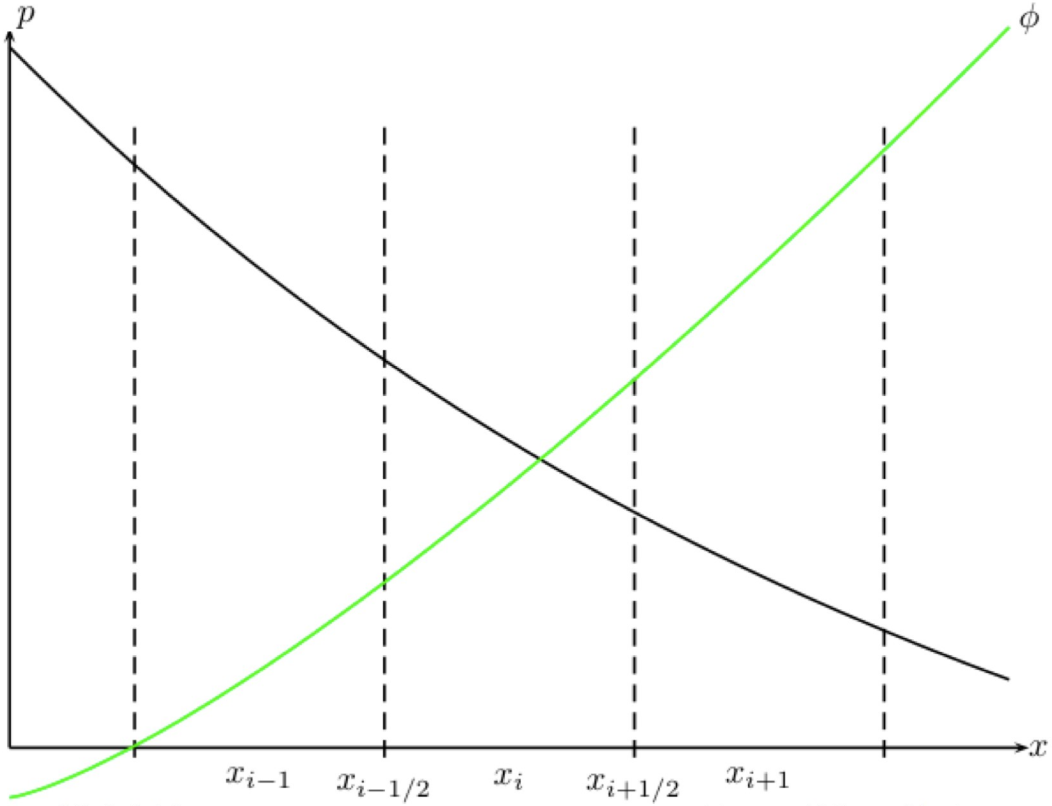
Discretise in cells $\left[x_{i-1/2}, x_{i+1/2}\right]$



Well-balanced scheme for HSE (3)

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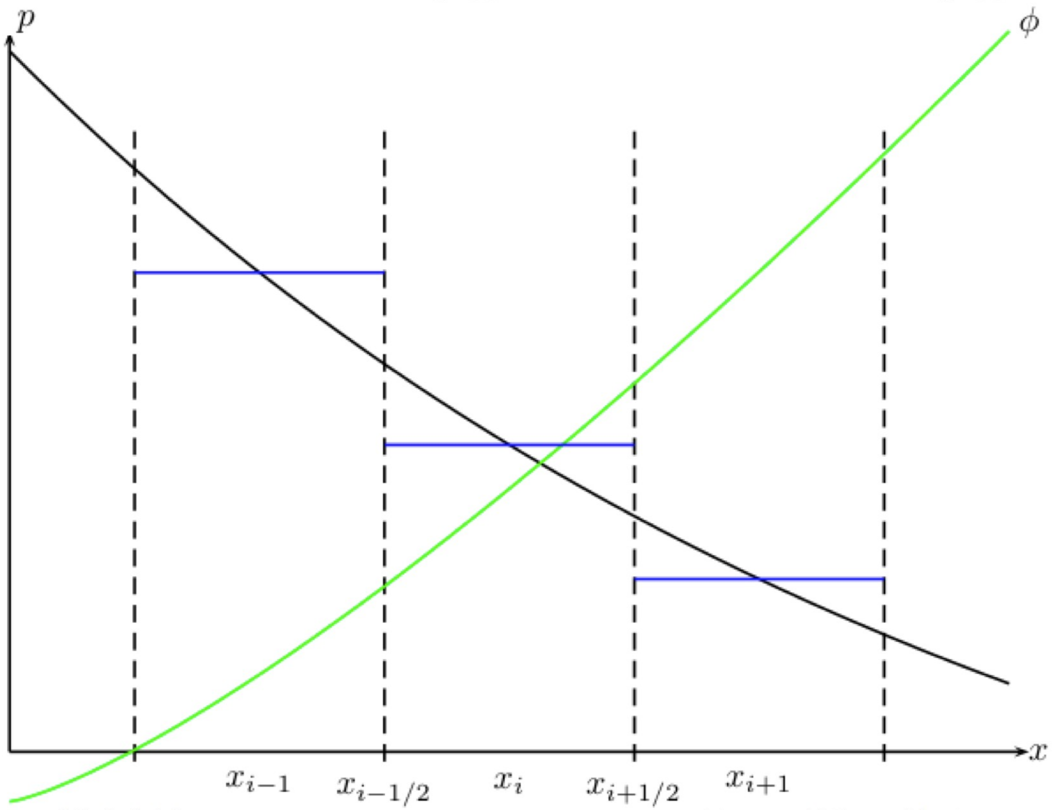
$$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \implies \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$



Discretise in cells $\left[x_{i-1/2}, x_{i+1/2}\right]$

Define cell averages

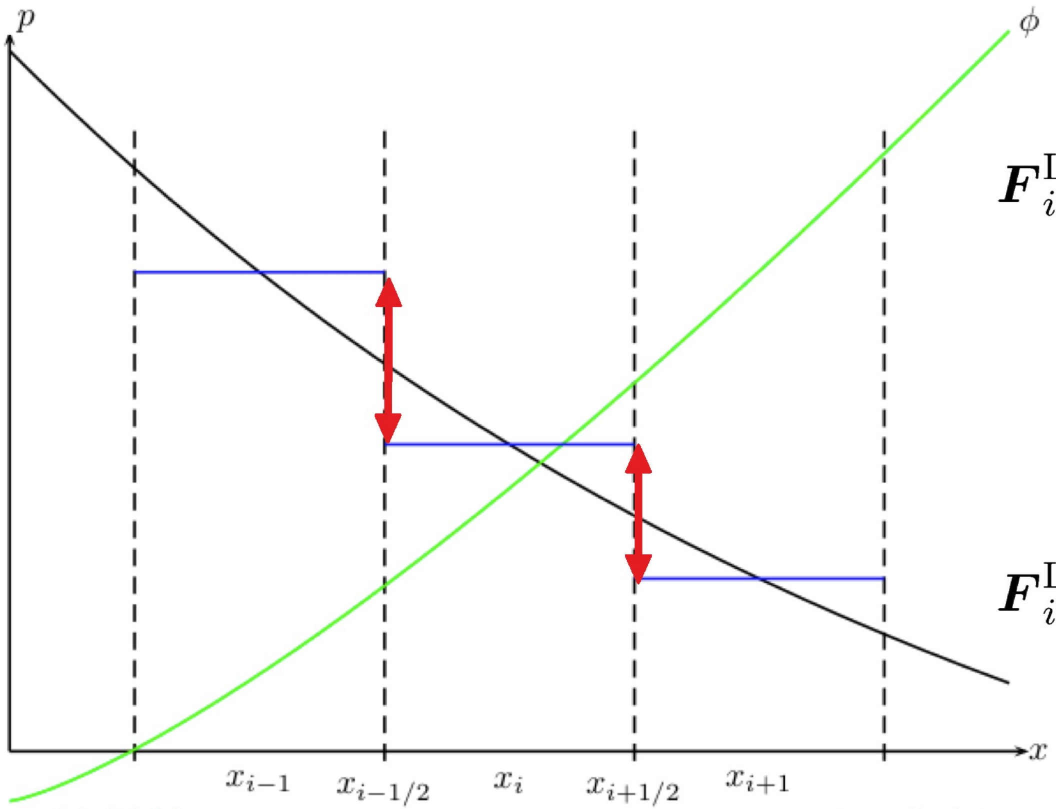
$$\boldsymbol{u}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}(x, t^n) \mathrm{d}x$$

$$\boldsymbol{S}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{S}\left(\boldsymbol{u}(x, t)\right) \mathrm{d}x$$

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) \overset{?}{=} \boldsymbol{S}^n_i$$



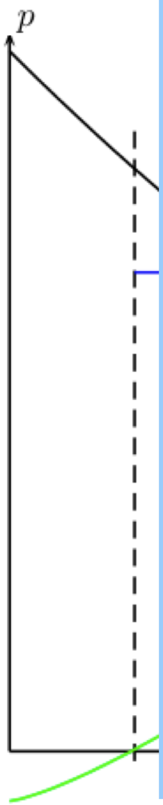
$$\boldsymbol{F}^{\mathrm{LxF}}_{i+1/2} = \frac{1}{2}\left(\boldsymbol{F}_i + \boldsymbol{F}_{i+1}\right) - \frac{S_{\max}}{2}\left(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i\right)$$

Contains also gravity induced gradient!

$$\boldsymbol{F}^{\mathrm{LxF}}_{i-1/2} = \frac{1}{2}\left(\boldsymbol{F}_{i-1} + \boldsymbol{F}_i\right) - \frac{S_{\max}}{2}\left(\boldsymbol{u}_i - \boldsymbol{u}_{i-1}\right)$$

Well-balanced scheme for HSE (3)

Interc
equil



Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \qquad \rho(x) = \left[\rho_0^{\gamma-1} - \frac{g}{K}\frac{\gamma-1}{\gamma}x\right]^{\frac{1}{\gamma-1}} \qquad p = \frac{p_0}{\rho_0^{\gamma}}\rho^{\gamma} = K\rho^{\gamma}$$

$$x \in [0, 2]$$

Error in pressure:

(after 2 sound crossing times)

N	1st	2ndTVD
128	2.1E-02	6.5E-05
256	1.1E-02	1.6E-05
512	5.3E-03	4.1E-06
1024	2.6E-03	1.0E-06
2048	1.3E-03	2.6E-07

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

HLLC numerical flux

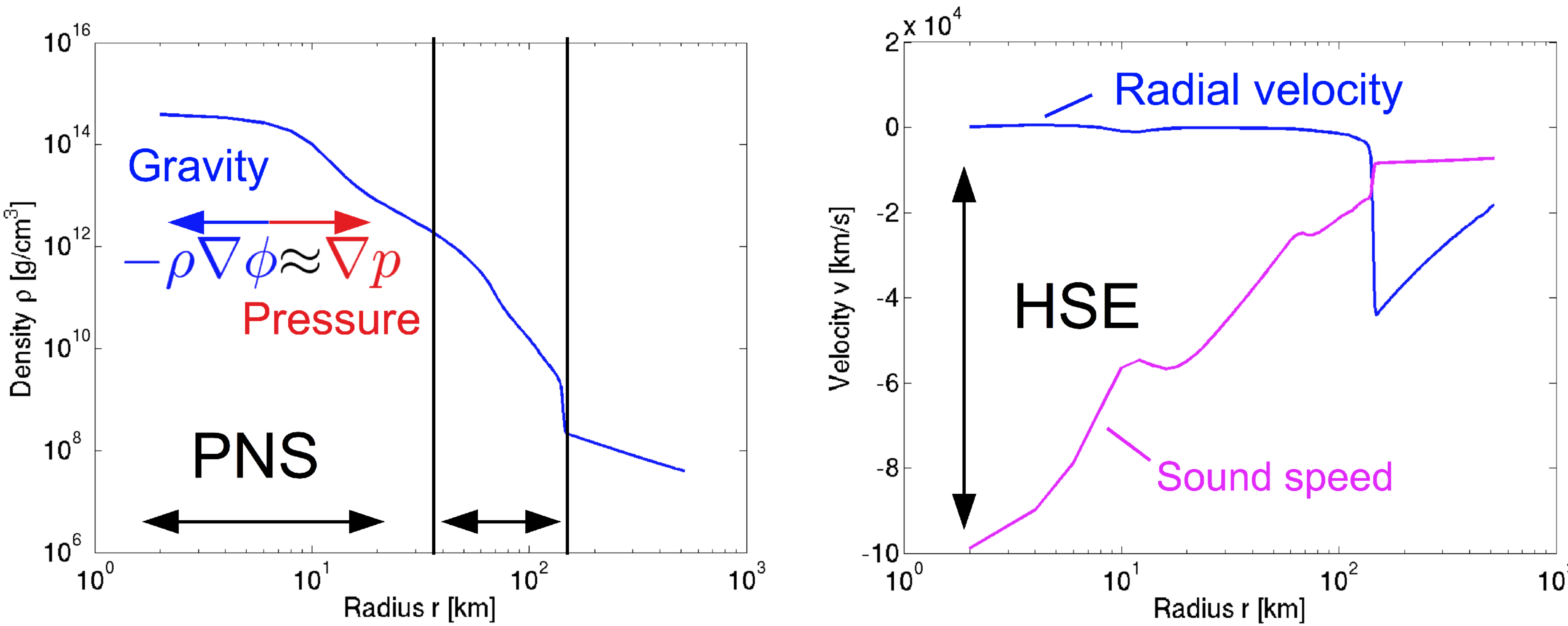
$_{-1} - \boldsymbol{u}_i)$

avity
!

$- \boldsymbol{u}_{i-1})$

Well-balanced scheme for HSE (4)

- The problem: (in our simulations)



Ability to maintain near hydrostatic equilibrium for a long time!

Well-balanced scheme for HSE (5)

- Solutions:
  - Define a **global** stationary state $u_0(r)$ **at each time step** and evolve $u(\boldsymbol{x}) - u_0(r)$
  - Steady state preserving reconstructions, well-balanced schemes e.g. LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010), Xing & Shu (2013)

Note: there are many, many more... especially for shallow-water eqs!!!

