Aspects of Stochastic Integration of Parker's Transport Equation

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We acknowledge helpful discussions with J. Kóta and J. GIacalone.

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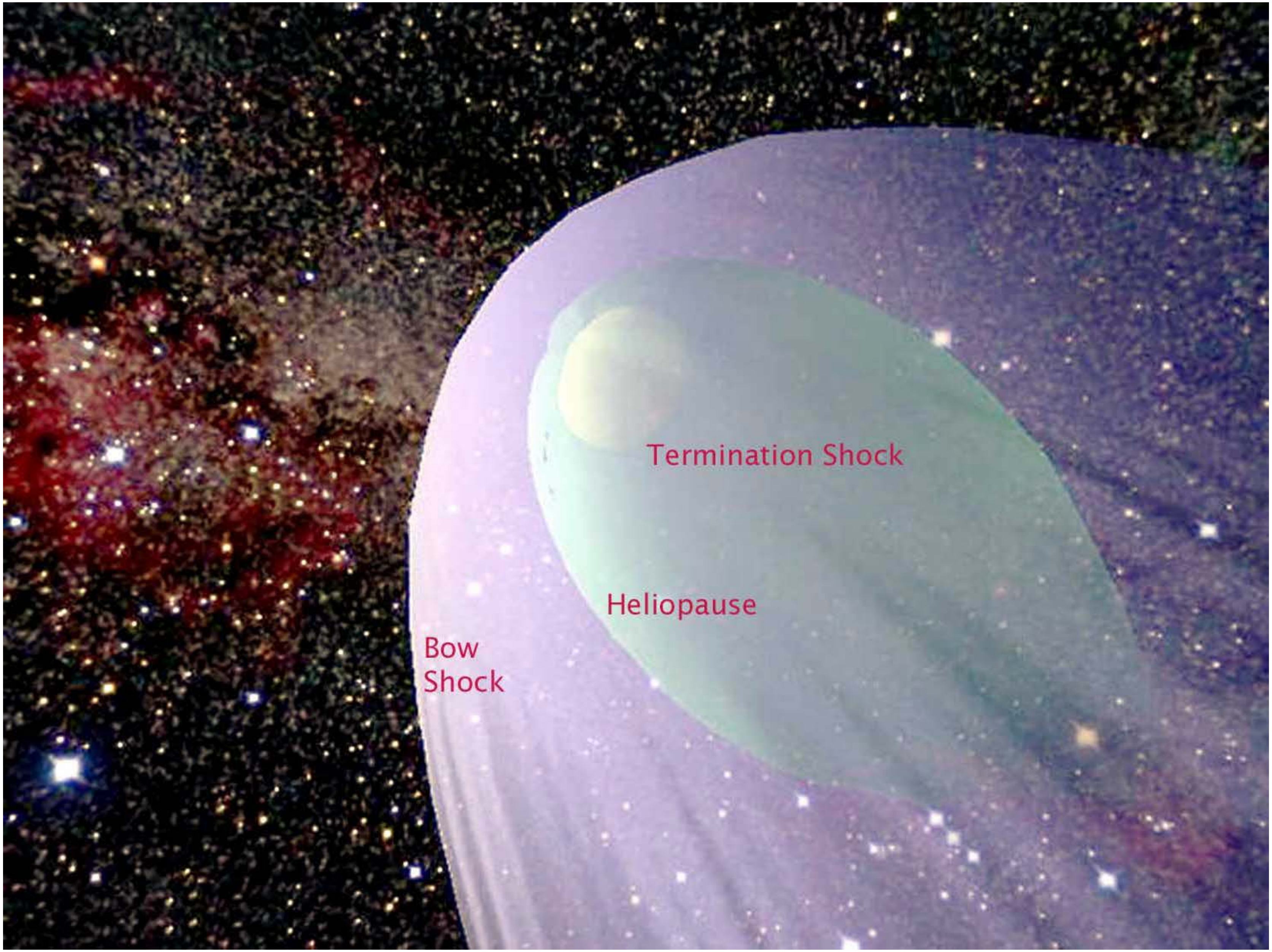
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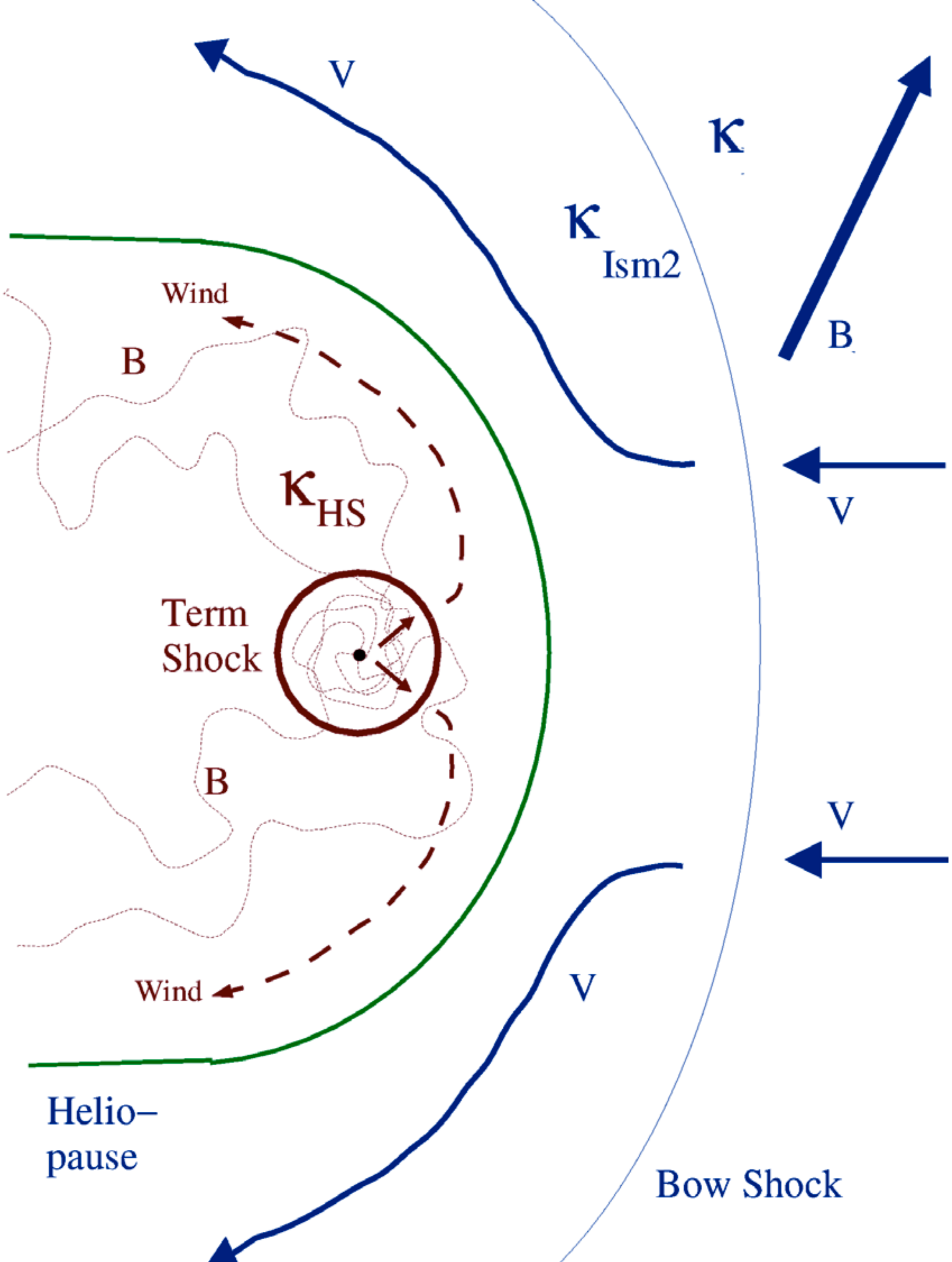
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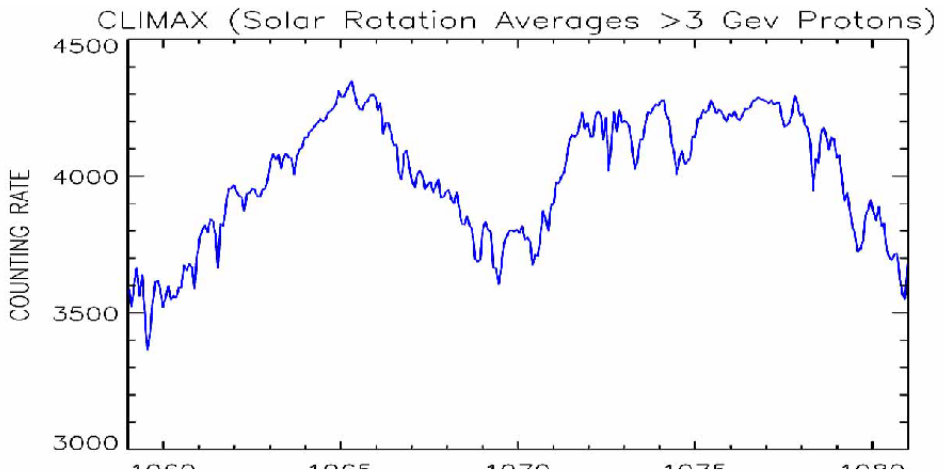
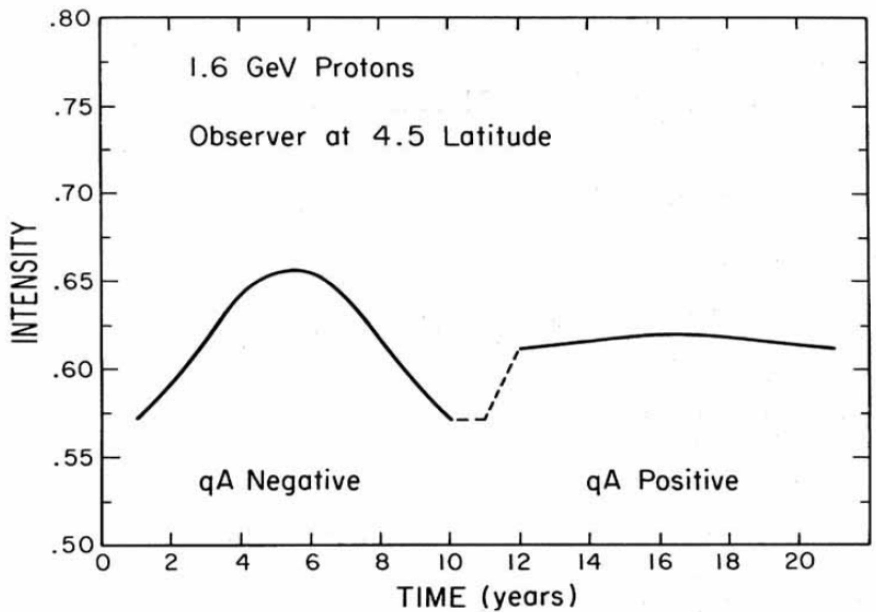


