Lecture #3

Solenoidal; Dipole; Quadrupole, Racetrack Coils

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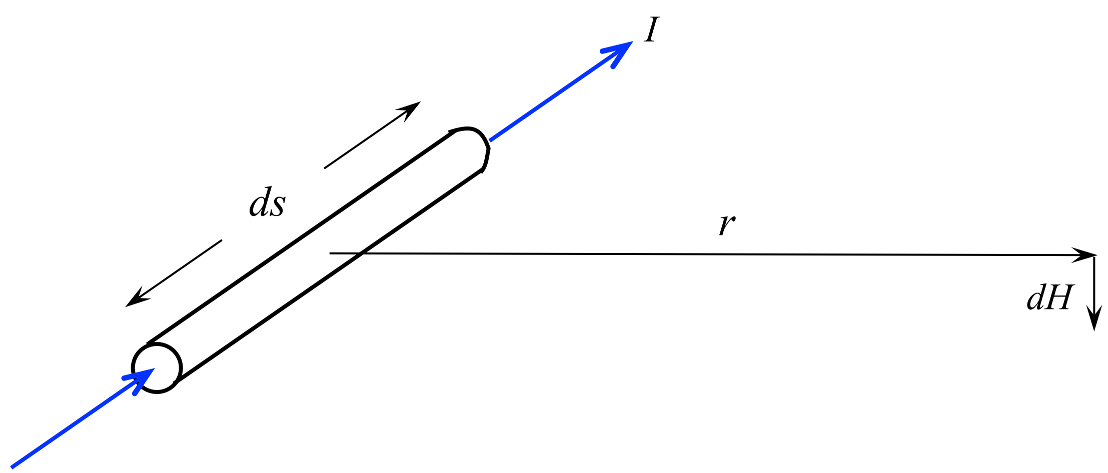
Outline

- Generation of magnetic field
- Solenoid
- Ideal dipole
- Ideal quadrupole
- Ideal racetrack
- Self inductances

Generation of Magnetic Field

Law of Biot-Savart

$$d\vec{H} = \frac{I\,d\vec{s} \times \vec{r}}{4\pi r^3}$$



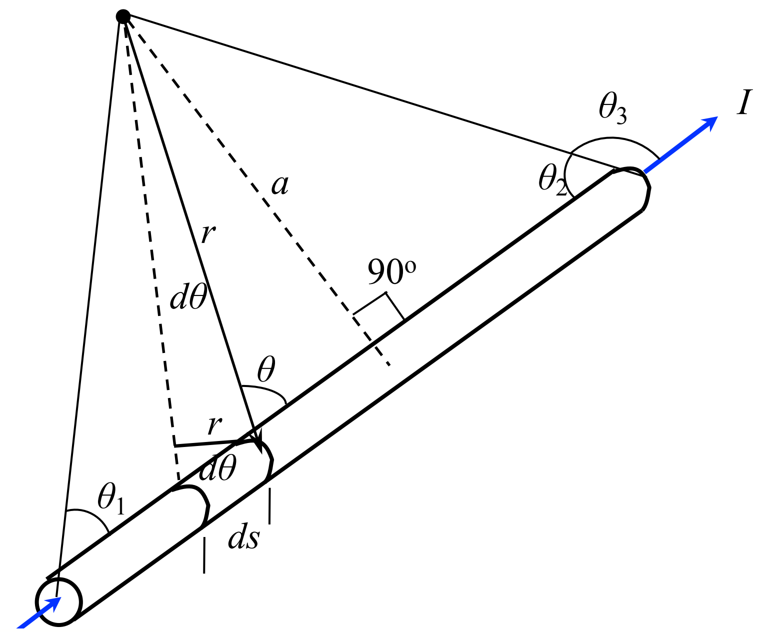
Straight Conductor of Finite Length

$$d\vec{H} = \frac{I\, d\vec{s} \times \vec{r}}{4\pi r^3} \;\rightarrow\; dH = \frac{I\, ds \sin\theta}{4\pi r^2}$$

$$\frac{r d\theta}{ds} = \sin\theta = \frac{a}{r} \qquad ds/r^2 = d\theta/a$$

$$H = \frac{I}{4\pi a}\int_{\theta_1}^{\theta_3} \sin\theta\, d\theta$$

$$H = \frac{I}{4\pi a}(-\cos\theta_3 + \cos\theta_1) = \frac{I}{4\pi a}(\cos\theta_2 + \cos\theta_1)$$



When *length* >> a

θ

$$\cos\theta_1 \rightarrow 1; \quad \cos\theta_2 \rightarrow 1$$

$$H = \frac{I}{2\pi a}$$

H varies as $1/r$

Current-Carrying Ring

$$d\vec{H} = \frac{I\, d\vec{s} \times \vec{r}}{4\pi r^3}$$

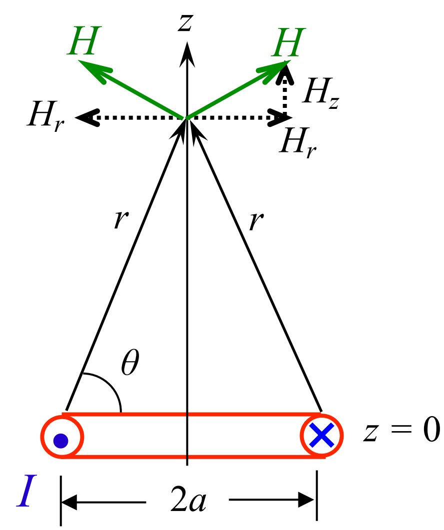
$$H_z = \frac{I(2\pi a) r \cos\theta}{4\pi r^3} = \frac{Ia\cos\theta}{2r^2}$$

$$\cos\theta = \frac{a}{r} \qquad H_z = \frac{a^2 I}{2r^3}$$

$$r^2 = a^2 + z^2$$

$$H_z(z,0) = \frac{a^2 I}{2(a^2+z^2)^{3/2}}$$

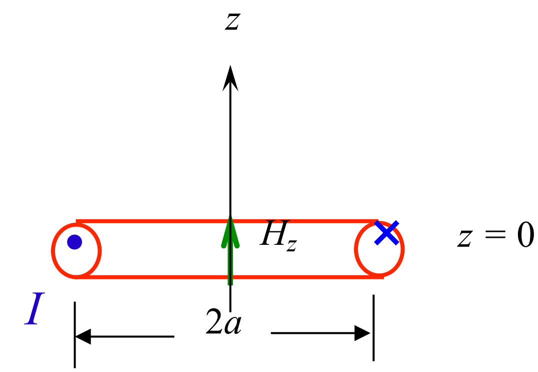
r =0; Here the r represents (r, z, θ) coordinates



Carrying-Current Ring (cont.)

$$H_z(z,0) = \frac{a^2 I}{2(a^2+z^2)^{3/2}}$$

$$H_z(z \gg a) = \frac{I}{2a}\left(\frac{a}{z}\right)^3 = H(0)\left(\frac{a}{z}\right)^3$$



Far away from the source, even a solenoid field varies as $1/r^3$

$$H_z(0) = \frac{I}{2a}$$

$$H = \frac{I}{2\pi a}$$ for a straight conductor

H increased by π when a straight piece of wire is folded into a circle

$$B_z(0) = \mu_\circ \frac{NI}{2a} \qquad \mu_\circ = 4\pi \times 10^{-7}\,\mathrm{H/m}$$

