On the existence and non-uniqueness of solutions of Riemann problems in ideal magnetohydrodynamics

Kazuya Takahashi @Waseda Univ.

Co-worker: Shoichi Yamada @Waseda Univ.

@ASTRONUM 2013

Outline

- **Introduction**: intermediate shocks in MHD

  Non-uniqueness; Evolutionary conditions;
  Intermediate shocks

- **Aim & Method**: Exact MHD Riemann solver

  Algorithm to handle the intermediate shock and find the solutions

- **Results**: intriguing solutions for some MHD Riemann problems

  *Uncountably many solutions that includes an intermediate shock for the Brio & Wu problem and its neighborhood;
  *Discovery of an initial condition that does not have any solutions without non-evolutionary shocks

Ideal MHD

MHD (MagnetoHydroDynamics)

- Plasma -> fluid; interaction with magnetic field
- Basic eqs. :
  - conservation laws
    - mass
    - momentum
    - energy
  - Maxwell eqs.
  - Ohm's law
- neglecting dissipation & infinite electrical conductivity

→ Ideal MHD eqs（nonlinear hyperbolic eqs）

Riemann problem

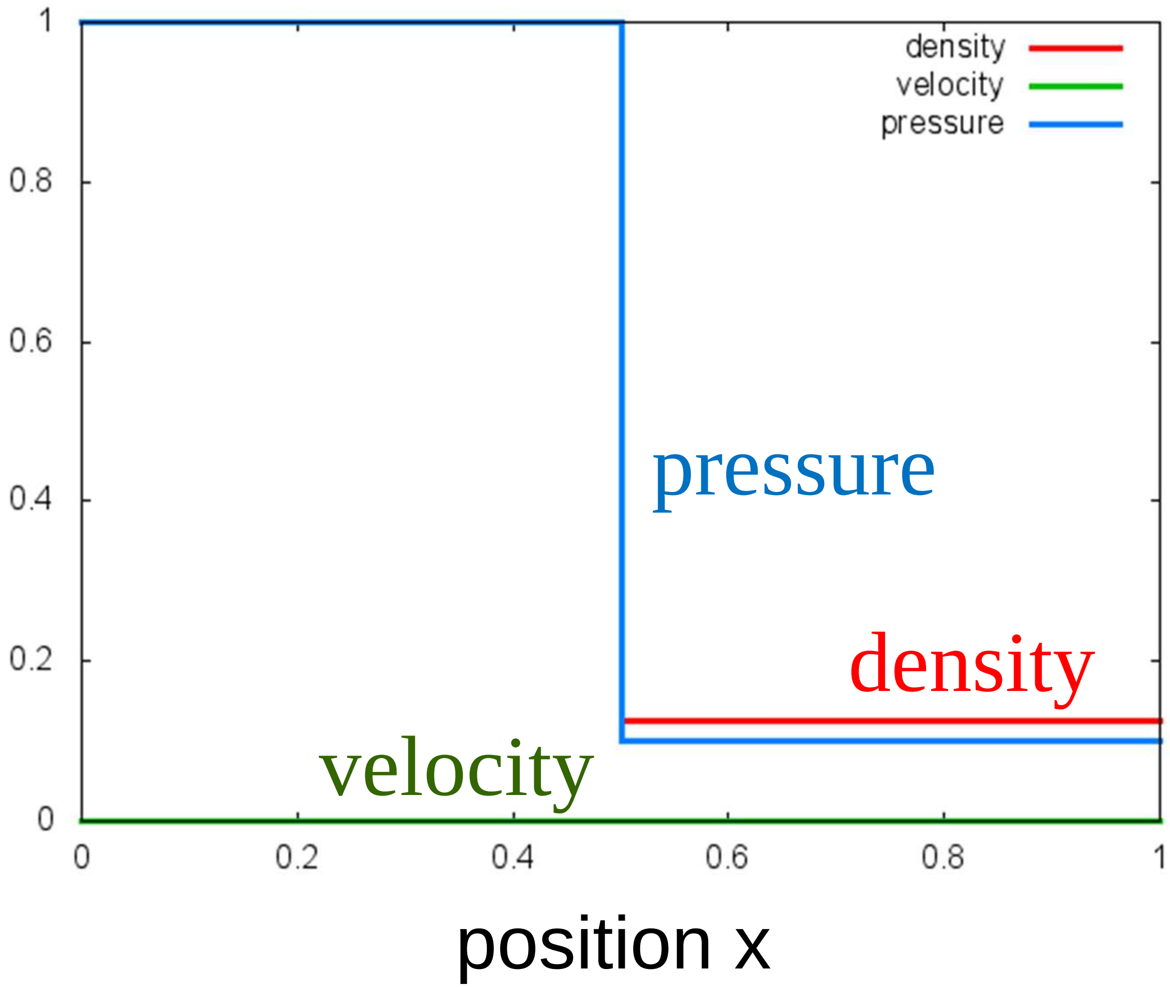
- An initial value problem for the initial conditions, which consist of the constant states divided by a discontinuity

discontinuity

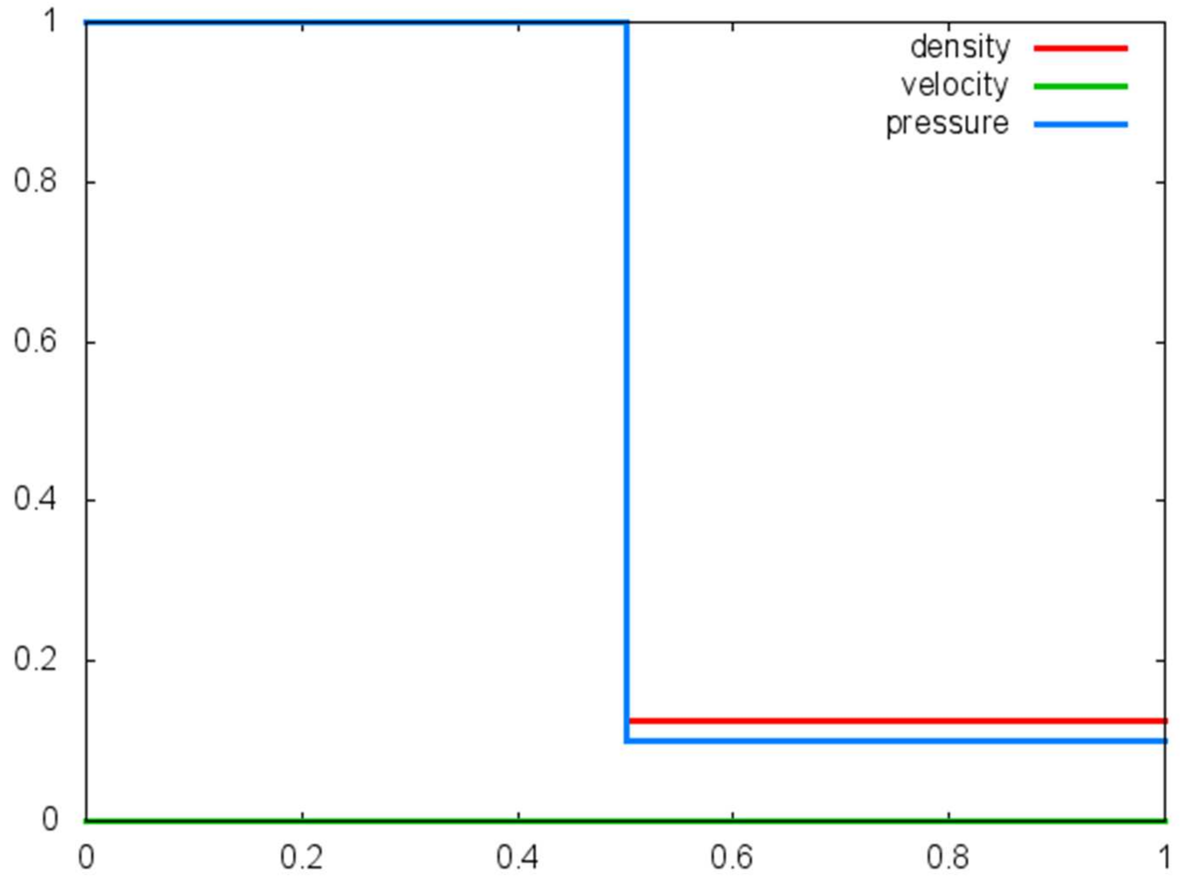
$$\rho_L, p_L, \boldsymbol{v}_L, \boldsymbol{B}_L \qquad | \qquad \rho_R, p_R, \boldsymbol{v}_R, \boldsymbol{B}_R$$