KÓTA AND JOKIPII (Ap J 1983)



The Parker Transport Equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x_i}\left[\kappa_{ij}^{(S)}\frac{\partial f}{\partial x_j}\right] \quad \Rightarrow \text{ Diffusion}$$

$$-\mathbf{U} \cdot \nabla f \quad \Rightarrow \text{ Convection w. plasma}$$

$$-\mathbf{V_d} \cdot \nabla f \quad \Rightarrow \text{ Grad \& Curvature Drift}$$

$$+\frac{1}{3}\nabla \cdot \mathbf{U}\left[\frac{\partial f}{\partial \ell n p}\right] \quad \Rightarrow \text{ Energy change}$$

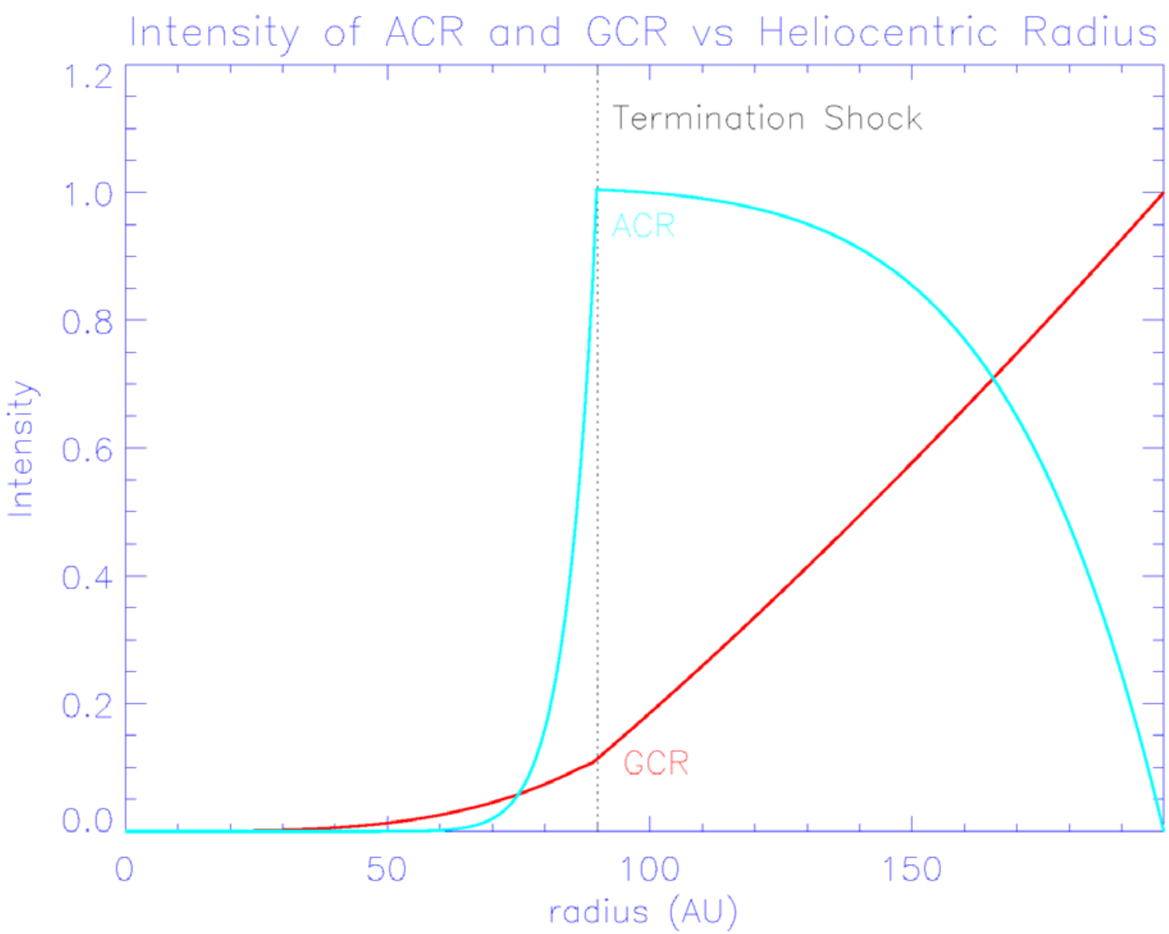
$$+Q \quad \Rightarrow \text{ Source}$$

Where the drift velocity due to the large scale curvature and gradient of the average magnetic field is:

$$\mathbf{V_d} = \frac{pcw}{3q}\nabla \times \left[\frac{\mathbf{B}}{B^2}\right], \qquad |\mathbf{V_d}| = O\left(\frac{r_c}{L}w\right)$$

It is difficult to overstate the importance of this equation. It is the basis of 95% or more analyses of energetic particles and cosmic rays – Sun, Heliosphere, galaxy, intergalactic, etc.

Applying Parker's equation to the heliosphere for a simple spherically symmetric approximation.



Stochastic Integration

- A numerical technique which has recently become popular.
- Basic idea: Diffusion is equivalent to the long-time limit of a random walk (t>> τ_c).
- Write

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left(\kappa \frac{\partial f}{\partial x}\right) \\
&= \frac{\partial^2}{\partial x^2}(\kappa f) - \frac{\partial}{\partial x}\left(\frac{\partial \kappa}{\partial x} f\right) \\
&= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{< \Delta x^2 >}{\Delta t} f\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{< \Delta x >}{\Delta t} f\right)
\end{aligned}
$$

- This is a Fokker-Planck equation with

$<\Delta x^2> = 2\kappa\Delta t$ and $<\Delta x> = (\partial\kappa / \partial x)\Delta t$.

- we may then follow ‘pseudo particles’ by incrementing their positions by

$$
\begin{aligned}
t_{n+1} &= t_n + dt \\
x_{n+1} &= x_n \pm \sqrt{2\kappa dt} + \frac{\partial \kappa}{\partial x} dt
\end{aligned}
$$

where $\pm$ denotes a random number with zero mean.

- If we have advection with velocity U in the x-direction, we add U dt to the righ of the x equation. We may also add energy change.
- The average over a large number of particles gives the solution for f. This is stochastic integration of a SDE.

Some Early Papers.

VOL. 80, NO. 10 JOURNAL OF GEOPHYSICAL RESEARCH APRIL 1, 1975

Implications of Observed Charge States of Low-Energy Solar Cosmic Rays

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Recent measurements of the charge states of low-energy (~100 keV/nucleon) solar cosmic rays at 1 AU are discussed. The measurements are consistent with models involving charge equilibrium with neutral matter at the sun only if the particles lose ~90% of their energy owing to adiabatic deceleration in the solar wind. Such an energy loss is shown to be possible only if the diffusion coefficient for 1-MeV/nucleon particles is smaller than 10^{20} cm^2 s^{-1}. The implications of these results for models of solar cosmic ray acceleration are discussed.

Recent observations of the nuclear composition of solar flare particles have established that the fraction of heavier nuclei is enhanced at lower energy [*Price et al.*, 1971; *Lanzerotti et al.*, 1972; *Mogro-Camparo and Simpson*, 1972; *Gloeckler et al.*, 1973; *Teegarden et al.*, 1973]. The models used to explain this enrichment of heavy nuclei usually invoke acceleration in regions of a large enough neutral hydrogen density for charge equilibrium to be established [*Ramadurai*, 1971; *Cartwright and Mogro-Camparo*, 1972].