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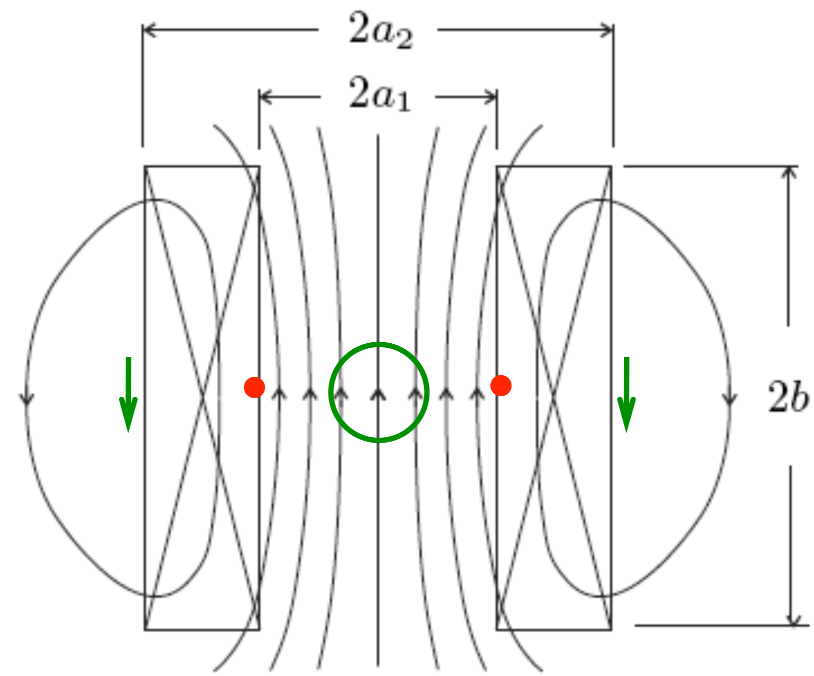
Solenoid

Generates a uniform axial field; most common magnet configuration

- Important dimensionless coil constants

$$\alpha = \frac{a_2}{a_1}; \quad \beta = \frac{b}{a_1}$$



Field Lines

- "Uniform" around the center
- Maximum at $r = a_1$, $z = 0$ (single coil)
- At $r = a_2$, $z = 0$, $B_z \approx$ 1/10 of $-B_o \equiv B_z(r = 0, z = 0) = B_z(0, 0)$

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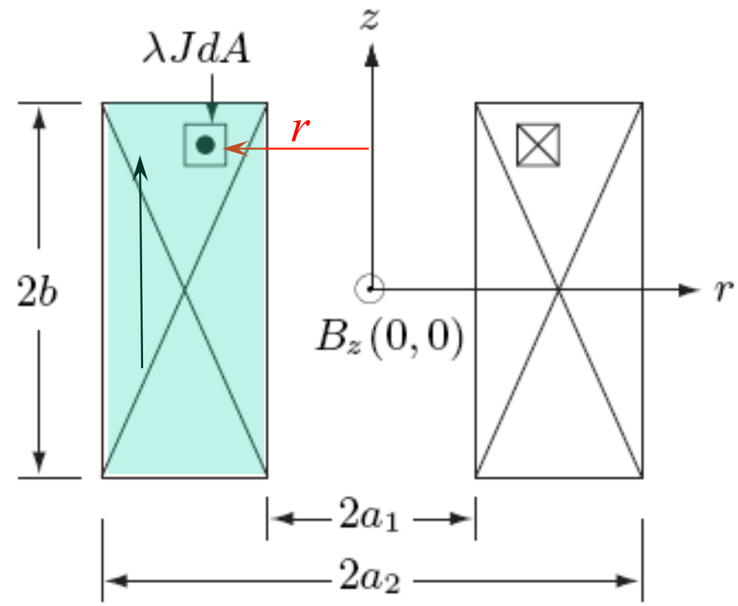
Solenoid: Filed Computation

Uniform-Current Density Solenoid

$$dB_z(0,0) = \frac{\mu_\circ r^2 \lambda J\, dA}{2(r^2+z^2)^{3/2}}$$

Winding total cross section



$$\lambda J = \frac{NI}{2b(a_2-a_1)}$$

N number of turns

Overall current density, J_{ov}

$$= \frac{NI}{2a_1^2\beta(\alpha-1)}$$

$B_z(0,0) = \mu_\circ \lambda J a_1 F(\alpha,\beta)$ $\quad F(\alpha,\beta)$: Field factor; depends only on α & β

$$F(\alpha,\beta) = \beta \ln\left(\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$$

$$B_z(0,0) = \frac{\mu_\circ NI}{2a_1(\alpha-1)} \ln\left(\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$$

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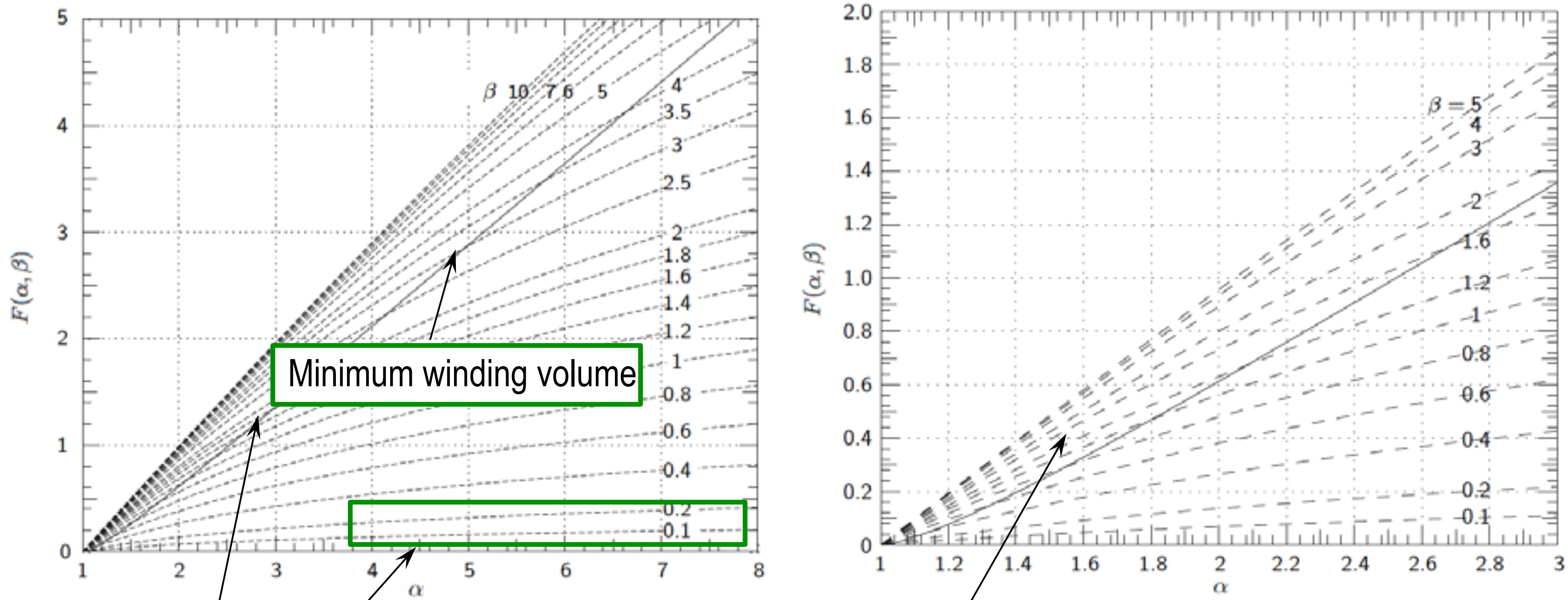
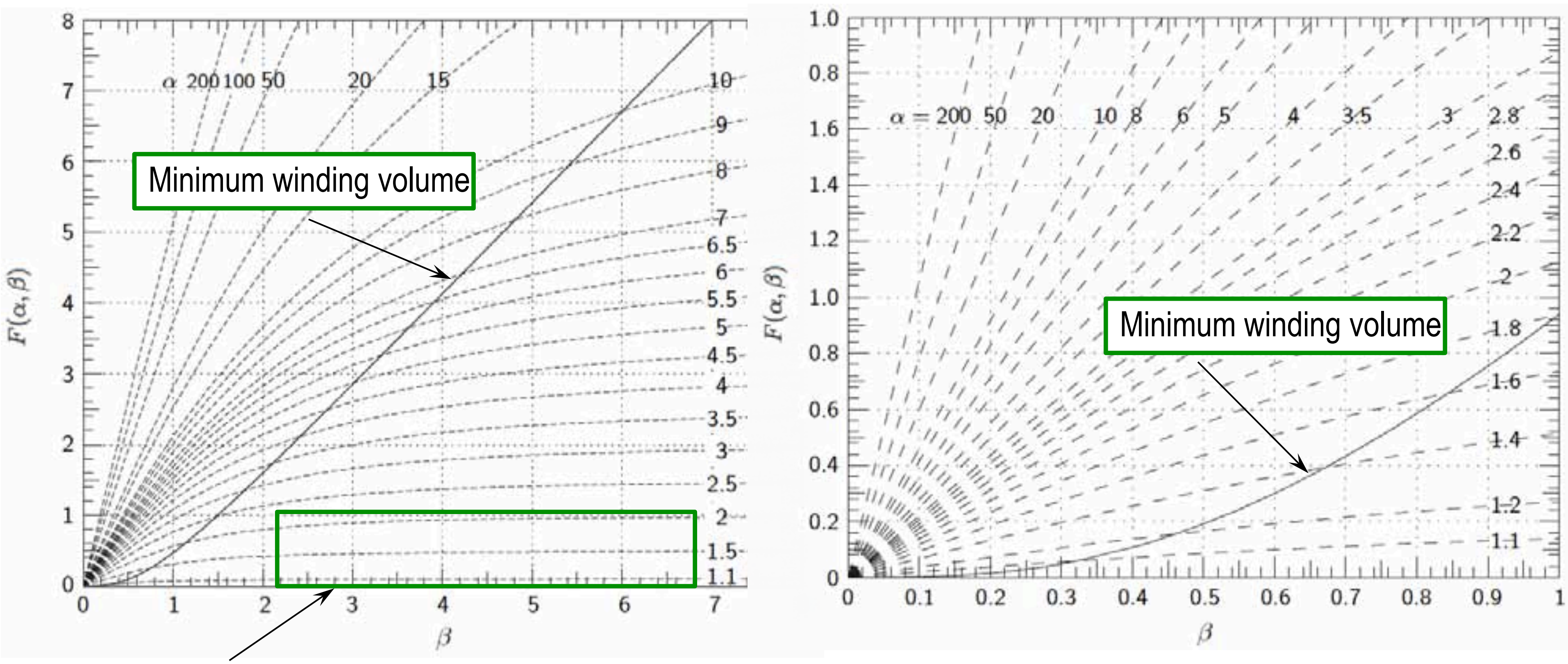
$F(\alpha, \beta)$ vs. α for selected values of β