Well-balanced scheme for HSE (5)

- Solutions:
  - Define a **global** stationary state $u_0(r)$ **at each time step** and evolve $u(x) - u_0(r)$
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Requirements

- Equilibrium not known in advance (self-gravity)
- Extensible for general EoS
- (At least) second order accuracy
- Preserve robustness of base shock capturing scheme

Well-balanced scheme for HSE (6)

- Hydrostatic equilibrium

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

Describes only a mechanical equilibrium...
Density and pressure not uniquely determined

$$p = p(\rho, s) = p(\rho, T)$$

s Entropy
T Temperature

Arbitrary entropy or temperature profiles not (physically) stable (convection!)

Well-balanced scheme for HSE (7)

- Consider constant entropy profile

- Using the thermodynamic relation

$$\mathrm{d}h = T\mathrm{d}s + \frac{\mathrm{d}p}{\rho} \qquad h = e + \frac{p}{\rho}$$ Enthalpy

- Hydrostatic eq.

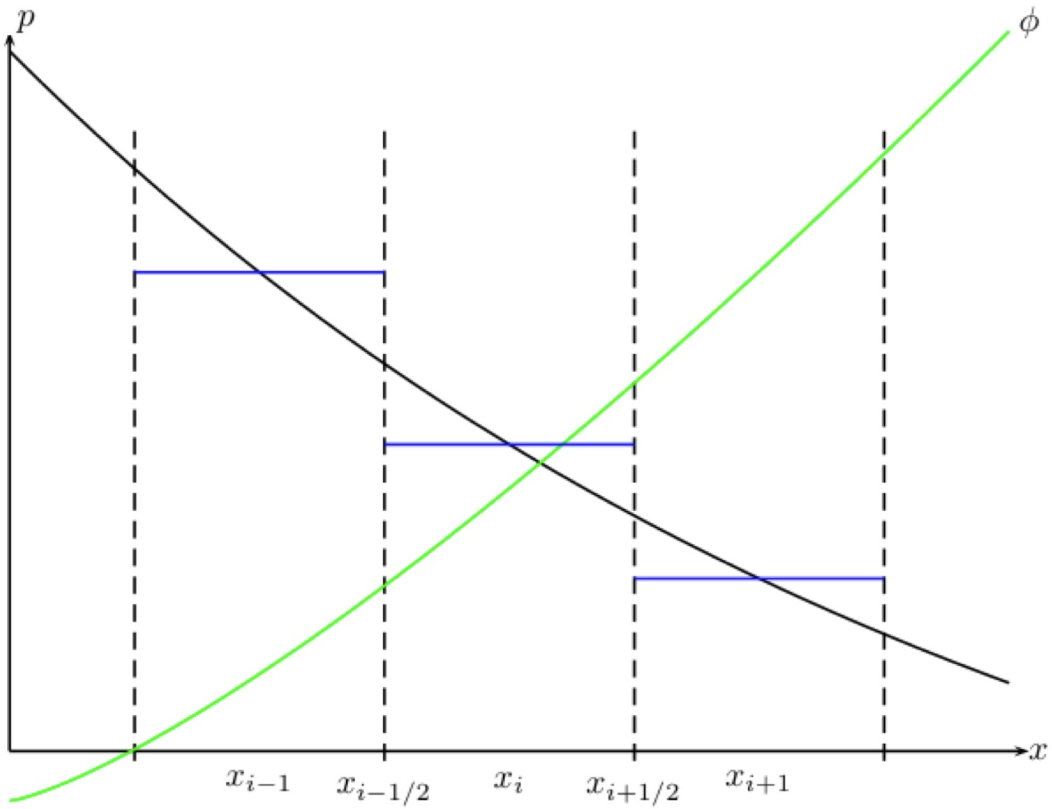
$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial h}{\partial x} = -\frac{\partial \phi}{\partial x}$$

- Or simply $h + \phi = const$

Well-balanced scheme for HSE (8)

Perform equilibrium reconstruction:

$$h + \phi = const$$



$$h_{0,i}(x) = h_i + \phi_i - \phi(x)$$

Eq. enthalpy

EoS $h_{0,i}(x) = h\left(s_i, p_{0,i}(x)\right)$

$p_{0,i}(x)$ & $\rho_{0,i}(x)$

$$\boldsymbol{w}^n_{i\pm1/2\mp} = \begin{bmatrix} \rho^n_{0,i}(x_{i\pm1/2}) \\ v^n_{x,i} \\ p^n_{0,i}(x_{i\pm1/2}) \end{bmatrix}$$

Eq. reconstructed primitive variables

Well-balanced scheme for HSE (9)

- Well-balanced discretization of momentum source term

$$S^n_{\rho v,i} = \frac{p^n_{0,i}(x_{i+1/2}) - p^n_{0,i}(x_{i-1/2})}{\Delta x} = -\int_{x_{i-1/2}}^{x_{i+1/2}} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x + O(\Delta x^2)$$

- Then for data satisfying $h + \phi = const$, $v_x = 0$ and any consistent numerical flux

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) = \boldsymbol{S}^n_i$$

Well-balanced wrt hydrostatic equilibrium!

Well-balanced scheme for HSE (10)

- Second order extension: $r_{1,i}(x_j) = r_j - r_{0,i}(x_j)$

$r =$ pressure, density

Eq. perturbation ($r_{1,i}(x_j)$), Data (r_j), Equilibrium ($r_{0,i}(x_j)$)

Stencil: $j = ..., i-1, i, i+1, ...$

$$r_{1,i}(x) = r_{1,i}(x_i) + Dr_{1,i}(x - x_i) = Dr_{1,i}(x - x_i)$$

$$Dr_{1,i} = \text{limiter}\left(\frac{r_{0,i}(x_{i-1}) - r_{i-1}}{\Delta x}, \frac{r_{i+1} - r_{0,i}(x_{i+1})}{\Delta x}\right)$$

Reconstruction in deviation from equilibrium

Similar to Botta et al. 2004,Fuchs et al. 2010

- Time stepping:

$$\begin{aligned} \boldsymbol{u}^* &= \boldsymbol{u}^n + \Delta t^n \boldsymbol{L}(\boldsymbol{u}^n) \\ \boldsymbol{u}^{**} &= \boldsymbol{u}^* + \Delta t^n \boldsymbol{L}(\boldsymbol{u}^*) \\ \boldsymbol{u}^{n+1} &= \frac{1}{2}(\boldsymbol{u}^n + \boldsymbol{u}^{**}) \end{aligned}$$

Strong Stability Preserving Runge-Kutta, Gottlieb et al. 2001

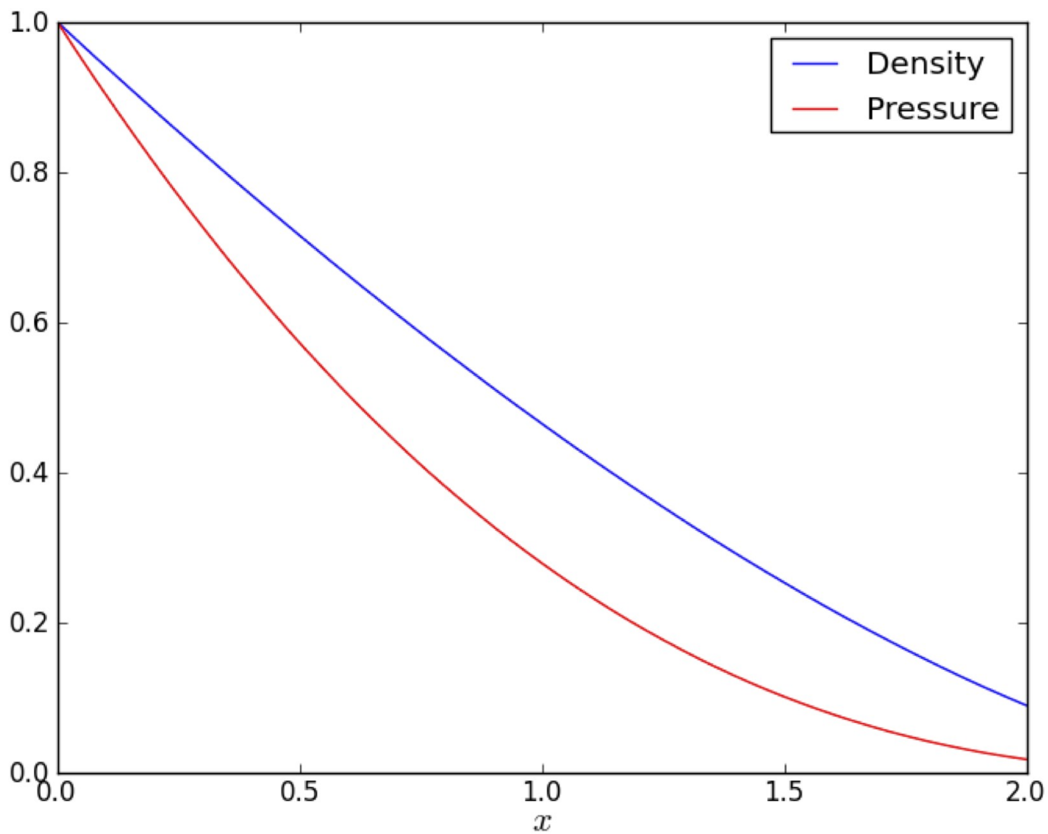
Example 1

Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \quad \rho(x) = \left[\rho_0^{\gamma-1} - \frac{g}{K}\frac{\gamma-1}{\gamma}x\right]^{\frac{1}{\gamma-1}} \quad p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma = K\rho^\gamma$$

$$x \in [0, 2]$$

$$h + \phi = const$$



Error in pressure:

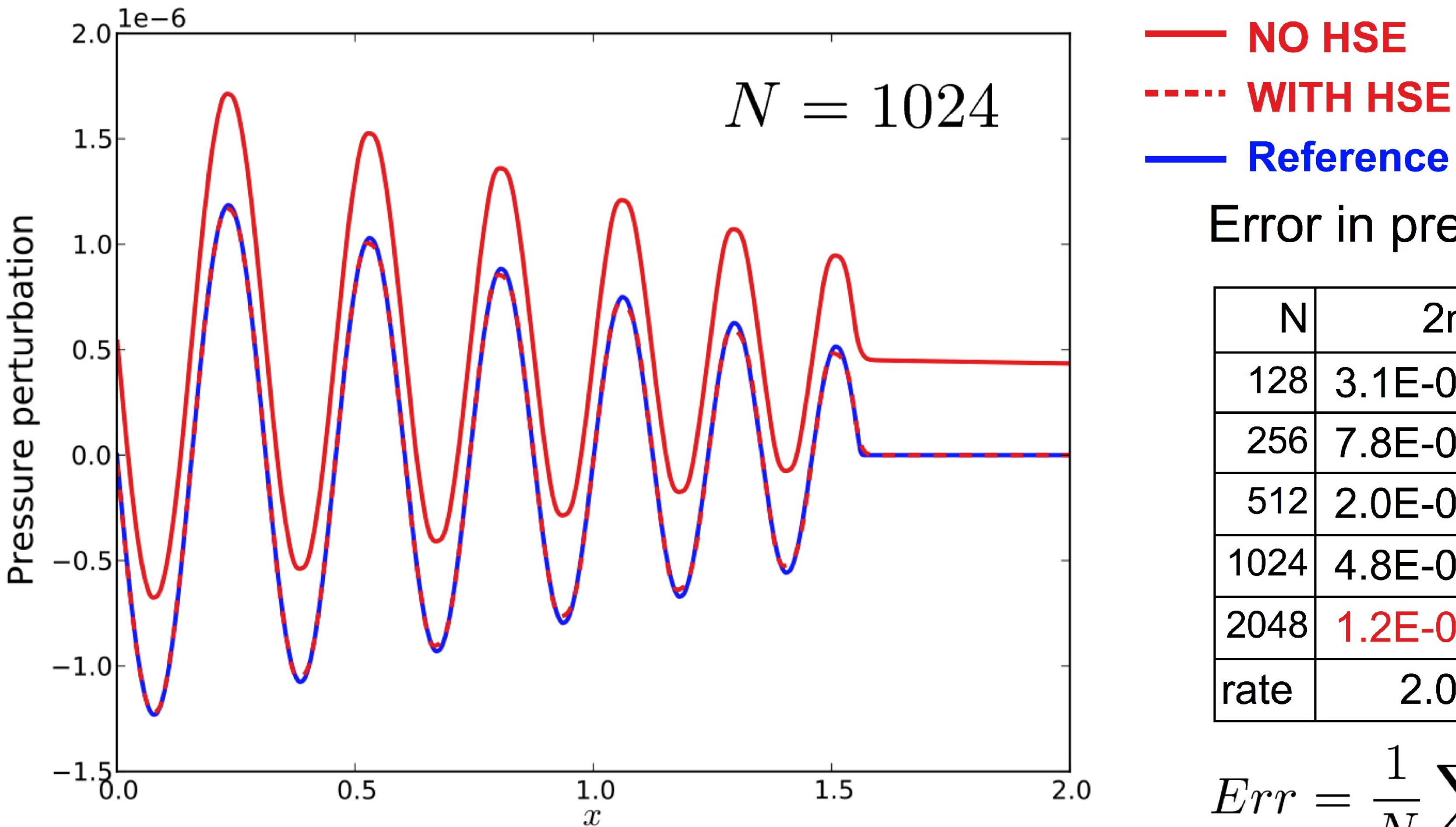
N	1st	2ndTVD
128	2.1E-02 / **1.3E-14**	6.5E-05 / **1.3E-14**
256	1.1E-02 / **3.6E-14**	1.6E-05 / **1.5E-14**
512	5.3E-03 / **7.7E-14**	4.1E-06 / **4.6E-14**
1024	2.6E-03 / **5.7E-14**	1.0E-06 / **6.1E-14**
2048	1.3E-03 / **1.2E-13**	2.6E-07 / **1.5E-14**
rate	1.00 / -	2.00 / -

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

Example 2

Hydrostatic atmosphere in a constant gravitational field

+ small perturbation $v(t, x=0) = 10^{-6} \sin(8\pi t)$



Error in pressure:

N	2ndTVD
128	3.1E-05 / **1.9E-07**
256	7.8E-06 / **6.8E-08**
512	2.0E-06 / **2.5E-08**
1024	4.8E-07 / **8.5E-09**
2048	1.2E-07 / **4.1E-09**
rate	2.01 / 1.40