Recently, *Gloeckler et al.* [1973] have reported observations of the charge states of low-energy solar flare particles. Their measurements at 100 keV/nucleon indicated average charges of 5.4 for carbon and 7.4 for oxygen, while the equilibrium charge states [*Bloom and Sauter*, 1971] for 100-keV/nucleon particles in neutral matter would be approximately 2.5 and 2.9, respectively. However, the observed charge states are indicative of the equilibrium charge states of 1- to 2-MeV/nucleon particles. Observations of iron reported by *Sullivan and Price* [1973] similarly indicate a much higher charge state than the equilibrium one. As Gloeckler et al. suggest, a possible explanation of the observations is that the particles detected at 100 keV/nucleon near earth are those that had an energy of 1 MeV/nucleon in the accelerating region near the sun. The observations are in accord with the equilibrium charge state in neutral matter being established during acceleration near the sun if the typical 1-MeV/nucleon particle loses about 90% of its energy during its propagation from the sun to the earth. In the next section we investigate this possibility and go on to discuss further the implications of the observations for models of the acceleration and propagation of solar flare particles.

MONTE CARLO SIMULATION OF SOLAR COSMIC RAY DIFFUSION

In order to investigate the propagation of solar cosmic rays we consider the simplest possible model that should give a reasonable estimate of the energy change. Assume that the cosmic ray omnidirectional intensity $U(r, T, t)$ is a spherically symmetric function of heliocentric radius r, particle kinetic energy per nucleon T, and time t. Then the cosmic ray diffusion equation for a constant-velocity solar wind and $\beta << 1$ is [*Parker*, 1965; *Gleeson and Axford*, 1967; *Jokipii*, 1971]

$$\frac{\partial U}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\kappa_{rr}\frac{\partial U}{\partial r} - r^2VU\right\} + \frac{4}{3}\frac{V}{r}\frac{\partial}{\partial T}(TU) \qquad (1)$$

where $V \approx 350$ km/s is the (radial) velocity of the solar wind and κ_{rr} is the radial cosmic ray diffusion coefficient. For simplicity take κ_{rr} to be independent of both r and T; the generalization to arbitrary κ_{rr} is straightforward. Define the new function

$$f = 4\pi r^2 U \qquad (2)$$

From (1) one obtains

$$\frac{\partial f}{\partial t} = \kappa_{rr}\frac{\partial^2 f}{\partial r^2} - \frac{\partial}{\partial r}\left[\left(V + \frac{2\kappa_{rr}}{r}\right)f\right] + \frac{\partial}{\partial T}\left[\left(\frac{4VT}{3r}\right)f\right] \qquad (3)$$

The number of particles in a spherical shell of thickness dr is then simply $f\,dr$.

Equation (3) is in the form of a simple diffusion equation and is identical to the equation obtained from one-dimensional random walk in radius with the random step size given by

$$(\Delta r)_{\rm diffusion} = (2\kappa_{rr}\Delta t)^{1/2} \qquad (4)$$

the radial convection given by

$$(\Delta r)_{\rm convection} = [V + 2\kappa_{rr}/r]\Delta t \qquad (5)$$

and energy-space 'convection' given by

$$\Delta T = -[4VT/3r]\,\Delta t \qquad (6)$$

[e.g., *Chandrasekhar*, 1943]. The cosmic ray diffusion can therefore be simulated by a Monte Carlo process in which a particle's position and energy at a time t_{n+1} are related to the parameters at t_n by

$$r_{n+1} = r_n + s(\Delta r)_{\rm diffusion} + (\Delta r)_{\rm convection} \qquad (7)$$

and

$$T_{n+1} = T_n + \Delta T \qquad (8)$$

Here s is a random sign, $s = 1$ or -1, and the step sizes for a time interval Δt are given by (4)–(6).

Our model for cosmic ray propagation from a solar flare is obtained as follows. A particle is released near the sun (at $r = R_0$) with kinetic energy T_0 at time $t = 0$ and is allowed to evolve

THE ASTROPHYSICAL JOURNAL, 213:L85–L88, 1977 April 15

EFFECTS OF PARTICLE DRIFTS ON THE SOLAR MODULATION OF GALACTIC COSMIC RAYS

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ABSTRACT

Gradient and curvature drifts in an Archimedean-spiral magnetic field are shown to produce a significant effect on the modulation of galactic cosmic rays by the solar wind. The net modulation, heliocentric radial gradient, and average energy change of particles which reach the inner solar system are significantly reduced. The effects of drifts are due to the fact that cosmic rays for which the drift velocity is comparable to the wind velocity or larger, have more rapid access to the inner solar system than in the absence of drifts.

Subject headings: cosmic rays: general — Sun: solar wind

I. INTRODUCTION

It has recently been suggested that inclusion of gradient and curvature drifts in the interplanetary magnetic field may result in considerable modification of present models of cosmic-ray transport in the solar wind (Jokipii, Levy, and Hubbard 1976). Those authors pointed out that although drifts are explicitly contained in standard transport theories (e.g., Parker 1965; Axford 1965; Jokipii and Parker 1970) they have been neglected in all models of galactic cosmic-ray modulation or solar-flare particle events.[1]

Simple magnetic-field models based on the Archimedean spiral of Parker were shown by Jokipii, Levy, and Hubbard (1976) to produce drift velocities which, for particles with rigidities greater than 0.3 GV, are greater than the solar wind velocity over much of the inner solar system. Furthermore, the magnitude of the *radial* component of the drift is comparable to or greater than the wind velocity. This, coupled with the fact that the divergence of the drift velocity is zero (Levy 1976*a*), indicates that cosmic rays are brought in from and out to the boundary of the modulating region much more rapidly than previously believed, and that the drifts therefore can considerably reduce the net modulation. Some indication that drifts can in fact reduce the heliocentric cosmic-ray gradient was reported by Jokipii, Levy, and Hubbard (1976), although they considered only extremely idealized cases.

This *Letter* reports the initial results of Monte Carlo simulations of cosmic-ray modulation by a spherically symmetric solar wind, which carries an Archimedean-spiral average magnetic field. The associated particle drifts are explicitly included. The simulations demonstrate that, in a reasonably realistic model, inclusion of particle drifts can substantially reduce the modulation, heliocentric gradient, and energy change of ~1 GV rigidity particles in the inner solar system.

II. THE METHOD OF CALCULATION

We use the general formulation of cosmic-ray transport written down by Jokipii and Parker (1970). Decompose the cosmic-ray diffusion tensor κ_{ij} into its symmetric and antisymmetric parts $\kappa_{ij}^{(S)}$ and $\kappa_{ij}^{(A)}$. Then, as noted by Jokipii, Levy, and Hubbard (1976), the average particle drifts may be written

$$v_{D,i} = +\frac{\partial}{\partial x_j}(\kappa_{ij}^{(A)})\,. \qquad (1)$$

Noting that $\nabla\cdot(\mathbf{v}_D) = 0$, one may write the equation for the cosmic-ray density U as a function of position $\mathbf{r}$, time t, and energy T as

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x_i}\left[\kappa_{ij}^{(S)}\frac{\partial U}{\partial x_j} - (V_{w,i} + v_{D,i})U\right] + \frac{1}{3}\frac{\partial V_{w,i}}{\partial x_i}\frac{\partial}{\partial T}(\alpha TU)\,, \qquad (2)$$

where $\alpha = (2m_0c^2 + T)/(m_0c^2 + T)$. For simplicity assume that the modulating region is symmetric about the Sun's rotation axis. To cast the equation into a form suitable for Monte Carlo solution, assume that κ_{rr}, $\kappa_{\theta\theta}$ are independent[2]

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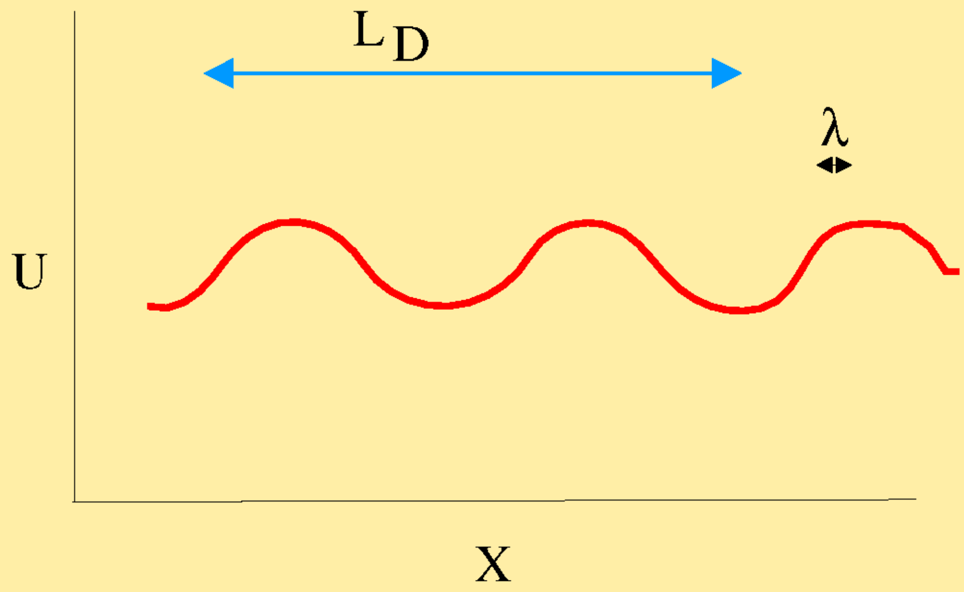
[1] Jokipii (1970), Levy (1975, 1976 *a*, *b*), and Barnden and Bercovitch (1975) pointed out some consequences of drifts, but did not construct complete models. Herein we will use the term "drift" to refer to gradient and curvature drifts, and not to the convection with the solar wind.

[2] Generalization to arbitrary κ_{rr}, $\kappa_{\theta\theta}$ is a straightforward but unnecessary complication at this stage.