- "Long" coils ($\beta > 1$): field increases with winding thickness, $a_1(\alpha - 1)$
- "Short" coils ($\beta << 1$): field ~independent of $a_1(\alpha - 1)$

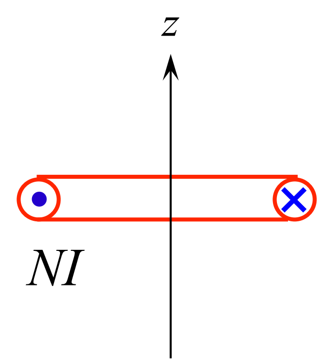
$F(\alpha, \beta)$ vs. β for selected values of α



- Field ~ independent of coil length $\beta > \sim 2$ for α up to ~2
  - From stress consideration $\alpha > 2$ rarely used
  - Field spatial homogeneity improves with ➚ β

  ➔ NMR & MRI magnets tend to be long

"Ring" Coil $(\alpha = 1; \beta = 0)$



$$B_z(0,0) = \frac{\mu_\circ NI}{2a_1(\alpha - 1)} \ln\left(\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{1 + \sqrt{1 + \beta^2}}\right)$$

$$\lim_{\beta \to 0} \ln\left(\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{1 + \sqrt{1 + \beta^2}}\right) = \ln \alpha \qquad \ln \alpha = \alpha - 1$$

$$B_z(0,0) = \frac{\mu_\circ NI}{2a_1(\alpha - 1)}(\alpha - 1) = \frac{\mu_\circ NI}{2a_1}$$

This expression very useful to get a feel of what NI would be for a given set of $B_z(0,0)$ and $2a_1$ (and sometimes I)

ISEULT

Parameters

$B_z(0) = 11.76$ T; $2a_1 = 1$ m; $2a_2 \approx 1.95$ m; $2b \approx 3.1$ m; $I_{op} \approx 1500$ A

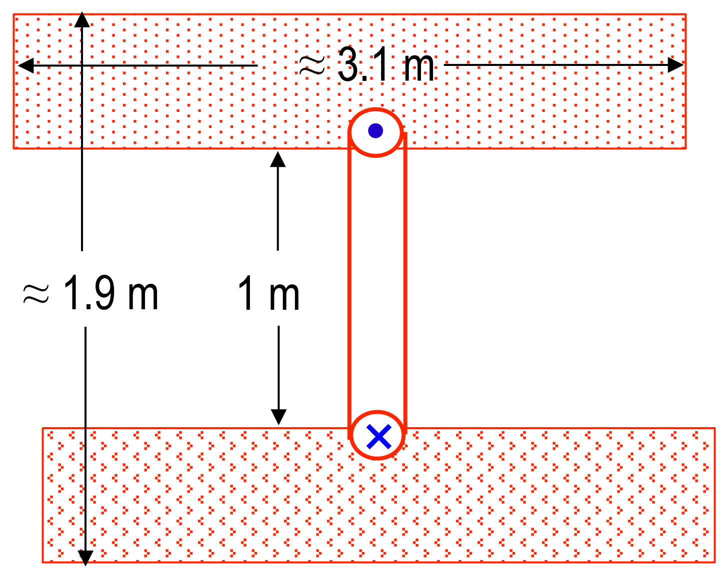
$$B_z(0) = \mu_\circ \frac{NI}{2a} \Rightarrow 11.76\ \text{T} = [(4\pi \times 10^{-7}\ \text{H/m})\ (\text{NI A})] / 1\ \text{m}]$$

Solve for NI [A]

$$NI = (11.76\ \text{T} \times 1\ \text{m}) / (4\pi \times 10^{-7}\ \text{H/m}) = 9.4\ \text{MA}$$

$I_{op} \approx 1500$ A ➔ $N \approx 6250$ turns

- Because obviously all 6250 turns *cannot be placed at a single center point*, N must be spread out over a wider space, making real *ISEULT* $N = 29920 > 6250$
- Note that the ***center ring*** generates the highest field at the magnet center, the rest less



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"Long" Solenoid $(\beta \gg \alpha)$

$$B_z(0,0) = \frac{\mu_\circ NI}{2a_1(\alpha - 1)} \ln\left(\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{1 + \sqrt{1 + \beta^2}}\right)$$

$$\lim_{\beta \gg \alpha} \ln\left(\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{1 + \sqrt{1 + \beta^2}}\right) = \ln\left(\frac{\alpha + \beta}{1 + \beta}\right) = \ln\left(\frac{\alpha/\beta + 1}{1/\beta + 1}\right)$$

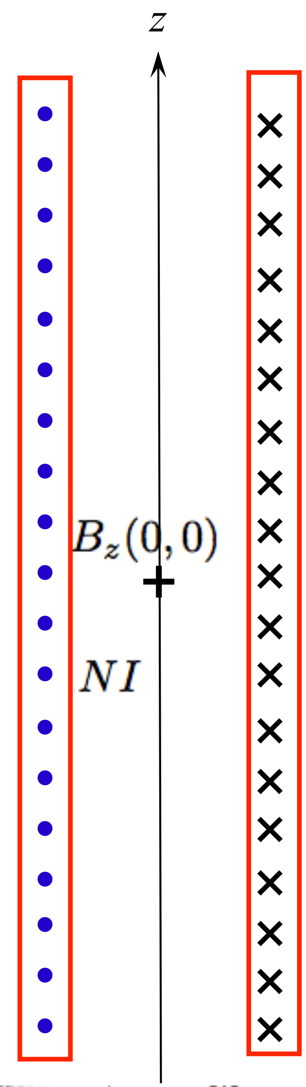
$$\lim_{\beta \gg \alpha} \ln\left(\frac{\alpha/\beta + 1}{1/\beta + 1}\right) \simeq \frac{\alpha}{\beta} - \frac{1}{\beta} = \frac{\alpha - 1}{\beta}$$