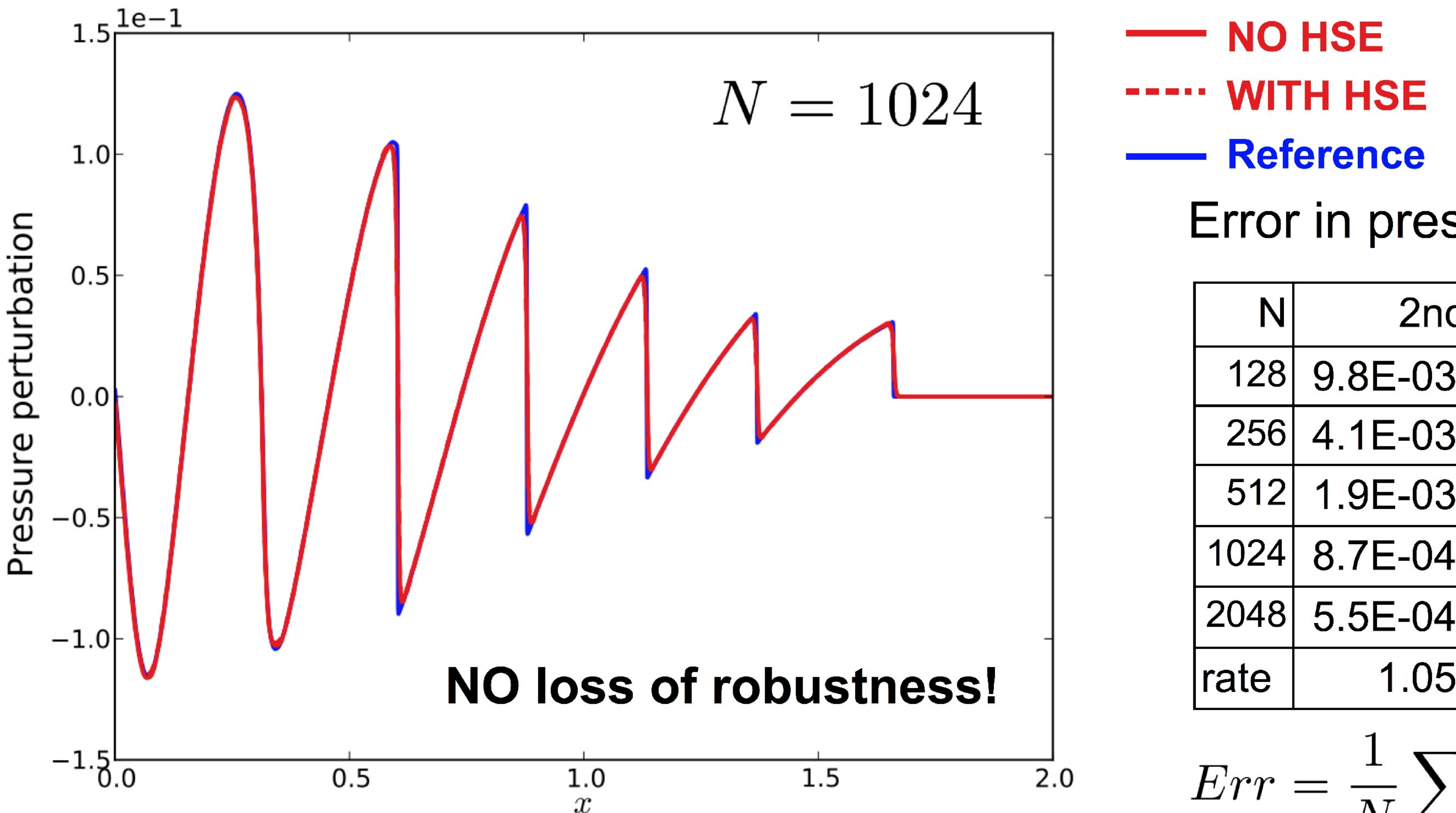
!

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

Example 3

Hydrostatic atmosphere in a constant gravitational field

+ large perturbation $v(t, x=0) = 0.1 \sin(8\pi t)$



Error in pressure:

N	2ndTVD
128	9.8E-03 / **1.1E-02**
256	4.1E-03 / **4.9E-03**
512	1.9E-03 / **2.0E-03**
1024	8.7E-04 / **8.0E-04**
2048	5.5E-04 / **3.3E-04**
rate	1.05 / 1.28

$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

Multi-dimensional extension

- Straight forward directional application of HydroStatic Reconstruction

$$\frac{\mathrm{d}\boldsymbol{u}_{i,j}}{\mathrm{d}t} = \boldsymbol{L}(\boldsymbol{u}) = -\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2,j} - \boldsymbol{F}_{i-1/2,j}\right) - \frac{1}{\Delta y}\left(\boldsymbol{G}_{i,j+1/2} - \boldsymbol{G}_{i,j-1/2}\right) + \boldsymbol{S}_{i,j}$$

- Hydrostatic equilibrium:

$$h + \phi = const$$

Example 4

Polytrope: model star (e.g. main sequence stars, white dwarfs, neutron stars)

HSE: $\nabla p = -\rho \nabla \phi$ Poisson equation: $\nabla^2 \phi = -4\pi G \rho$

Equation of state $p = K \rho^\gamma$ $K = 1$

Take $\gamma = 2$ ~ neutron stars

Then there's an exact solution:

Central density

$$\rho(\boldsymbol{x}) = \rho_c \frac{\sin(\alpha r)}{r}$$

$$\phi(\boldsymbol{x}) = -\gamma K \rho(\boldsymbol{x})$$

$$\alpha = \sqrt{\frac{2K}{4\pi G}} \quad r = \sqrt{x^2 + y^2 + z^2}$$

Example 4

Po

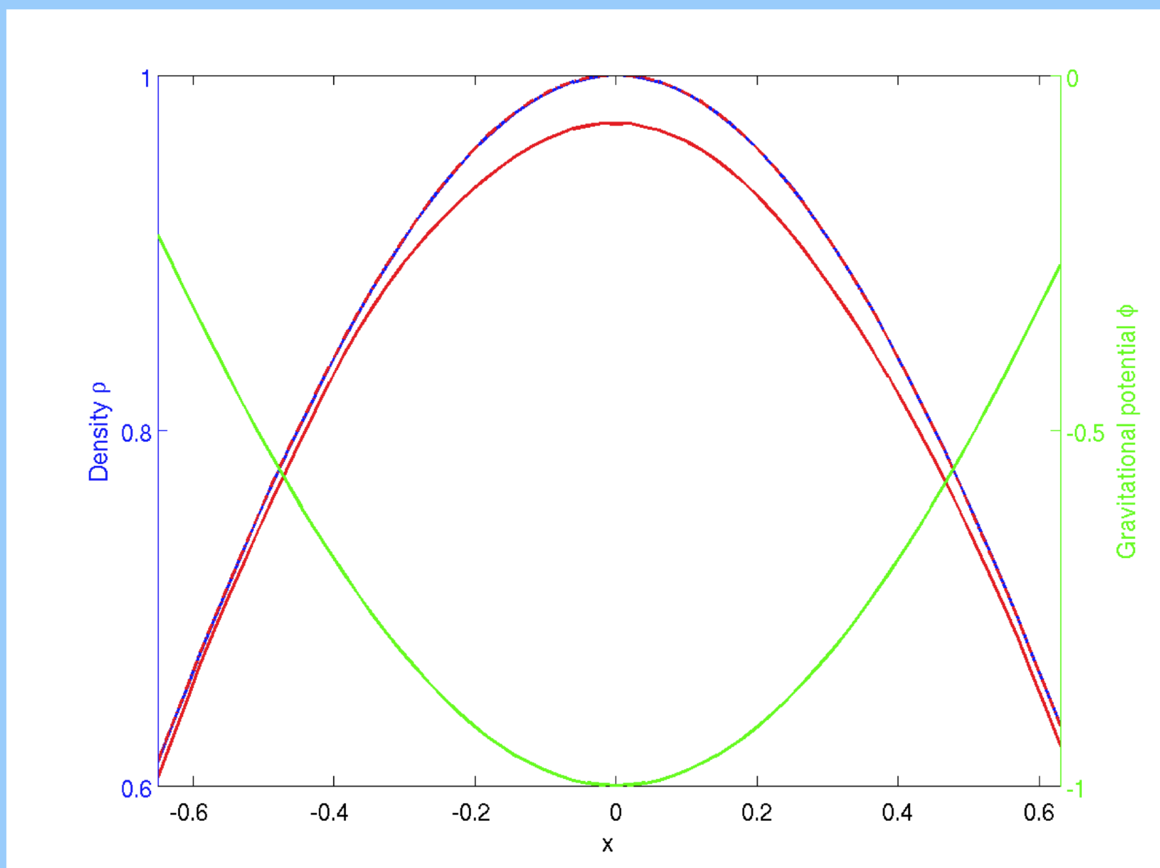
H ρ

E

Ta

Th

Evolution for 20 “sound crossing” times



NO HSE

WITH HSE

Reference

Error in density:

N	2ndTVD
32	1.3E-02 / **1.5E-14**
64	3.6E-03 / **3.0E-14**
128	1.0E-03 / **5.6E-14**
rate	1.82 / **-**

Example 4

Po

Small perturbation $\rho(\boldsymbol{x}) = \rho_0(\boldsymbol{x}) + Ae^{-100\boldsymbol{x}^2} \quad A = 10^{-3}$

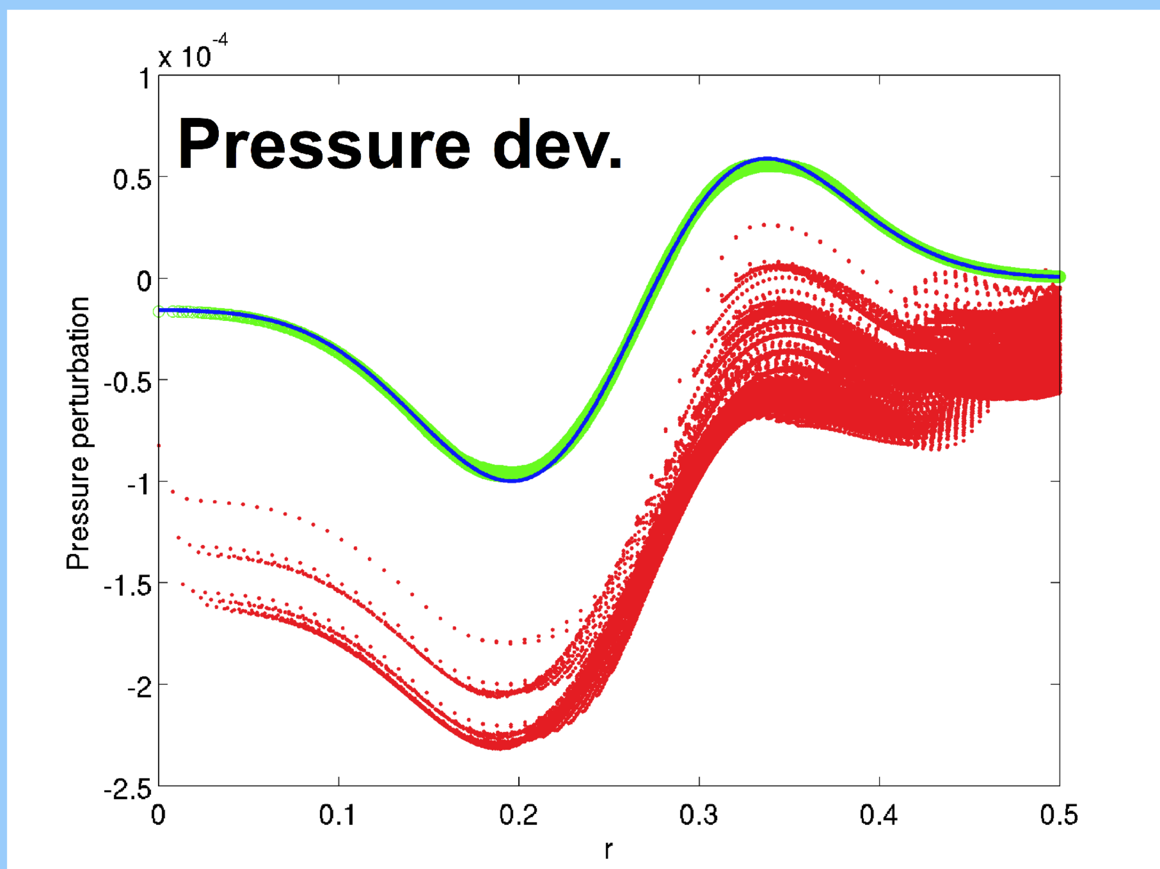
H

E

Ta

Th

ρ



NO HSE
WITH HSE
Reference

Error in pressure:

N	2ndTVD
32	8.4E-04 / **1.1E-06**
64	2.1E-04 / **3.7E-07**
128	5.1E-05 / **1.1E-07**
rate	2.02 / **1.67**

!