Compression Acceleration Has Recently Received Attention

- Solve Parker's equation in *1 spatial dimension*, x, – periodic with scale L.
- The velocity U(x) is taken to be sinusoidal with period 2 π. Momentum change dp/dt = -(p/3) dU/dx.
- The diffusion coefficient is constant in space and is large (κ >> LU)for the tail and small (κ << LU) for the thermal core.
- Advance pseudoparticle by the rule ($\pm$ denotes a random number with unit variance and zero mean)

$$
\begin{aligned}
x_{n+1} &= x_n \pm \sqrt{(2\kappa dt)} + U dt \\
p_{n+1} &= p_n(1 - \frac{1}{3}\frac{dU}{dx} dt)
\end{aligned}
$$



Fisk and Gloeckler (2010) considered this picture to discuss an analytical formulation. This is controvertial. Jokipii and Lee (2010) showed that their equation did not conserve particles.

This can be checked easily using stochastic integration.

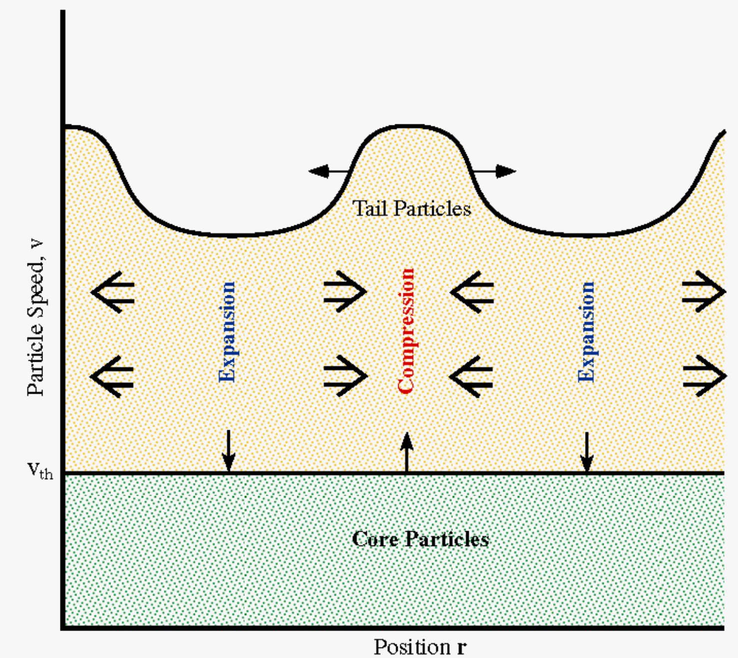
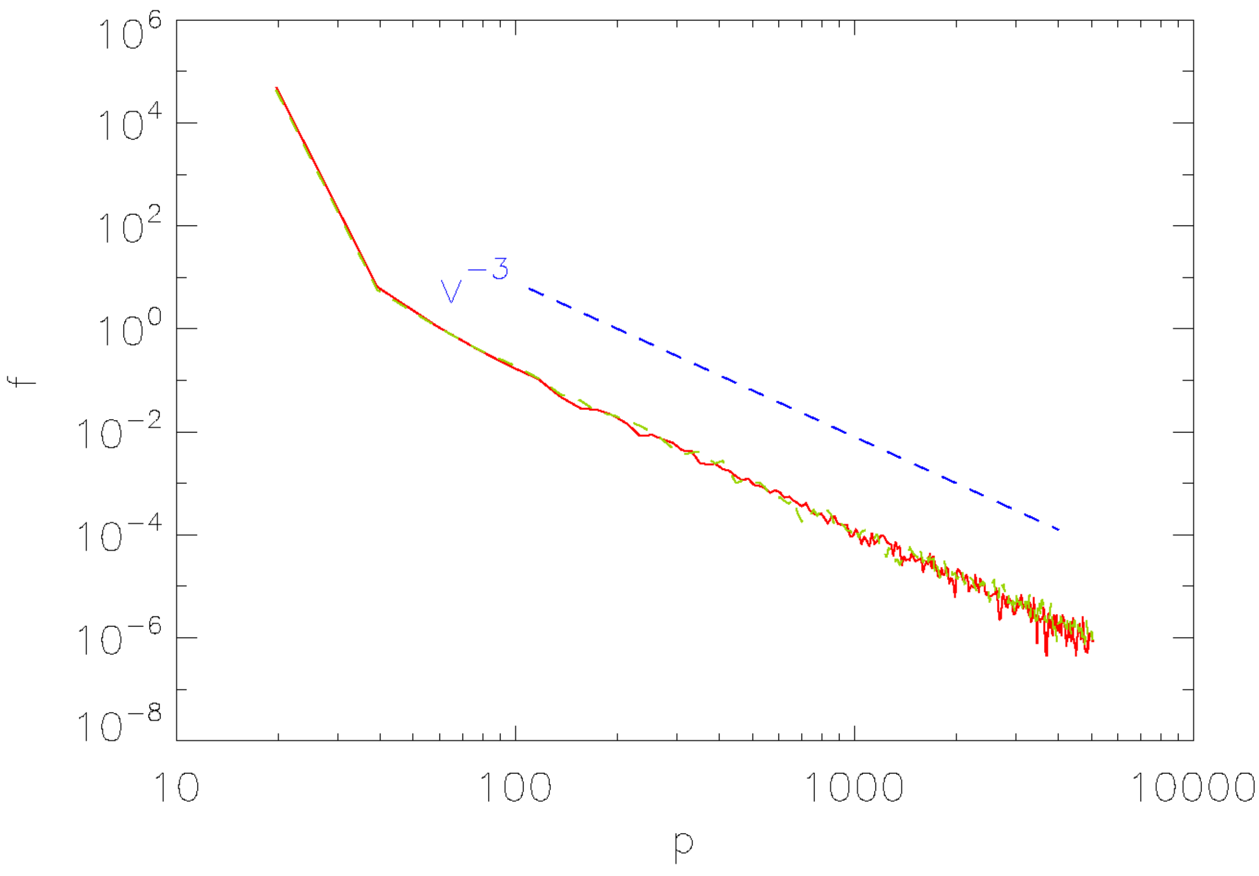


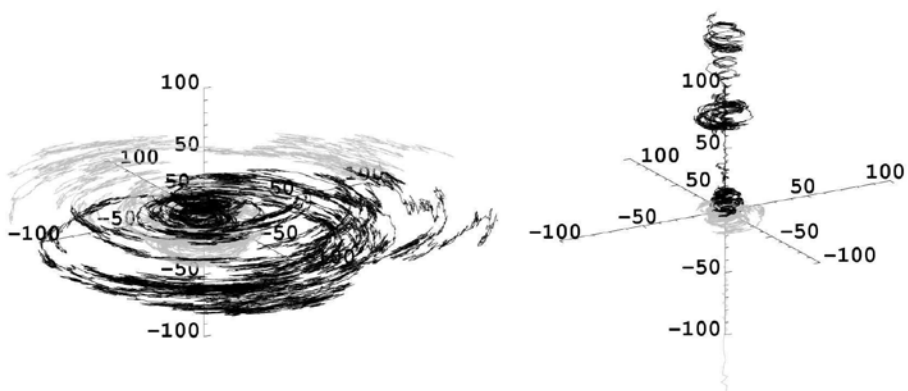
Figure 1. Schematic description of the basic principles behind the acceleration mechanism of Fisk & Gloeckler. Particle speed is plotted on the vertical axis and position on the horizontal axis. There is a core distribution of particles with speeds greater than the thermal speed of the bulk plasma and less than an upper threshold speed, $v \leqslant v_{th}$. The bulk thermal plasma contains random compressions and expansions, which randomly and adiabatically compress and expand the particles as shown. Particles with speeds above the threshold speed v_{th} are the tail particles. The distinction between the core and the tail particles is that the tail particles can diffuse spatially.

The momentum (velocity) spectrum for a constant spatial diffusion coefficient κ, velocity amplitude A = .4 in units of κ / L. Varying A over factors of 2 made little difference. The V^{-3} power-law slope is what Lee-Jokipii quasilinear analysis (2010) would give. But this solution is for very large amplitude.



Recently, this has been applied in many papers concerning cosmic rays in the heliosphere.