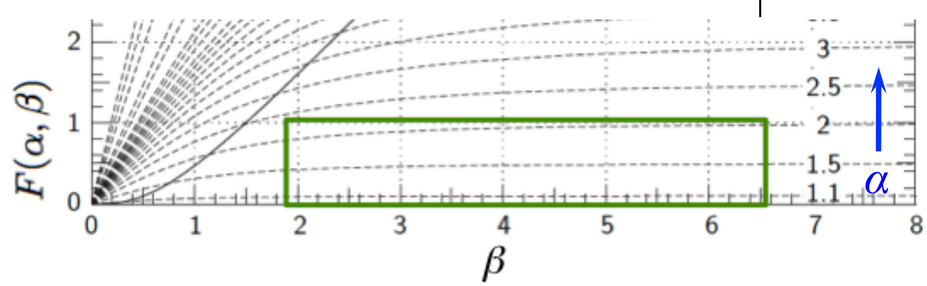
$$B_z(0,0) = \frac{\mu_\circ NI}{2b}$$

Turns/length:
$B_z(0,0)$ length ($2b$)-*independent*

$$B_z(0,0) = \mu_\circ \lambda J a_1(\alpha - 1)$$

$B_z(0,0) \propto$ to winding build





"Thin-Walled" Solenoid $(\alpha \to 1)$

$$F(\alpha,\beta) = \beta \ln\left(\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$$

$$\lim_{\alpha\to 1} \ln\left(\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right) = \frac{\alpha - 1}{\sqrt{1+\beta^2}}$$

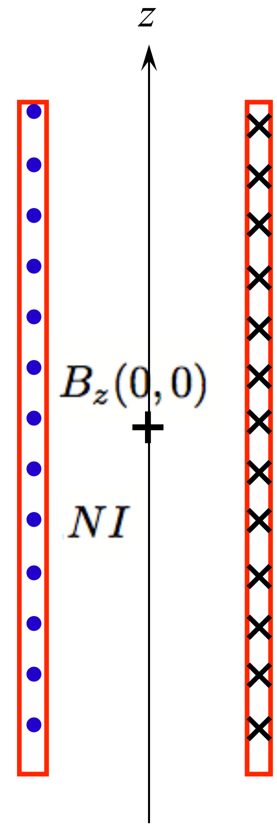
$$\lim_{\alpha\to 1} F(\alpha,\beta) = \beta \frac{\epsilon}{\sqrt{1+\beta^2}} = \frac{\beta(\alpha-1)}{\sqrt{1+\beta^2}}$$

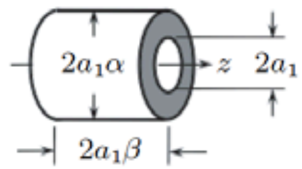
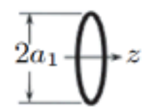
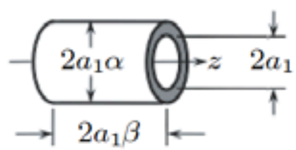
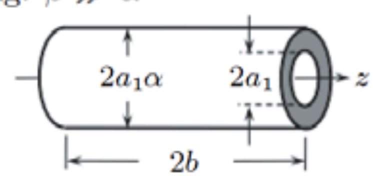
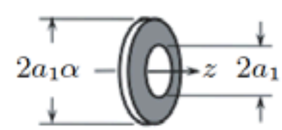
$$B_z(0,0) = \frac{\mu_\circ NI}{2a_1(\alpha-1)} \ln\left(\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$$

$$B_z(0,0) = \frac{\mu_\circ NI}{2a_1}\left(\frac{1}{\sqrt{1+\beta^2}}\right) = \frac{\mu_\circ NI}{2b}$$

A "long" coil, $\beta >> 1$, becomes β-independent: Note that: $2a_1\beta = 2b$

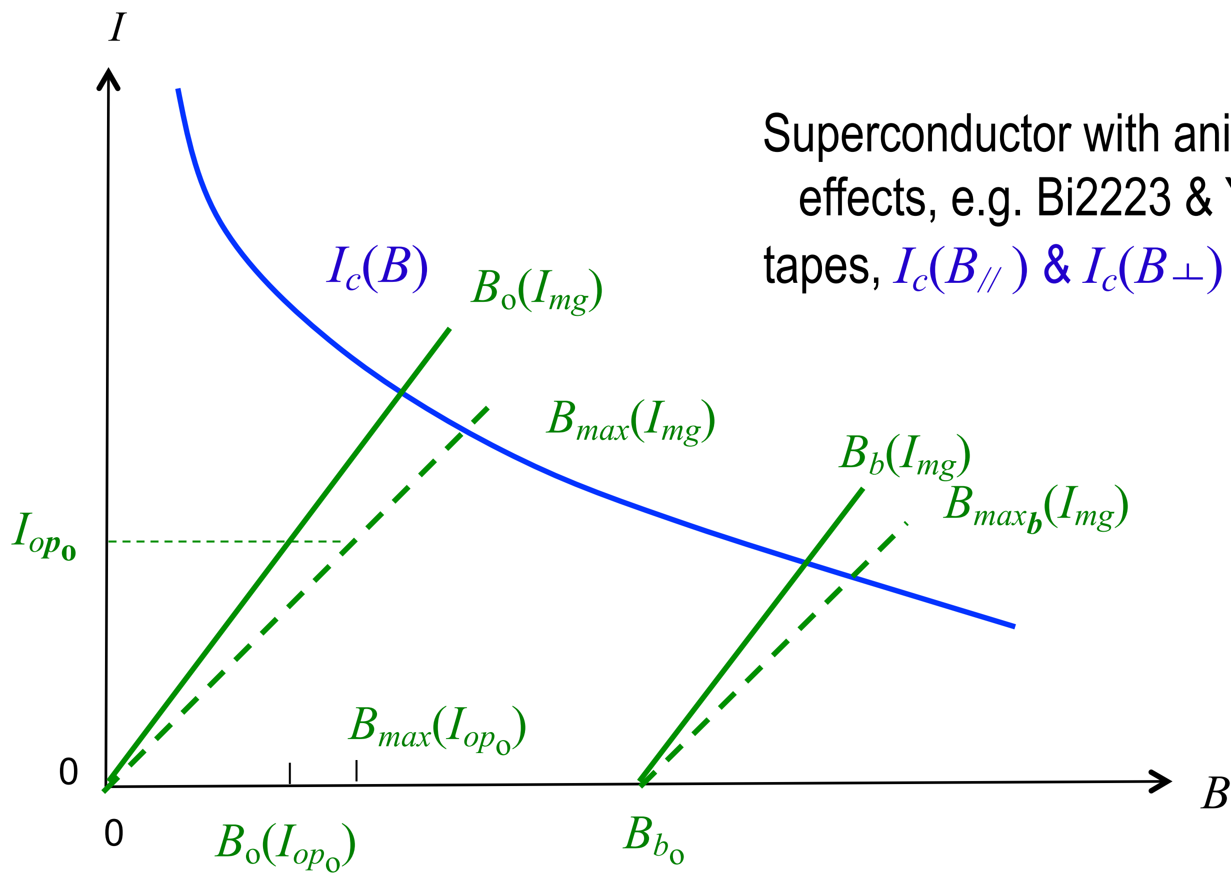
$$B_z(0,0) = \mu_\circ \lambda J a_1(\alpha-1)\frac{\beta}{\sqrt{1+\beta^2}}$$
$$= \mu_\circ \lambda J a_1(\alpha-1)$$



Solenoid Type	Center (0,0,0) Field B_{z0}
General: α, β 	$B_{z0} = \dfrac{\mu_\circ NI}{2a_1(\alpha-1)} \ln\left(\dfrac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$ $= \mu_\circ \lambda J a_1 F(\alpha,\beta)$ $F(\alpha,\beta) = \beta \ln\left(\dfrac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$
Ring: $\alpha=1$, $\beta=0$ 	$B_{z0} = \dfrac{\mu_\circ NI}{2a_1}$
Thin-Walled: $\alpha \to 1$, β 	$B_{z0} = \dfrac{\mu_\circ NI}{2a_1}\left(\dfrac{1}{\sqrt{1+\beta^2}}\right)$ $= \mu_\circ \lambda J a_1 (\alpha-1) \dfrac{\beta}{\sqrt{1+\beta^2}}$
Long: $\beta \gg \alpha$ 	$B_{z0} = \dfrac{\mu_\circ NI}{2b}$ $= \mu_\circ \lambda J a_1 (\alpha-1)$
Short (pancake): $\alpha, \beta \to 0$ 	$B_{z0} = \dfrac{\mu_\circ NI}{2a_1}\left(\dfrac{\ln\alpha}{\alpha-1}\right)$