ρ :density p :pressure $\boldsymbol{v}$:velocity $\boldsymbol{B}$:magnetic field

Riemann problem in hydrodynamics

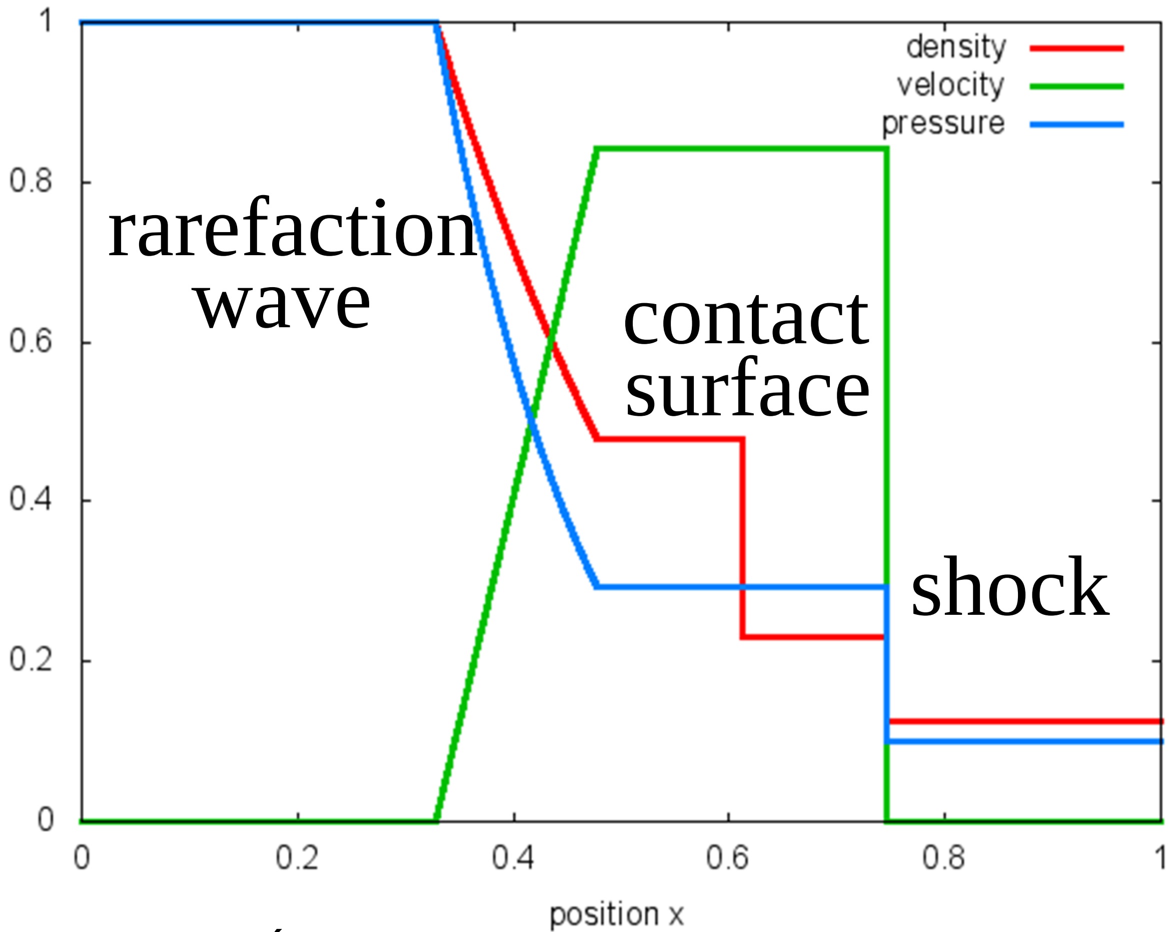


Riemann problem in hydrodynamics



position x

Solutions of Riemann problem



- Self-similar solution（∵there is no typical scale of time nor space）
- The solutions consist of discontinuities and centered simple waves
- The maximum number of waves = #(eigenvalues of the system)

Non-uniqueness of the solutions

The solutions of Riemann problems are ***generally not unique!***

In HD…

By introducing the entropy condition and discarding the manifestly unphysical waves (e.g. expanding shocks), the solution is uniquely determined.

Non-uniqueness of the solutions

The solutions of Riemann problems are ***generally not unique!***

In HD…

By introducing the entropy condition and discarding the manifestly unphysical waves (e.g. expanding shocks), the solution is uniquely determined.

However, in MHD…

The entropy condition is insufficient!

i.e., ***more than one solutions satisfy the entropy condition.***

(as you see later)

Non-uniqueness of the solutions

The solutions of Riemann problems are ***generally not unique!***

In HD…

By introducing the entropy condition and discarding the manifestly unphysical waves (e.g. expanding shocks), the solution is uniquely determined.

However, in MHD…

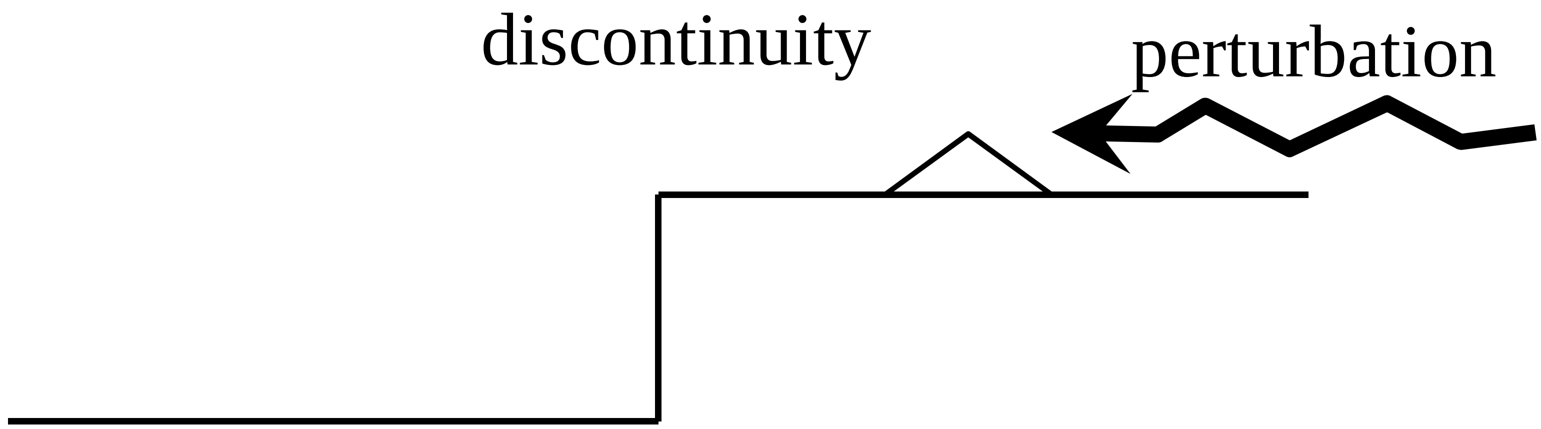
The entropy condition is insufficient!

i.e., ***more than one solutions satisfy the entropy condition.***

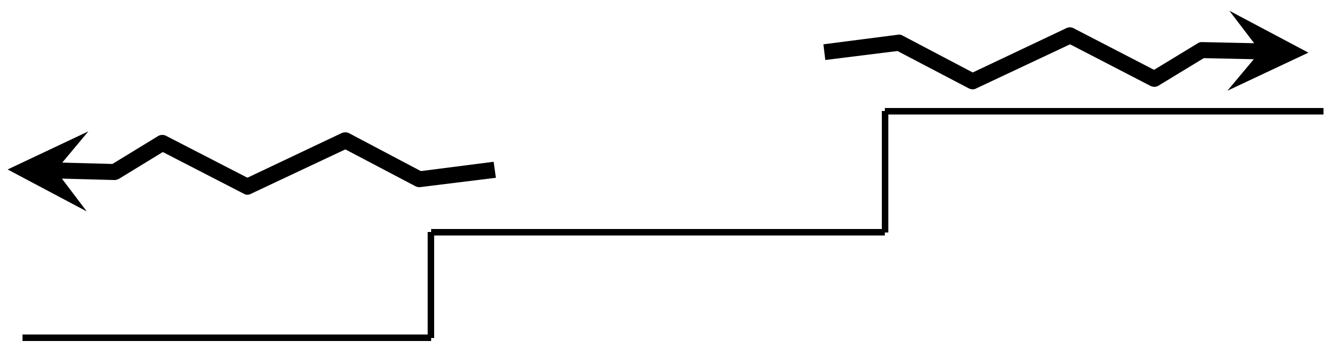
How can we single out the solution uniquely?
→ Evolutionary conditions

Evolutionary conditions (e.g. Jeffrey & Taniuti 1964)

Discontinuities should be structurally stable

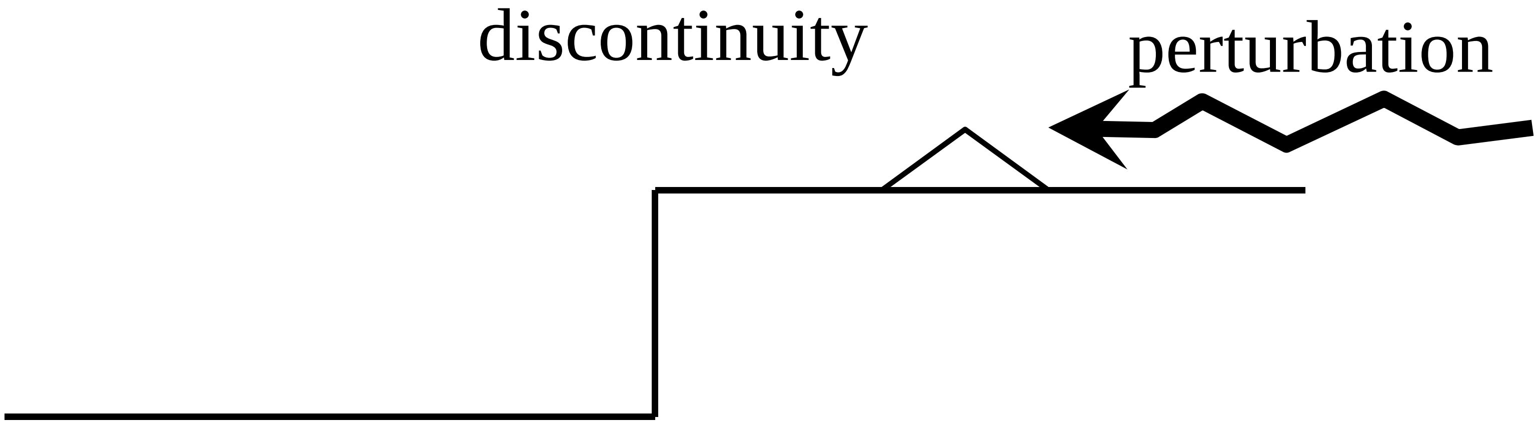


Structurally unstable: instantaneously split into other waves

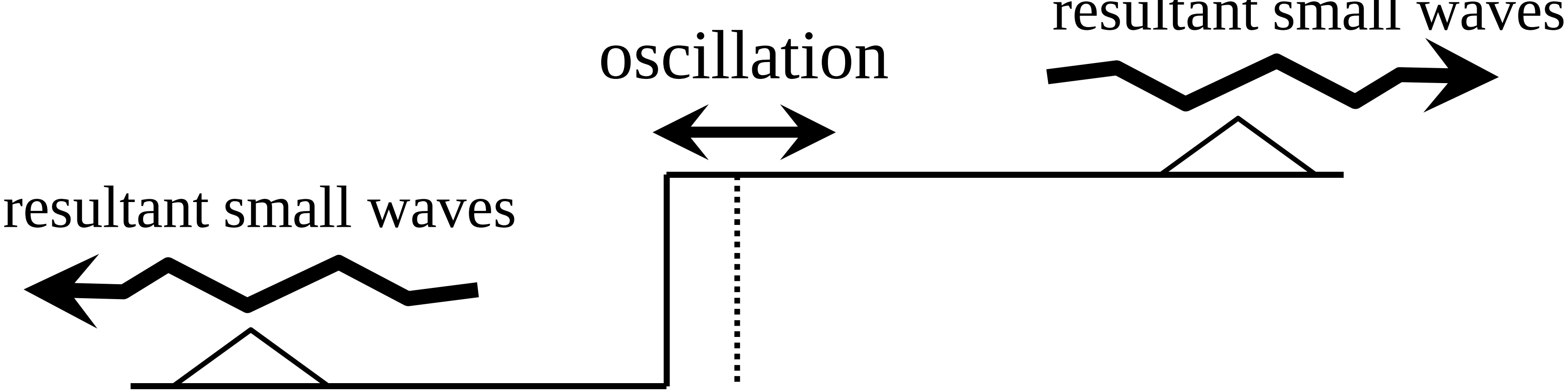


Evolutionary conditions (e.g. Jeffrey & Taniuti 1964)

Discontinuities should be structurally stable



Structurally stable: maintain the structure



Formulation of the evolutionary conditions

The Rankine-Hugoniot relations:

$$S[\![\boldsymbol{U}]\!] - [\![\boldsymbol{F}]\!] = \boldsymbol{0}$$

$$[\![X]\!] := X_{\mathrm{front}} - X_{\mathrm{behind}}$$

S :velocity of the discontinuity

linearization:

$$\delta S = \delta s\, e^{i\omega t}$$

$$\delta \boldsymbol{U} = \sum_{k=1}^{n} \delta a_k \boldsymbol{r}_k \exp\left[i\omega \left(t - \frac{x}{\mu_k} \right) \right]$$

$\mu_k, \boldsymbol{r}_k$:eigenvalues and eigenvectors

The evolutionary conditions require that the amplitudes of the resultant waves and oscillation are uniquely determined

After some calculations… It is concluded:

Evolutionary conditions (e.g. Jeffrey & Taniuti 1964)

1. #(characteristics fanning out from the discontinuity) = $N-1$,

   where N = #(independent linearized Rankine-Hugoniot eqs.)