Example 4

Po

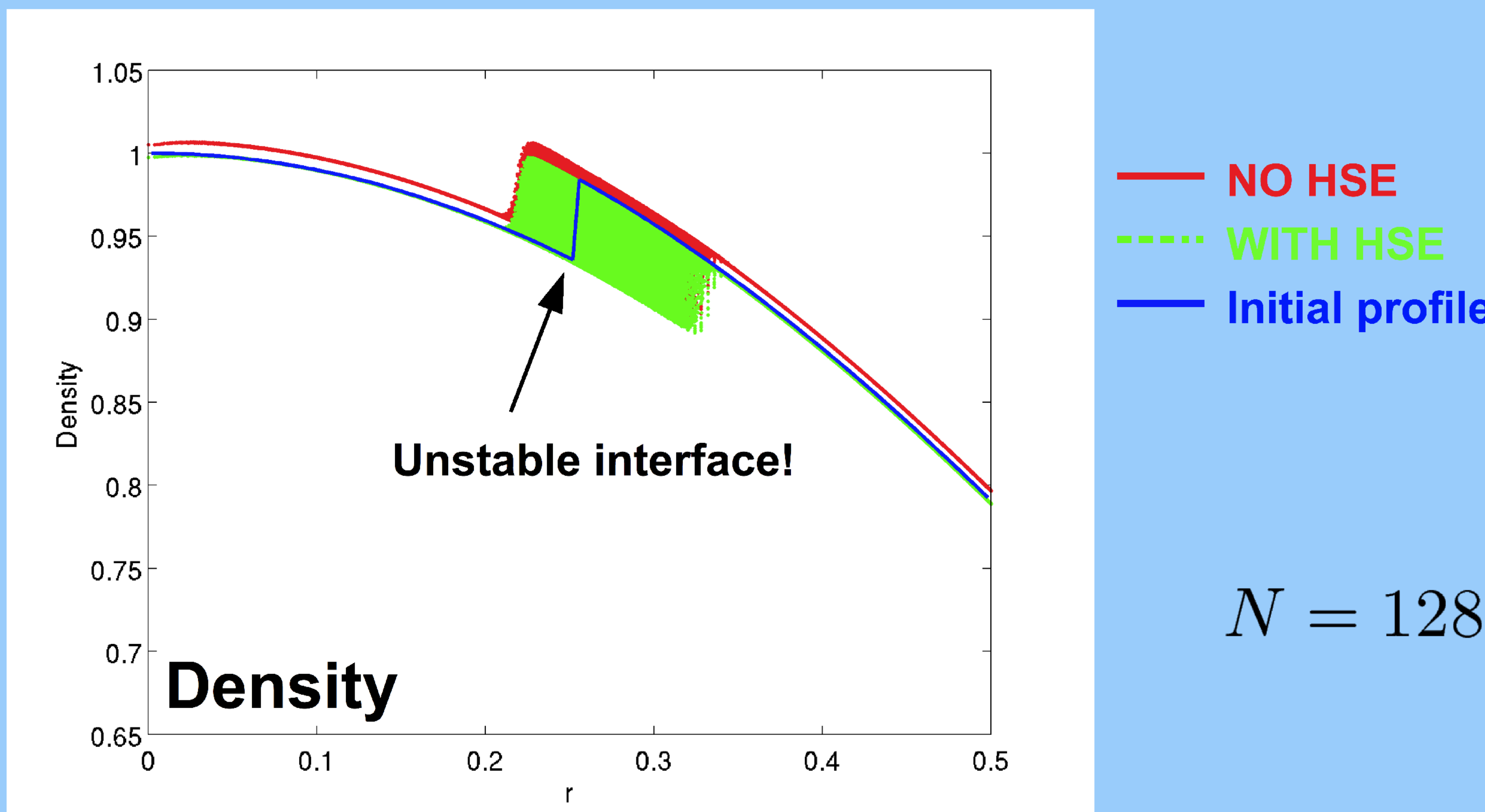
H ρ

E

Ta

Th

Rayleigh-Taylor instability



$$N = 128$$

Example 4

Po

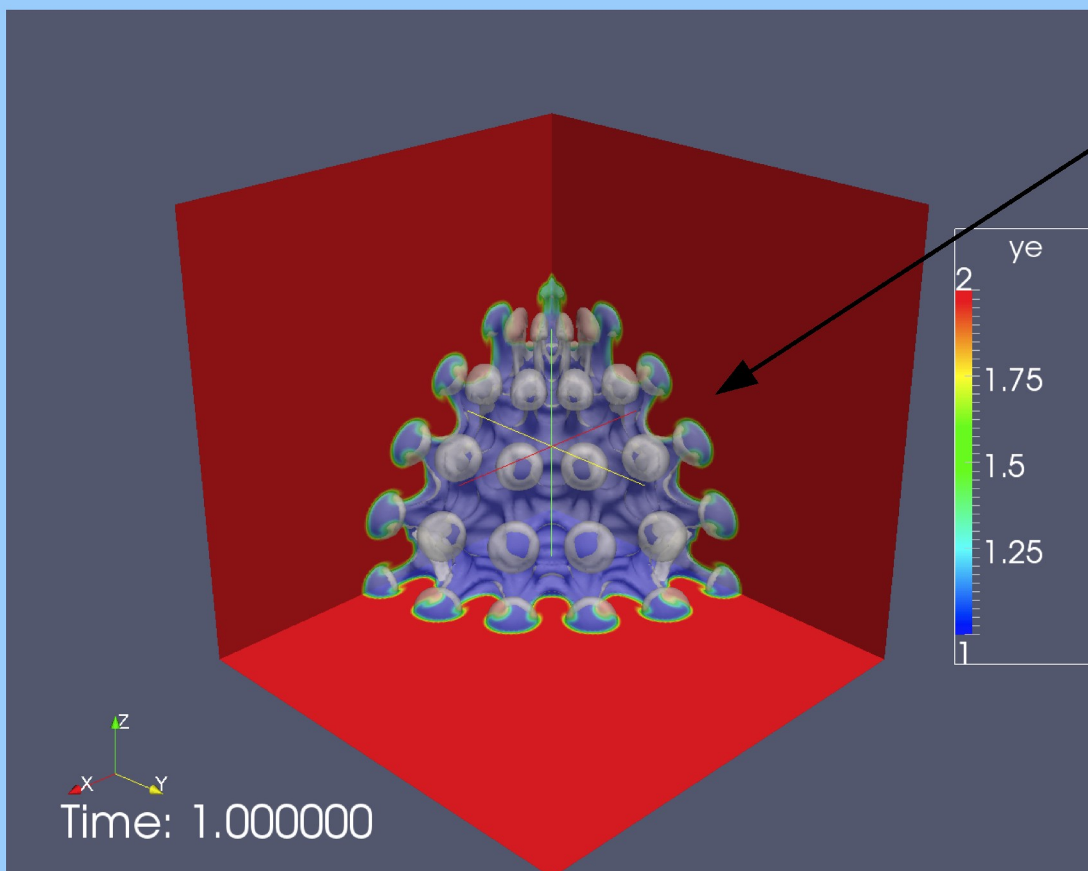
H ρ

E

Ta

Th

Rayleigh-Taylor instability



Rayleigh-Taylor "mushrooms"

$$N = 128$$

Structures

- Examples
  - Divergence of the magnetic field constraint

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times \boldsymbol{v} \times \boldsymbol{B} = 0 \longrightarrow \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) = 0$$

  - (Pseudo-) vorticity conservation (inviscid)

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \rho \boldsymbol{v}) + \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \boldsymbol{v})] = 0$$

  - Angular momentum conservation

$$\frac{\partial}{\partial t}(\boldsymbol{x} \times \rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot [\boldsymbol{x} \times (\rho \boldsymbol{v} \boldsymbol{v} + p\boldsymbol{I})] = 0$$

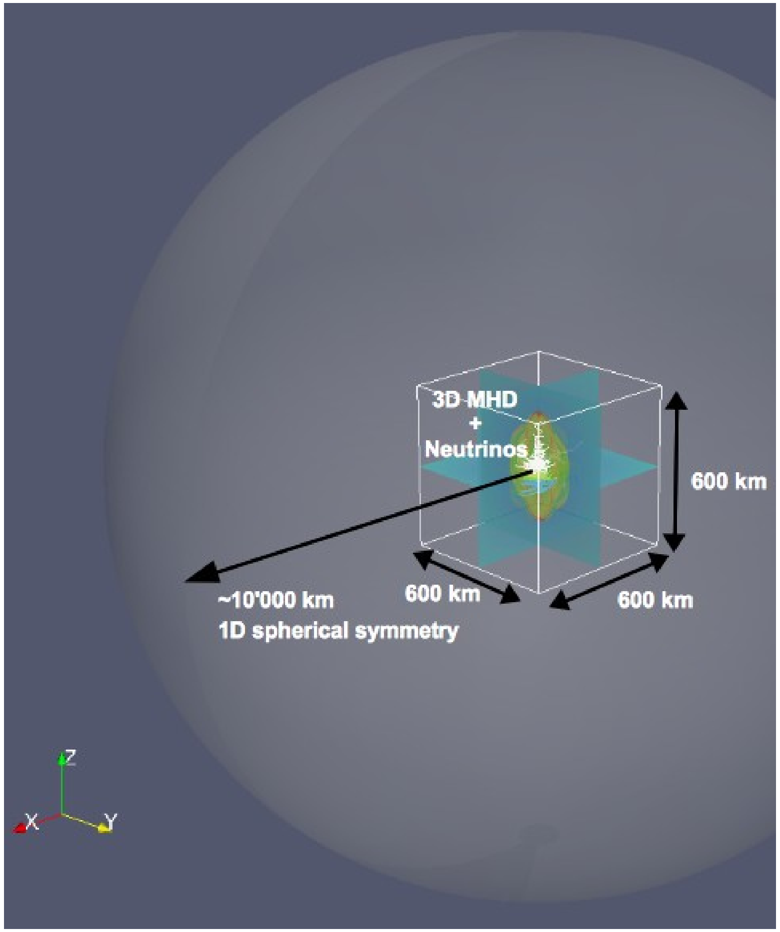
  - Steady/stationary solutions

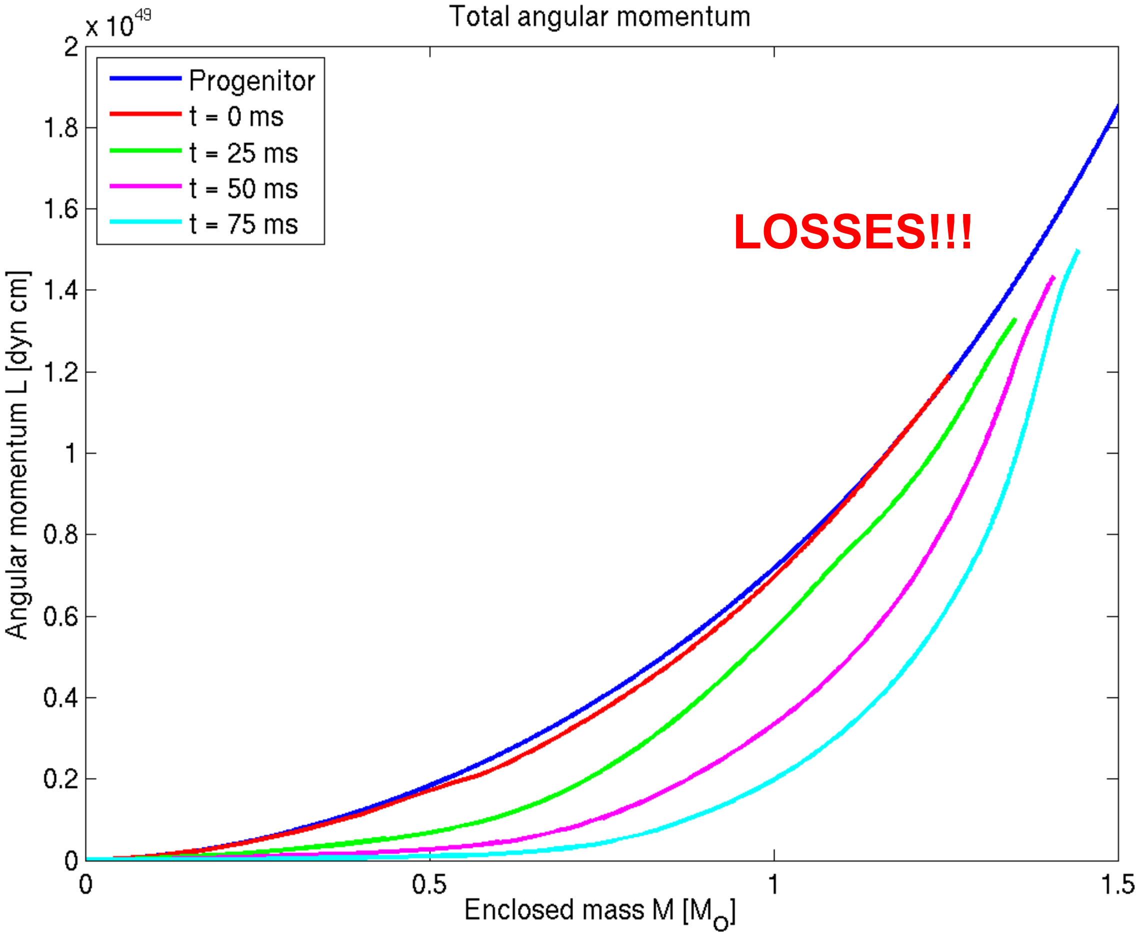
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{S} \longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{S}$$

Discretely preserved structures?

Angular momentum preserving schemes

- The problem: (in our simulations)





Angular momentum preserving schemes

- The problem:

  well-known for Godunov-type schemes

- Solution:
  - Adopt a coordinate system with symmetry axis (cylindrical or spherical) and rewrite azimuthal momentum component in “angular momentum” form

- But what in general?