Strauss etal 2011



Miyake etal 2011

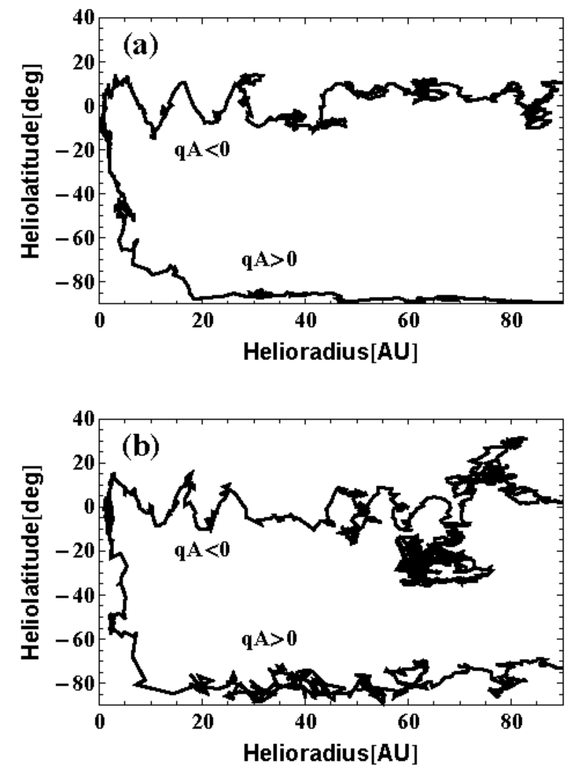
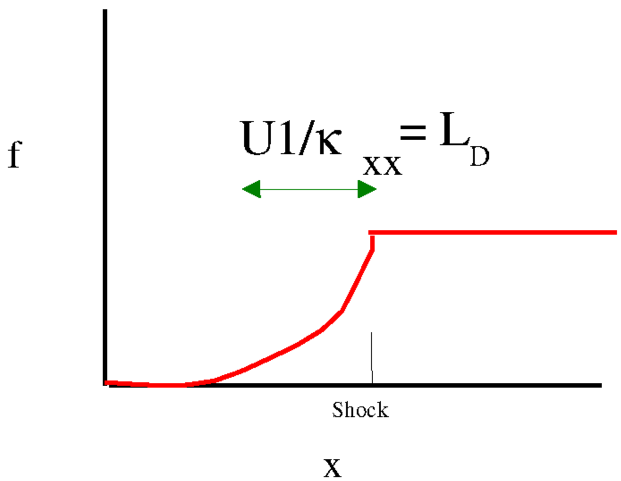


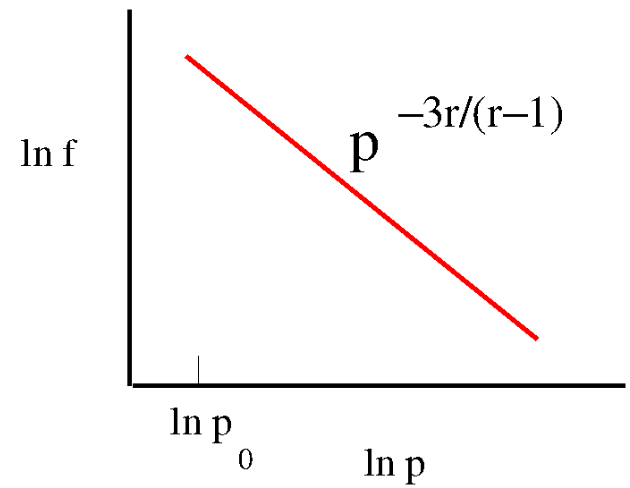
Figure 3: (a) Sample trajectory of protons with energy of 1 GeV at Earth in a Parker HMF.; (b) the same for a Fisk-type HMF.

Application to Discontinuities:

We often wish to deal with discontinuities in cosmic-ray transport: shocks or current sheets are commonly found.

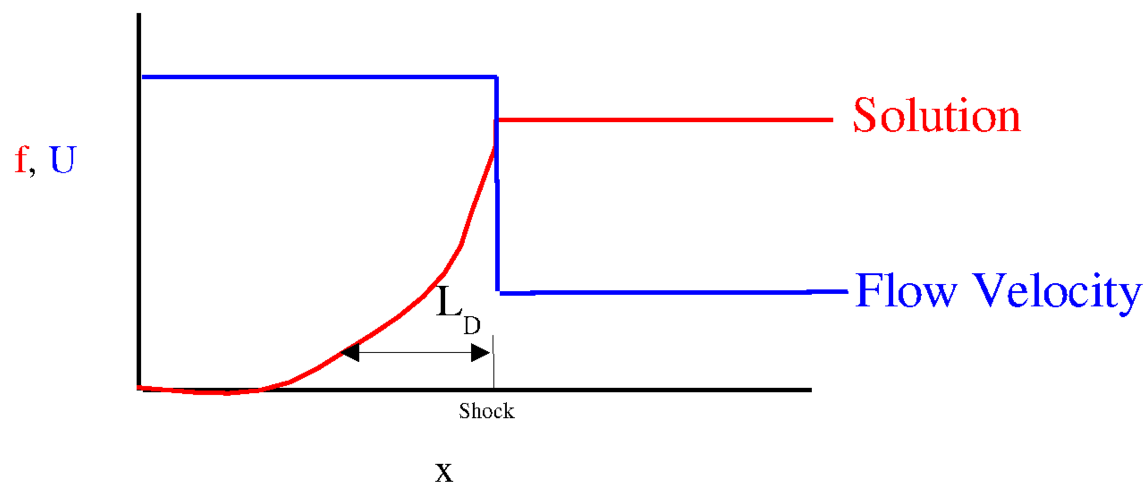
Diffusive shock acceleration has many desirable properties.



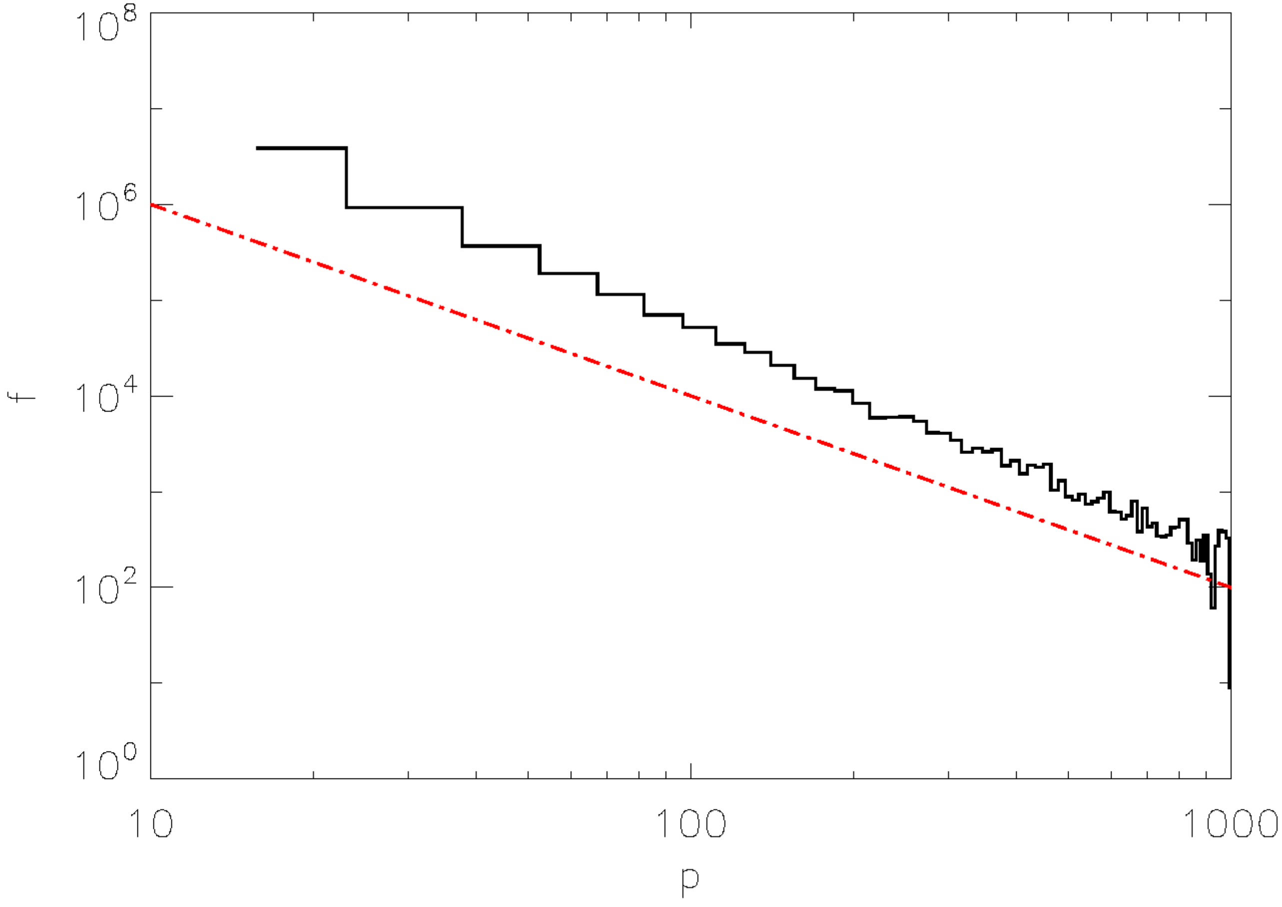


How do we treat discontinuities?