2. Eigenvectors corresponding to the fanning out characteristics and jumps of the conservative quantities at the discontinuity $[\![U]\!]$ are linearly independent

Condition 1. is deduced from

#(eqs) =

#(unknown variables = amp. of oscillation and resultant waves)

‖

#(outgoing characteristics)

Condition 2. is deduced from the coefficient matrix is not singular

Discontinuities in ideal MHD $(B_n \neq 0)$

Rankine-Hugoniot relations for 1D ideal MHD :

$$m = \text{const.},$$

$$m^2[\![v]\!] + \left[\!\!\left[p + \frac{\boldsymbol{B}_t^2}{2} \right]\!\!\right] = 0,$$

$$m[\![\boldsymbol{v}_t]\!] - B_n[\![\boldsymbol{B}_t]\!] = \boldsymbol{0},$$

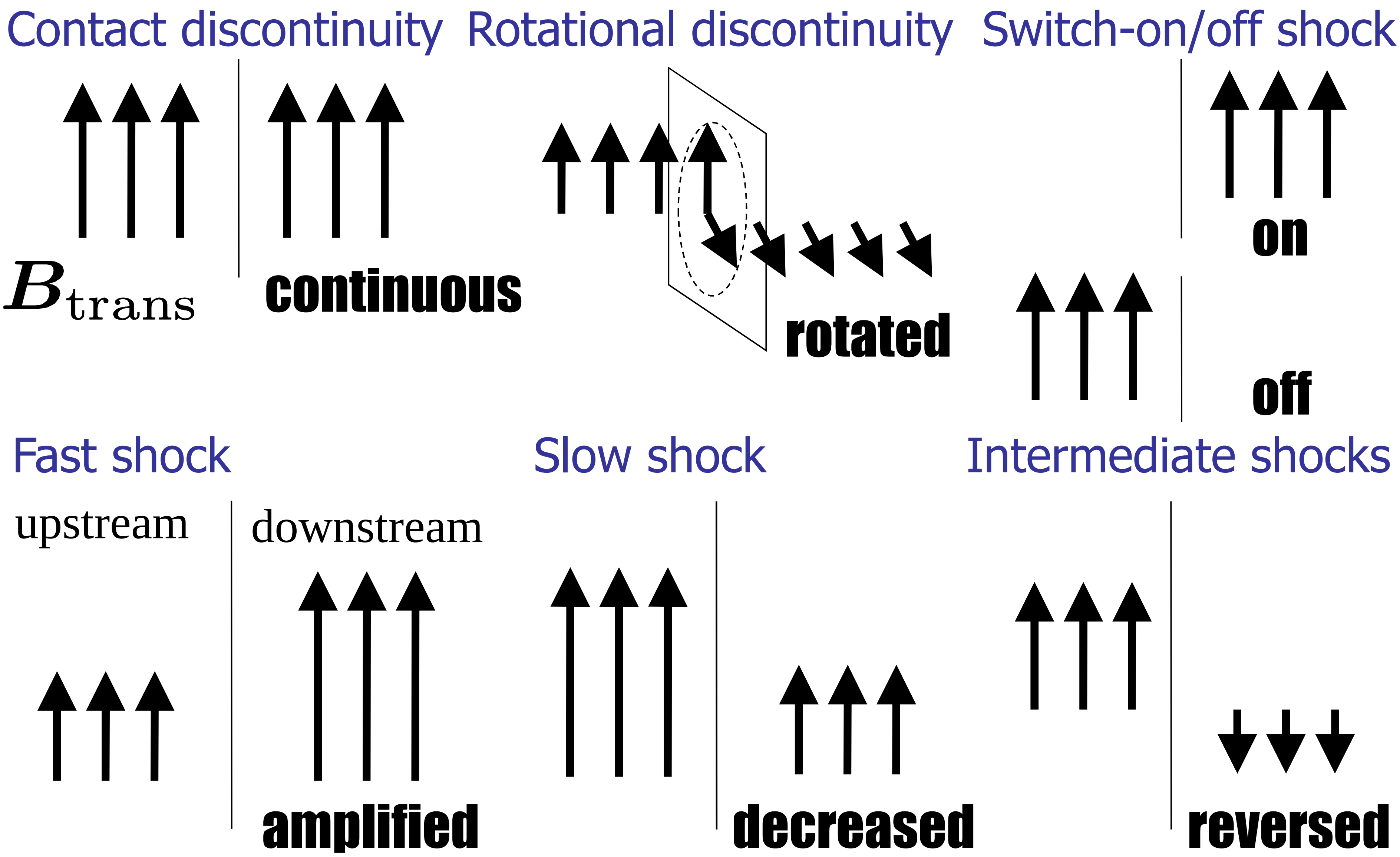
$$m[\![v\boldsymbol{B}_t]\!] - B_n[\![\boldsymbol{v}_t]\!] = \boldsymbol{0},$$

$$m\left(\left[\!\!\left[\frac{pv}{\gamma - 1} \right]\!\!\right] + \langle p \rangle [\![v]\!] + \frac{1}{4}[\![v]\!][\![\boldsymbol{B}_t]\!]^2 \right) = 0,$$

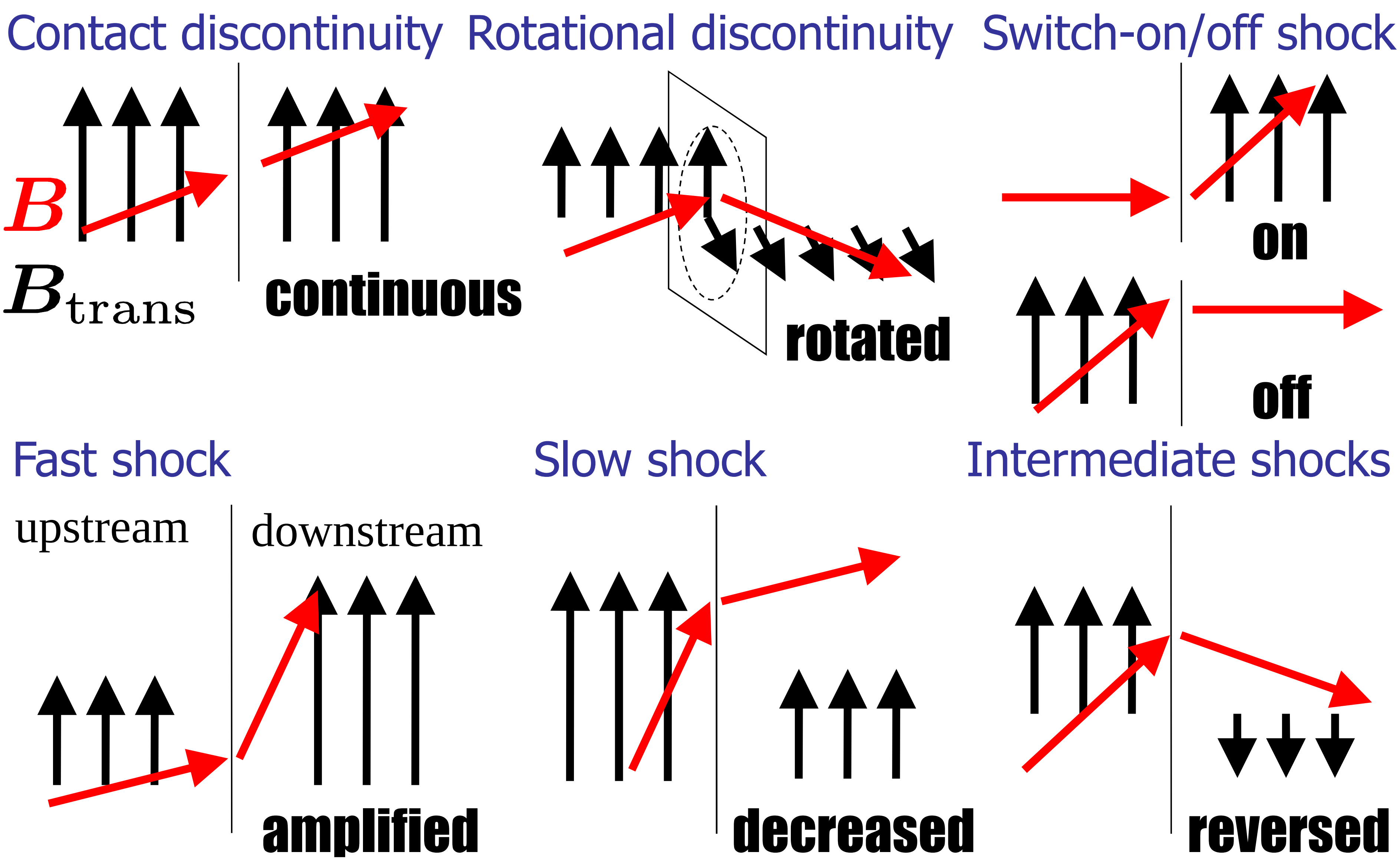
$$m := \rho_0 v_{n0} = \rho_1 v_{n1} \qquad v := 1/\rho$$

$$\langle X \rangle := (X_0 + X_1)/2$$

Discontinuities in ideal MHD $(B_n \neq 0)$



Discontinuities in ideal MHD $(B_n \neq 0)$



Classification of MHD shocks

- MHD shocks are classified based on the upstream and downstream flow speeds

$$1 > c_f > 2 > c_A > 3 > c_s > 4$$

e.g. $v_{\rm front} > c_{f,\rm front}$ and $c_{A,\rm behind} > v_{\rm behind} > c_{s,\rm behind}$

then the shock is called **1 → 3 shock**

- Fast shock = $1 \rightarrow 2$ shock
- Slow shock = $3 \rightarrow 4$ shock

Classification of MHD shocks

- Flow speed of intermediate shocks changes from super-Alfvenic to sub-Alfvenic.