→ Structure preserving schemes

Angular momentum preserving schemes

- Angular momentum density $\boldsymbol{j} = \boldsymbol{x} \times \rho \boldsymbol{v}$

- Conservation from linear momentum equation

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I}) = 0 \quad \Big| \quad \boldsymbol{x}\times$$

$$\frac{\partial}{\partial t}(\boldsymbol{x} \times \rho \boldsymbol{v}) + \boldsymbol{x} \times \boldsymbol{\nabla} \cdot \underbrace{(\rho \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I})}_{\text{Symmetric!}} = 0$$

$$\frac{\partial}{\partial t}(\boldsymbol{x} \times \rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot [\boldsymbol{x} \times (\rho \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I})] = 0$$

Angular momentum preserving schemes

- Angular momentum density $j^z = j = \rho\left(v^y x - v^x y\right)$

- Conservation from linear momentum equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial \boldsymbol{G}}{\partial y} = 0$$

$$\rightarrow \quad \frac{\partial j}{\partial t} + x\frac{\partial F_3}{\partial x} - y\frac{\partial F_2}{\partial x} + x\frac{\partial G_3}{\partial y} - y\frac{\partial G_2}{\partial y} = 0$$

Symmetry

$$\frac{\partial j}{\partial t} + \frac{\partial}{\partial x}\left(xF_3 - yF_2\right) + \frac{\partial}{\partial y}\left(xG_3 - yG_2\right) = F_3 - G_2 = 0$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v^x \\ \rho v^y \\ E \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} \rho v^x \\ \rho (v^x)^2 + p \\ \rho v^y v^y \\ (E+p)v^x \end{bmatrix} \qquad \boldsymbol{G} = \begin{bmatrix} \rho v^y \\ \rho v^x v^y \\ \rho (v^y)^2 + p \\ (E+p)v^y \end{bmatrix}$$

Angular momentum preserving schemes

- Angular momentum density $j_{i,j} = \rho_{i,j}\left(v^y_{i,j} x_i - v^x_{i,j} y_j\right)$

$$\approx \frac{1}{V}\iint_V \rho\left(v^y x - v^x y\right) \mathrm{d}x\mathrm{d}y$$

- Conservation from linear momentum equation

$$\frac{\mathrm{d}\boldsymbol{u}_{i,j}}{\mathrm{d}t} = -\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2,j} - \boldsymbol{F}_{i-1/2,j}\right) - \frac{1}{\Delta y}\left(\boldsymbol{G}_{i,j+1/2} - \boldsymbol{G}_{i,j-1/2}\right)$$

→ …

Discrete symmetry

$$\frac{1}{2}\left((F_3)_{i-1/2,j} + (F_3)_{i+1/2,j}\right) - \frac{1}{2}\left((G_2)_{i,j-1/2} + (G_2)_{i,j+1/2}\right) = 0$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v^x \\ \rho v^y \\ E \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} \rho v^x \\ \rho (v^x)^2 + p \\ \rho v^y v^y \\ (E+p) v^x \end{bmatrix} \qquad \boldsymbol{G} = \begin{bmatrix} \rho v^y \\ \rho v^x v^y \\ \rho (v^y)^2 + p \\ (E+p) v^y \end{bmatrix}$$

Angular momentum preserving schemes

- Potential-based schemes   Mishra & Tadmor 2011, 2011, 2012

Define "numerical potentials" at each vertex

$$\phi_{i+1/2,j+1/2} \quad \psi_{i+1/2,j+1/2}$$

Consistency

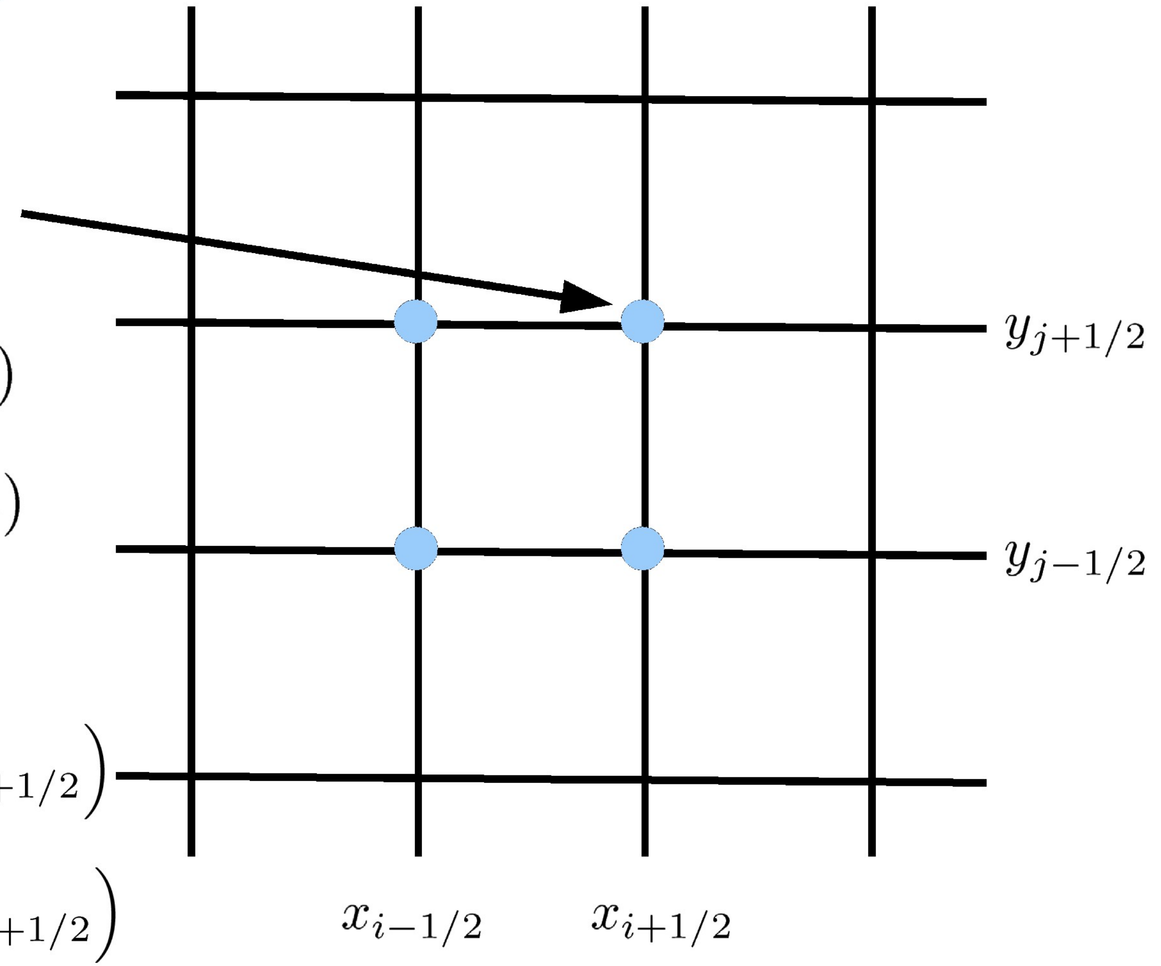
$$\phi_{i+1/2,j+1/2}(\boldsymbol{u},\dots,\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u})$$

$$\psi_{i+1/2,j+1/2}(\boldsymbol{u},\dots,\boldsymbol{u}) = \boldsymbol{G}(\boldsymbol{u})$$

Numerical fluxes

$$\boldsymbol{F}_{i+1/2,j} = \frac{1}{2}\left(\phi_{i+1/2,j-1/2} + \phi_{i+1/2,j+1/2}\right)$$

$$\boldsymbol{G}_{i,j+1/2} = \frac{1}{2}\left(\psi_{i-1/2,j+1/2} + \psi_{i+1/2,j+1/2}\right)$$



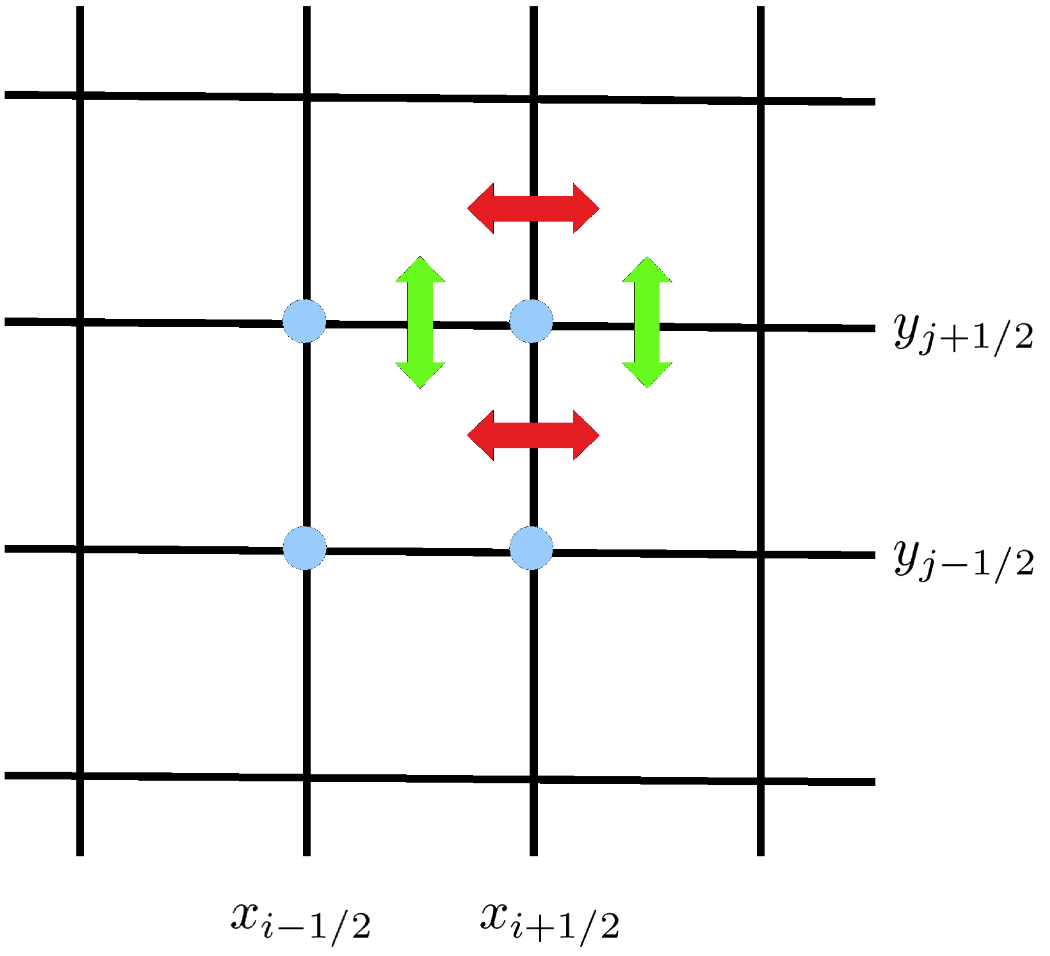
Angular momentum preserving schemes

- Potential-based schemes Mishra & Tadmor 2011, 2011, 2012

Several possibilities to construct numerical potentials, e.g. "symmetric potentials":

$$\phi_{i+1/2,j+1/2} = \frac{1}{2}\left(\boldsymbol{F}_{i+1/2,j} + \boldsymbol{F}_{i+1/2,j+1}\right)$$

$$\psi_{i+1/2,j+1/2} = \frac{1}{2}\left(\boldsymbol{G}_{i,j+1/2} + \boldsymbol{G}_{i+1,j+1/2}\right)$$



Angular momentum preserving schemes

- Angular momentum preserving potential-based schemes

$$\phi_{i+1/2,j+1/2} = \begin{bmatrix} (\phi_1)_{i+1/2,j+1/2} \\ (\phi_2)_{i+1/2,j+1/2} \\ (\phi_3)_{i+1/2,j+1/2} \\ (\phi_4)_{i+1/2,j+1/2} \end{bmatrix}$$

$$\psi_{i+1/2,j+1/2} = \begin{bmatrix} (\psi_1)_{i+1/2,j+1/2} \\ (\psi_2)_{i+1/2,j+1/2} \\ (\psi_3)_{i+1/2,j+1/2} \\ (\psi_4)_{i+1/2,j+1/2} \end{bmatrix}$$

Angular momentum preserving schemes

- Angular momentum preserving potential-based schemes

$$\phi_{i+1/2,j+1/2} = \begin{bmatrix} (\phi_1)_{i+1/2,j+1/2} \\ (\phi_2)_{i+1/2,j+1/2} \\ \chi_{i+1/2,j+1/2} \\ (\phi_4)_{i+1/2,j+1/2} \end{bmatrix}$$

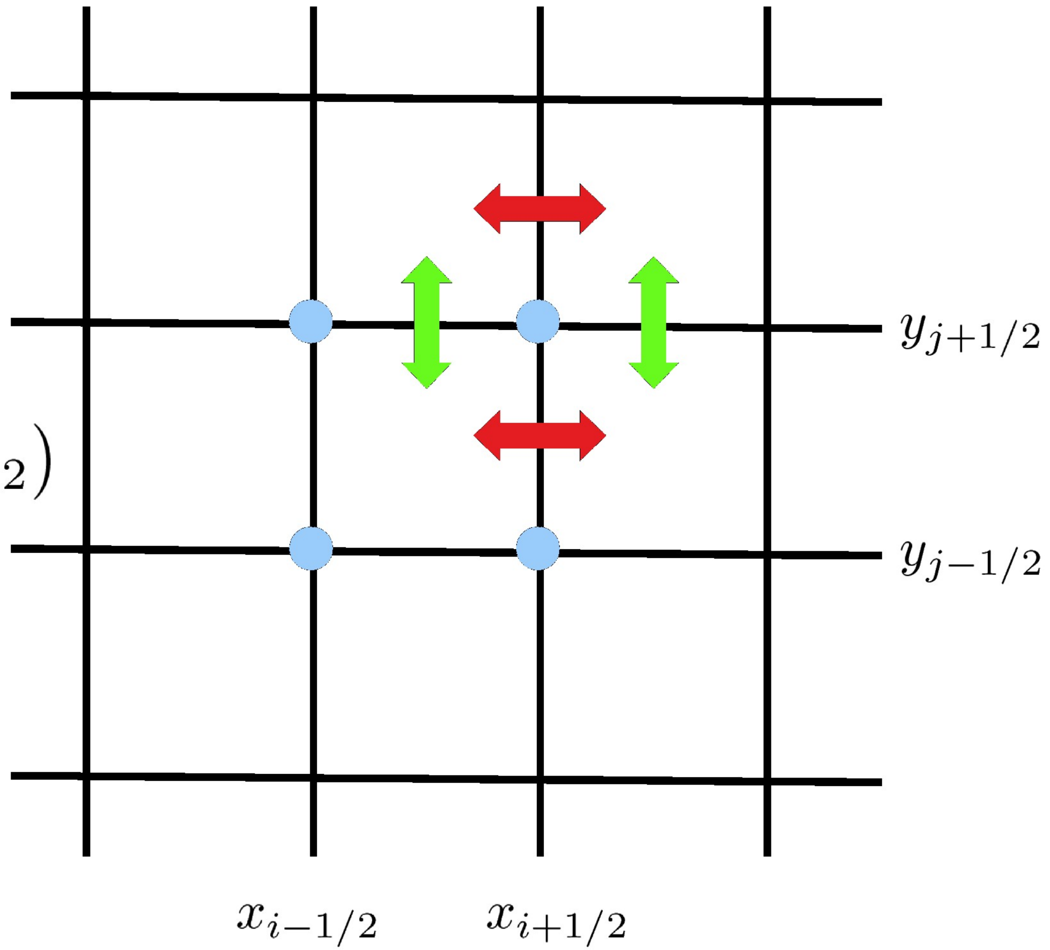
$$\psi_{i+1/2,j+1/2} = \begin{bmatrix} (\psi_1)_{i+1/2,j+1/2} \\ \chi_{i+1/2,j+1/2} \\ (\psi_3)_{i+1/2,j+1/2} \\ (\psi_4)_{i+1/2,j+1/2} \end{bmatrix}$$

