Supplementary Material
Shallow Water Analogue of the Standing Accretion Shock Instability: Experimental Demonstration and Two-Dimensional Model

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Dimensionless inviscid formulation

The set of inviscid shallow water equations are written in a dimensionless form by defining $\tilde r \equiv r/r_{\rm jp}$, $\tilde v \equiv v/v_{\rm ff}^{\rm jp}$, $\tilde c^2 \equiv gH/(v_{\rm ff}^{\rm jp})^2$, and $\tilde t \equiv t/t_{\rm ff}^{\rm jp}$. Denoting with $\tilde\nabla$ the differential operator with respect to the dimensionless coordinates, this system becomes:

$$\frac{\partial \tilde c^2}{\partial \tilde t} + \tilde\nabla\cdot(\tilde c^2 \tilde v) = 0, \quad (1)$$

$$\frac{\partial \tilde v}{\partial \tilde t} + (\tilde\nabla\times\tilde v)\times v + \tilde\nabla\left(\frac{\tilde v^2}{2} + \tilde c^2 - \frac{1}{2\tilde r}\right) = 0. \quad (2)$$

It can be compared to the dimensionless differential system describing an inviscid adiabatic gas falling in a potential $\Phi = -GM_{\rm NS}/r$, where $M_{\rm NS}$ is the mass of the proto-neutron star. The free fall velocity at the shock radius is defined by $(v_{\rm ff}^{\rm sh})^2/2 \equiv \Phi(\infty) - \Phi(r_{\rm sh})$. The normalized variables are $\tilde r \equiv r/r_{\rm sh}$, $\tilde v \equiv v/v_{\rm ff}^{\rm sh}$, $\tilde c \equiv c/v_{\rm ff}^{\rm sh}$, and $\tilde t \equiv t/t_{\rm ff}^{\rm sh}$. Defining the dimensionless entropy as $S \equiv \log(P/\rho^\gamma)/(\gamma-1)$, where γ is the adiabatic index of the gas and P its pressure, the sound speed is $c^2 \equiv \gamma P/\rho$ and the density ρ is proportional to $\rho \propto \tilde c^{2/(\gamma-1)}{\rm e}^{-S}$. In the adiabatic approximation, the equation of mass conservation and the Euler equation can be written as follows:

$$\frac{\partial}{\partial \tilde t}\tilde c^{\frac{2}{\gamma-1}} + \tilde\nabla\cdot\left(\tilde c^{\frac{2}{\gamma-1}}\tilde v\right) = 0 \quad (3)$$

$$\frac{\partial \tilde v}{\partial \tilde t} + (\tilde\nabla\times\tilde v)\times v + \tilde\nabla\left(\frac{\tilde v^2}{2} + \frac{\tilde c^2}{\gamma-1} - \frac{1}{2\tilde r}\right) = \frac{\tilde c^2}{\gamma}\tilde\nabla S \quad (4)$$

The inviscid system of shallow water Eqs.(1,2) is thus formally identical to the system of Eqs. (3,4) describing an isentropic gas ($\nabla S = 0$) with an adiabatic index $\gamma = 2$.

The solution of the shallow water model is fully determined by specifying the size of the jump radius relative to the size of the inner boundary $r_{\rm jp}/r_*$, the pre-shock velocity relative to the free fall velocity $v_1/v_{\rm ff}^{\rm jp}$ and the Froude number ${\rm Fr}_1$ ahead of the shock. Once these dimensionless parameters are chosen, the experimental solution can be scaled to astrophysical proportions by using the scaling factors $r_{\rm sh}/r_{\rm jp}$, $t_{\rm ff}^{\rm sh}/t_{\rm ff}^{\rm jp}$ and $v_{\rm ff}^{\rm sh}/v_{\rm ff}^{\rm jp}$ for distances, timescales and velocities.