  ⇔ transverse magnetic field is reversed

  which follows the Rankine-Hugoniot relation,

$$\boldsymbol{B}_{t,1} = \frac{v_0^2 - c_{A,0}^2}{v_1^2 - c_{A,1}^2} \boldsymbol{B}_{t,0}$$

$$1 > c_f > 2 > c_A > 3 > c_s > 4$$

Intermediate shocks are largely classified into the 4 families

$1 \rightarrow 3$, $1 \rightarrow 4$, $2 \rightarrow 3$, $2 \rightarrow 4$

sub-class, including switch-on/off shocks

- It is happened that the flow speed is equal to some characteristic speed

$$1 > c_f > 2 > c_A > 3 > c_s > 4$$

e.g. $v_{\mathrm{front}} > c_{f,\mathrm{front}}$ and $v_{\mathrm{behind}} = c_{s,\mathrm{behind}}$

then the shock is called **1 → 3,4 shock**

1→ 3,4   2→ 3,4   1,2→ 3
1,2 →4   1,2 → 3,4   } intermediate shocks

1→ 2,3 and **2,3 →4** ⇒ switch-on/off shocks

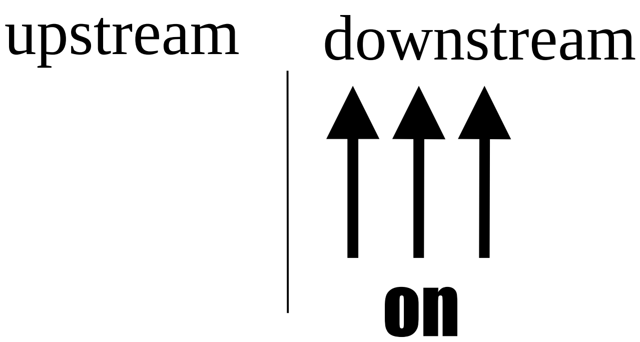
※ **1,2 → 3,4** and **2,3 → 3,4** do not exist in MHD Rankine-Hugoniot relations

Switch-on/off shocks

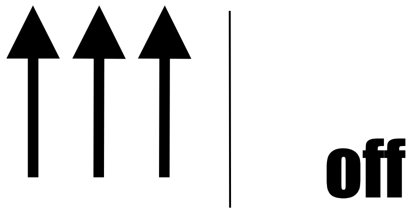
† From the Rankine-Hugoniot relations:

$$(v_1^2 - c_{A,1}^2)\boldsymbol{B}_{t,1} = (v_0^2 - c_{A,0}^2)\boldsymbol{B}_{t,0}$$

- Switch-on shock (1 → 2,3 shock)

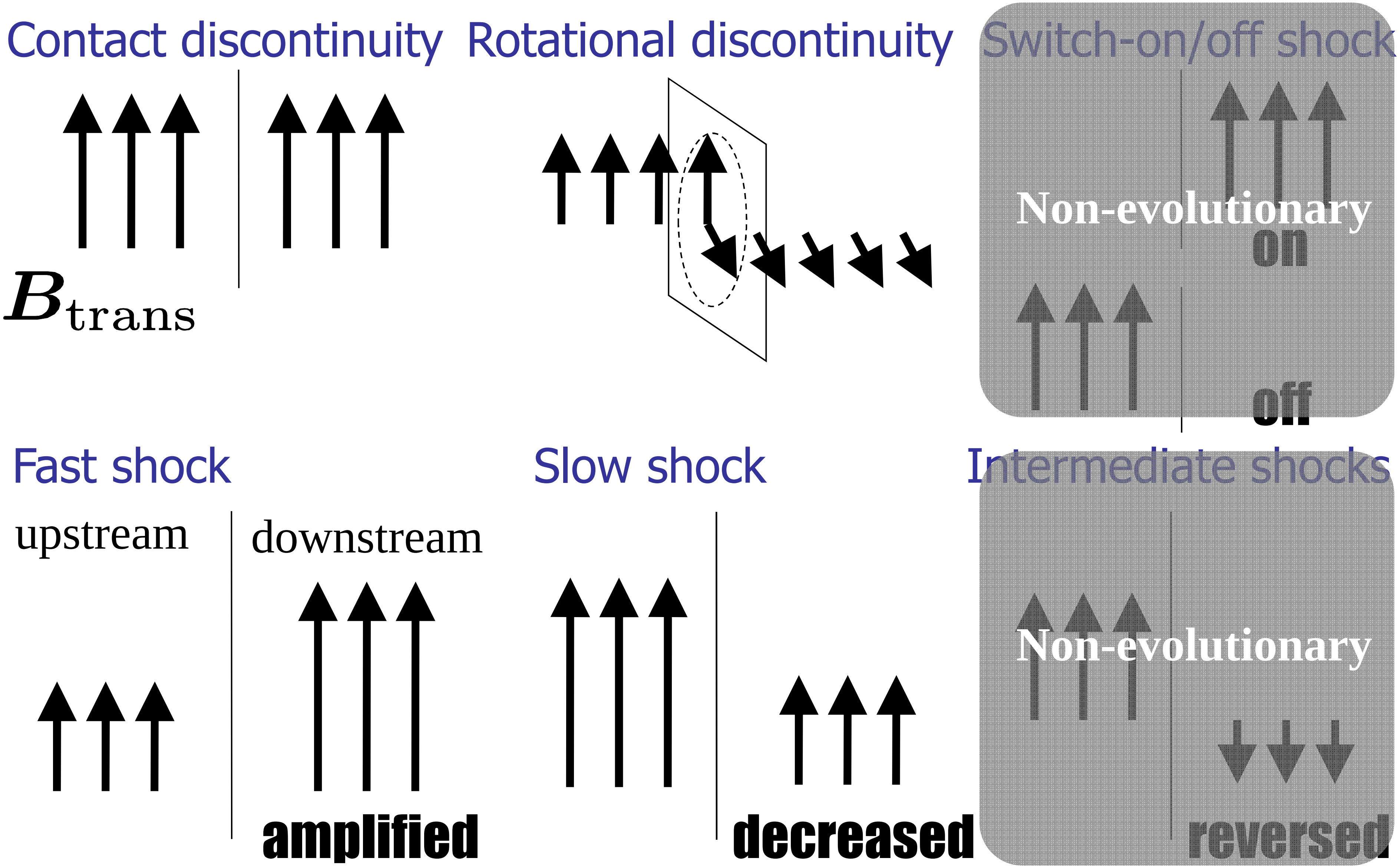


- Switch-off shock (2,3 → 4 shock)



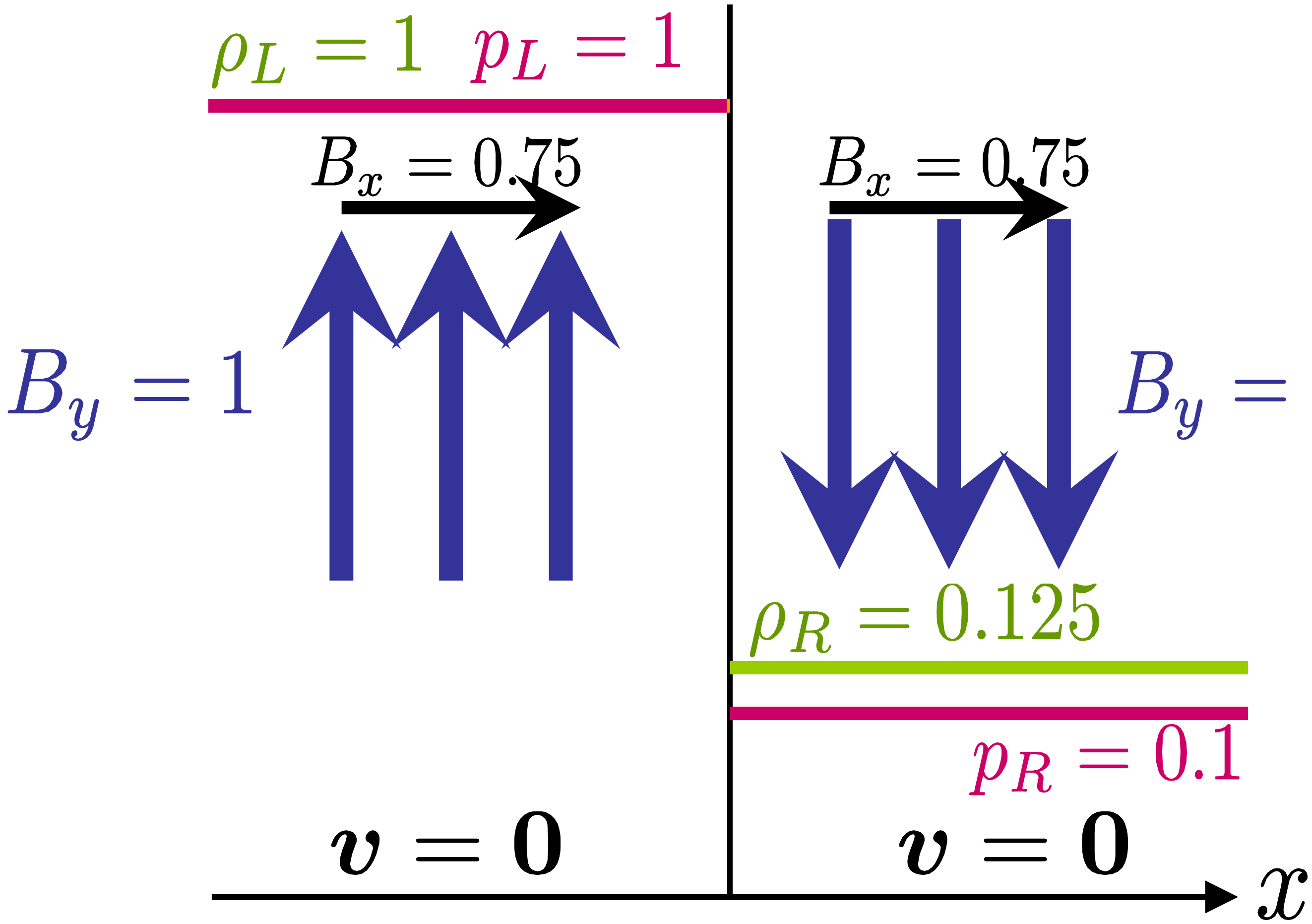
Switch-off shock lies at the boundary
between the slow shock and intermediate shock