Angular momentum preserving schemes

- Angular momentum preserving potential-based schemes

Several possibilities to construct, $\chi_{i+1/2,j+1/2}$
e.g. "symmetric"

$$\chi_{i+1/2,j+1/2} =$$
$$\frac{1}{2}\left((\phi_3)_{i+1/2,j+1/2} + (\psi_2)_{i+1/2,j+1/2}\right)$$



Angular momentum preserving schemes

- Isentropic vortex e.g. Yee et al. 1999

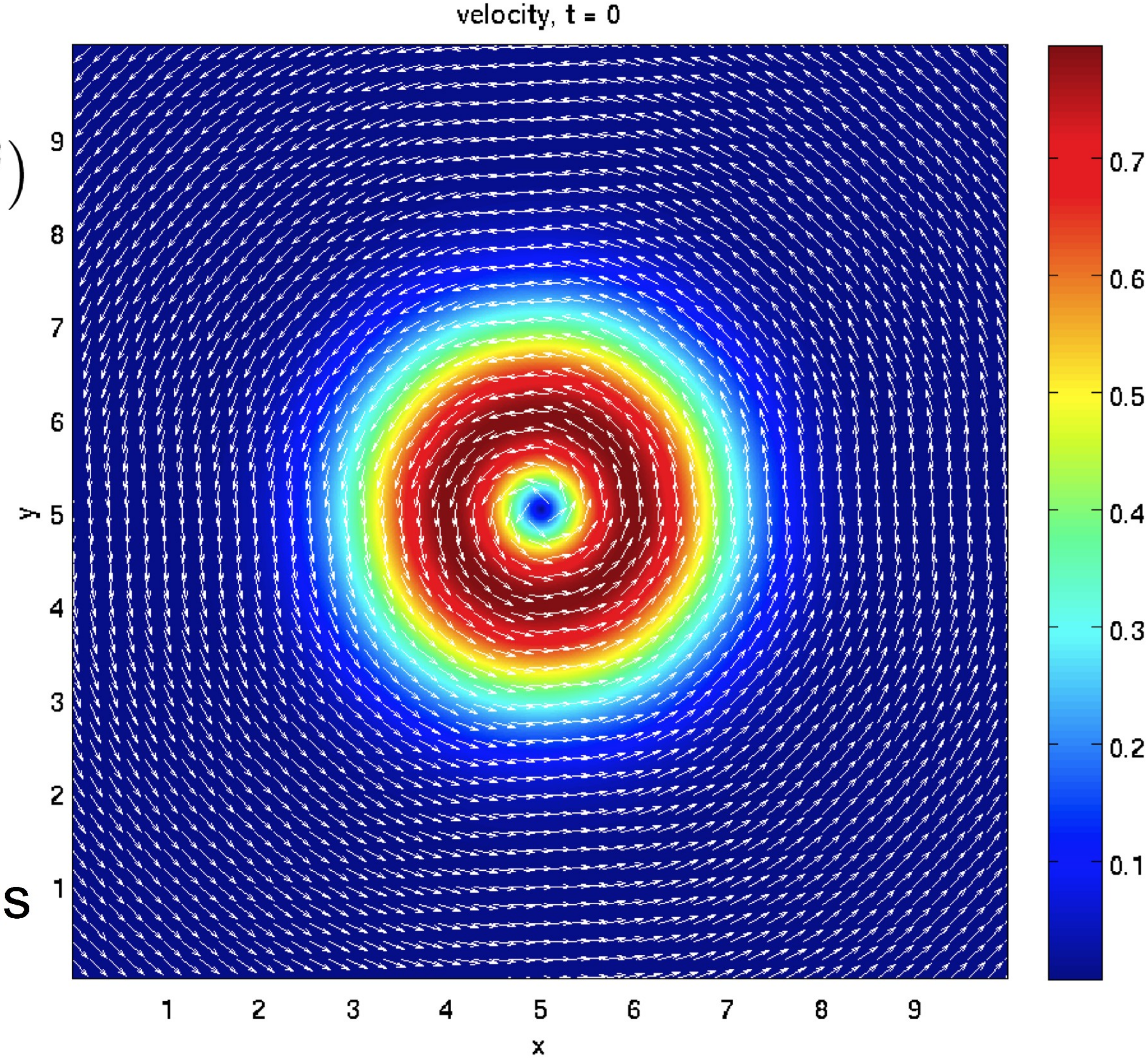
$$T = T_0 - \frac{\beta^2}{8\pi^2}\frac{\gamma - 1}{\gamma}\exp\left(1 - r^2\right)$$

$$v_x = -\frac{\beta}{2\pi}\exp\left(\frac{1 - r^2}{2}\right)y$$

$$v_y = +\frac{\beta}{2\pi}\exp\left(\frac{1 - r^2}{2}\right)x$$

$$\beta = 5, \gamma = 1.4$$

+ solid wall boundary conditions



Angular momentum preserving schemes

- Isentropic vortex e.g. Yee et al. 1999

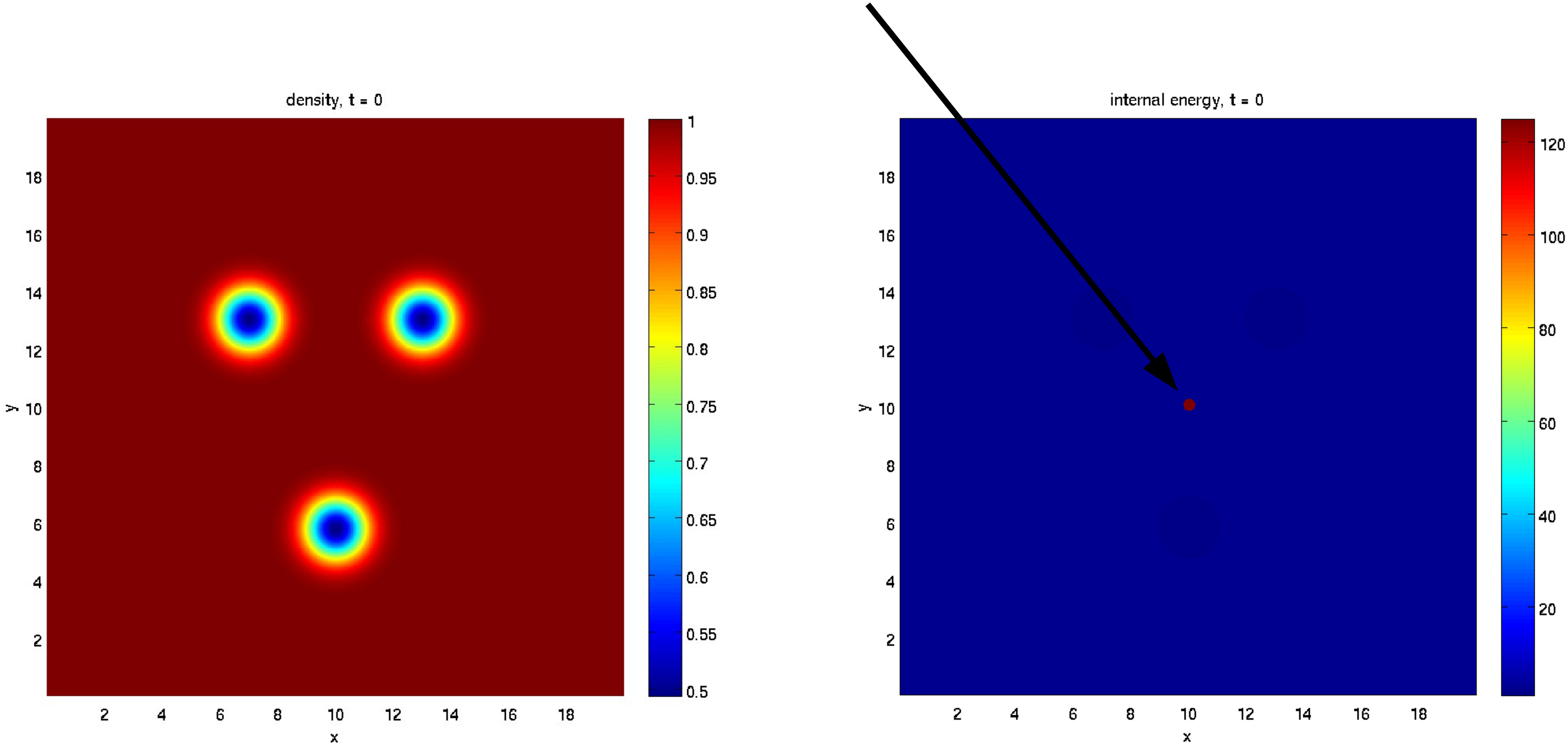
Angular momentum conservation: $\left| \frac{L_z(t_{\mathrm{end}})}{L_z(0)} - 1 \right|$

N	DS1	Unsplit1	GMD1	GMD1CP	DS2	Unsplit2	GMD2	GMD2CP
32	7.55E-02	7.83E-02	7.87E-02	**4.44E-16**	2.75E-02	2.64E-02	1.67E-02	**1.67E-15**
64	3.39E-02	4.07E-02	4.08E-02	**1.78E-15**	3.08E-03	2.83E-03	1.33E-03	**3.33E-15**
128	2.02E-02	2.80E-02	2.80E-02	**1.55E-15**	4.10E-04	6.13E-04	5.24E-04	**3.33E-15**
256	1.19E-02	1.87E-02	1.87E-02	**4.44E-16**	1.65E-04	3.37E-04	7.94E-05	**8.44E-15**
512	6.43E-03	1.13E-02	1.13E-02	**3.77E-15**	9.18E-05	1.17E-04	2.09E-05	**1.27E-14**
1024	3.32E-03	6.23E-03	6.23E-03	**3.38E-14**	3.05E-05	4.36E-05	4.08E-06	**2.24E-14**

1st order (DS1, Unsplit1, GMD1, GMD1CP) **2nd order** (DS2, Unsplit2, GMD2, GMD2CP)