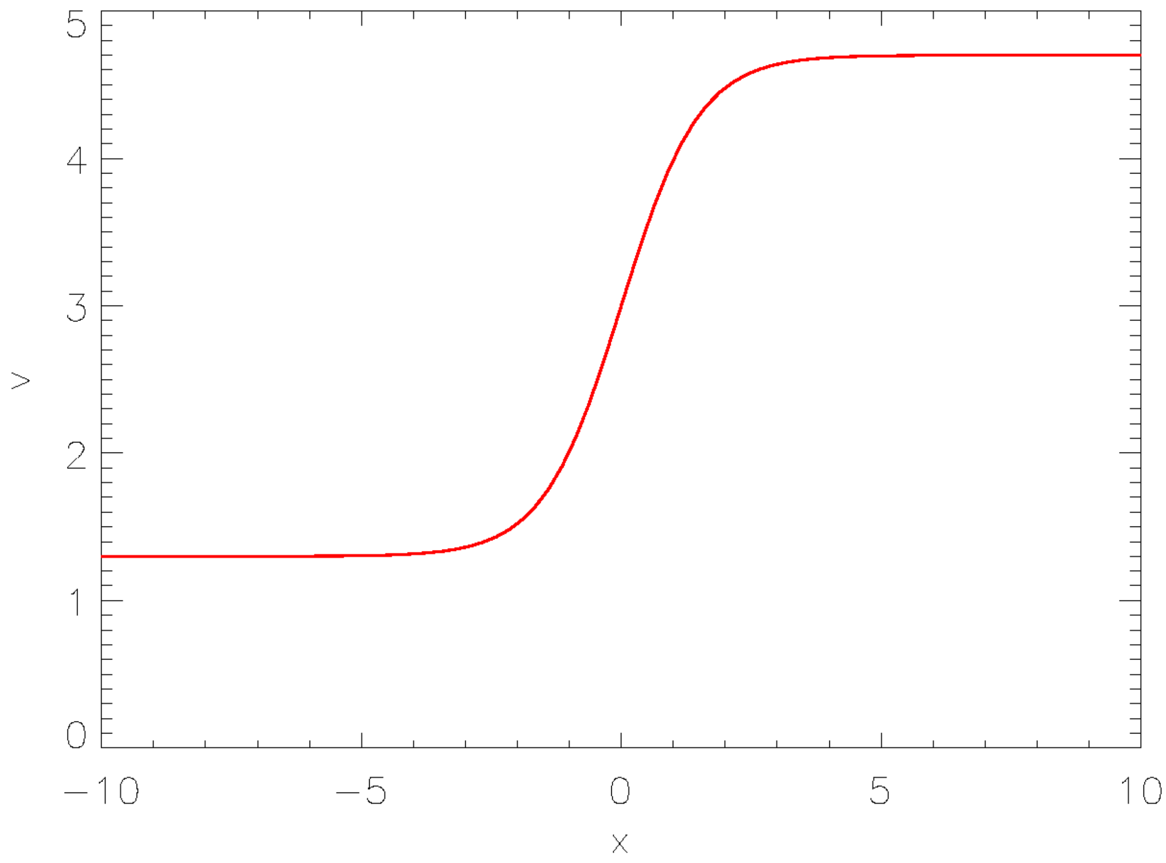
As in finite difference methods, one must use care.



If one just blindly proceeds and ignores the fact that the shock changes the spatial transport, but increment p at the shock crossing one obtains a poor approximation.



This problem is often circumvented by smoothing the shock as below, and using small-enough time steps to resolve the shock. This is expensive.



Discontinuities and Stochastic Integration.

- In its simplest form, one-dimensional diffusion from one boundary to another exhibits the problem quite well.

- Consider this first.

THE ASTROPHYSICAL JOURNAL, 541:428–435, 2000 September 20

CALCULATION OF DIFFUSIVE SHOCK ACCELERATION OF CHARGED PARTICLES BY SKEW BROWNIAN MOTION

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ABSTRACT

In the study of diffusive shock acceleration of charged particles, Fokker-Planck diffusion equations can be replaced by stochastic differential equations that describe the trajectory of the guiding center and the momentum of randomly walking individual particles. Numerical solution of stochastic differential equations is much easier to achieve, and very complicated shock acceleration cases can be simulated. However, the divergence of plasma velocity is a δ-function at the shock, resulting in a singularity for the momentum gain rate. The straightforward way of calculating shock acceleration is very slow because it requires that the shock be treated with finite thickness and the particles diffuse many steps inside the velocity gradient region. To overcome this difficulty, we suggest the use of skew Brownian motion, a diffusion process that has asymmetric reflection probability on both sides of the shock. The skew Brownian motion can be solved by a scaling method that eliminates the δ-function in the stochastic differential equation. The particle momentum gain is proportional to the *local time* spent by the diffusion process at the shock. In this way, the shock can be treated as infinitely thin, and thus the speed of numerical simulation is greatly improved. This method has been applied to a few cases of shock acceleration models, and results from the stochastic process simulation completely agree with analytical calculation. In addition, we have outlined a method using time backward stochastic processes to solve general diffusive shock acceleration problems with an extended source of particle injection.

Subject headings: acceleration of particles — cosmic rays — diffusion — shock waves

1. INTRODUCTION

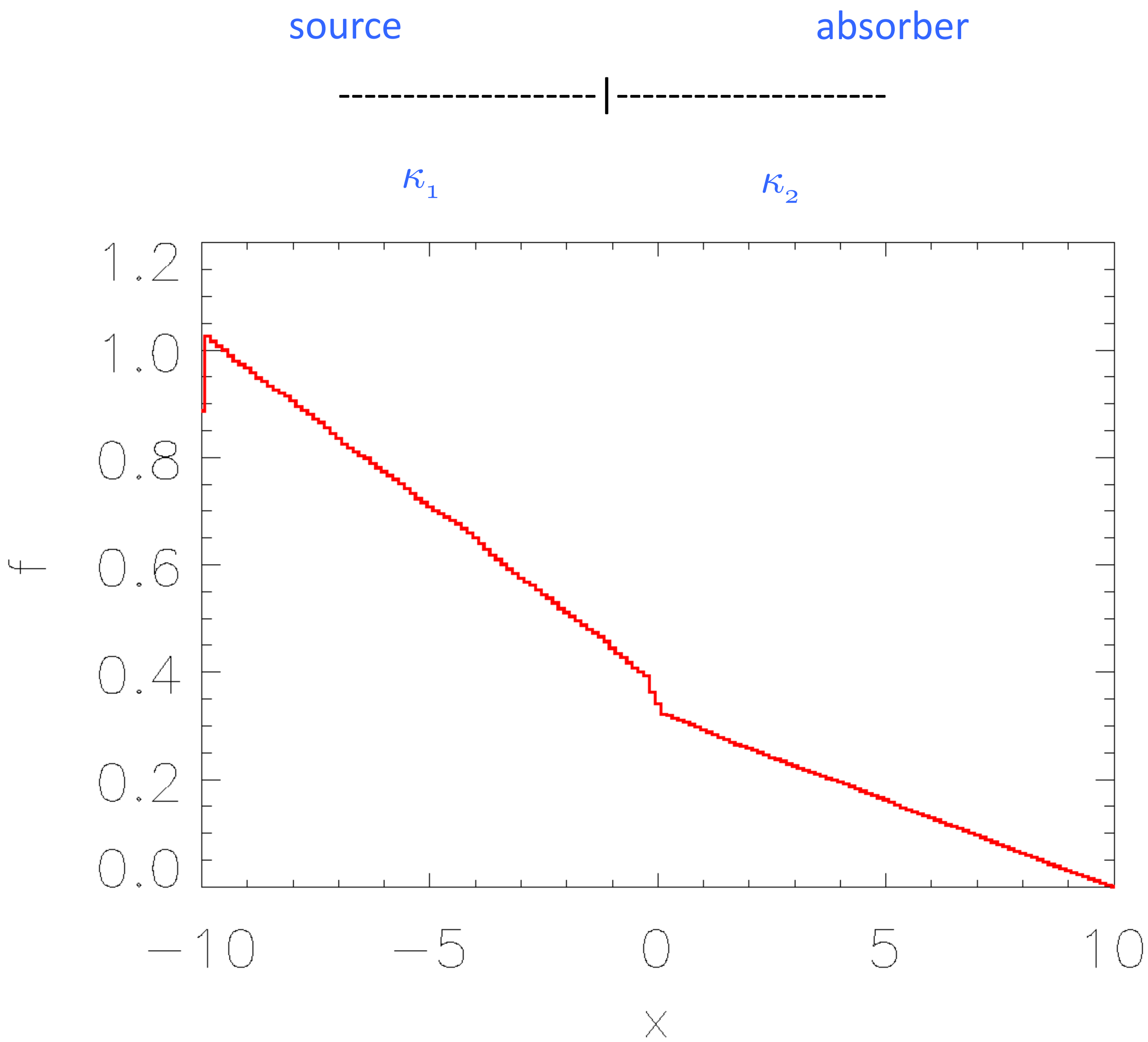
Shock acceleration of charged particles is often considered in the framework of diffusion approximation when there is sufficient scattering by magnetic irregularities in the media on both sides of the shock and the phase-space distribution of particles is nearly isotropic. In this approach, particles are accelerated in part by the convergence motion of magnetic scattering centers in the upstream region relative to downstream flow and vice versa (Drury 1983) or in part by drift along the shock front if the magnetic field has a component perpendicular to shock normal (Decker 1988). Upon each passage of the shock, particles gain a small amount of energy. By multiple scattering across the shock back and forth, particles may eventually achieve substantially high energy. This mechanism can account for many high-energy phenomena in astrophysics, such as cosmic rays in the Galaxy, anomalous cosmic rays accelerated by the solar wind termination shock, large gradual solar energetic particle events, and so on.

A typical way of calculating diffusive shock acceleration involves the use of Fokker-Planck diffusion equation for the particle distribution function. A large number of papers reporting analytical and numerical solutions to the Fokker-Planck equation have been published in the literature (see, e.g., review by Blandford & Eichler 1987). The Fokker-Planck equation is a second-order partial differential equation; a few analytical solutions are possible only when parameters in the equation (i.e., diffusion coefficients, convection speed profile) are given by trivial dependency on spatial coordinates and particle momentum. When these parameters are not trivial and second-order Fermi acceleration or other energy loss mechanisms have to be considered, only numerical solutions are feasible ways, but they are usually computationally expensive.