Load Lines

Superconductor with anisotropic effects, e.g. Bi2223 & YBCO tapes, $I_c(B_{//})$ & $I_c(B_{\perp})$ required



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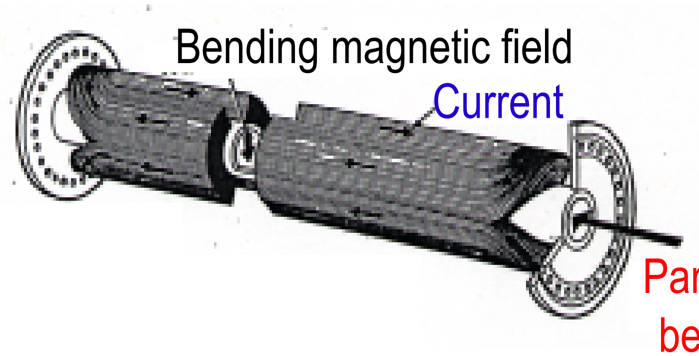
Dipole

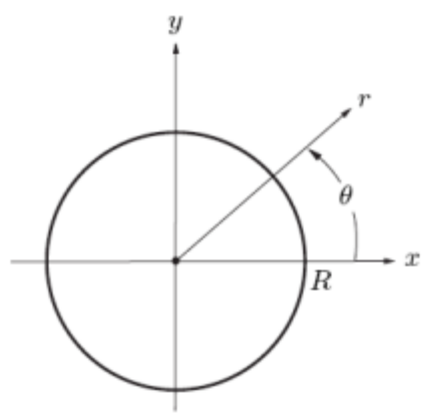
Ideal Dipole

- Circular cross section (radius R) of "zero" winding thickness
  - Infinitely long, i.e., no end effects



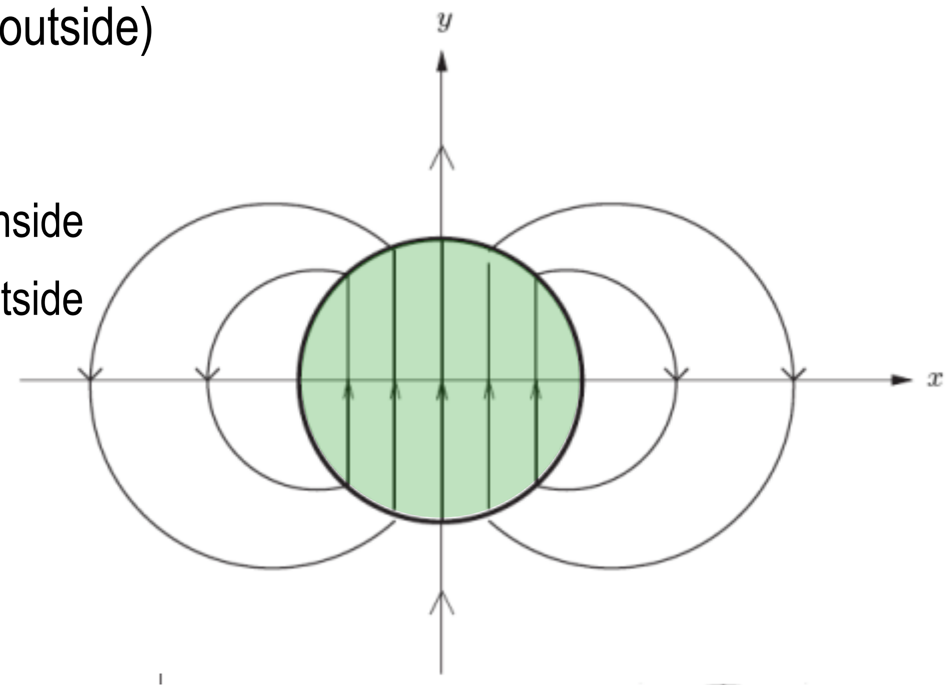
$$\vec{H}_{d1} = H_0(\sin\theta\,\vec{\imath}_r + \cos\theta\,\vec{\imath}_\theta) \qquad \text{(inside)}$$

$$\vec{H}_{d2} = H_0\left(\frac{R}{r}\right)^2(\sin\theta\,\vec{\imath}_r - \cos\theta\,\vec{\imath}_\theta) \qquad \text{(outside)}$$



Uniform H inside

$H \propto 1/r^2$ outside

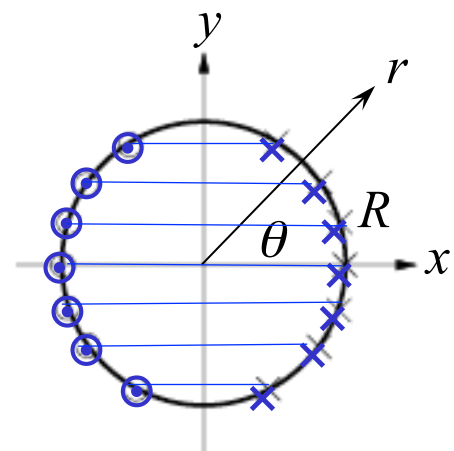


Ideal Dipole (continuation)

- Surface current density at $r = R$ sinusoidal:

$$\vec{K}_f = -2H_0 \cos\theta \, \vec{\imath}_z$$

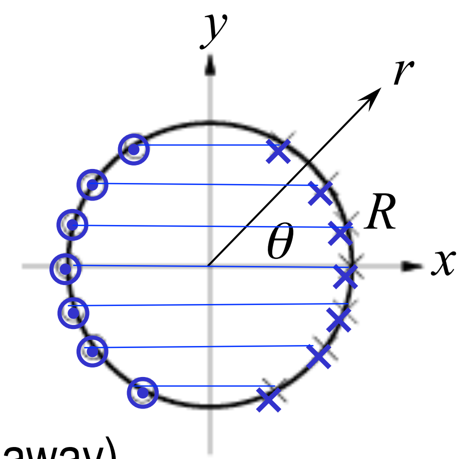
where H_0 is the dipole field strength



Because surface current density varies as $\cos\theta$, this is called a "cosine dipole"

Although an ideal dipole different from the real-world dipole used in HEP devices, most ballpark numbers may be obtained from an ideal dipole

Ideal Dipole (continuation)



$$\vec{K}_f = -2H_0 \cos\theta\, \vec{\imath}_z$$

What should K_f be for $\mu_o H_0$ = 8.33 T*? (LHC)

* Neglects field contribution of steel yoke (because yoke a bit "far" away)

$|K_f| = 2H_0 = 2(8.33\text{ T})/(4\pi \times 10^{-7}\text{ H/m}) = 13.3\text{ MA/m}$

For $2R_i = 56$ mm & $2R_o = 120.5$ mm, $2R_{av} = 88.25$ mm:

Ampere-turns: $$\int_{-90^\circ}^{90^\circ} 2H_0 R_{av} \cos\theta\, d\theta$$

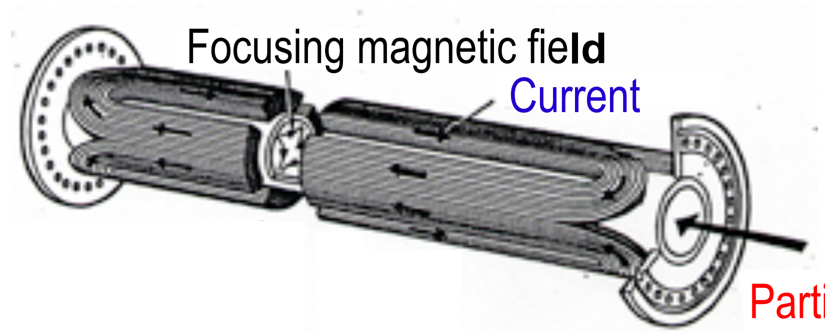
$$= (2H_0)(2R_{av}) = 1.17\text{ MA}$$

For $I_{op} = 11.8$ kA, the LHC dipole winding requires ~100 turns

Quadrupole

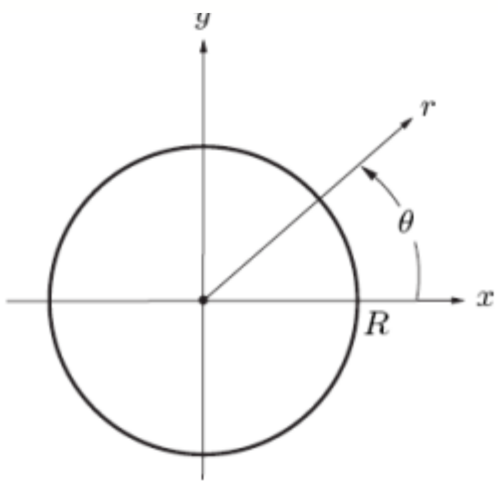
Ideal Quadrupole

- Circular cross section (radius R) of "zero" winding thickness
- Infinitely long, i.e., no end effects



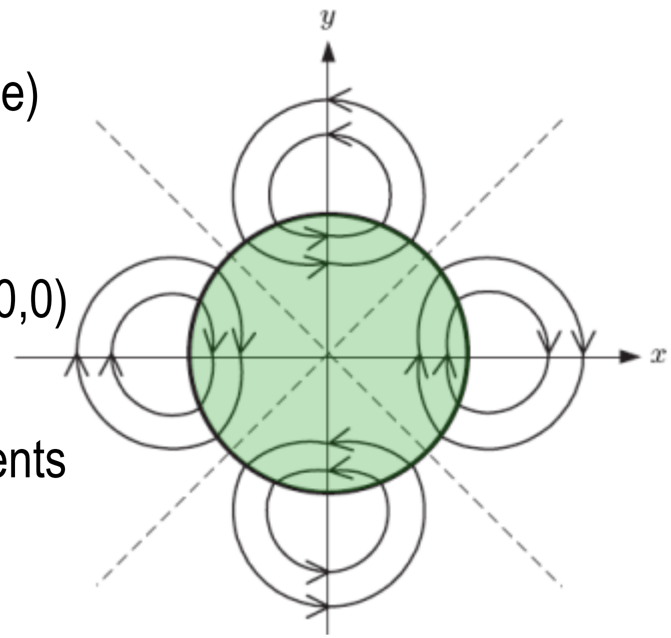
$$\vec{H}_{q1} = H_0 \left(\frac{r}{R}\right) (\sin 2\theta\, \vec{\imath}_r + \cos 2\theta\, \vec{\imath}_\theta) \quad \text{(Inside)}$$

$$\vec{H}_{q2} = H_0 \left(\frac{R}{r}\right)^3 (\sin 2\theta\, \vec{\imath}_r - \cos 2\theta\, \vec{\imath}_\theta) \quad \text{(Outside)}$$



At the center (0,0)

- Zero field
- Fields gradients



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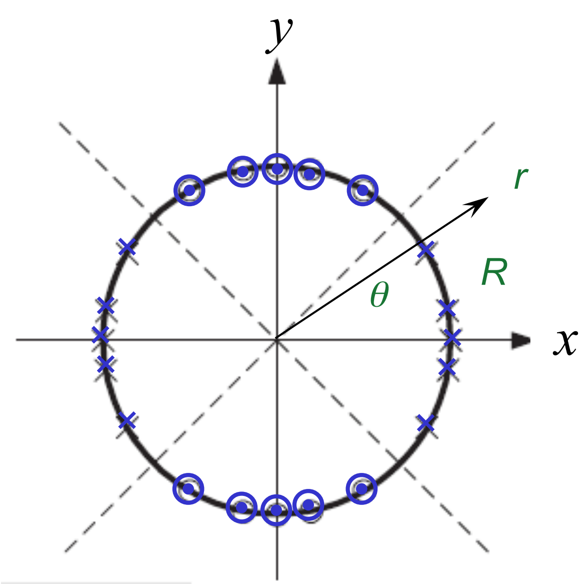
Ideal Quadrupole (continuation)



- Surface current density at $r = R$ sinusoidal:

$$\vec{K}_f = -2H_0 \cos 2\theta \, \vec{\imath}_z$$

where H_0 is the dipole field strength

Magnetic spring constant in the x-direction, k_{Lx}

$$k_{Lx} = -\frac{\partial F_{Lx}}{\partial x}$$

$$F_{Lx} \simeq [q(c\,\vec{\imath}_z) \times \mu_\circ H_{q1}\,\vec{\imath}_\theta]_{\theta=0}$$

$$\simeq -qc\mu_\circ H_0 \left(\frac{r}{R}\right) \vec{\imath}_x$$

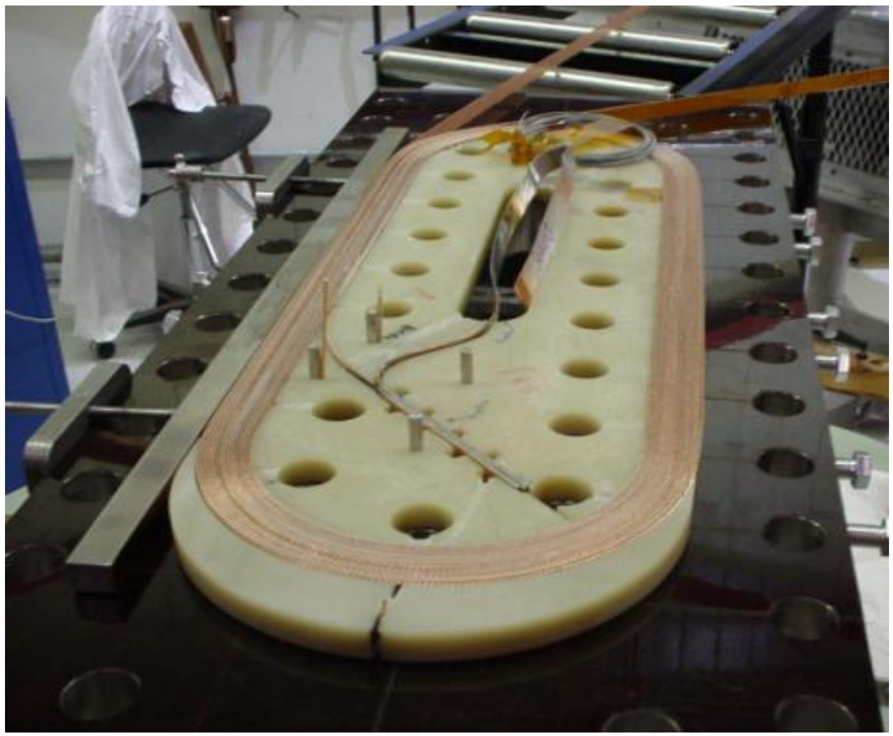
$$k_{Lx} \simeq \frac{qc\mu_\circ H_0}{R}$$

Focusing in the $\pm\, x$ directions

Racetrack Coil

A magnet resembling a racetrack, wound in a plane, each turn having two parallel sides, joined by a semi-circle at each end; Two or more such coils separated by a gap generate a field approximating that of a dipole magnet

- *See the 2nd Edition of my textbook for analytical treatment of an ideal racetrack*



[Arnaud Devred (CEA), 2002]

Self Inductance

Total flux linking a coil, Φ, $\propto$ to the coil current, I

$$\Phi = LI$$

L the coil's self inductance; a circuit element that relates to a field quantity

L also related to the coil's magnetic energy, E_m:

$$E_m = \frac{1}{2}LI^2$$

Generally L computed from Φ, then used to compute E_m

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Inductance Matrix—An Example