Angular momentum preserving schemes

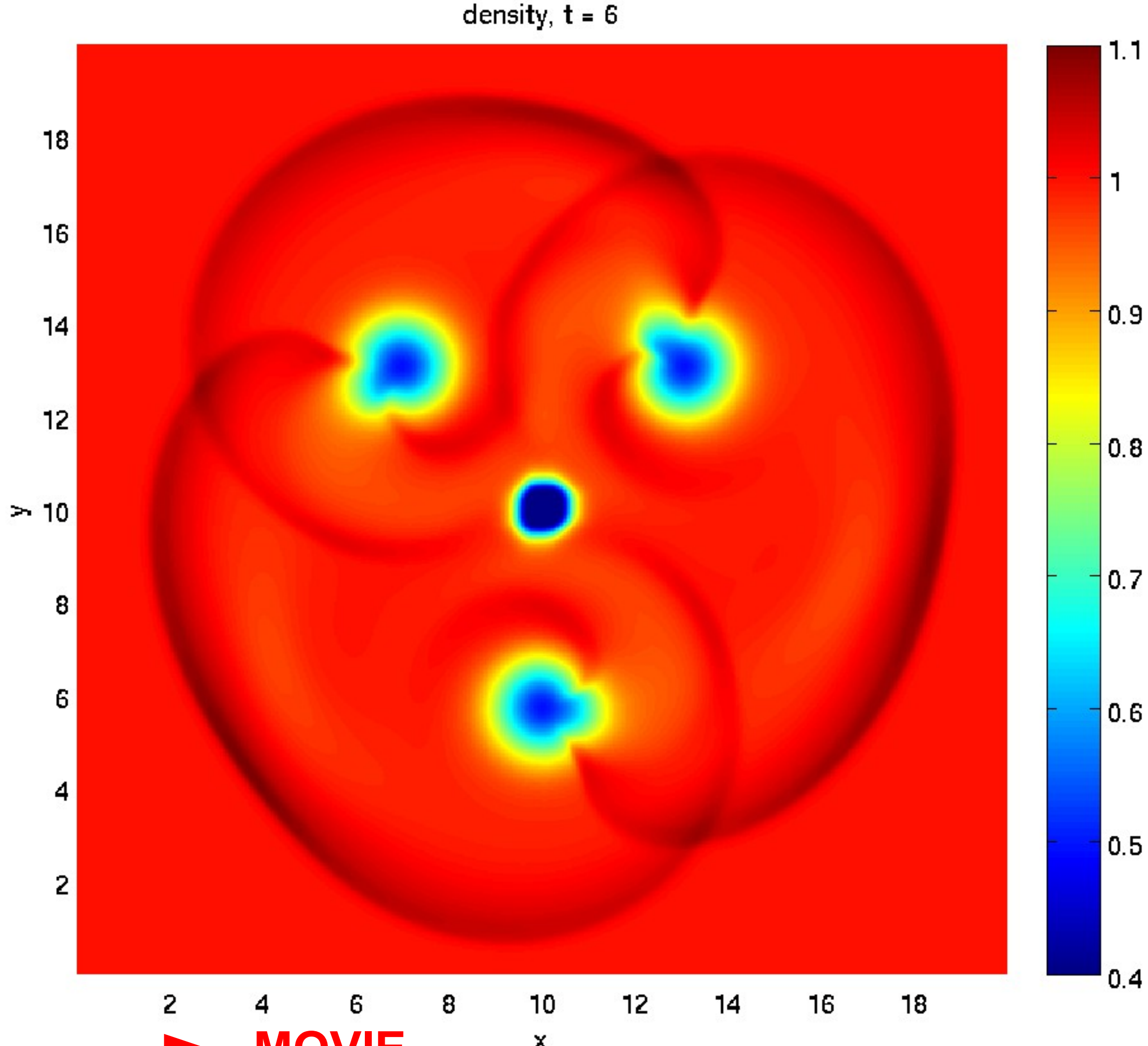
- Isentropic vortices + explosion



→ **MOVIE**

Angular momentum preserving schemes

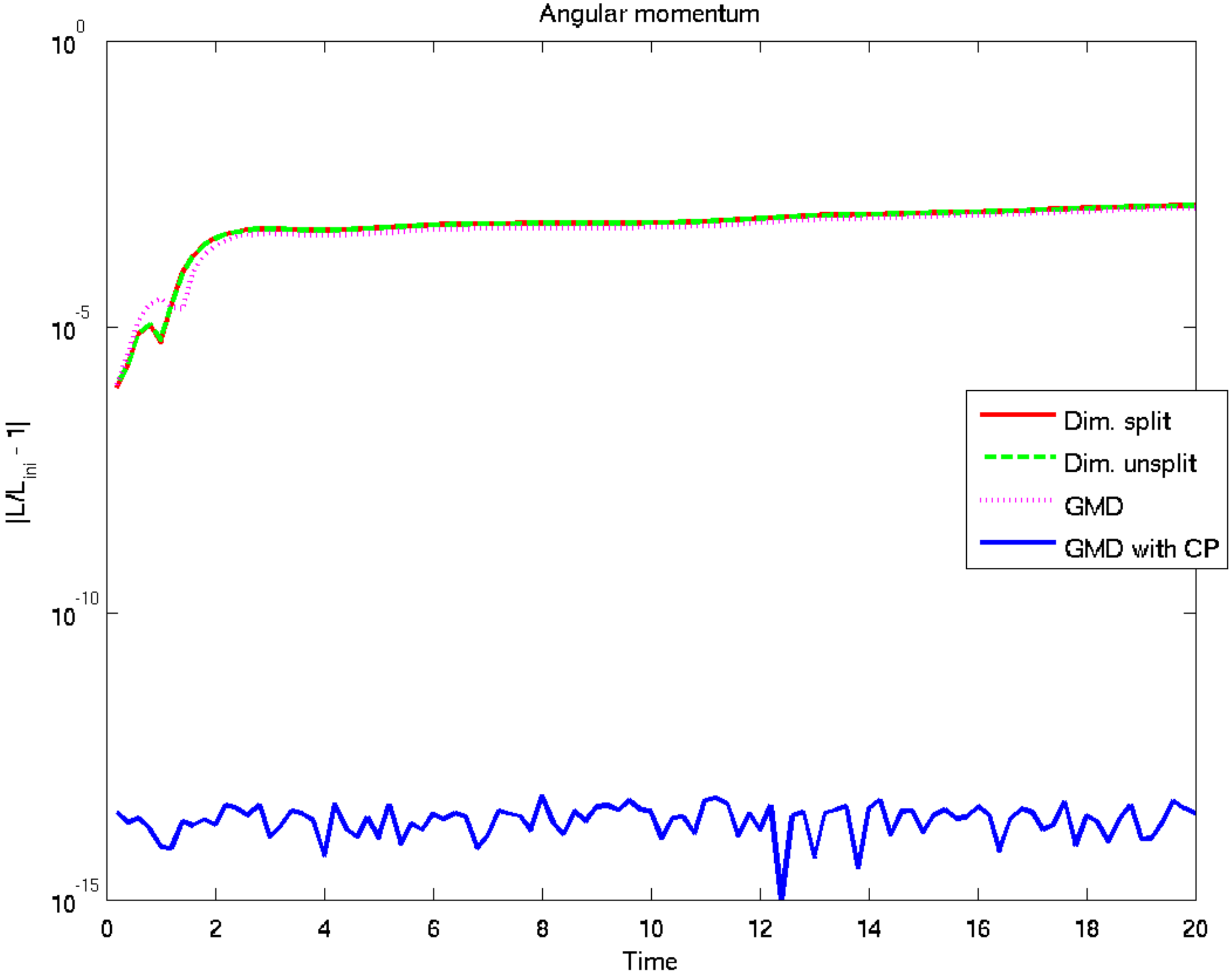
- Isentropic vortices + explosion



MOVIE

Angular momentum preserving schemes

- Isentropic vortices + explosion



Conclusions

- Preserving certain structures discretely is essential in many relevant (astrophysical) applications
- Preserving certain (geometric) symmetry properties seems to be crucial
- Multi-D well-balanced scheme for (isentropic) hydrostatic equilibrium (for general Equation of State)

Käppeli & Mishra Submitted
http://www.sam.math.ethz.ch/sam_reports/index.php?id=2013-05

- A Godunov-type finite volume scheme which conserves angular momentum (to machine precision)
- Including (self-) gravitational source terms and extension to MHD, (RHD, RMHD) ... is ongoing

Käppeli et al. in prep.

- Higher-order, mesh refinement ...

Thank you for your attention!!!

Example 4

Po

H

E

Ta

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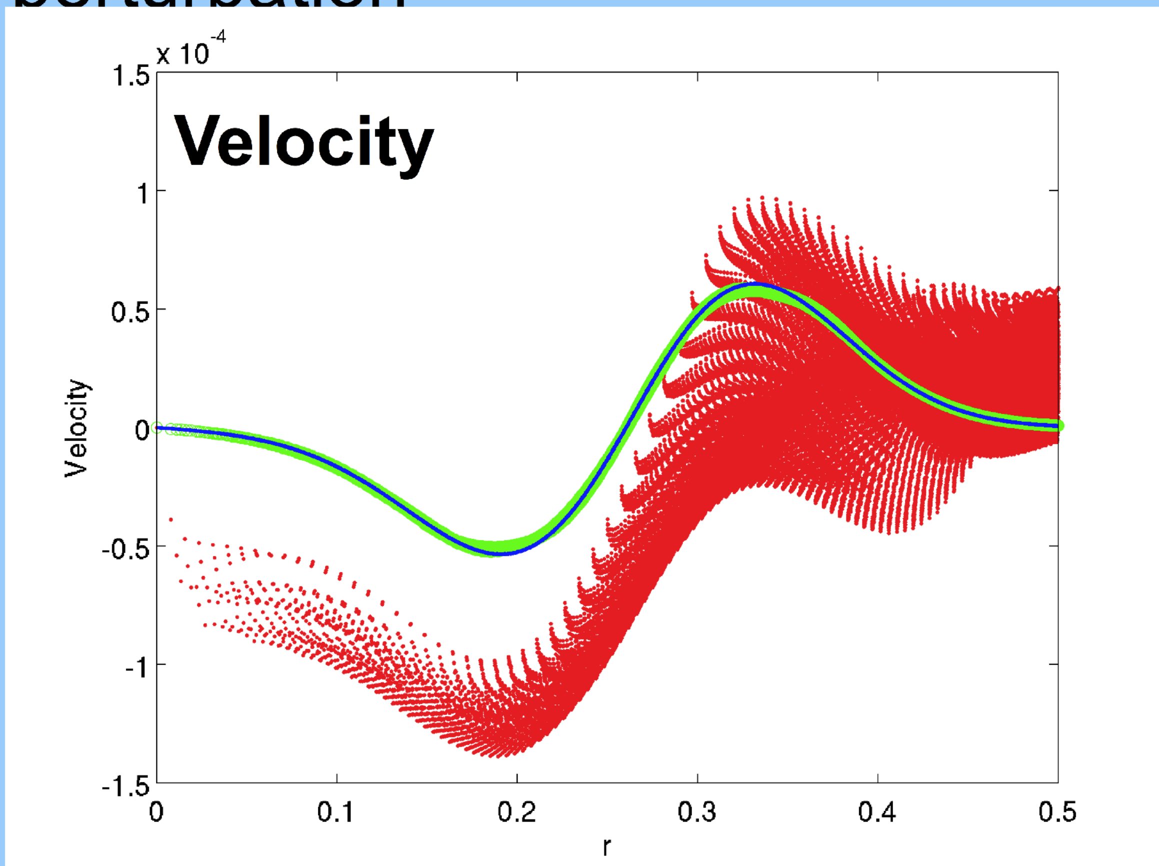
ρ

Small perturbation

$$\rho(\boldsymbol{x}) = \rho_0(\boldsymbol{x}) + Ae^{-100\boldsymbol{x}^2} \quad A = 10^{-3}$$



NO HSE

WITH HSE

Reference

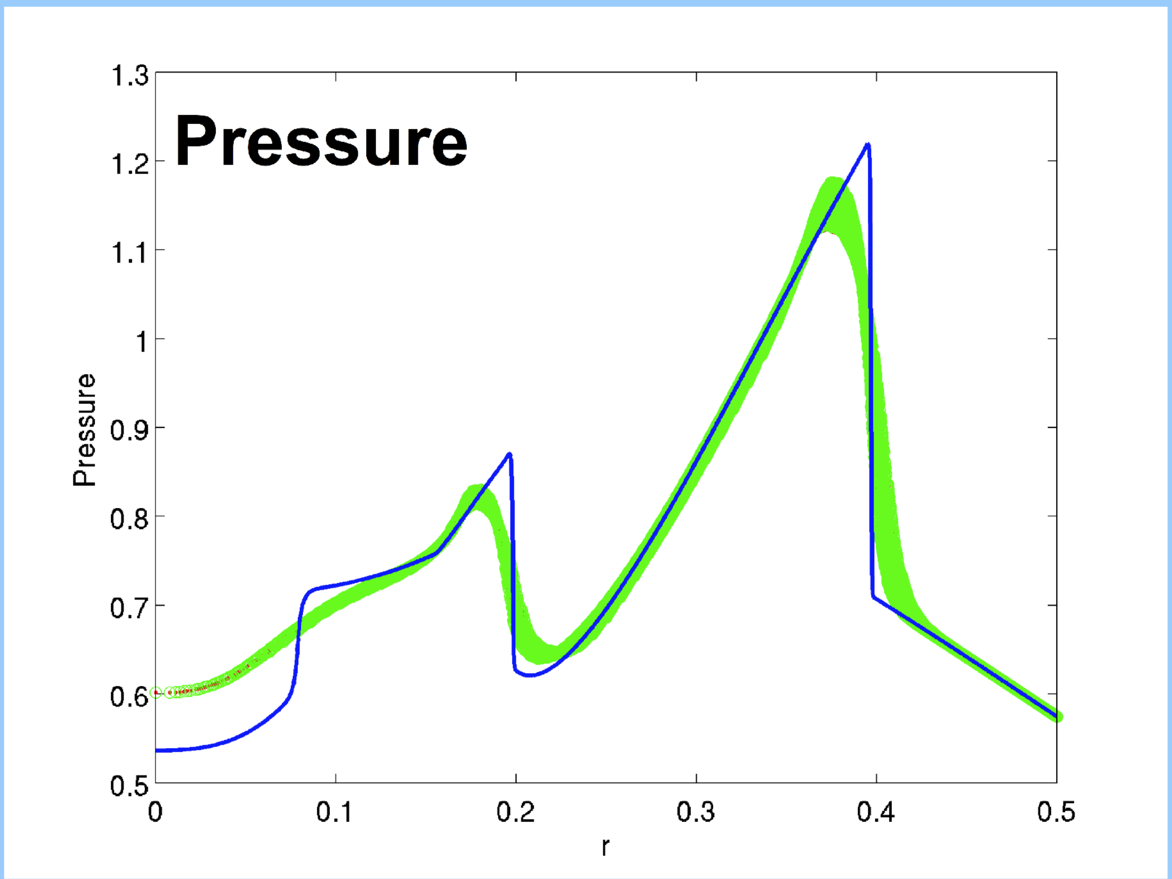
Error in velocity:

N	2ndTVD
32	6.0E-04 / **8.8E-07**
64	1.6E-04 / **3.0E-07**
128	4.1E-05 / **8.5E-08**
rate	1.92 / **1.69**

!

Example 4

Large perturbation (detonation!)



NO HSE
WITH HSE
Reference

Error in pressure:

N	2ndTVD
32	3.3E-02 / **3.3E-02**
64	1.8E-02 / **1.8E-02**
128	9.6E-03 / **9.5E-03**
rate	0.90 / **0.89**

Example 4

Po

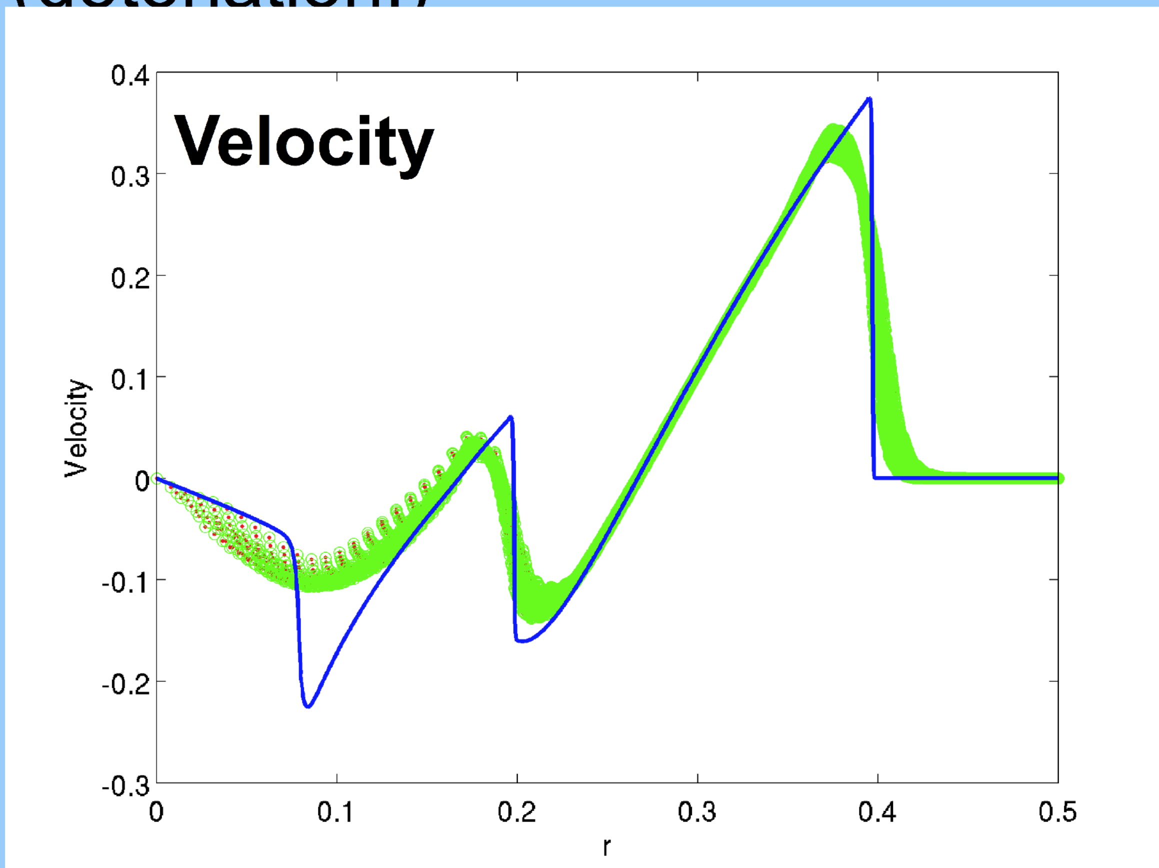
H ρ

E

Ta

Th

Large perturbation (detonation!)