Stochastic differential equations (SDEs) are known to be equivalent to the Fokker-Planck equations (Gardiner 1983). Applications of SDEs have been made to study heliospheric cosmic-ray modulation (Zhang 1999) and to study shock acceleration (Krulls & Achterberg 1994). These papers have shown that results from Monte Carlo simulations of stochastic processes completely agree with those by calculation with the Fokker-Planck equation. An SDE is much easier to solve than the Fokker-Planck equation because an SDE is like a first-order ordinary differential equation, and the parameters and the geometry in the diffusion problem can be given very complicated form almost without extra computational burden. The SDE approach is particularly advantageous if the problems have to be treated in high dimensions.

However, there is a difficulty with the SDE approach when shock acceleration is concerned. At the shock the convection speed and in some cases the diffusion coefficient change abruptly. To high-energy particles with mean free path much larger than the size of shocked speed gradient, the divergence of the convection speed becomes very large in a very thin region. The time step of stochastic process simulation has to be chosen very small near the shock so that the simulated particles do not miss the sharp gradient at the shock. This significantly slows down the entire simulation, which is a big limitation in the Monte Carlo simulation of shock acceleration by Krulls & Achterberg (1994).

In this paper, we introduce the use of stochastic processes with skew reflection at shock. We will derive the momentum equation in terms of these stochastic processes. This eliminates the necessity to treat the shock as being of finite thickness. For simplicity we will demonstrate the calculation in one dimension with a planar shock set at $x = 0$. Acceleration by shocks with other types of geometry can be worked out in similar manner.


source
absorber
κ_1
κ_2
1.2
1.0
0.8
0.6
0.4
0.2
0.0
f
−10
−5
0
5
10
X

Consider a skewed step at crossing if κ changes.

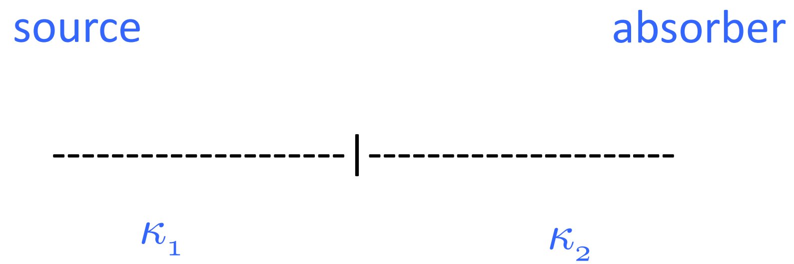
Based on the fact that if κ varies, there is an advecting term $\partial\kappa / \partial x$.

Let $\alpha = \kappa 2^{0.5}/(\kappa 1^{0.5} + \kappa 2^{0.5})$

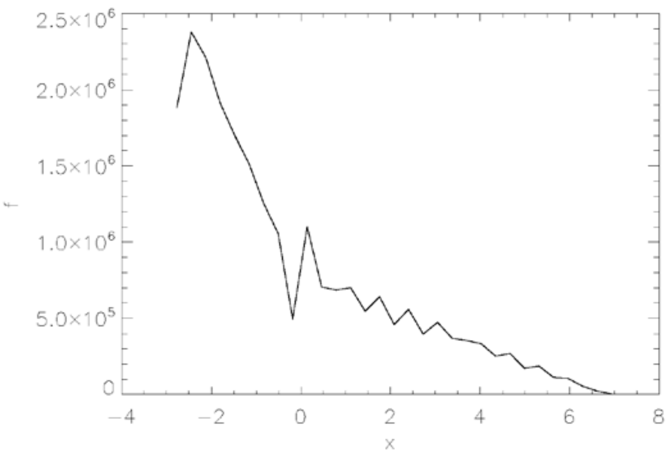
If the discontinuity is crossed, compute a uniform random number on (0,1). If it is less than α, go to region 1, else go to region 2.

Really only works properly if the particle exactly lands on the discontinuity. But implementing this is difficult.

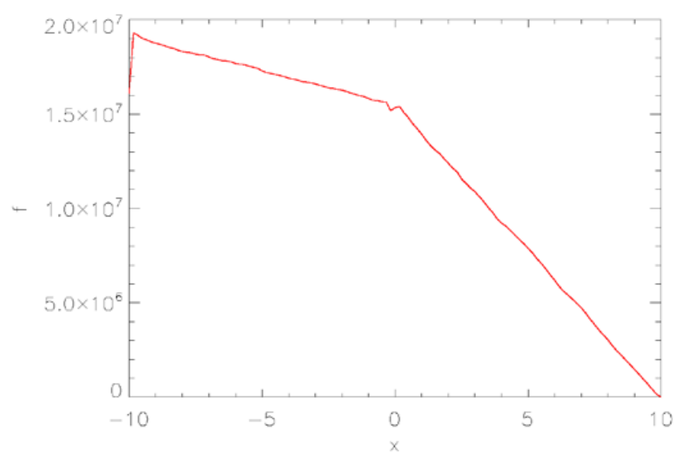
Alternatively, one can use a gaussian distribution. This is better.



Exact, uniform steps



Gaussian



How to do better?

Try to estimate, statisticlly, when the discontinuity was crossed.

Use conditional statistics. Given the first estimate, obtain the best estimate of the actual position. Use this.

This can be done using Bayesian statistics – quite non-intuitive to most people.

Using Bayesian Statistics.

$$p(x_{zT}|b) = \frac{p(x_{zT}, b)}{p(b)}$$

$$p(x_{zT}, b) = \frac{1}{\sqrt{2\pi\kappa z T}} e^{-\frac{((x_{zT}-0)-uzT)^2}{2\kappa z T}} \times \frac{1}{\sqrt{2\pi\kappa(1-z)T}} e^{-\frac{((b-x_{zT})-u(1-z)T)^2}{2\kappa(1-z)T}}$$

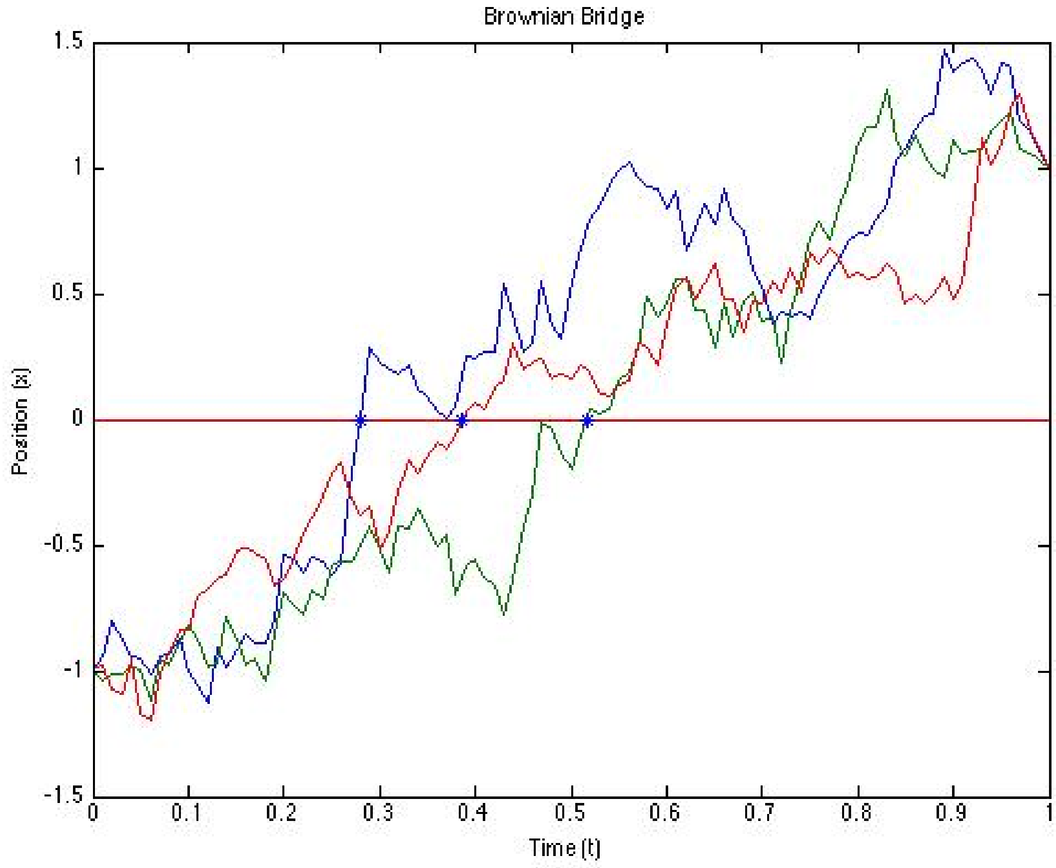
$$p(b) = \frac{1}{\sqrt{2\pi\kappa T}} e^{-\frac{((b-0)-uT)^2}{2\kappa T}}$$

Which simplifies to:

$$p(x_{zT}|b) = \frac{1}{\sqrt{2\pi\kappa z(1-z)T}} e^{-\frac{(x_{zT}-z(b-0))^2}{2\kappa z(1-z)T}}$$

→ Normally Distributed with mean zb variance $\kappa z(1-z)T$

Distribution of First Passage Times



Suppose that the shock *at x=0* is crossed, so that $x_i\ x_{i+1} < 0$. How to estimate the real x_{i+1}.

Compute the time $T_1 = \xi/(1+\xi)$ where $\xi = IG(\lambda^2/(\kappa\ dt)$.

Compute $T_f = T_i + (dt-T_1)\beta$ where β is sampled from the arcsin distribution.