Evolutionarity

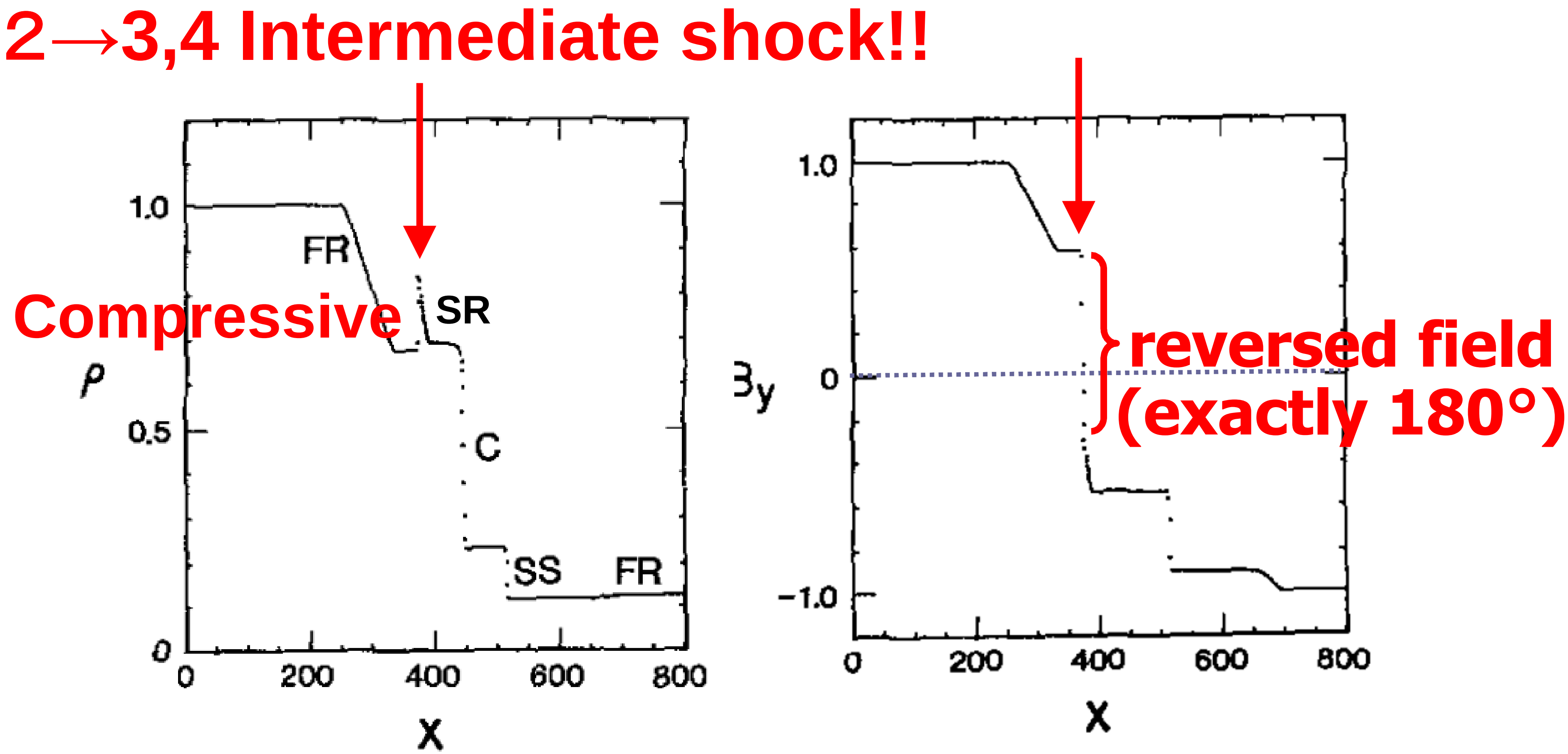


The initial condition of Brio & Wu (1988)

- The intermediate shock is commonly realized in numerical simulations (see next slide)



The intermediate shock in Brio & Wu



※ The rotational discontinuity had been expected for reversing the magnetic field

Why non-evolutionary waves are evolutionary?

Some authors focused on the numerical **dissipation**, noting that the evolutionary condition is valid in IDEAL MHD

cf. Wu (1988a,b; 1990), Wu & Kennel (1992), Hada (1994), Inoue & Inutsuka (2007)

- Intermediate shocks are not determined uniquely by the boundary (the Rankine-Hugoniot) conditions, but they possess the internal structure and corresponding degree of freedom
  ⇒ they are persistent against the wave interactions
- In dissipative MHD, some dissipative modes arise and number of outgoing characteristics is different from ideal MHD
  ⇒ dissipative modes change evolutionarity of the intemediate shocks

Uniqueness?

By the way… when the solution is uniquely determined only by entropy condition?
When should we consider the possibility of the intermediate shocks?

- Torrilhon (2002) "Exact solver and uniqueness conditions for Riemann problems of ideal magnetohydrodynamics"
- Torrilhon (2003) "Uniqueness conditions for Riemann problems of ideal magnetohydrodynamics"

They investigated the uniqueness conditions for solutions in ideal MHD Riemann problems

Lack of Torrilhon (2002;2003)

Their discussions are *incomplete*:

★ Considering only the cases with finite magnetic field.

i.e. neglecting

- initial conditions with vanishing transverse magnetic field
- switch-on/off shocks, switch-on/off rarefactions

★ Analyzing only the case that either of a **1→3 or 2→4** intermediate shock exists in the solution.

☆ No proof of a conjecture, " there is always a unique regular solution for any Riemann problem" .

∗ *regular solutions:*
Including only evolutionary shocks

∗ *non-regular solutions:*
Including non-evolutionary shocks:
intermediate shocks, switch-on/off shocks

Our focus

Aim :

*To test the conjecture that there is always a unique regular solution for MHD Riemann problem

*To investigate the structure of the solution space of MHD Riemann problems

Method:

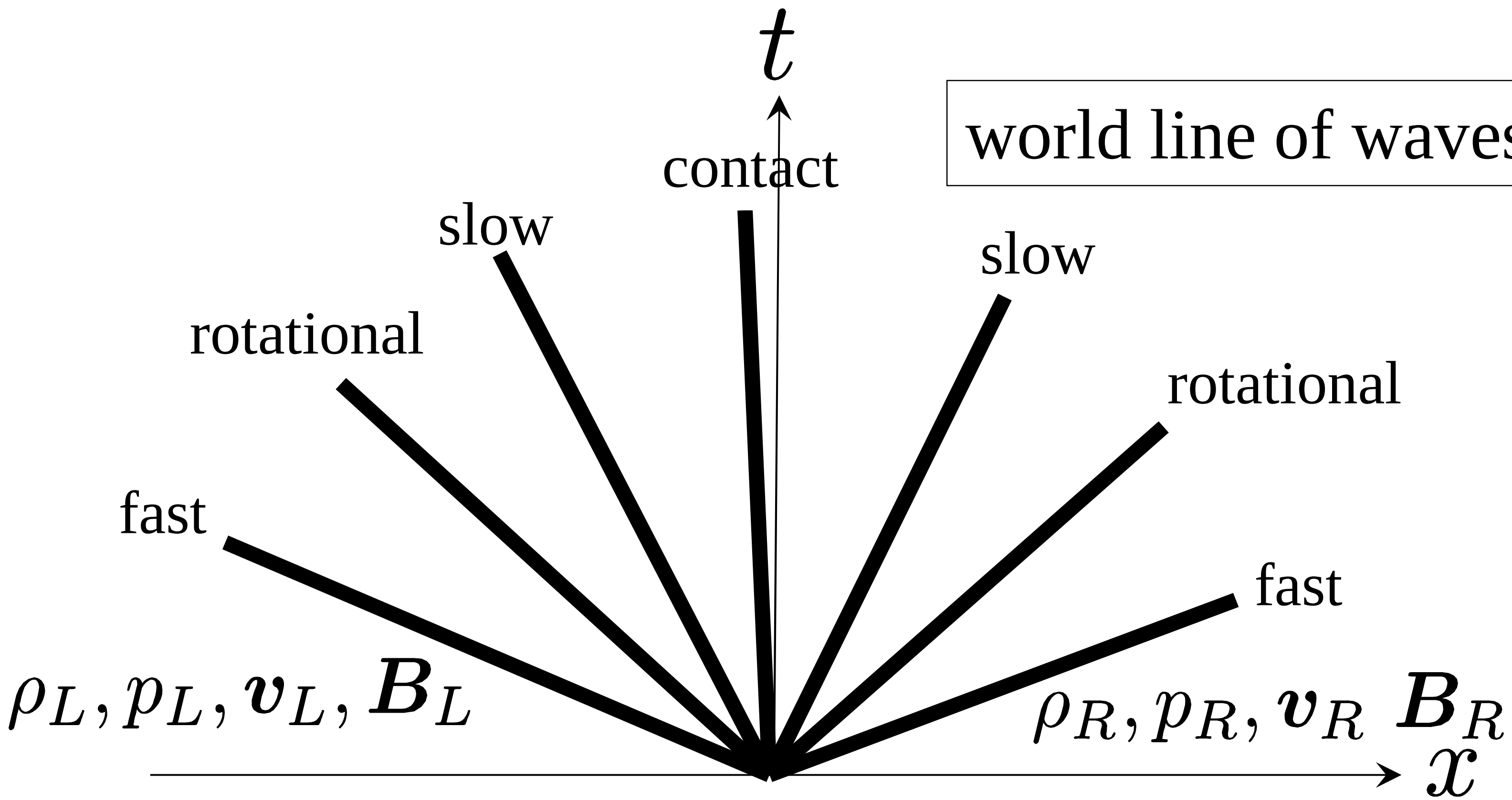
Construct the exact solutions of MHD Riemann problems by the new exact Riemann solver

- can solve any Riemann problem
- can get all of regular and non-regular solutions

※ For our purpose, conventional Riemann solvers are useless since they always realize a single solution