NO HSE

WITH HSE

Reference

Error in velocity:

N	2ndTVD
32	2.9E-02 / **2.8E-02**
64	1.5E-02 / **1.4E-02**
128	7.6E-03 / **7.6E-03**
rate	0.96 / **0.94**

Angular momentum preserving schemes

- Isentropic vortex e.g. Yee et al. 1999

$$j_z = \rho(x v_y - y v_x)$$

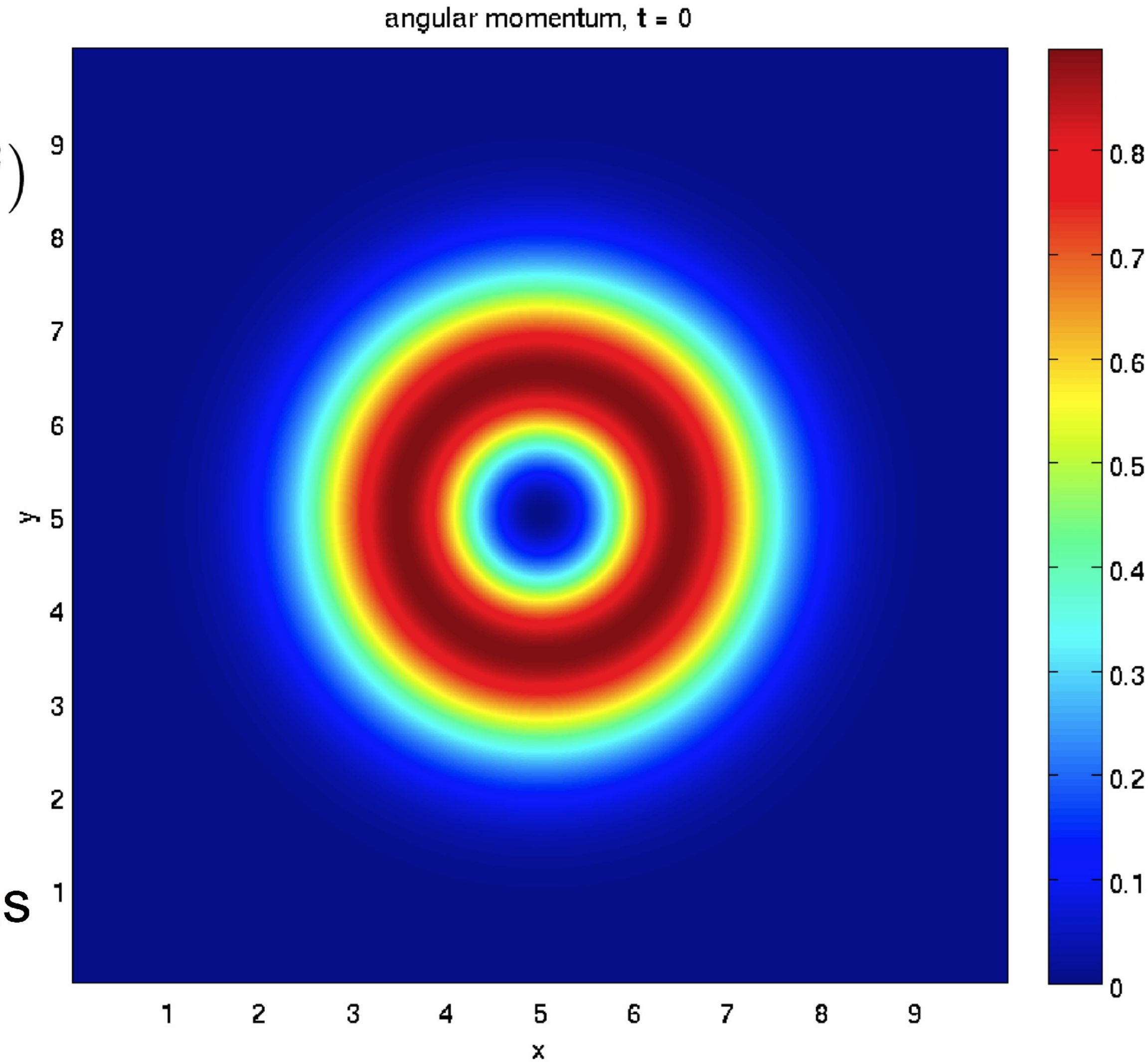
$$T = T_0 - \frac{\beta^2}{8\pi^2}\frac{\gamma - 1}{\gamma}\exp\left(1 - r^2\right)$$

$$v_x = -\frac{\beta}{2\pi}\exp\left(\frac{1 - r^2}{2}\right) y$$

$$v_y = +\frac{\beta}{2\pi}\exp\left(\frac{1 - r^2}{2}\right) x$$

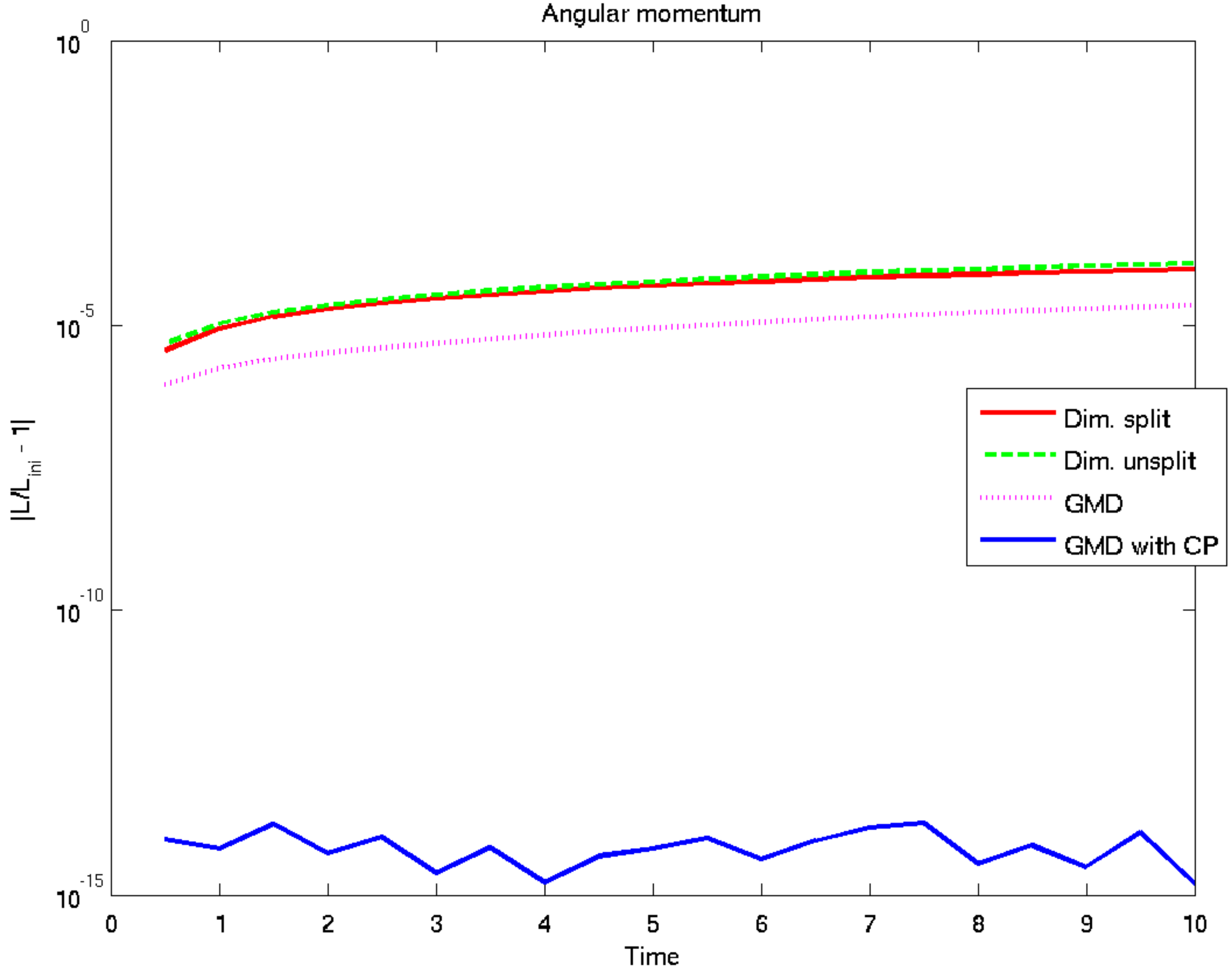
$$\beta = 5, \gamma = 1.4$$

+ solid wall boundary conditions



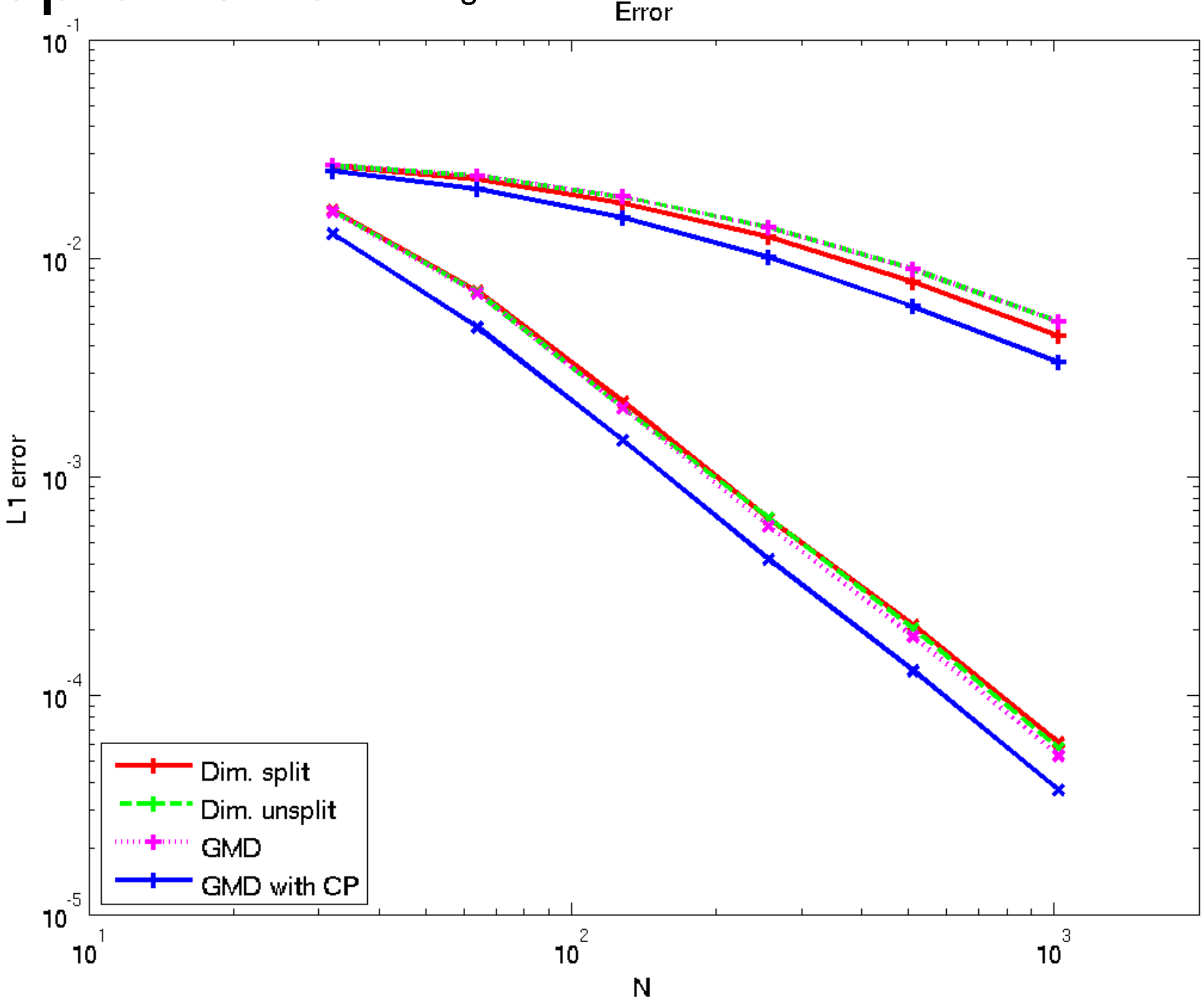
Angular momentum preserving schemes

- Isentropic vortex e.g. Yee et al. 1999



Angular momentum preserving schemes

- Isentropic vortex e.g. Yee et al. 1999



Angular momentum preserving schemes

$$\frac{\partial j}{\partial t} + \frac{\partial}{\partial x}\left(xF_3 - yF_2\right) + \frac{\partial}{\partial y}\left(xG_3 - yG_2\right) = F_3 - G_2 = 0$$

...

$$\begin{aligned}\frac{\mathrm{d}j_{i,j}}{\mathrm{d}t} &= -\frac{1}{\Delta x}\left(x_{i+1/2}(F_3)_{i+1/2,j} - x_{i-1/2}(F_3)_{i-1/2,j}\right) - \frac{1}{\Delta x}\left(y_j(F_2)_{i+1/2,j} - y_j(F_2)_{i-1/2,j}\right)\\ &- \frac{1}{\Delta y}\left(x_i(G_3)_{i,j+1/2} - x_i(G_3)_{i,j-1/2}\right) - \frac{1}{\Delta y}\left(y_{j+1/2}(G_2)_{i,j+1/2} - y_{j-1/2}(G_2)_{i,j-1/2}\right)\\ &+ \frac{1}{2}\left((F_3)_{i-1/2,j} + (F_3)_{i+1/2,j}\right) - \frac{1}{2}\left((G_2)_{i,j-1/2} + (G_2)_{i,j+1/2}\right)\end{aligned}$$