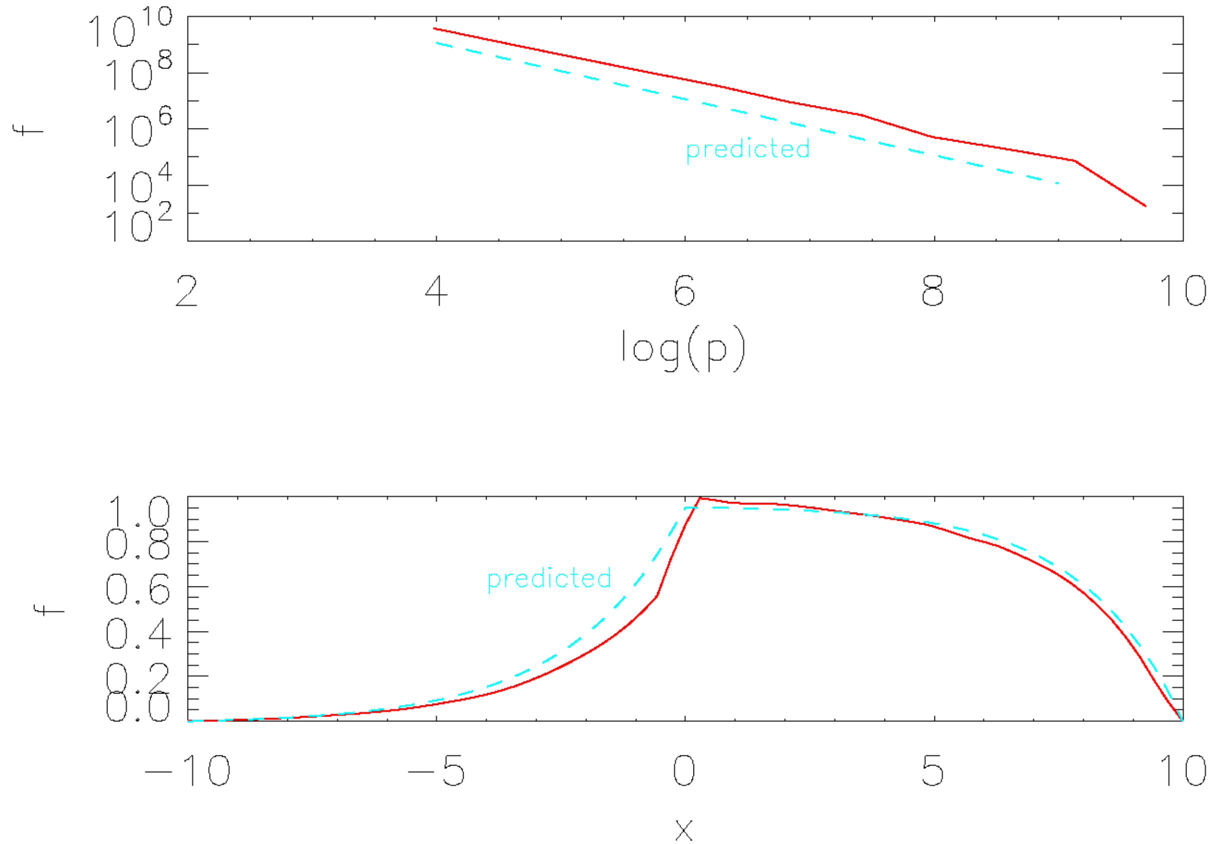
The estimated position is found from the following:

Sample R from the Rayleigh Distribution. Sample U from the uniform distribution over 0,1.

If $U < \kappa 2^{0.5} /(\kappa 1^{0.5} + \kappa 2^{0.5})$ $\epsilon = \kappa 2^{0.5}$ else $\epsilon = \kappa 1^{.5}$.

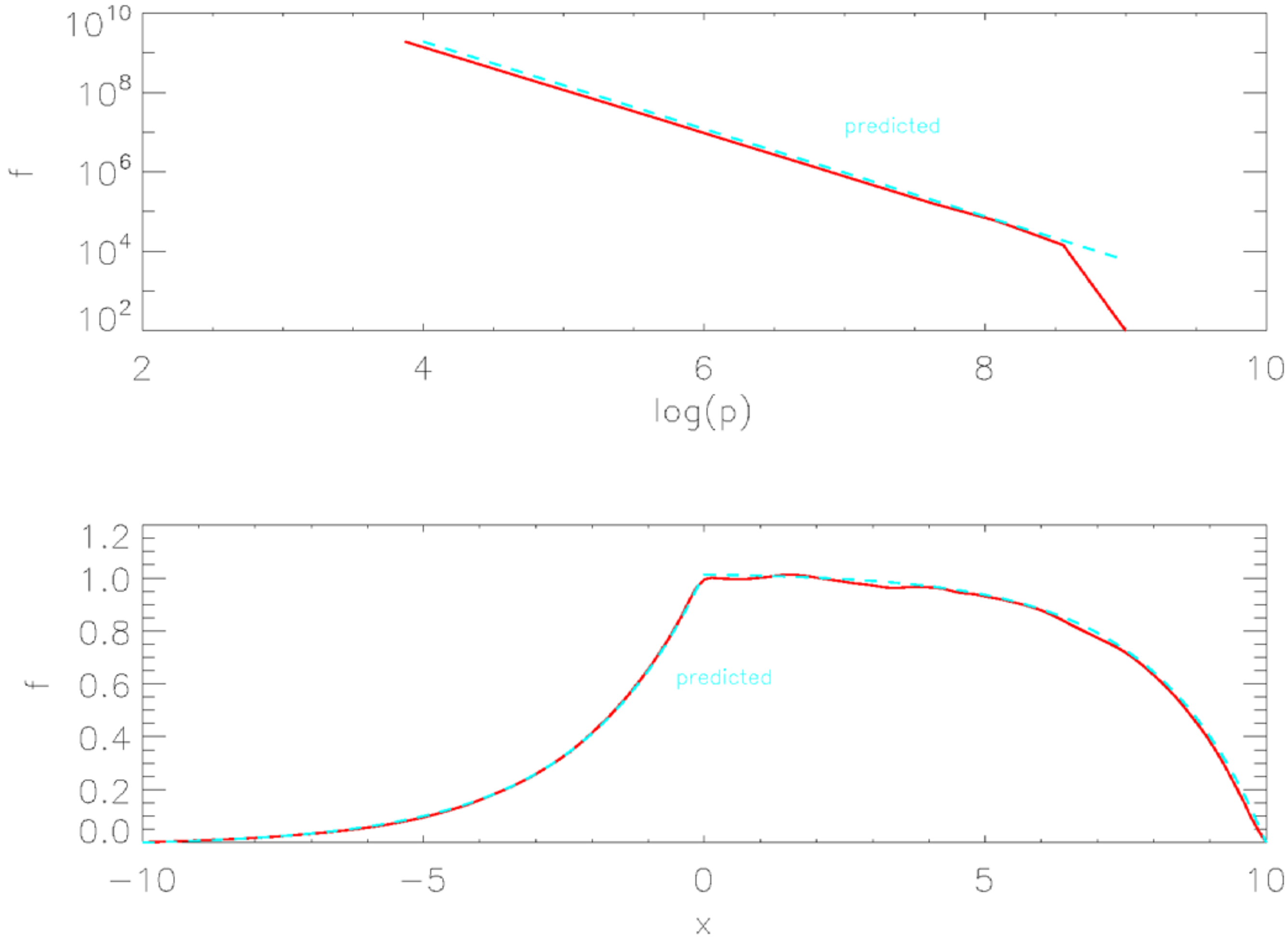
Final $X_{i+1} = 0 + \epsilon\ R\ (dt-T_f)0.5$

```
            if (sign(1,x1/x).lt.0)  then   ! crossed shock

c       do erica's approach to handle crossing of shock

        deltalp1 = (v1-v2)/(3.*vstoch1)
        deltalp2 = (v1-v2)/(3.*vstoch2)
        alp = alp + (deltalp1 + deltalp1)/2.

            amu = abs(x/x1)    ! set up gaussian dist
            if (x1.gt.0) alam = x**2/(akap1*2.)
            if (x1.le.0) alam = x**2/(akap2*2.)
            xx = rand(0)
            yy = rand(0)
            ann = sqrt(-2.0*alog(xx))*cos(tpi*yy)  !gaussian nn
       z = amu*ann**2
            s1 = amu+(amu*x)/(2.*alam)-(amu/(2.*alam))*sqrt(4.*alam*z+z**2)
            uu = rand(0)
            cond = amu/(amu+s1)
        if (uu.le.cond) xsi = s1
        if (uu.gt.cond) xsi = amu**2/s1
            t1 = xsi*dt/(1.+xsi)
            u2 = rand(0)
            beta = (sin(pi*u2/2.))**2
            tf = t1+(dt-t1)*beta
            uuu = rand(0)
            rr = sqrt(-2.*alog(uuu))
            yyy = rand(0)
             if(yyy.lt.alpha) eps = sqrt(akap2)
             if(yyy.ge.alpha) eps = -sqrt(akap1)
       x1 = 0.0 + eps*rr*sqrt(dt-tf)
```



10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
f
predicted
2
4
6
8
10
log(p)
1.0
0.8
0.6
0.4
0.2
0.0
f
predicted
−10
−5
0
5
10
x

Non-Bayes



With full bayes

Summary

- Using skewed steps with Baysian statistics seems to provide a usable algorithm for treating discontinuies in stochastic integration.

- Applying this to a simple planar shock gives good results.

- Further refinement is underway.