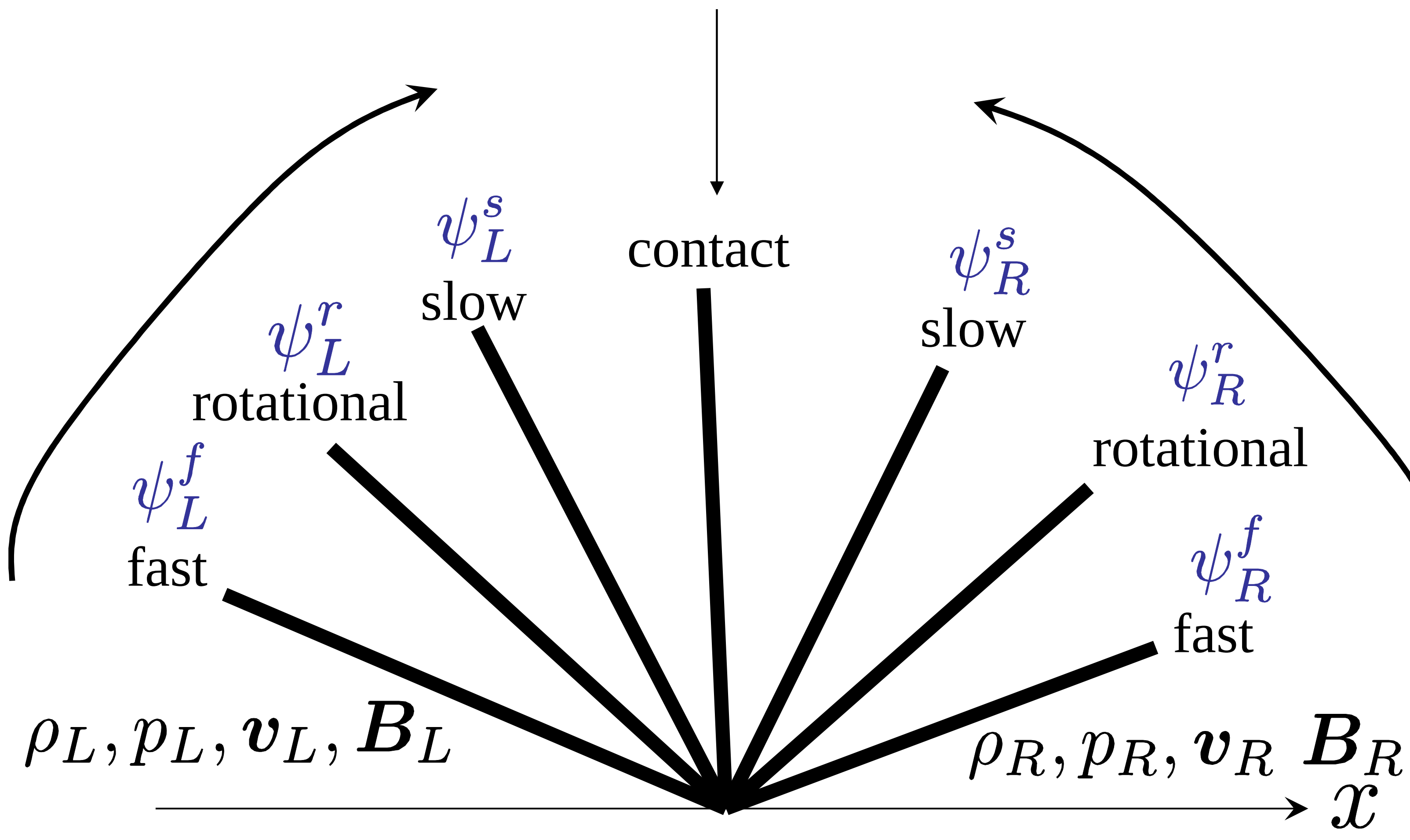
How to find the regular solutions

Without intermediate shocks and switch-on/off waves, the structure of the solutions is ***a priori*** known.



Determine the parameters associated with each wave so that the quantities other than density is continuous across the contact discontinuity.

$$p, v_x, v_y, v_z, B_y, B_z \text{ :continuous}$$

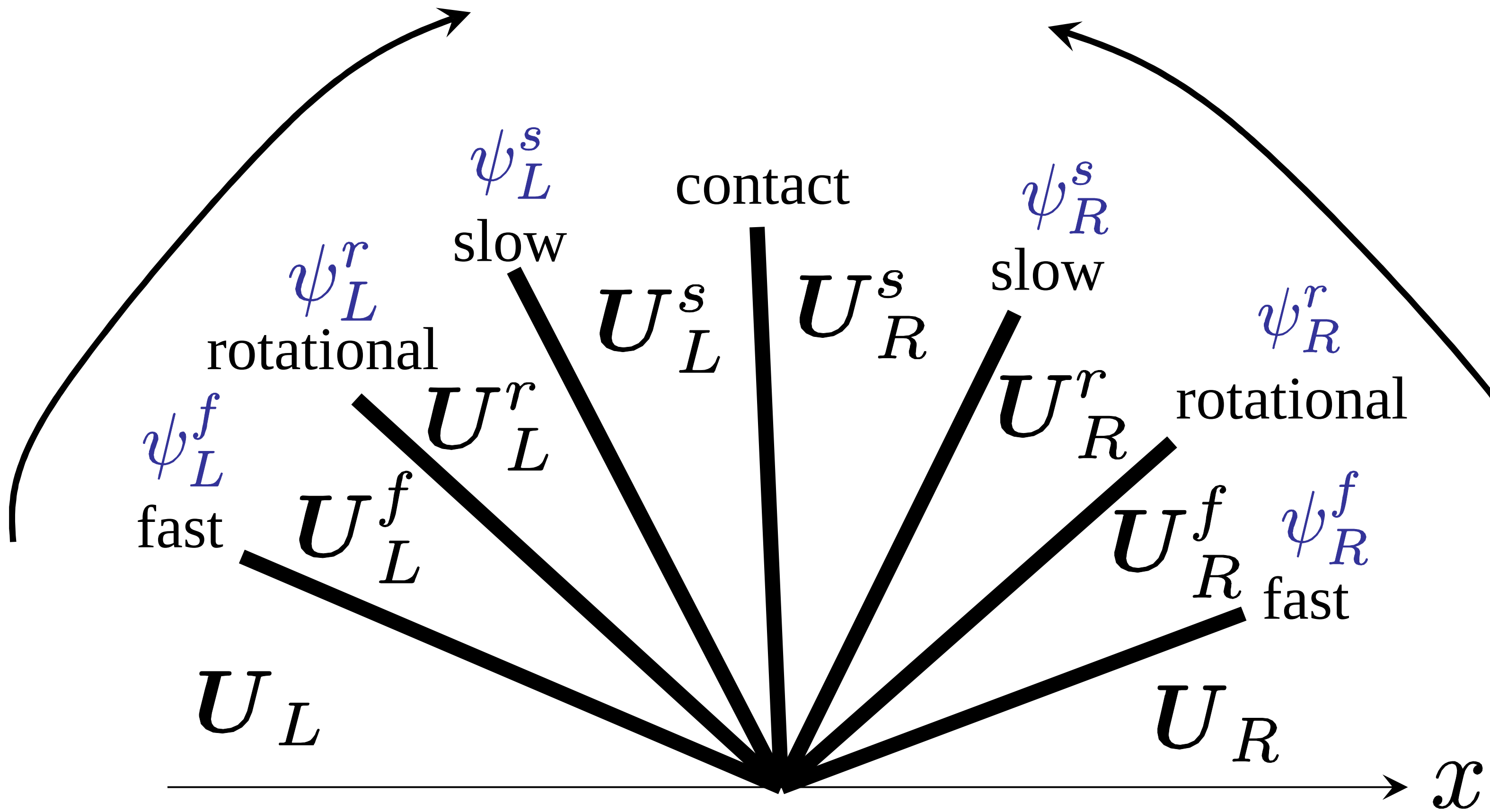


We solve the system of eqs. by Newton-Raphson method.

$$\boldsymbol{W}_L^s(\psi_L^s;\boldsymbol{U}_L^r(\psi_L^r;\boldsymbol{U}_L^f(\psi_L^f;\boldsymbol{U}_L))) - \boldsymbol{W}_R^s(\psi_R^s;\boldsymbol{U}_R^r(\psi_R^r;\boldsymbol{U}_R^f(\psi_R^f;\boldsymbol{U}_R))) = \boldsymbol{0}$$

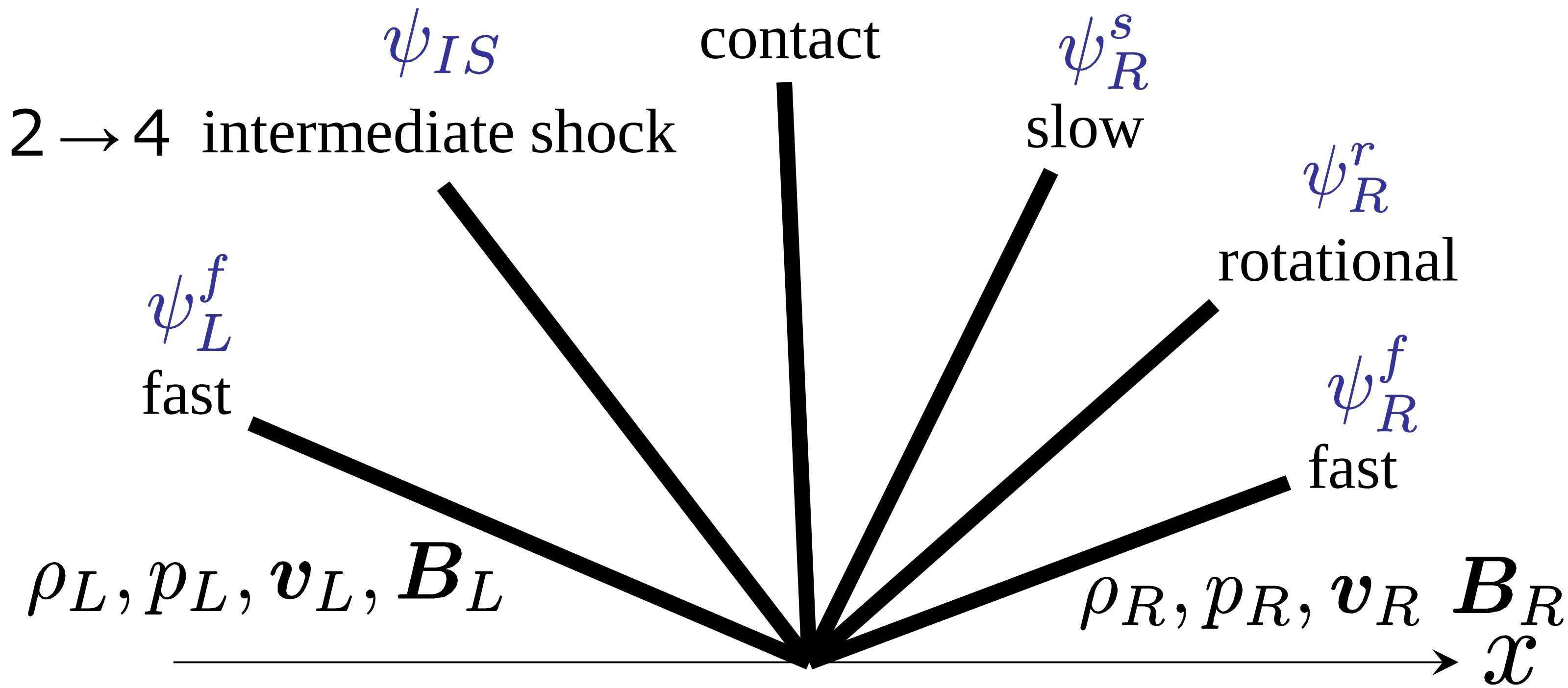
where

$\boldsymbol{W} = (p, v_x, v_y, v_z, B_y, B_z)$:vector of variables that should be continuous



How to find non-regular solutions

The structure of non-regular solutions is **not *a priori* known**
since intermediate shocks skip some waves,
and the kind of skipped waves depends upon
the kind of the intermediate shocks will exist in the solutions



How to find non-regular solutions

The structure of non-regular solutions is **not *a priori* known**.