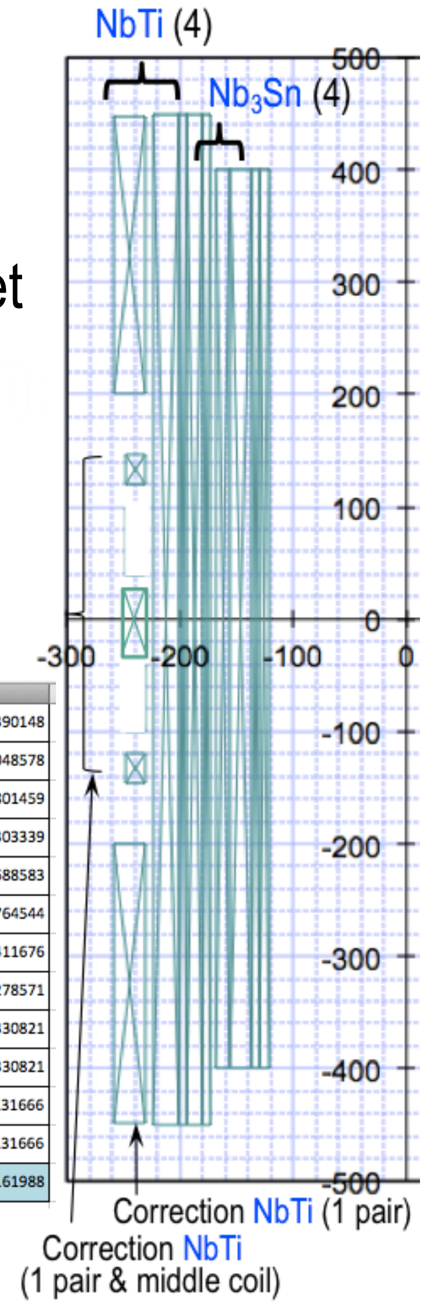
***L500* (JASTEC)**

500-MHz (11.7 T)/237-mm cold bore NMR magnet:

LTS part of the MIT 1.3-GHz (30.5 T) LTS/HTS NMR magnet

- Composed of 13* separate coils: 8 main (4 Nb_3Sn; 4 NbT
  5 correction (NbTi; 2 pairs and a middle coil)

  * Actually 14 with the last middle coil split into a pair



L500 Inductance Matrix (Total $L = 147.185$ H)

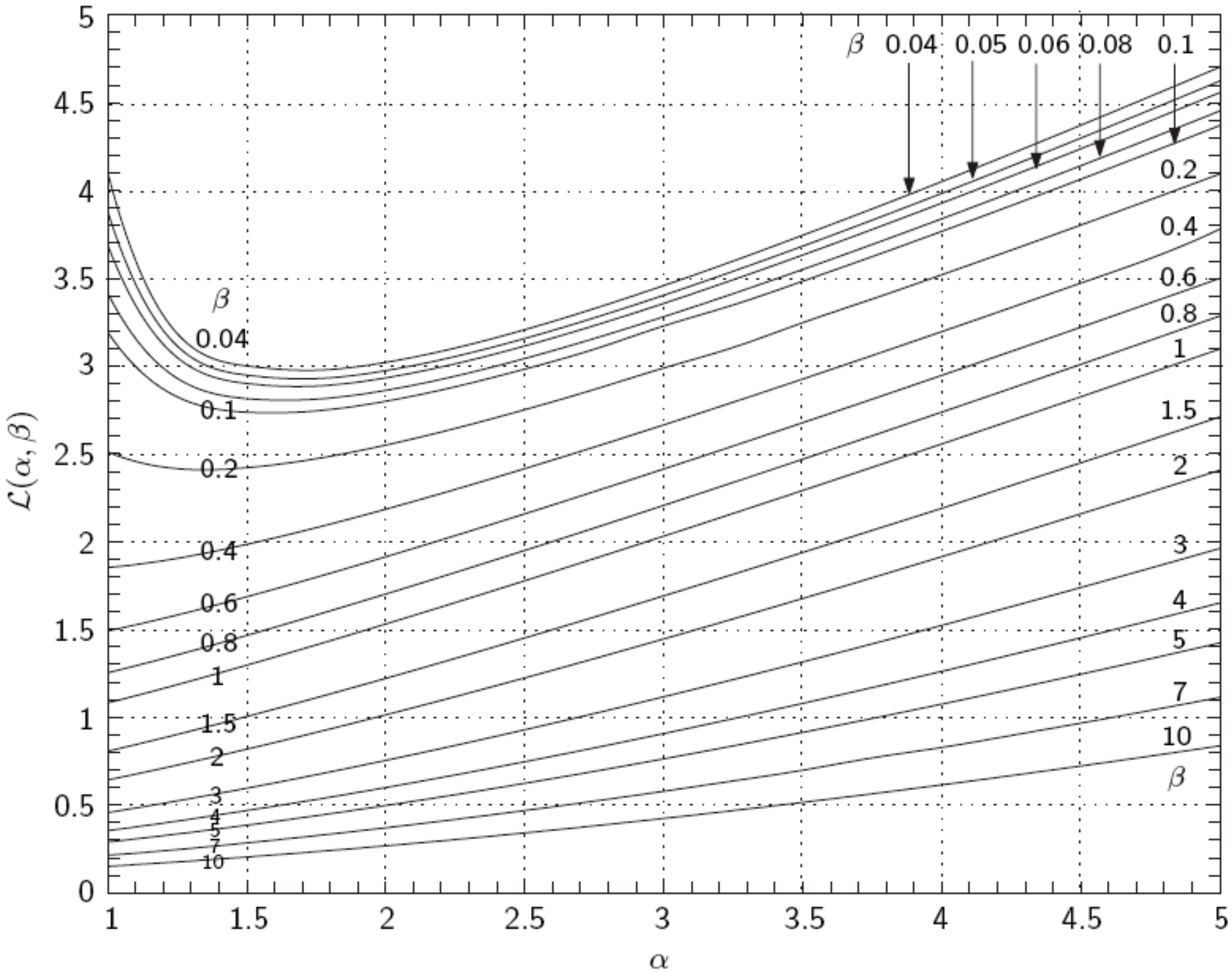
	A	B	C	D	E	F	G	H	I	J	K	L	M
1	0.300075559	0.318608485	0.742363862	0.442794726	0.211890862	0.568950953	0.337059981	1.04646977	0.359747712	0.359747712	-0.03297875	-0.03297875	0.079390148
2	0.318608485	0.37343513	0.891376893	0.531338093	0.253985181	0.681936473	0.403966195	1.254075001	0.431246035	0.431246035	-0.039502157	-0.039502157	0.095048578
3	0.742363862	0.891376893	2.473219875	1.548955383	0.738604483	1.982915107	1.174497393	3.645502171	1.254362163	1.254362163	-0.11472094	-0.11472094	0.275801459
4	0.442794726	0.531338093	1.548955383	1.127351895	0.5485133	1.472451569	0.872039262	2.706210928	0.931895089	0.931895089	-0.085062479	-0.085062479	0.204303339
5	0.211890862	0.253985181	0.738604483	0.5485133	0.323703483	0.878933207	0.519804017	1.611459234	0.583535021	0.583535021	-0.048195114	-0.048195114	0.114688583
6	0.568950953	0.681936473	1.982915107	1.472451569	0.878933207	2.591051807	1.577043012	4.886075245	1.771063235	1.771063235	-0.145778828	-0.145778828	0.346764544
7	0.337059981	0.403966195	1.174497393	0.872039262	0.519804017	1.577043012	1.048792362	3.300478187	1.197637925	1.197637925	-0.09815822	-0.09815822	0.233411676
8	1.04646977	1.254075001	3.645502171	2.706210928	1.611459234	4.886075245	3.300478187	11.66980298	4.419823388	4.419823388	-0.359131258	-0.359131258	0.855278571
9	0.359747712	0.431246035	1.254362163	0.931895089	0.583535021	1.771063235	1.197637925	4.419823388	5.927316197	0.251466252	-0.1623768	-0.035340583	0.164330821
10	0.359747712	0.431246035	1.254362163	0.931895089	0.583535021	1.771063235	1.197637925	4.419823388	0.251466252	5.927316197	-0.035340583	-0.1623768	0.164330821
11	-0.03297875	-0.039502157	-0.11472094	-0.085062479	-0.048195114	-0.145778828	-0.09815822	-0.359131258	-0.1623768	-0.035340583	0.059391112	0.006239589	-0.035131666
12	-0.03297875	-0.039502157	-0.11472094	-0.085062479	-0.048195114	-0.145778828	-0.09815822	-0.359131258	-0.035340583	-0.1623768	0.006239589	0.059391112	-0.035131666
13	0.079390148	0.095048578	0.275801459	0.204303339	0.114688583	0.346764544	0.233411676	0.855278571	0.164330821	0.164330821	-0.035131666	-0.035131666	0.254161988