Perturbative analysis

Denoting by δ the Eulerian perturbation, we adopt the variables $(\delta f, \delta h)$ defined by

$$\delta f \equiv v_r\delta v_r + g\delta H, \quad (5)$$

$$\delta h \equiv \frac{\delta v_r}{v_r} + \frac{\delta H}{H}. \quad (6)$$

The differential system satisfied by the perturbations in the inviscid approximation is formally identical to the system describing the perturbations of an isothermal flow [1]. Taking into account a laminar viscous drag, this differential system can be rewritten as follows:

$$\frac{{\rm d}\delta f}{{\rm d}r} = \frac{i\omega v_r}{1-{\rm Fr}^2}\left(\delta h - \frac{\delta f}{c^2}\right) + \frac{\bar\nu v_r}{H^2(1-{\rm Fr}^2)}\left[3\frac{\delta f}{c^2} - (1+2{\rm Fr}^2)\delta h\right], \quad (7)$$

$$\frac{{\rm d}\delta h}{{\rm d}r} = \frac{i\omega}{v_r(1-{\rm Fr}^2)}\left(\frac{\delta f}{c^2} - {\rm Fr}^2\delta h\right) - \frac{im}{r^2 v_r}r\delta v_\theta, \quad (8)$$

$$\frac{{\rm d}r\delta v_\theta}{{\rm d}r} = \frac{imv_r}{1-{\rm Fr}^2}\left(\delta h - \frac{\delta f}{v_r^2}\right) + \left(\frac{i\omega}{v_r} - \frac{\bar\nu}{v_r H^2}\right)r\delta v_\theta \quad (9)$$

The jump equations are deduced from the conservation of mass flux and momentum flux across the jump. Let us define $\Delta v = -i\omega\Delta r$, and use the subscripts "jp" and "1" for post-jump and pre-jump quantities:

$$\delta f_{\rm jp} = \left(1 - \frac{H_1}{H_{\rm jp}}\right)\left\{-v_{\rm jp}\Delta v + \Delta\zeta\left[\frac{\partial\Phi}{\partial r} - \frac{v_1 v_{\rm jp}}{r_{\rm jp}} - \frac{\bar\nu v_1^2}{H_{\rm jp}}\left(1 + \frac{v_{\rm jp}}{v_1}\right)\right]\right\} \quad (10)$$

$$\delta h_{\rm jp} = \frac{\Delta v}{v_{\rm jp}}\left(1 - \frac{H_1}{H_{\rm jp}}\right), \quad (11)$$

$$r\delta v_{\theta,{\rm jp}} = imv_1\Delta\zeta\left(1 - \frac{H_1}{H_{\rm jp}}\right). \quad (12)$$

Since water is in free fall after passing the upper edge of the central cylinder, the inner boundary condition is modeled as a nozzle where the flow velocity reaches the wave velocity. Denoting by H_* the depth of the fluid over the upper edge, it is related to the flow rate Q as follows:

$$Q = 2\pi R_* g^{\frac{1}{2}} H_*^{\frac{3}{2}}. \quad (13)$$

The corresponding boundary condition is defined by the regularity at the critical point:

$$\delta f_* = c_\mathrm{s}^2 \delta h_*, \tag{14}$$

$$c_s^2 \equiv \left(\frac{Qg}{2\pi R_*} \right)^{\frac{2}{3}}. \tag{15}$$

The boundary condition should be modified when the flow rate is so large that the inner cylinder is completely submerged.

Numerical simulations of the experiment

The hydrocode used for the numerical simulations is the JUPITER code [2]. We used a polar mesh with even radial and azimuthal spacing. Our fiducial resolution has 800 radial zones by 800 azimuthal zones. The outer boundary is set at $r = 32$cm, where a torrential inflow boundary condition is applied (equivalent to supersonic in the terminology of shallow water). The inner radius of the mesh is set at r=4cm, and has a sonic point boundary condition, analogous to that described in the perturbative analysis. The numerical scheme is based on a higher order Godunov method. A shallow water Riemann problem is solved at each interface. The left and right state of this problem are constructed using a characteristic method. The problem is then solved using a two-jump Riemann solver, together with an exact sampling of the solution to infer the interface state (in a strict analogy with the two-shock Riemann solver [3] for an ideal gas). The density and momenta fluxes thus obtained are used to update the fluid's height and velocities. The gradient of the bottom topography is used as a source term in the velocity equations. Godunov methods are known to handle stationary flows with source terms in an inaccurate manner [4]. The resolution that we adopt, however, together with the mild gradient of the bottom topography, is found to yield a satisfactory accuracy on the initial, stationary flow, and no further special treatment of the source term is required. The code is parallelized using the Message Passing Interface (MPI) and a slab domain decomposition.

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