⟶ **Case Study**

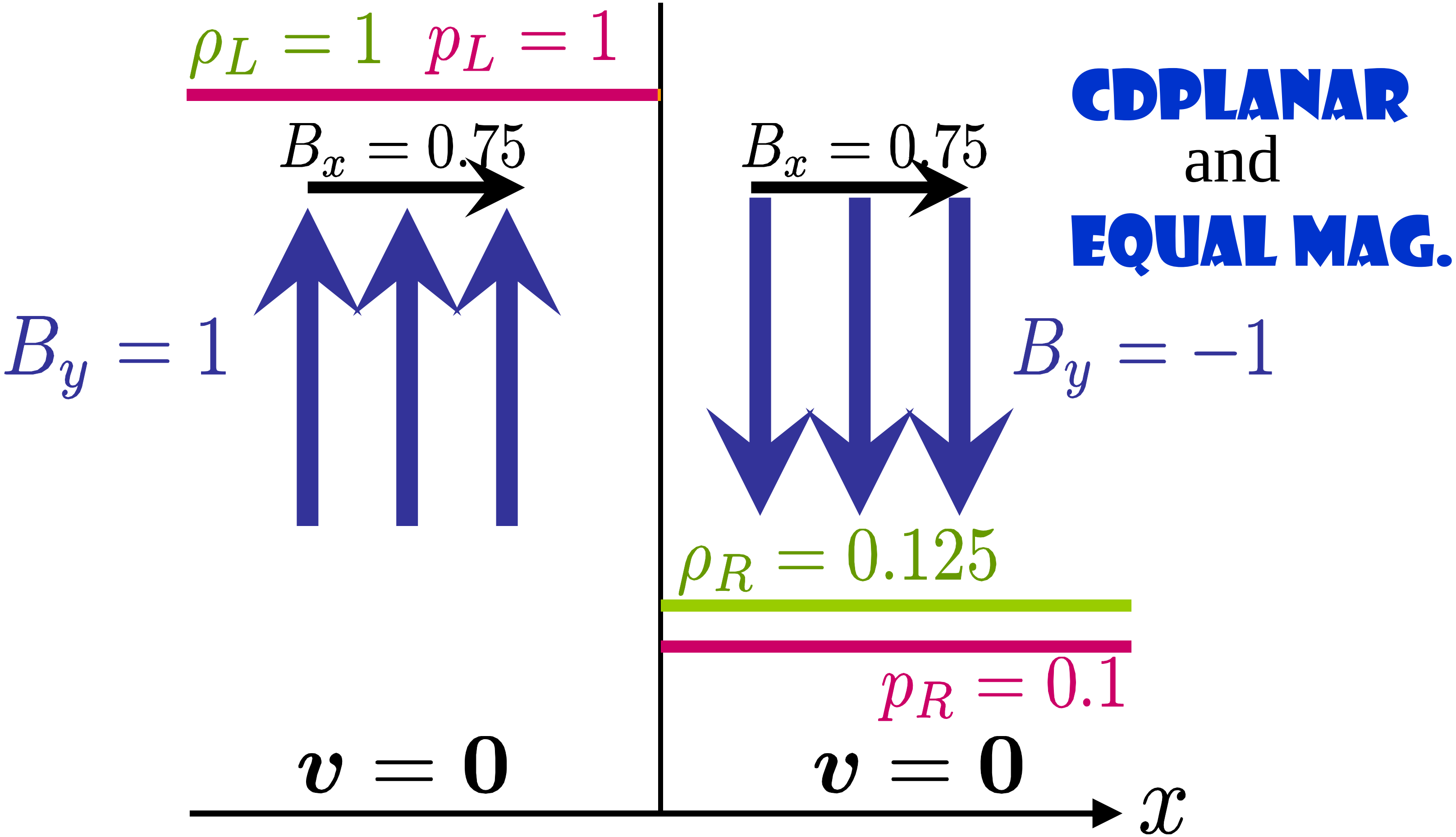
(if-else-if ladder)

Results

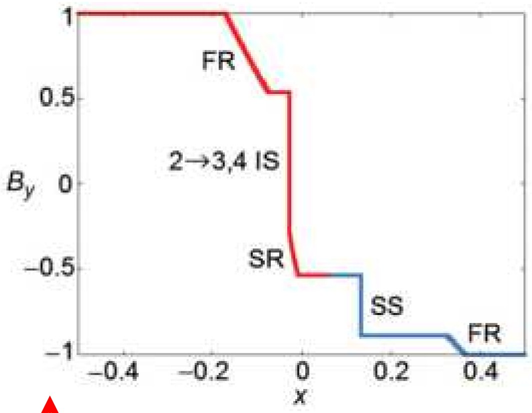
- Uncountably many non-regular solutions
  for the Brio & Wu problem

- An initial condition where no regular solution exists

Initial condition of Brio & Wu (1988)



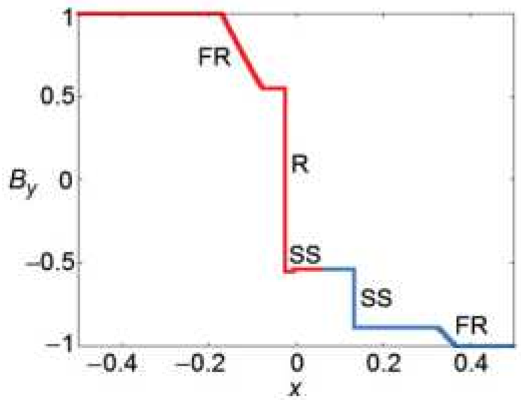
Uncountably many solutions



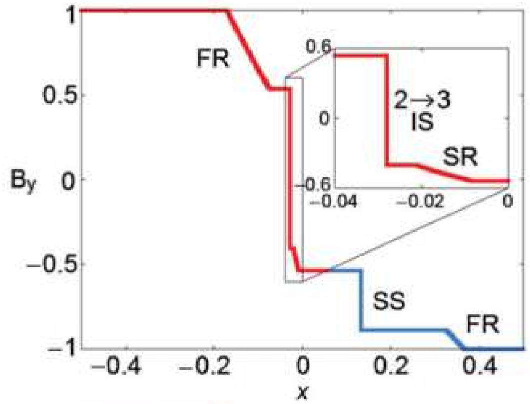
The famous non-regular solution, which is realized in numerical simulations

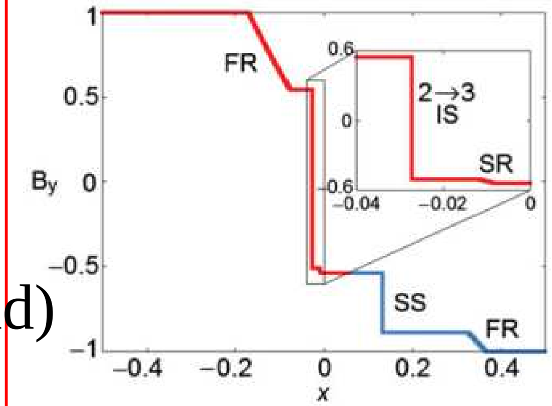
Uncountably many non-regular solutions, which connects known two solutions

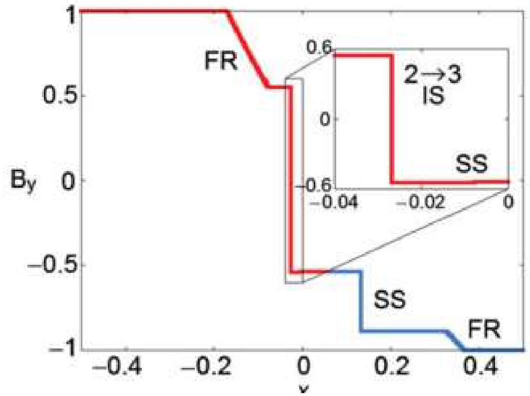
parameter: strength of 2→3 intermediate shock
(the magnitude of transverse magnetic field behind)



The evolutionary regular solution





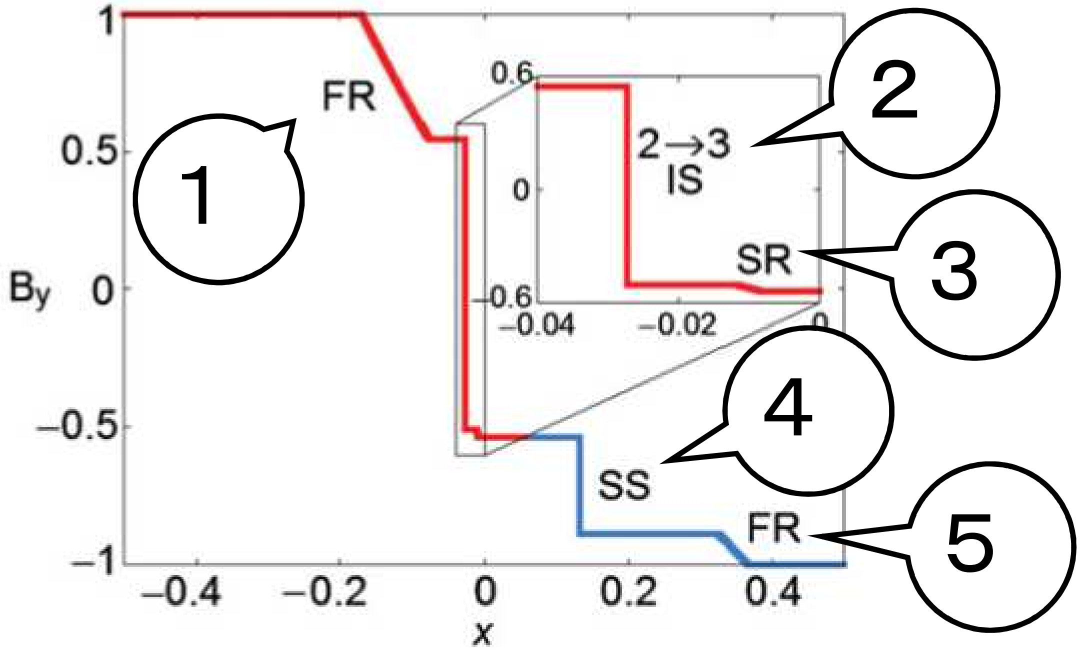


Why uncountably many?

Brio & Wu problem : reduced system (z-component ≡ 0)

→ the number of the conditions imposed at contact discontinuity is **4** (continuity of p, v_x, v_y, B_y)

On the other hand… the number of the wave-parameters is **5**



Impact of the result

- **New question:**

Why other non-regular solutions that contains a $2 \to 3$ intermediate shock are not realized in numerical simulations?