[Dongkeun Park (Former Postdocs, FBML)]

Solenoid Self Inductance

$$L = \mu_\circ a_1 N^2 \mathcal{L}(\alpha, \beta)$$

$\mathcal{L}(\alpha, \beta)$ a dimensionless parameter that depends on coil shape, α, β



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Selected Inductance Formulas

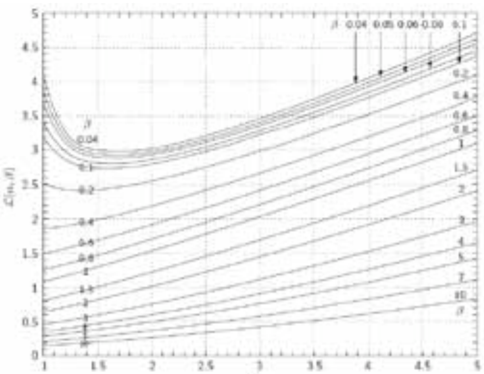
Solenoid (a_1, α, β, N) "Very Long" ($\beta >> 1$) "Pancake" ($\beta << 1$)	$L = \mu_\circ a_1 N^2 \mathcal{L}(\alpha, \beta)$  $L = \mu_\circ a_1 N^2 \left(\frac{\pi}{2\beta}\right)$ $L \simeq \mu_\circ a_1 N^2 \left(\frac{\alpha+1}{2}\right) \times$ $\left\{ \ln\left[\frac{4(\alpha+1)}{\alpha-1}\right]\left[1+\tfrac{1}{24}\frac{(\alpha-1)^2}{(\alpha+1)^2}\right] - \tfrac{1}{2}\left[1-\tfrac{43}{144}\frac{(\alpha-1)^2}{(\alpha+1)^2}\right] \right\}$
"Ideal" Dipole	$L_\ell = \tfrac{1}{8}\mu_\circ \pi N^2$ [H/m] *Independent of* R
"Ideal" Quadrupole	$L_\ell = \tfrac{1}{16}\mu_\circ \pi N^2$ [H/m] *Independent of* R
"Ideal" circular shape toroid	$L = \mu_\circ R N^2 \left[1 - \sqrt{1-\left(\frac{a}{R}\right)^2}\right] \simeq \mu_\circ a N^2 \left(\frac{a}{2R}\right)$ $(a \ll R)$
"Ideal" rectangular shape toroid	$L = \mu_\circ b N^2 \left[\frac{1}{\pi}\ln\left(\frac{R+a}{R-a}\right)\right] \simeq \mu_\circ b N^2 \left(\frac{2a}{\pi R}\right)$ $(a \ll R)$

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Illustration

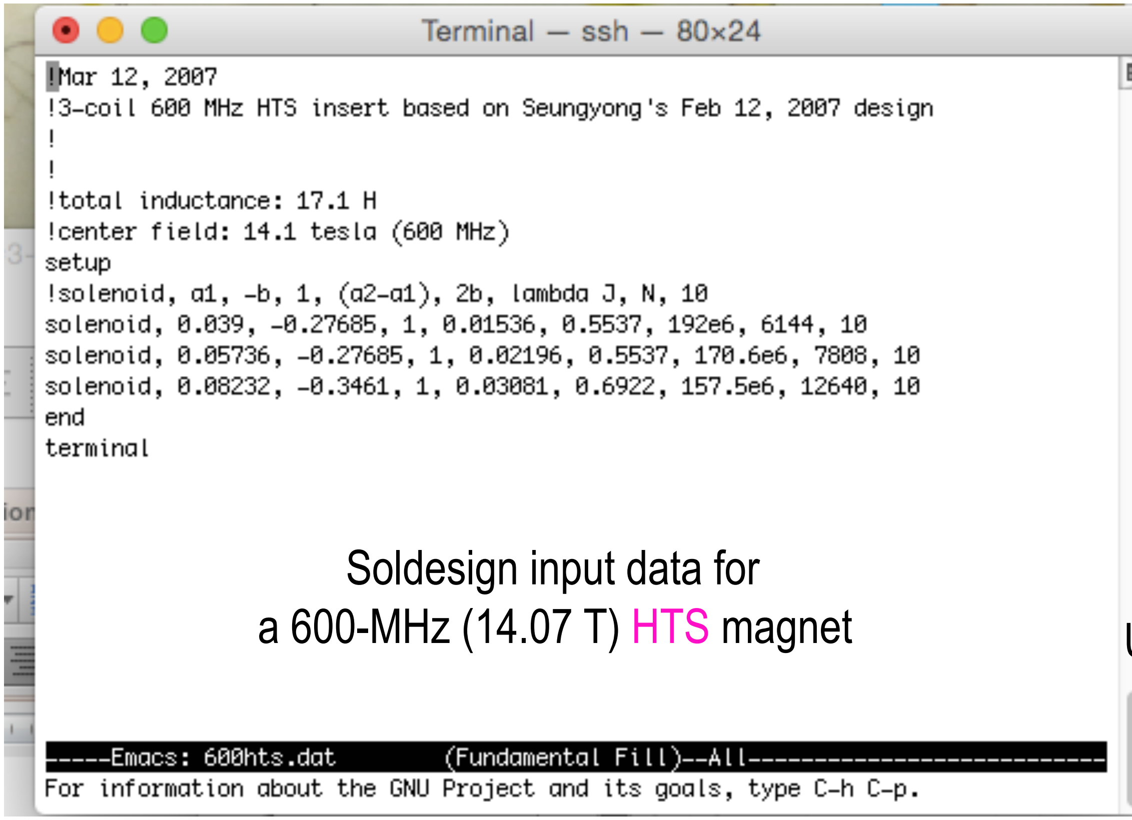
Compute *Iseult* L

$a_1 = 0.5$ m; $\alpha = 1.9$; $\beta = 3.1$; $N = 29{,}920$; $\mathcal{L}(\alpha, \beta) = 0.575$

$$L = \mu_\circ a_1 N^2 \mathcal{L}(\alpha, \beta) \quad = 321.6 \text{ H}$$

[323 H (Thierry Schild)]

Computation with MIT Soldesign (through Internet Access to MIT)



```
!Mar 12, 2007
!3-coil 600 MHz HTS insert based on Seungyong's Feb 12, 2007 design
!
!
!total inductance: 17.1 H
!center field: 14.1 tesla (600 MHz)
setup
!solenoid, a1, -b, 1, (a2-a1), 2b, lambda J, N, 10
solenoid, 0.039, -0.27685, 1, 0.01536, 0.5537, 192e6, 6144, 10
solenoid, 0.05736, -0.27685, 1, 0.02196, 0.5537, 170.6e6, 7808, 10
solenoid, 0.08232, -0.3461, 1, 0.03081, 0.6922, 157.5e6, 12640, 10
end
terminal

-----Emacs: 600hts.dat          (Fundamental Fill)--All---------------------------
For information about the GNU Project and its goals, type C-h C-p.
```


Soldesign input data for
a 600-MHz (14.07 T) HTS magnet

Using the inductance formula and figure given in Slide 25, compute inductances | Soldesign

Coil	a_1 [m]	$2b$ [mm]	$a_2 - a_1$ [mm]	α	β	N	$\mathcal{L}(\alpha, \beta)$	L [H]	L [H]
1	0.0390	553.70	0.01536	1.394	7.10	6144			0.488
2	0.05736	553.70	0.02196	1.383	4.83	7808			1.636
3	0.08232	692.20	0.03001	1.374	4.20	12640			6.920

Rendez-vous le 5 Juillet!