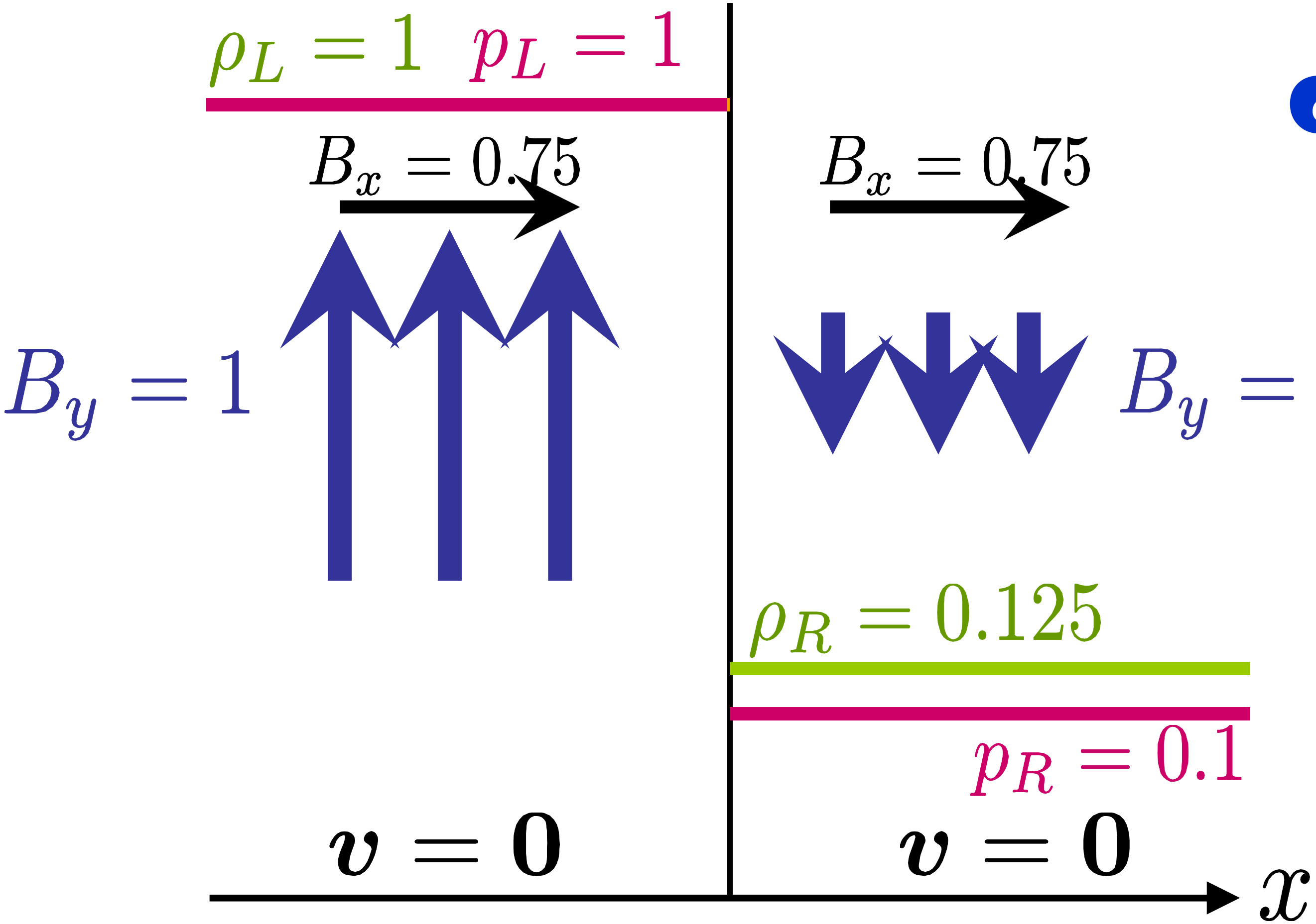
This might be an important question when we discuss the stability of intermediate shocks and the relevancy of numerical results.

Results

- Uncountably many non-regular solutions for the Brio & Wu problem

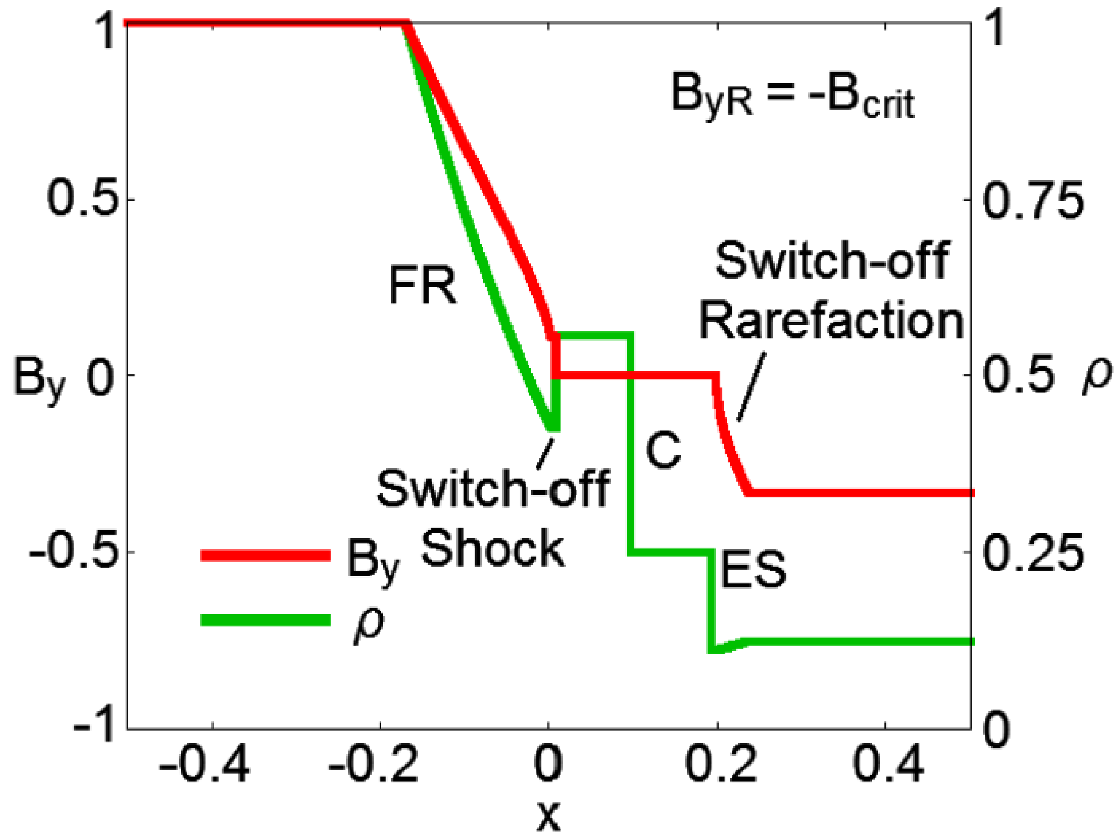
- An initial condition where no regular solution exists

We modified the magnetic field in Brio & Wu



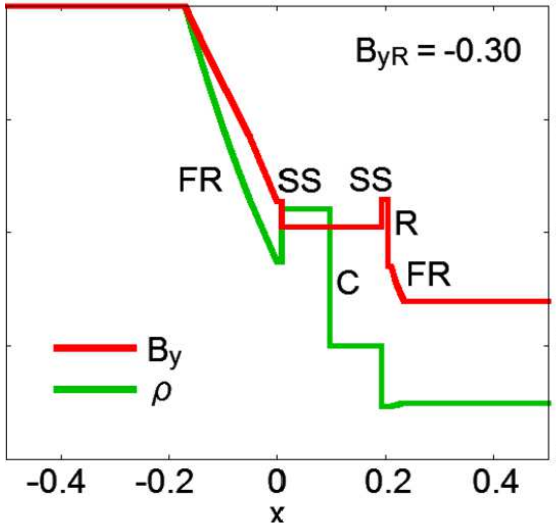
COPLANAR
but
SMALLER

Switch-off solution



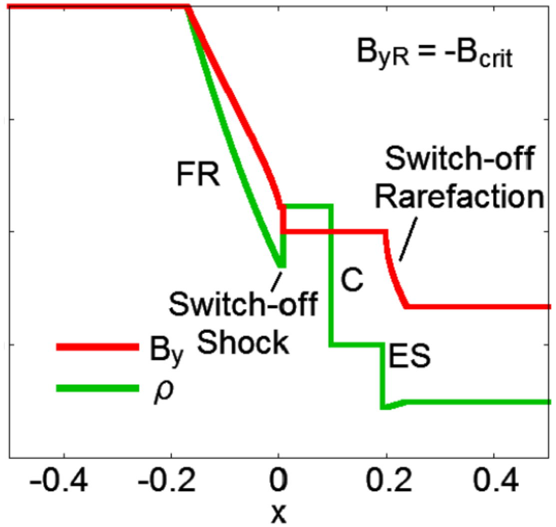
Switch-off shock is not evolutionary

No regular solution exists



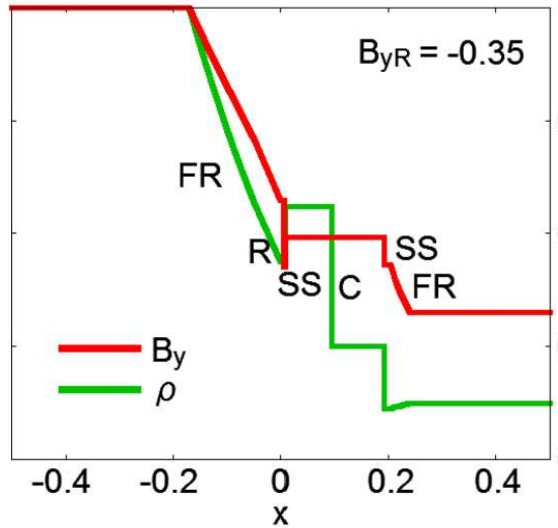
regular solution for

$$B_{yR} = -0.30$$



Switch-off solution

$$B_{yR} = -0.329875$$



regular solution for

$$B_{yR} = -0.35$$

- The regular solutions of neighboring initial conditions connect ***continuously*** the switch-off solution

Impact of the result

- **The first counter-example of classical belief, "there is always a unique regular solution".**

➡ Ideal MHD Riemann problem cannot be solved only by evolutionary waves

- Note, however, since our solver is based on the Newton-Raphson method, some regular solution might possibly elude our search although it should be away from the neighboring solutions, if any.

Summary

PROBLEM

Non-uniqueness of the solutions of MHD Riemann problems

- which is highly associated with intermediate shocks

METHOD

New exact Riemann solver

- can handle all types of intermediate shocks & switch-on/off waves
- can find all the solution for a given initial condition

RESULTS

- Uncountably many solutions for the Brio & Wu problem

→ Why numerical simulations always realize a particular one?

- An initial condition with no regular solution

→ MHD Riemann problems are not solved only by evolutionary waves